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Analysis of Double Ring Resonators using Method of Equating Fields

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Abstract

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Optical ring resonators have the potential to be integral parts of large scale photonic circuits. My thesis theoretically analyzes parallel coupled double ring resonators (DRRs) in detail. The analysis is performed using the method of equating fields (MEF) which provides an in depth understanding about the transmitted and reflected light paths in the structure. Equations for the transmitted and reflected fields are derived; these equations allow for unequal ring lengths and coupling coefficients. Sanity checks including comparison with previously studied structures are performed in the final chapter in order to prove the correctness of the obtained results.

Acknowlegements

Let me begin by thanking almighty Allah for giving me the opportunity to return to college and continue my education after being a mother.

I would like to dedicate my sincere gratitude to my thesis advisor Prof. Drew Maywar for his continuous support and guidance throughout my graduate study at RIT. I will always be grateful to him for helping me identify my research interests and inspiring me to take up the challenge of doing a thesis. His dedication and sincerity has always been an inspiration to me. I consider my research experience with him as a valuable asset as I embark on my doctoral studies in the next academic year.

Special thanks to Prof. Indelicato and Prof. Carlos Diaz Acosta for taking time from their busy schedule to be on my thesis committee. Thanks again to Prof. Indelicato for giving me the opportunity to work as his teaching assistant.

Next I would like to thank my husband without whose support none of this would have been possible. Thanks to him for believing in me. Thanks to my son for letting me be with my books even when he needed me.

Thanks to my family, especially my father who brought up his daughters with their education as the top priority. Thanks to my mother, for her prayers.

Finally a special thanks to my sis and bro-in law and to my little niece who brings a smile to my face even when I am stressed out with my work.

Dedication

To my ikka.. thank you.. for all that you are and for all that you do

To my Rehaan, you complete me

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Chapter 1

Ring Resonator, Single Bus Waveguide

1.1 Introduction

The study and characterization of optical ring resonators have developed as a major area of research due to their potential to become integral components of large scale photonic circuits. Ring resonator devices are comprised of bus waveguides and rings made out of bent optical waveguides. The path where the bus and the ring come close to one another can be regarded as a coupler, where the interaction path (coupler length) of the optical field in the ring and bus waveguide can be short. The principle of operation of coupling can be based on evanescent-field coupling, although multimode interference also works.

Different configurations of multiple ring resonators are found to exhibit filtering characteristics which are desirable in WDM applications. They find applications as wavelength selective filters such as add-drop filters or band rejection filters in photonic circuits. The ease of fabrication into intergrated circuits has been recognized as the major advantage of optical filters based on ring resonators.

1.2 Background

A theoretical analysis of a basic ring resonator configuration consisting of one waveguide and a single ring was presented by A.Yariv in 2000. The fundamental working equations describing the general behavior of a basic resonator filter were derived and the applicability as filters were discussed. The analysis was performed using matrix relations considering single polarization under lossless conditions [1].

In 2004, a new type of reflector consisting of a circular array of micro ring resonators coupled to a waveguide was proposed and analyzed [2]. The transmittance and reflectance of the structure was computed using transfer matrix analysis. It was proved that the structure acted as an all pass filter for an even number of rings. Although different geometries of multiple ring resonators have been studied in detail since then, almost all of these prior works are based on matrix analysis.

Double ring-resonators (DRRs) composed of just two micro-rings and a straight waveguide which are coupled to each other was studied for the first time in 2005 and was proved to be a good reflector [4]. Scattering matrix formalism was used to prove that a range of reflectivity profiles can be obtained by tuning the coupling coefficients. Another study of DRRs with rings having slightly different radii was reported a year later [3] [5].

DRRs have many advantages over the previously reported structures. As it consists of only two rings, which simplifies the fabrication process, this resonator can replace distributed Bragg reflectors used in realizing tunable laser diodes [9] [6] [7] [8]. In order to exploit all the functionalities of this structure, a deep understanding of its characteristics is necessary.

The method of equating fields (MEF) provides a deeper understanding of the light path and the propagation of the electric fields taking place in the structure. The method yields several equations consisting of components representing the path followed by light as it travels along the structure. The results of the analysis includes equations for transmission and reflection coefficients $\tilde{t}$, $\tilde{r}$. The circumference of the rings L_1 is not assumed to be equal to L_2 and coupling coefficients ϵ_1 is not equal to ϵ_2 in the solution. An analysis is performed on simpler structures in the early chapters in order to compare with the final equations for the DRR as a sanity check in the final chapter.

1.3 Single Ring Resonator

In the basic configuration, a ring resonator consists of a single bus waveguide coupled to a single ring made of a bent waveguide as shown in Figure 1.1. The arrow indicates the direction of propagation of light and the thin line indicates the coupling region.

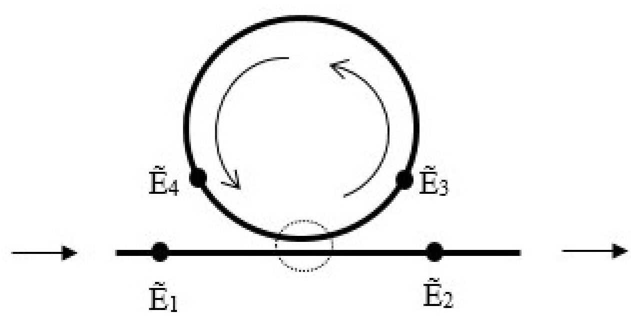

Figure 1.1: Ring resonator with single ring and single bus waveguide

When light through the bus waveguide enters the region where the ring is close to the bus waveguide it is coupled to the ring resonator through evanescent coupling. The light entering the ring travels along the circumference of the ring and enters the coupling region again. Exchange of power takes place at the coupling region due to the interaction between the optical field in the ring and the bus waveguide. Here $\tilde{E}_1$ is the input electric field to the coupler and $\tilde{E}_4$ is the input from the ring to the coupler. $\tilde{E}_3$ is the output electric field from the coupler along the ring and $\tilde{E}_2$ is the output from the coupling region along the bus waveguide.

1.4 Output Electric Field

The output electric field $\tilde{E}_2$ is given as

$$\tilde{E}_2 = \tilde{E}_1 \, \sqrt{1-\epsilon} + \tilde{E}_4 \; i \, \sqrt{\epsilon}, \tag{1.1}$$

$$\tilde{E}_4 = \tilde{E}_3 \; e^{i\beta L}, \tag{1.2}$$

$$\tilde{E}_3 = i\sqrt{\epsilon} \; \tilde{E}_1 + \sqrt{1-\epsilon} \; \tilde{E}_4, \tag{1.3}$$

where L is the circumference of the entire ring.

