Department of Computer Science
Rochester Institute of Technology
Master's Thesis Proposal

# Computing Ramsey Numbers

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## Committee

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# Abstract

Ramsey Theory deals with the fact that true disorder is unattainable. More specifically it involves looking at large combinatorial objects for certain given smaller combinatorial objects which must be present. Ramsey numbers help to quantify many of the existing theories regarding Ramsey Theory. The Ramsey Number $R(s,t)$ is defined to be the smallest integer $n$ such that any graph on $n$ vertices must contain either a cliche of size $s$ or an independent subgraph of size $t$. This thesis will be aimed at determining the Ramsey Number $R(W_5, K_5)$, utilizing a combinatorial approach. $W_5$ is a wheel of size 5, which can be pictured as a wheel having four spokes or as a cycle of length 4, with all four vertices being adjacent to another vertex. $K_5$ is defined to be the independent graph on five vertices. Currently the bounds on this case our $27 \leq R(W_5, K_5) \leq 29$.


## Overview

Ramsey Theory was initiated by Professor Frank P. Ramsey in 1930. In his work "On a Problem of Formal Logic" he ecknowledges that Ramsey's theorem had independent interest, but was mainly concerned with its application to logic [7]. In fact, only 7 pages of this work actually serve to introduce Ramsey theory. Unfortunately, Professor Ramsey died shortly after publishing this important work. The field was subsequently developed by the notable mathematician Paul Erdos. In fact, in their Book "Ramsey Theory", its authors Ronald Graham, Bruce Rothschild, and Joel Spencer credit Erdos as the father of modern Ramsey Theory [7]. While there was progress in the field of Ramsey Theory, progress on determining the values of actual Ramsey numbers was quite slow until computer algorithms were recently employed. It has only been in the relatively short past that Ramsey Theory has emerged as a cohesive subdiscipline of combinatorial analysis [7].

To prove a lower bound for a given Ramsey number case it suffices to construct a graph on $n$ vertices that contains no such cliche of order $s$ or independent set of order $t$. Such a graph typically denoted $(s,t,n)$ is considered a critical graph on $n$ vertices. If a critical graph can be constructed than a lower bound of $n+1$ will be established. Working with upper bounds is a great deal more difficult since they can only be lowered by means of a constructive proof that provides an argument, given certain properties of the graph, against there be any such graph on $n$ vertices, which doesn't contain either a cliche of order $s$ or an independent set of order $t$.

In his inital paper Ramsey gives an upper bound of $R(n) \leq 2^{n(n-1)/2}$ and then improves it to $R(n) \leq n!$. He goes on to say that this value is still much too high [7]. Let me now introduce a trivial relation over Ramsey numbers: $R(k,l) \leq R(k,l-1) + R(k-1,l)$, now if both $R(k,l-1)$, $R(k-1,l)$ are even that this relation becomes a strict inequality. Notice that this relationship when expressed recursively will yield a simple formula for determining weak upper bounds. Now for my specific case we have,

$R(W_5, K_5) \leq R(C_4, K_5) + R(W_5, K_4)$

Then since, $R(C_4, K_5) = 14$ and $R(W_5, K_4) = 17$

We get, $R(W_5, K_5) \leq 14 + 17$ and finally, $R(W_5, K_5) \leq 31$

But once again notice that the current best upper bound for this case is 29. [Hendry] is credit with producing this upper bound. Unfortunately the proof of this upper bound was never published and is not currently available.

By taking a disjoint union of two critical graphs we arrive at $R(k,p) \geq s$ and $R(k,q) \geq t$ implies $R(k, p+q-1) \geq s+t-1$ [3]. This has been improved to yield better lower bounds $R(k, p+q-1) \geq s+t+k-3$. Then since $R(W_5, K_3) = 11$ we have $R(W_5, K_5) \geq 22$. Once more notice that the current lower bound for this case is 27. Thus the naive bounds for this case our 22 and 31 respectable so already the bounds for this case have been improved significantly.

## Functional Specification

My approach in this thesis will be two fold. I will primarily be developing a gluing algorithm in an attempt to glue together $(C_4, K_5)$ and $(W_5, K_4)$ graphs to construct an exhaustive set of $(W_5, K_5)$ graphs. Next utilizing one vertex extension algorithms I will attempt to determine the size of a graph at which no more critical graphs exist. Secondly, I will be attempting to produce construction proofs to further lower the current best upper bounds for this case. In fact, I will begin my analysis of the upper bounds by attempting to recreate the proofs that originally lowered the current upper bound.