Combining Equations (1.2) and (1.3) yields

$$\tilde{E}_4 = i\sqrt{\epsilon} \; e^{i\beta L} \; \tilde{E}_1 + \sqrt{1-\epsilon} \; e^{i\beta L} \; \tilde{E}_4,$$

$$(1 - \sqrt{1-\epsilon} \; e^{i\beta L}) \; \tilde{E}_4 = i\sqrt{\epsilon} \; e^{i\beta L} \; \tilde{E}_1,$$

$$\tilde{E}_4 = \frac{i\sqrt{\epsilon} \; e^{i\beta L}}{(1 - \sqrt{1-\epsilon} \; e^{i\beta L})} \; \tilde{E}_1. \tag{1.4}$$

The equation for $\tilde{E}_2$ becomes

$$\tilde{E}_2 = \tilde{E}_1[\sqrt{1-\epsilon} + \frac{i\ \sqrt{\epsilon}\ i\sqrt{\epsilon}\ e^{i\beta L}}{1-\sqrt{1-\epsilon}\ e^{i\beta L}}],$$

$$\tilde{E}_2 = \tilde{E}_1[\frac{\sqrt{1-\epsilon} - (1-\epsilon)\ e^{i\beta L} - \epsilon\ e^{i\beta L}}{1-\sqrt{1-\epsilon}\ e^{i\beta L}}],$$

$$\tilde{E}_2 = \tilde{E}_1[\frac{\sqrt{1-\epsilon} - e^{i\beta L} + \epsilon\ e^{i\beta L} - \epsilon\ e^{i\beta L}}{1-\sqrt{1-\epsilon}\ e^{i\beta L}}],$$

$$\tilde{E}_2 = \tilde{E}_1[\frac{\sqrt{1-\epsilon} - e^{i\beta L}}{1-\sqrt{1-\epsilon}\ e^{i\beta L}}],$$

$$\tilde{E}_2 = \tilde{E}_1[\frac{\sqrt{1-\epsilon}\ e^{-i\beta L} - 1}{1-\sqrt{1-\epsilon}\ e^{i\beta L}}]\ e^{i\beta L},$$

$$\tilde{E}_2 = \tilde{E}_1[\frac{1-\sqrt{1-\epsilon}\ e^{-i\beta L}}{1-\sqrt{1-\epsilon}\ e^{i\beta L}}]\ e^{i\beta L}\ e^{i\pi}. \tag{1.5}$$

We know

$$\tilde{E}_2 = \tilde{t}\ \tilde{E}_1$$

Therefore the transmission coefficient $\tilde{t}$ is

$$\tilde{t} = [\frac{1 - \sqrt{1-\epsilon}\ e^{-i\beta L}}{1 - \sqrt{1-\epsilon}\ e^{i\beta L}}]\ e^{i\beta L}\ e^{i\pi}. \tag{1.6}$$

1.5 Transmittivity

Transmittivity of a medium is defined as the ratio of transmitted power to the incident power:

$$T = |\frac{\tilde{E}_2}{\tilde{E}_1}|^2 = |\tilde{t}|^2. \tag{1.7}$$

The above equation is rewritten as,

$$T = \frac{(\sqrt{1-\epsilon} - e^{i\beta L})(\sqrt{1-\epsilon} - e^{-i\beta L})}{(1 - \sqrt{1-\epsilon}\ e^{i\beta L})(1 - \sqrt{1-\epsilon}\ e^{-i\beta L})},$$

$$T = \frac{(1-\epsilon) + (1 - \sqrt{1-\epsilon}\ e^{i\beta L}) - (\sqrt{1-\epsilon}\ e^{-i\beta L})}{1 + (1-\epsilon) - (\sqrt{1-\epsilon}\ e^{i\beta L}) - (\sqrt{1-\epsilon}\ e^{-i\beta L})},$$

$$T = \frac{2 - \epsilon - 2\sqrt{1-\epsilon}\ \cos(\beta L)}{2 - \epsilon - 2\sqrt{1-\epsilon}\ \cos(\beta L)} = 1. \tag{1.8}$$

Here, transmittivity is equal to one which means that the total power entering and leaving the ring resonator are equal. The unity value of transmittivity shows that under loss-less conditions, there are no spectral features or no resonances and all the light that enters the coupling region of the ring resonator passes through it.

1.5.1 Transmittivity plot for single ring resonator as a function of wavelength

A sanity check on the derived equations is performed by comparing the graphs obtained by plotting transmittivity T based on both analytical and numerical techniques.

Transmittivity plot based on analytic expression T_A

$$T_A = \frac{(\sqrt{1-\epsilon} - e^{i\beta L})(\sqrt{1-\epsilon} - e^{-i\beta L})}{(1-\sqrt{1-\epsilon}\ e^{i\beta L})(1-\sqrt{1-\epsilon}\ e^{-i\beta L})}.$$

The figure shows the transmittivity of a single ring resonator plotted as a function of the normalised product βL where β is the wavenumber and L is the circumference of the ring.

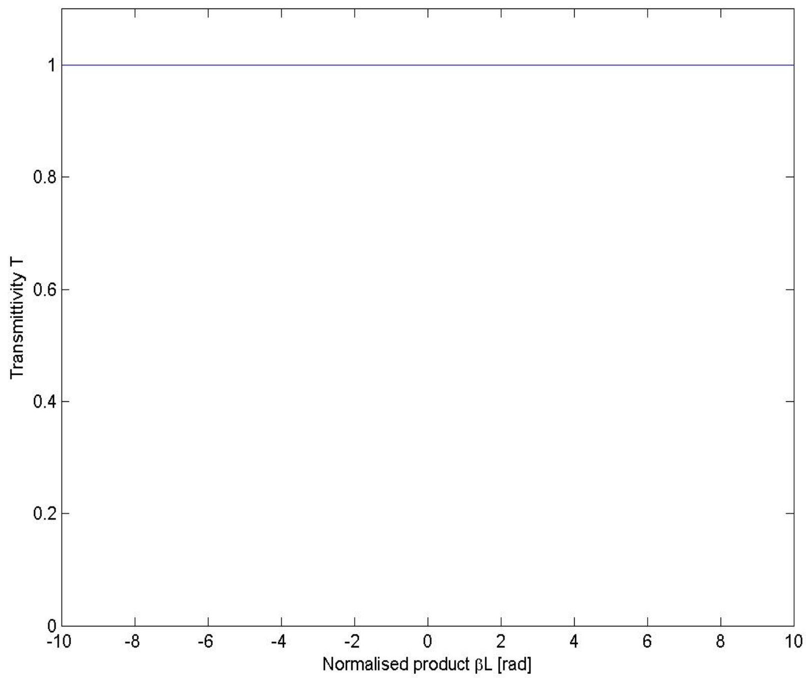

Figure 1.2: Transmittivity plot of a ring resonator as a function of wavenumber under zero loss condition.

The equation derived for transmittivity is used for plotting the graph. We can see that under lossless conditions, the transmittivity is equal to one which indicates that the total power entering and leaving the coupling region are equal.

Transmittivity plot based on numerical expression T_N

We know transmittivity is the product of the transmission coefficient $\tilde{t}$ and its conjugate $\tilde{t}^{\star}$.

$$T_N = \tilde{t} \ \times \ \tilde{t}^{\star}.$$

where,

$$\tilde{t} = [\frac{1 - \sqrt{1-\epsilon} \ e^{-i\beta L}}{1 - \sqrt{1-\epsilon} \ e^{i\beta L}}] \ e^{i\beta L} \ e^{i\pi}.$$

The figure shows the response obtained when $\tilde{t} \times \tilde{t}^{\star}$ is plotted against βL. We can see that the plots from both the analytic and numerical expressions are the same.

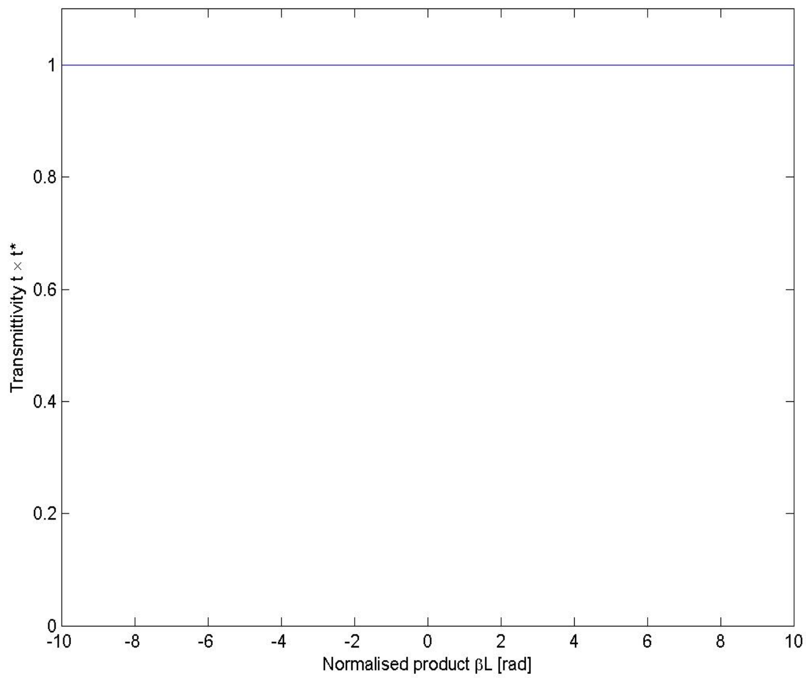

Figure 1.3: Transmittivity plot of a ring resonator where tansmittivity is calculated as $\tilde{t} \times \tilde{t}^{\star}$

1.6 Phase Transfer Function

The phase transfer function of the ring resonator is composed of three parts:

$$\phi_T = arg\left(\frac{\tilde{E}_2}{\tilde{E}_1}\right),$$

$$\phi_T = x + \beta L + \pi,$$

where x is the phase of the bracketed term B in equation (1.5), where

$$B = \frac{1 - \sqrt{1-\epsilon}\ e^{-i\beta L}}{1 - \sqrt{1-\epsilon}\ e^{i\beta L}}.$$

The phase x can be determined by multiplication of B and a unity ratio whose denominator turns the resulting denominator into a real quantity.

$$B = \frac{(1 - \sqrt{1-\epsilon}\ e^{-i\beta L})}{(1 - \sqrt{1-\epsilon}\ e^{i\beta L})} * \frac{(1 - \sqrt{1-\epsilon}\ e^{-i\beta L})}{(1 - \sqrt{1-\epsilon}\ e^{-i\beta L})}$$

$$B = \frac{(1 - \sqrt{1-\epsilon}\ e^{-i\beta L})^2}{(2 - \epsilon - 2\sqrt{1-\epsilon}\cos(\beta L))},$$

$$B = \frac{(1 - \sqrt{1-\epsilon}\ \cos(\beta L) + i\ \sqrt{1-\epsilon}\ \sin(\beta L))^2}{D},$$

where,

$$D = 2 - \epsilon - 2\sqrt{1-\epsilon}\cos(\beta L).$$

B can be written as $(a + i\ b)^2$, where

$$a = \frac{1 - \sqrt{1-\epsilon}\ \cos(\beta L)}{\sqrt{D}},$$
$$b = \frac{\sqrt{1-\epsilon}\ \sin(\beta L)}{\sqrt{D}}.$$

We desire to know the phase x of B, where

$$B = (a + ib)^2,$$

but x is related to the phase of

$$\sqrt{B} = a + ib.$$

In general, the Cartesian and phasor representations of a complex quantity are given as:

$$a + ib = m\ e^{i\theta},$$
$$(a + ib)^2 = (m\ e^{i\theta})^2 = m^2\ e^{i2\theta} = G\ e^{ix},$$

where,

$$x = 2\theta.$$

Thus, the square of a complex quantity is equivalent to a new complex quantity whose phase x is twice that of the original quantity θ. The phase x of

B can be found as

$$\tan(\theta) = \frac{b}{a} = \frac{\sqrt{1-\epsilon}\ sin(\beta L)}{1-\sqrt{1-\epsilon}\ \cos(\beta L)},$$

$$\theta = \tan^{-1}(\frac{\sqrt{1-\epsilon}\ \sin(\beta L)}{1-\sqrt{1-\epsilon}\ \cos(\beta L)}),$$

$$x = 2\theta = 2\tan^{-1}\left(\frac{\sqrt{1-\epsilon}\ \sin(\beta L)}{1-\sqrt{1-\epsilon}\ \cos(\beta L)}\right). \tag{1.9}$$

The phase transfer function is

$$\phi_T = x + \pi + \beta L.$$

Substituting x from equation (1.8), yields

$$\phi_T = 2\tan^{-1}\left(\frac{\sqrt{1-\epsilon}\ \sin(\beta L)}{1-\sqrt{1-\epsilon}\ \cos(\beta L)}\right) + \pi + \beta L. \tag{1.10}$$

1.6.1 Phase Transfer Function plot for single ring resonator as a function of wavelength

A sanity check on the derived equation is performed by comparing the graphs obtained by plotting phase transfer function ϕ_T based on both analytic expression ϕ_{T_A} and numerical expression ϕ_{T_N} against wavenumber.

Phase plot based on analytic expression ϕ_{T_A}

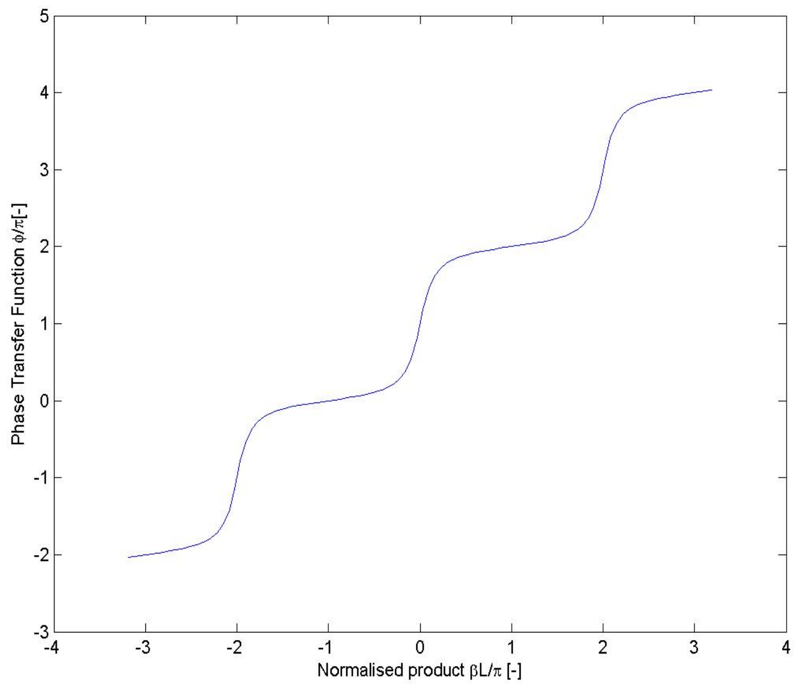

Figure 1.4: Phase plot of a single ring resonator as a function of wavenumberunder zero loss condition

The figure shows the phase transfer function of a single ring resonator based on the analytic expression (1.9), plotted as a function of the normalised product βL where β is the wavenumber and L is the circumference of the ring. We can see that the phase increases in a staircase-type fashion.