I will be making extensive use of the gtools package made availabe by Brendan McKay. It is a package, which can be used generate and check isomorphism of graphs, written in a highly portable subset of C. It effectively abstracts away the storing and manipulating of individual graphs. Also take a moment to appreciate the genius behind checking isomorphism. Given Brendan McKay software package I am guaranteed to be working with graphs which are not isomorphic to each other.

Of specific use to my efforts was the program's geng and shortg contained in the nauty software package. Geng is a powerful program for generating all possible graphs for a given small size of vertices. Shortg is a program which will remove all isomorphic graphs from a given graph file. All graphs were stored utilizing a compact notation known as graph6 format. Graph6 format writes only the upper right triangle of the adjaceny matrix as a bit vector $x$ of length $n(n-1)/2$. The format also includes the size of the graph in its output. The following ordering is used: $(0,1), (0,2), (1,2), (0,3), (1,3), (2,3), ..., (n-1, n)$. Then the graph is represented as N(n) R(x). Consider the following example to illustrate graph6 format:

Suppose n=5 and G has edges 0-2, 0-4, 1-3 and 3-4.

x = 0 10 010 1001

Then N(n) = 68 and R(x) = R(010010 100100) = 81 99. So, the graph is 68 81 99.

To perform one-vertex extensions all possible edge combinations for a new edge were added to the adjacent matrix. Thus, when adding one vertex to a graph containing 10 vertices there are $2^{10}$ possible edge configurations that need to be explored. Of course since many of these new graphs are isomorphic the program shortg was then run to remove all isomorphs.

Now let me introduce some important notation. If $F$ is a graph, $v \in VF$ and $W \subseteq VF$, then $N_F(v, W) = w \in W \mid vw \in EF$. The subgraph of $F$ induced by $W$ will be denoted by $F[W]$. The special case $F[VF - v]$ will also be written as $F - v$. Suppose that $x$ is a vertex of $F$. Define the induced subgraphs $G_x = G_x(F) = F[N_F(x, VF)]$ and $H_x = H_x(F) = F[VF - N_F(x, VF) - x]$. If $F$ is a $(W_5, K_5, X) - graph$, and $x \in VF$ has degree $d$, it is obvious that $G_x$ is a $(C_4, K_5, d) - graph$ and $H_x$ is a $(W_5, K_4, X - 1 - d) - graph$.

# Schedule

Finish proposal 3/20/03

Enumerate all $(C_4, K_5)$ and $(W_5, K_4)$ graphs 4/1/03

Develop gluing algorithm 4/7/03

Implement gluing algorithm 4/28/03

Run experiment 5/5/03

Analyize experimental results 5/23/03

Finish write up 6/23/03

Thesis defense 7/1/03

# List of Deliverables

Formal Thesis Document (PDF) Description of algorithms utilized Progress Recorded

Various Program Source Codes Gluing Algorithm Construction of Critical Graphs

## Status of Work at Present

Currently I have already enumerated all $(C_4, K_5)$ graphs and all $(W_5, K_4, n)$ graphs for values of $n < 11$. There are over 1 million $(W_5, K_4, 10) - graphs$ and as such any attempt at one-vertex extensions is unfeasible. As a result I will enumerate $(W_5, K_4) - graphs$ for vertices between 11 and 16 by gluing together smaller $(C_4, K_4)$ and $(W_5, K_3)$ graphs. I have familiarized myself with the gluing process and feel confident that soon I will develop an appropriate algorithm. I recognized a subgraph of $C_4$ by simply checking every two vertices to see if any two vertices are adjacent to the same two vertices. To recognize both $K_4$ and $K_5$ I first enumerated all $K_3$ graphs and then check their intersections to see if any resulted in a $K_5$ or $K_4$. Finally to recognize $W_5$ I used my previous technique for determining $C_4$, for once I have recognized $C_4$ I simply check if any vertice in the graph is adjacent to each vertex in $C_4$. Below are graphs illustrating the critical graphs for both $(C_4, K_4)$ and $(W_5, K_3)$ respectable.

| vertices | num graphs | time (sec) |
|---|---|---|
| 1 | 1 | 0.00 |
| 2 | 2 | 0.00 |
| 3 | 4 | 0.00 |
| 4 | 11 | 0.00 |
| 5 | 34 | 0.00 |
| 6 | 156 | 0.01 |
| 7 | 1044 | 0.03 |
| 8 | 12346 | 0.42 |
| 9 | 274668 | 10.45 |
| 10 | 12005168 | 450.85 |