Phase Transfer function based on numerical expression ϕ_{T_N}

We know,

$$\phi_{T_N} = angle(\tilde{t})$$

where,

$$\tilde{t} = [\frac{1-\sqrt{1-\epsilon}\ e^{-i\beta L}}{1-\sqrt{1-\epsilon}\ e^{i\beta L}}]\ e^{i\beta L}\ e^{i\pi}.$$

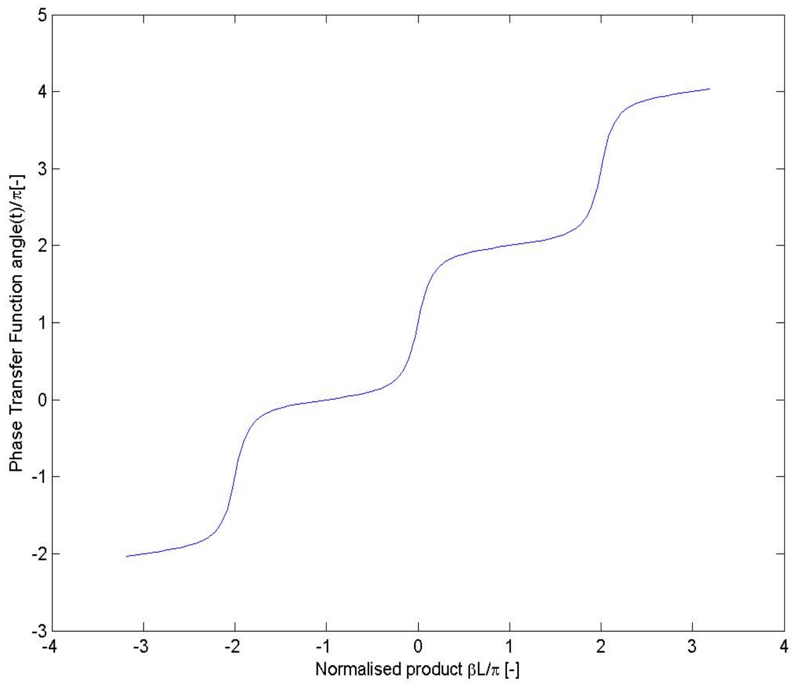

Figure 1.5: Phase plot of a single ring resonator where phase is calculated as $angle(\tilde{t})$

The figure shows the response obtained when $angle(\tilde{t})$ is plotted against βL. We can see that the plots obtained using both analytic and numerical expressions are same.

Chapter 2

Uncoupled Double Ring Resonator,Single Bus Waveguide

Double Ring Resonators are coupled ring resonator devices consisting of two ring waveguides coupled to one or more bus waveguides. The Double Ring Resonator shown in Figure 2.1 is comprised of two ring waveguides of circumference L_1 and L_2 coupled to a same single bus waveguide. The rings are spaced apart and are not coupled to each other. The space where the rings are close to the bus waveguide creates two coupling regions separated by a distance L_s along the bus waveguide.

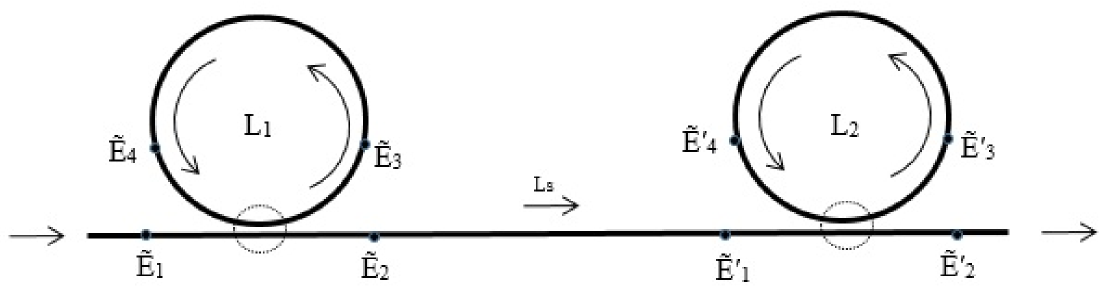

Figure 2.1: Uncoupled Double Ring Resonator with single bus waveguide

Light entering the bus waveguide is coupled to the first ring through evanescent coupling and travels along the circumference L_1 of the ring before entering the same coupling region again. Transfer of power takes place at the coupling region. The light leaving the region travels a distance of L_s along the bus waveguide to enter the region where the second ring is coupled to the bus waveguide. Through evanescent coupling light enters the second ring, travels along its circumference L_2 and re enters the coupling region.

Here $\tilde{E_1}$ is the input electric field to the coupler and $\tilde{E_4}$ is the input from the first ring to the coupler. $\tilde{E_3}$ is the output electric field from the coupler along the first ring and $\tilde{E_2}$ is the output from the first coupling region along the bus waveguide. $\tilde{E_1'}$ is the input electric field to the second coupling region and $\tilde{E_4'}$ is the input from the second ring to the coupler. $\tilde{E_3'}$ is the output electric field from the coupler along the second ring and $\tilde{E_2'}$ is the output from the second coupling region along the bus waveguide.

2.1 Output Electric Field

The electric fields $\tilde{E}_1'$,$\tilde{E}_2$ and $\tilde{E}_2'$ are given as,

$$\tilde{E}_2 = \tilde{t}_{L_1}\ \tilde{E}_1$$

$$\tilde{E}_1' = \tilde{t}_{L_s}\ \tilde{E}_2$$

$$\tilde{E}_2' = \tilde{t}_{L_2}\ \tilde{E}_1'$$

where $\tilde{t}_{L_1}$, $\tilde{t}_{L_s}$ and $\tilde{t}_{L_2}$ are transmission coefficients of the first coupling region, the space between the two coupling regions and the second coupling region respectively. Therefore, $\tilde{E}_2'$ is

$$\tilde{E}_2' = t\ \tilde{E}_1. \tag{2.1}$$

where

$$\tilde{t} = \tilde{t}_{L_1}.\ \tilde{t}_{L_s}.\ \tilde{t}_{L_2}. \tag{2.2}$$

2.1.1 Bus Wave guide transmission Coefficient

The equation for electric field $\tilde{E}_1'$ is

$$\tilde{E}_1' = \tilde{E}_2\ e^{i\beta L_s}.$$

Therefore

$$\tilde{t}_{L_s} = e^{i\beta L_s}. \tag{2.3}$$

2.1.2 Ring Resonator transmission Coefficient

We have

$$\tilde{E}_2 = \tilde{E}_1 \left(\frac{1 - \sqrt{(1-\epsilon_1)}\ e^{-i\beta L_1}}{1 - \sqrt{(1-\epsilon_1)}\ e^{i\beta L_1}} \right)\ e^{i\beta L_1}\ e^{i\pi}.$$

Therefore

$$\tilde{t}_{L_1} = \left(\frac{1 - \sqrt{(1-\epsilon_1)}\ e^{-i\beta L_1}}{1 - \sqrt{(1-\epsilon_1)}\ e^{i\beta L_1}} \right)\ e^{i\beta L_1}\ e^{i\pi}. \tag{2.4}$$

Likewise,

$$\tilde{E}_2' = \tilde{E}_1' \left(\frac{1 - \sqrt{(1-\epsilon_2)}\ e^{-i\beta L_2}}{1 - \sqrt{(1-\epsilon_2)}\ e^{i\beta L_2}} \right)\ e^{i\beta L_2}\ e^{i\pi}. \tag{2.5}$$

Therefore

$$\tilde{t}_{L_2} = \left(\frac{1 - \sqrt{(1-\epsilon_2)}\ e^{-i\beta L_2}}{1 - \sqrt{(1-\epsilon_2)}\ e^{i\beta L_2}} \right)\ e^{i\beta L_2}\ e^{i\pi}. \tag{2.6}$$

2.1.3 Total transmission Coefficient

$$\tilde{t} = \tilde{t}_{L_1}.\ \tilde{t}_{L_s}.\ \tilde{t}_{L_2}.$$

Substituting the values of $\tilde{t}_{L_1}$,$\tilde{t}_{L_s}$ and $\tilde{t}_{L_2}$ in the above equation yields

$$\tilde{t} = \left(\frac{1 - \sqrt{(1-\epsilon_1)}\ e^{-i\beta L_1}}{1 - \sqrt{(1-\epsilon_1)}\ e^{i\beta L_1}} \right)\ e^{i\beta L_1}\ e^{i\pi}\ e^{i\beta L_s} \left(\frac{1 - \sqrt{(1-\epsilon_2)}\ e^{-i\beta L_2}}{1 - \sqrt{(1-\epsilon_2)}\ e^{i\beta L_2}} \right)\ e^{i\beta L_2}\ e^{i\pi},$$

$$\tilde{t} = \left(\frac{1 - \sqrt{(1-\epsilon_1)}\ e^{-i\beta L_1}}{1 - \sqrt{(1-\epsilon_1)}\ e^{i\beta L_1}} \right)\ e^{i\beta(L_1+L_s+L_2)}\ e^{i2\pi} \left(\frac{1 - \sqrt{(1-\epsilon_2)}\ e^{-i\beta L_2}}{1 - \sqrt{(1-\epsilon_2)}\ e^{i\beta L_2}} \right),$$

$$\tilde{t} = \left(\frac{1 - \sqrt{(1-\epsilon_1)}\ e^{-i\beta L_1}}{1 - \sqrt{(1-\epsilon_1)}\ e^{i\beta L_1}} \right)\ e^{i\beta(L_1+L_s+L_2)}\ \left(\frac{1 - \sqrt{(1-\epsilon_2)}\ e^{-i\beta L_2}}{1 - \sqrt{(1-\epsilon_2)}\ e^{i\beta L_2}} \right). \tag{2.7}$$

2.2 Transmittivity

Transmittivity is the ratio of transmitted power to incident power:

$$T = |\frac{\tilde{E}'_2}{\tilde{E}_1}|^2. \tag{2.8}$$

The total transmittivity is related to the constituent transmittivities as follows

$$T = T_2.T_s.T_1 \tag{2.9}$$

where

$$T_1 = \frac{(\sqrt{1-\epsilon_1} - e^{i\beta L_1})\,(\sqrt{1-\epsilon_1} - e^{-i\beta L_1})}{(1-\sqrt{1-\epsilon_1}\; e^{i\beta L_1})\,(1-\sqrt{1-\epsilon_1}\; e^{i\beta L_1})} = 1,$$

$$T_2 = \frac{(\sqrt{1-\epsilon_2} - e^{i\beta L_2})\,(\sqrt{1-\epsilon_2} - e^{-i\beta L_2})}{(1-\sqrt{1-\epsilon_2}\; e^{i\beta L_2})\,(1-\sqrt{1-\epsilon_2}\; e^{i\beta L_2})} = 1,$$

$$T_s = (e^{i\beta L_s})\,(e^{-i\beta L_s}) = 1.$$

Therefore the total transmittivity is equal to one under loss less conditions.

2.2.1 Transmittivity plot for uncoupled double ring resonator as a function of wavelength

A sanity check on the derived equations is performed by comparing the graphs obtained by plotting transmittivity T based on both analytical and numerical expressions.

Transmittivity plot based on analytic expression T_A

$$T = T_{A_2}.T_{A_s}.T_{A_1} \tag{2.10}$$

where

$$T_{A_1} = \frac{(\sqrt{1-\epsilon_1} - e^{i\beta L_1})\ (\sqrt{1-\epsilon_1} - e^{-i\beta L_1})}{(1 - \sqrt{1-\epsilon_1}\ e^{i\beta L_1})\ (1 - \sqrt{1-\epsilon_1}\ e^{i\beta L_1})},$$

$$T_{A_2} = \frac{(\sqrt{1-\epsilon_2} - e^{i\beta L_2})\ (\sqrt{1-\epsilon_2} - e^{-i\beta L_2})}{(1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2})\ (1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2})},$$

$$T_{A_s} = (e^{i\beta L_s})\ (e^{-i\beta L_s}).$$

The figure shows the transmittivity plot of an uncoupled double ring resonator plotted as a function of the normalised product βL where β is the wavenumber and L is the circumference of the ring.

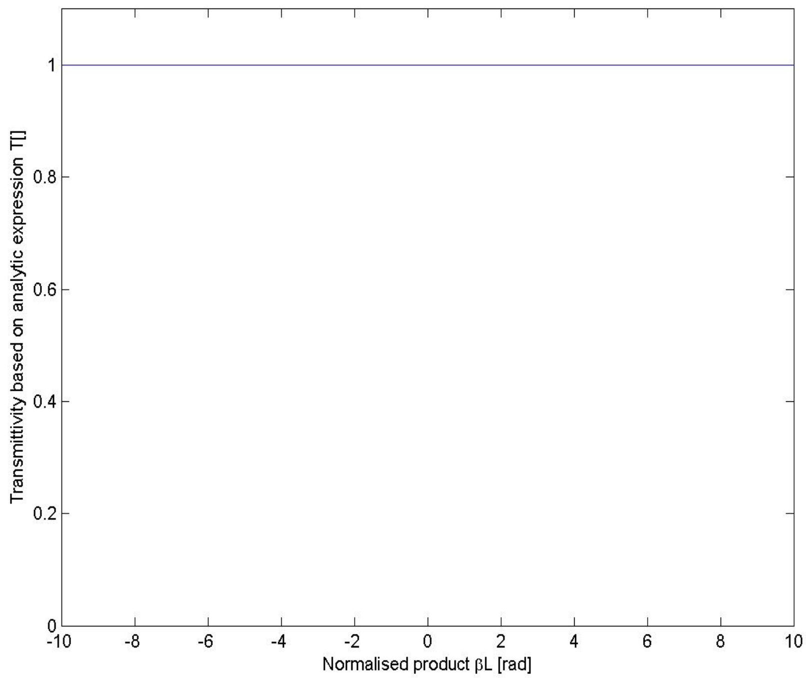

Figure 2.2: Transmittivity plot of an uncoupled double ring resonator based on analytic expression

Transmittivity plot based on numerical expression T_N

We know transmittivity is the product of the transmission coefficient $\tilde{t}$ and its conjugate $\tilde{t}^{\star}$.

$$T_N = \tilde{t} \ \times \ \tilde{t}^{\star}.$$

where,

$$\tilde{t} = \left(\frac{1 - \sqrt{(1-\epsilon_1)}\; e^{-i\beta L_1}}{1 - \sqrt{(1-\epsilon_1)}\; e^{i\beta L_1}} \right) \left(\frac{1 - \sqrt{(1-\epsilon_2)}\; e^{-i\beta L_2}}{1 - \sqrt{(1-\epsilon_2)}\; e^{i\beta L_2}} \right) \; e^{i\beta(L_1+L_s+L_2)}.$$