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| e | | | | | | | | | | | | | | |
| 0 | 1 | 1 | 1 | 1 | | | | | | | | | | 4 |
| 1 | | 1 | 1 | 1 | 1 | | | | | | | | | 4 |
| 2 | | | 1 | 2 | 2 | 1 | | | | | | | | 6 |
| 3 | | | 1 | 3 | 4 | 4 | 1 | | | | | | | 13 |
| 4 | | | | 1 | 5 | 7 | 3 | 1 | | | | | | 17 |
| 5 | | | | | 4 | 11 | 10 | 2 | | | | | | 27 |
| 6 | | | | | 1 | 11 | 22 | 9 | 1 | | | | | 44 |
| 7 | | | | | | 4 | 27 | 27 | 4 | | | | | 62 |
| 8 | | | | | | | 17 | 53 | 16 | 1 | | | | 87 |
| 9 | | | | | | | 5 | 62 | 50 | 5 | | | | 122 |
| 10 | | | | | | | | 31 | 108 | 18 | 1 | | | 158 |
| 11 | | | | | | | | 5 | 130 | 55 | 3 | | | 193 |
| 12 | | | | | | | | | 66 | 138 | 10 | 1 | | 215 |
| 13 | | | | | | | | | 10 | 200 | 32 | 1 | | 243 |
| 14 | | | | | | | | | | 126 | 75 | 3 | | 204 |
| 15 | | | | | | | | | | 29 | 129 | 9 | | 167 |
| 16 | | | | | | | | | | 2 | 139 | 15 | | 156 |
| 17 | | | | | | | | | | | 59 | 22 | | 81 |
| 18 | | | | | | | | | | | 9 | 33 | | 42 |
| 19 | | | | | | | | | | | | 25 | | 25 |
| 20 | | | | | | | | | | | | 14 | | 14 |
| 21 | | | | | | | | | | | | 3 | | 3 |
| 22 | | | | | | | | | | | | | | 0 |
| 23 | | | | | | | | | | | | | | 0 |
| 24 | | | | | | | | | | | | | 1 | 1 |
| total: | 1 | 2 | 4 | 8 | 17 | 38 | 85 | 190 | 385 | 574 | 457 | 126 | 1 | 1888 |

| n | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| e | | | | | | | | | | |
| 0 | 1 | 1 | 1 | | | | | | | |
| 1 | | 1 | 1 | 1 | | | | | | |
| 2 | | | 1 | 2 | 2 | | | | | |
| 3 | | | 1 | 3 | 4 | 1 | | | | |
| 4 | | | | 2 | 6 | 3 | | | | |
| 5 | | | | 1 | 6 | 12 | 1 | | | |
| 6 | | | | 1 | 6 | 20 | 4 | | | |
| 7 | | | | | 4 | 24 | 25 | 1 | | |
| 8 | | | | | 2 | 24 | 51 | 4 | | |
| 9 | | | | | 1 | 21 | 102 | 25 | 1 | |
| 10 | | | | | | 9 | 126 | 85 | 2 | |
| 11 | | | | | | 3 | 104 | 245 | 14 | |
| 12 | | | | | | 1 | 63 | 501 | 58 | 2 |
| 13 | | | | | | | 23 | 603 | 229 | 15 |
| 14 | | | | | | | 8 | 499 | 771 | 142 |
| 15 | | | | | | | 2 | 318 | 1819 | 362 |
| 16 | | | | | | | | 128 | 2844 | 2050 |
| 17 | | | | | | | | 29 | 3274 | 7845 |
| 18 | | | | | | | | 5 | 2257 | 25629 |
| 19 | | | | | | | | | 1065 | 69728 |
| 20 | | | | | | | | | 299 | 119282 |
| 21 | | | | | | | | | 43 | 202581 |
| 22 | | | | | | | | | 4 | 263529 |
| 23 | | | | | | | | | | 267414 |
| 24 | | | | | | | | | | 205841 |
| 25 | | | | | | | | | | 111435 |
| 26 | | | | | | | | | | 45897 |
| 27 | | | | | | | | | | 14177 |
| 28 | | | | | | | | | | 3149 |
| 29 | | | | | | | | | | 485 |
| 30 | | | | | | | | | | 46 |
| 31 | | | | | | | | | | 3 |
| total: | 1 | 2 | 4 | 10 | 31 | 118 | 509 | 2443 | 12680 | 1339612 |

## References


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[7] R.L. Graham, B.L. Rothschild and J.H. Spencer, *Ramsey Theory*, John Wiley & Sons, 1990.