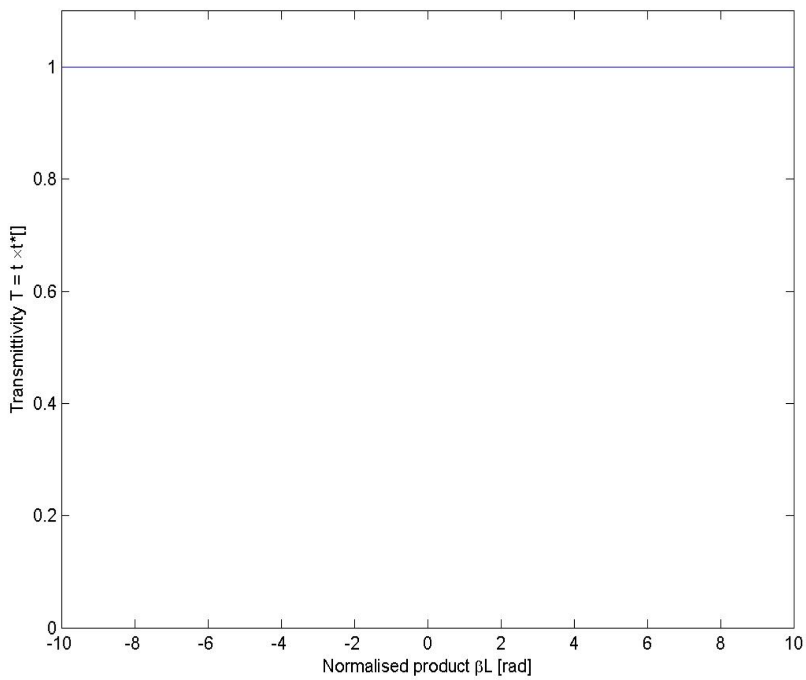

Figure 2.3: Transmittivity plot of an uncoupled double ring resonator where transmittivity is calculated as $\tilde{t} \times \tilde{t}^{\star}$

The figure shows the response obtained when $\tilde{t} \times \tilde{t}^{\star}$ is plotted against βL. We can see that the plots from both the analytic and numerical expressions are same.

Chapter 3

Serially Coupled Double Ring Resonator, Single Bus Waveguide

The structure consists of two ring optical waveguides coupled serially to a single bus waveguide. The first ring of circumference $L_1$is directly coupled to the waveguide while the second ring of circumference L_2 is coupled to the first ring through evanascent coupling. This creates two coupling regions -one between the first ring and the bus waveguide and one between the two rings. Light entering the waveguide is coupled to the first ring at the region where the ring is close to the waveguide. Light coupled to the ring through evanascence, travels along its circumference and enters the second coupling region where the second ring is in close contact with the first ring. Through

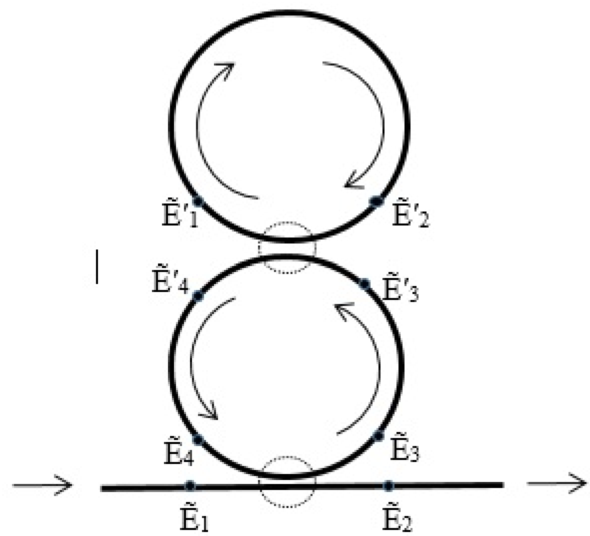

Figure 3.1: Double Ring Resonator coupled serially to a single bus waveguide

evanascent coupling light enters the second ring, travels along its circumference L_2 and re-enters the second coupling region to be coupled back to the first ring.It travels along the circumference L_1 of the first ring and re enters the first coupling region and exits along the bus waveguide.

Here $\tilde{E}_1$ is the input electric field to the coupler and $\tilde{E}_4$ is the input from the first ring to the coupler. $\tilde{E}_3$ is the output electric field from the coupler along the first ring and $\tilde{E}_2$ is the output from the first coupling region along the bus waveguide. $\tilde{E}_3'$ is the input electric field to the second coupling region from the first ring and $\tilde{E}_2'$ is the input from the second ring to the second coupling region. $\tilde{E}_1'$ is the output electric field from the second coupler along the second ring and $\tilde{E}_2'$ is the output from the second coupling region along the first ring.

3.1 Output Electric Field

The equations for electric fields at the two coupling regions are given as

$$\tilde{E}_3 = i\sqrt{\epsilon_1}\ \tilde{E}_1 + \sqrt{1-\epsilon_1}\ \tilde{E}_4, \tag{3.1}$$

$$\tilde{E}_2 = \tilde{E}_1\ \sqrt{1-\epsilon_1} + \tilde{E}_4\ i\ \sqrt{\epsilon_1}, \tag{3.2}$$

$$\tilde{E}_1{}' = \sqrt{1-\epsilon_2}\ \tilde{E}_2{}' + i\sqrt{\epsilon_2}\ \tilde{E}_3{}', \tag{3.3}$$

$$\tilde{E}_2{}' = \tilde{E}_1{}'\ e^{i\beta L_2}, \tag{3.4}$$

$$\tilde{E}_4{}' = i\sqrt{\epsilon_2}\ \tilde{E}_2{}' + \sqrt{1-\epsilon_2}\ \tilde{E}_3{}'. \tag{3.5}$$

$$\tilde{E}_3{}' = \tilde{E}_3\ e^{i\beta L_1/2}. \tag{3.6}$$

$$\tilde{E}_4 = \tilde{E}_4{}'\ e^{i\beta L_1/2}. \tag{3.7}$$

Combining Equations (3.5) and (3.7) yields

$$\tilde{E}_4 = (i\sqrt{\epsilon_2}\ \tilde{E}_2{}' + \sqrt{1-\epsilon_2}\ \tilde{E}_3{}')\ e^{i\beta L_1/2}. \tag{3.8}$$

Combining Equations (3.6) and (3.8) yields

$$\tilde{E}_4 = i\sqrt{\epsilon_2}\ \tilde{E}_2{}'\ \ e^{i\beta L_1/2} + \sqrt{1-\epsilon_2}\ \tilde{E}_3\ e^{i\beta L_1}. \tag{3.9}$$

Replacing $\tilde{E}_3$ in the above equation with Equation (3.1) gives

$$\begin{aligned}\tilde{E}_4 = i\sqrt{\epsilon_2}\ \tilde{E}_2{}'\ e^{i\beta L_1/2} &+ i\sqrt{\epsilon_1}\ \sqrt{1-\epsilon_2}\tilde{E}_1\ e^{i\beta L_1} \\ &+ \sqrt{1-\epsilon_1}\ \sqrt{1-\epsilon_2}\ \tilde{E}_4\ e^{i\beta L_1},\end{aligned}$$

$$\tilde{E}_4(1-\sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_1}) = i\sqrt{\epsilon_2}\ \tilde{E}_2{}'\ e^{i\beta L_1/2} + i\sqrt{\epsilon_1}\ \sqrt{1-\epsilon_2}\ \tilde{E}_1\ e^{i\beta L_1}. \tag{3.10}$$

Combining Equations (3.3) and (3.4) yields

$$\tilde{E_2}' = (\sqrt{1-\epsilon_2}\ \tilde{E_2}' + i\sqrt{\epsilon_2}\ \tilde{E_3}'\ e^{i\beta L_2},$$

$$\tilde{E_2}' = \frac{i\sqrt{\epsilon_2}\ \tilde{E_3}'\ e^{i\beta L_2}}{1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2}}.$$

Replacing $\tilde{E_3}'$ in the above equation with Equation (3.6) yields

$$\tilde{E_2}' = \frac{i\sqrt{\epsilon_2}\ \tilde{E_3}\ e^{i\beta L_1/2} e^{i\beta L_2}}{1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2}}. \tag{3.11}$$

Combining Equations (3.1) and (3.11) yields

$$\tilde{E_2}' = \frac{-\sqrt{\epsilon_2\ \epsilon_1}\ e^{i\beta L_1/2}\ e^{i\beta L_2}\ \tilde{E_1} + 1\sqrt{\epsilon_2(1-\epsilon_1)}\ e^{i\beta L_1/2}\ e^{i\beta L_2}\ \tilde{E_4}}{1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2}}. \tag{3.12}$$

Combining Equations (3.12) and (3.10) yields

$$\tilde{E}_4\ (1-\sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_1}) = i\sqrt{\epsilon_1\ (1-\epsilon_2)}\ \tilde{E}_1\ e^{i\beta L_1} +$$
$$i\sqrt{\epsilon_2}\ e^{i\beta L_1/2}\left(\frac{i\sqrt{\epsilon_2(1-\epsilon_1)}\ e^{i\beta L_2}\ e^{i\beta L_1/2}\ \tilde{E}_4 - \sqrt{\epsilon_1\ \epsilon_2}\ e^{i\beta L_2}\ e^{1\beta L_1/2}\ \tilde{E}_1}{1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2}}\right),$$

$$\tilde{E}_4 = \frac{\begin{array}{c} i\sqrt{(1-\epsilon_2)\epsilon_1}\ e^{i\beta L_1}\ \tilde{E}_1(1-\sqrt{1-\epsilon_2}e^{i\beta L_2}) \\ -\ \epsilon_2\sqrt{1-\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2}\ \tilde{E}_4 - i\sqrt{\epsilon_1}\ \epsilon_2\ e^{i\beta L_1}\ e^{i\beta L_2}\ \tilde{E}_1 \end{array}}{(1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2})\ (1-\sqrt{(1-\epsilon_1)(1-\epsilon_2)}\ e^{i\beta L_1})},$$

$$\tilde{E}_4\left(1+\frac{\epsilon_2\sqrt{1-\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2}}{(1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2})\ (1-\sqrt{(1-\epsilon_1)(1-\epsilon_2)}\ e^{i\beta L_1})}\right) =$$
$$\tilde{E}_1\left(\frac{i\sqrt{\epsilon_1(1-\epsilon_2)}\ e^{i\beta L_1} - i(1-\epsilon_2)\ \sqrt{\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2} - i\sqrt{\epsilon_1}\ \epsilon_2\ e^{i\beta L_1}\ e^{i\beta L_2}}{(1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2})\ (1-\sqrt{(1-\epsilon_1)(1-\epsilon_2)}\ e^{i\beta L_1})}\right),$$

$$\tilde{E}_4\left(\frac{(1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2})\ (1-\sqrt{(1-\epsilon_1)(1-\epsilon_2)}\ e^{i\beta L_1}) + \epsilon_2\sqrt{1-\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2}}{(1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2})\ (1-\sqrt{(1-\epsilon_1)(1-\epsilon_2)}\ e^{i\beta L_1})}\right) =$$
$$\tilde{E}_1\left(\frac{i\sqrt{\epsilon_1(1-\epsilon_2)}\ e^{i\beta L_1} - i(1-\epsilon_2)\ \sqrt{\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2} - i\sqrt{\epsilon_1}\ \epsilon_2\ e^{i\beta L_1}\ e^{i\beta L_2}}{(1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2})\ (1-\sqrt{(1-\epsilon_1)(1-\epsilon_2)}\ e^{i\beta L_1})}\right),$$

$$\tilde{E}_4 = \tilde{E}_1\left(\frac{i\sqrt{\epsilon_1(1-\epsilon_2)}\ e^{i\beta L_1} - i(1-\epsilon_2)\ \sqrt{\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2} - i\sqrt{\epsilon_1}\ \epsilon_2\ e^{i\beta L_1}\ e^{i\beta L_2}}{(1-\sqrt{1-\epsilon_2}\ e^{i\beta L_2})\ (1-\sqrt{(1-\epsilon_1)(1-\epsilon_2)}\ e^{i\beta L_1}) + \epsilon_2\sqrt{1-\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2}}\right),$$

$$\tilde{E}_4 = \tilde{E}_1 \left(\frac{i\sqrt{\epsilon_1(1-\epsilon_2)}\ e^{i\beta L_1} - i\sqrt{\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2}}{1 - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta L_2}\ e^{i\beta L_1}} \right). \tag{3.13}$$

We know

$$\tilde{E}_2 = \tilde{E}_1\ \sqrt{1-\epsilon_1} + \tilde{E}_4\ i\ \sqrt{\epsilon_1}. \tag{3.14}$$

Combining Equations (3.13) with the Equation for $\tilde{E}_2$ yields

$$\tilde{E}_2 = i\ \sqrt{\epsilon_1}\ \tilde{E}_1 \left(\frac{i\sqrt{\epsilon_1(1-\epsilon_2)}\ e^{i\beta L_1} - i\sqrt{\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2}}{1 - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta L_2}\ e^{i\beta L_1}} \right) + \tilde{E}_1\ \sqrt{1-\epsilon_1},$$

$$\tilde{E}_2 = \tilde{E}_1 \left(\frac{\sqrt{1-\epsilon_1}\left(1 - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2}\right) + i\sqrt{\epsilon}\ \left(i\sqrt{\epsilon_1(1-\epsilon_2)}\ e^{i\beta L_1} - i\sqrt{\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta L_2}\right)}{1 - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta L_2}\ e^{i\beta L_1}} \right),$$

$$\tilde{E}_2 = \tilde{E}_1 \left(\frac{\sqrt{1-\epsilon_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_1} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_2} + e^{i\beta L_2}\ e^{i\beta L_1}}{1 - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta L_2}\ e^{i\beta L_1}} \right),$$

$$\tilde{E_2} = \tilde{E_1}\left(\frac{\sqrt{1-\epsilon_1}\ e^{-i\beta L_2}\ e^{-i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{-i\beta L_2} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{-i\beta L_1} + 1}{1 - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta L_2}\ e^{i\beta L_1}}\right)\ e^{i\beta(L_1+L_2)},$$

$$\tilde{E_2} = \tilde{E_1}\left(\frac{\sqrt{1-\epsilon_2}\ e^{-i\beta L_2} - \sqrt{1-\epsilon_1}\ e^{-i\beta L_2}e^{-i\beta L_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{-i\beta L_1} - 1}{1 - \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta L_2}\ e^{i\beta L_1}}\right)\ e^{i\beta(L_1+L_2)}\ e^{i\pi}. \tag{3.15}$$

We know

$$\tilde{E_2} = \tilde{t}\ \tilde{E_1}$$

Therefore the transmission coefficient $\tilde{t}$ is

$$\tilde{t} = \left(\frac{\sqrt{1-\epsilon_2}\ e^{-i\beta L_2} - \sqrt{1-\epsilon_1}\ e^{-i\beta L_2}e^{-i\beta L_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{-i\beta L_1} - 1}{1 - \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta L_2}\ e^{i\beta L_1}}\right)\ e^{i\beta(L_1+L_2)}\ e^{i\pi}. \tag{3.16}$$

When $\epsilon_2 = 0$, the output electric field becomes,

$$\tilde{E_2} = \tilde{E_1}\left(\frac{\sqrt{1-\epsilon_1} - e^{i\beta L_1} - \sqrt{1-\epsilon_1}\ e^{i\beta L_2} + e^{i\beta(L_1+L_2)}}{1 - \sqrt{1-\epsilon_1}\ e^{i\beta L_1} - e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta(L_1+L_2)}}\right),$$

$$\tilde{E_2} = \tilde{E_1}\left(\frac{\sqrt{1-\epsilon_1}(1-e^{i\beta L_2}) - e^{i\beta L_1}(1-e^{i\beta L_2})}{(1-e^{i\beta L_2}) - \sqrt{1-\epsilon_1}\ e^{i\beta L_1}(1-e^{ibeta L_2})}\right),$$

$$\tilde{E_2} = \tilde{E_1}\left(\frac{1 - \sqrt{1-\epsilon_1}\ e^{-i\beta L_1}}{1 - \sqrt{1-\epsilon_1}\ e^{i\beta L_1}}\right) e^{i\beta L_1}\ e^{i\pi}.$$

which is same as Equation(1.5) which gives the output electric field of a single ring resonator coupled to a single bus waveguide.

When $\epsilon_1 = 0$, the output electric field becomes,

$$\tilde{E}_2 = \tilde{E}_1 \left(\frac{1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + e^{i\beta(L_1+L_2)}}{1 - \sqrt{1-\epsilon_1}\ e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + e^{i\beta(L_1+L_2)}} \right)$$

That is,

$$\tilde{E}_2 = \tilde{E}_1.$$

3.2 Transmittivity

Transmittivity is the ratio of transmitted power to the incident power:

$$T = |\frac{\tilde{E}_2}{\tilde{E}_1}|^2. \tag{3.17}$$

The above equation is rewritten as,

$$T = \left(\frac{\sqrt{1-\epsilon_2}\ e^{-i\beta L_2} - \sqrt{1-\epsilon_1}\ e^{-i\beta(L_1+L_2)} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)} e^{-i\beta L_1} - 1}{1 - \sqrt{(1-\epsilon_1)(1-\epsilon_2)} e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta(L_1+L_2)}} \right) \times$$

$$\left(\frac{\sqrt{1-\epsilon_2}\ e^{i\beta L_2} - \sqrt{1-\epsilon_1}\ e^{i\beta(L_1+L_2)} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)} e^{i\beta L_1} - 1}{1 - \sqrt{(1-\epsilon_1)(1-\epsilon_2)} e^{-i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{-i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{-i\beta(L_1+L_2)}} \right),$$

$$T = \frac{\begin{array}{r} (1-\epsilon_2) - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}(e^{i\beta L_1} + e^{-i\beta L_1}) + (1-\epsilon_2)\sqrt{1-\epsilon_1}(e^{i\beta(L_1-L_2)} + e^{-i\beta(L_1-L_2)}) \\ -\sqrt{1-\epsilon_2}(e^{i\beta L_2} + e^{-i\beta L_2}) + (1-\epsilon_1) \\ -(1-\epsilon_1)\sqrt{1-\epsilon_2}(e^{i\beta L_2} + e^{-i\beta L_2}) + (1-\epsilon_1)(1-\epsilon_2) \\ +\sqrt{1-\epsilon_1}(e^{i\beta(L_1+L_2)} + e^{-i\beta(L_1+L_2)}) \\ - \sqrt{(1-\epsilon_1)(1-\epsilon_2)}(e^{i\beta L_1} + e^{-i\beta L_1}) + 1 \end{array}}{\begin{array}{r} 1 - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}(e^{i\beta L_1} + e^{-i\beta L_1}) - \sqrt{1-\epsilon_2}(e^{i\beta L_2} + e^{-i\beta L_2}) \\ +\sqrt{(1-\epsilon_1)}(e^{i\beta(L_1+L_2)} + e^{-i\beta(L_1+L_2)}) - (1-\epsilon_1)\sqrt{1-\epsilon_2}(e^{i\beta L_2} + e^{-i\beta L_2}) \\ +(1-\epsilon_1)(1-\epsilon_2) + (\sqrt{1-\epsilon_1})(1-\epsilon_2)(e^{i\beta(L_1-L_2)} + e^{-i\beta(L_1-L_2)} \\ +(1-\epsilon_2) + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}(e^{i\beta L_1} + e^{-i\beta L_1}) \\ + (1-\epsilon_1) \end{array}},$$

$$T = \frac{\begin{array}{r} 3 - \epsilon_1 - \epsilon_2 + (1-\epsilon_1)(1-\epsilon_2) + (1-\epsilon_2)\sqrt{1-\epsilon_1}\ \cos(\beta(L_1-L_2)) - \sqrt{1-\epsilon_2}\ \cos(\beta L_2) \\ - (1-\epsilon_1)\sqrt{1-\epsilon_2}\ \cos(\beta L_2) + \sqrt{1-\epsilon_1}\cos(\beta(L_1+L_2)) \end{array}}{\begin{array}{r} 3 - \epsilon_1 - \epsilon_2 + (1-\epsilon_1)(1-\epsilon_2) + (1-\epsilon_2)\sqrt{1-\epsilon_1}\ \cos(\beta(L_1-L_2)) - \sqrt{1-\epsilon_2}\ \cos(\beta L_2) \\ - (1-\epsilon_1)\sqrt{1-\epsilon_2}\ \cos(\beta L_2) + \sqrt{1-\epsilon_1}\cos(\beta(L_1+L_2)) \end{array}} = 1.$$

which shows that under loss less conditions, total power entering the coupling region is equal to the total power leaving the region.

3.2.1 Transmittivity plot for single ring resonator as a function of wavelength

A sanity check on the derived equations is performed by comparing the graphs obtained by plotting transmittivity T based on both analytical and numerical techniques.

Transmittivity plot based on analytic expression T_A

$$T = \left(\frac{\sqrt{1-\epsilon_2}\; e^{-i\beta L_2} - \sqrt{1-\epsilon_1}\; e^{-i\beta(L_1+L_2)} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)} e^{-i\beta L_1} - 1}{1 - \sqrt{(1-\epsilon_1)(1-\epsilon_2)} e^{i\beta L_1} - \sqrt{1-\epsilon_2}\; e^{i\beta L_2} + \sqrt{1-\epsilon_1}\; e^{i\beta(L_1+L_2)}} \right) \times$$

$$\left(\frac{\sqrt{1-\epsilon_2}\; e^{i\beta L_2} - \sqrt{1-\epsilon_1}\; e^{i\beta(L_1+L_2)} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)} e^{i\beta L_1} - 1}{1 - \sqrt{(1-\epsilon_1)(1-\epsilon_2)} e^{-i\beta L_1} - \sqrt{1-\epsilon_2}\; e^{-i\beta L_2} + \sqrt{1-\epsilon_1}\; e^{-i\beta(L_1+L_2)}} \right).$$

The figure shows the transmittivity of a single ring resonator plotted as a function of the normalised product βL where β is the wavenumber and L is the circumference of the ring.

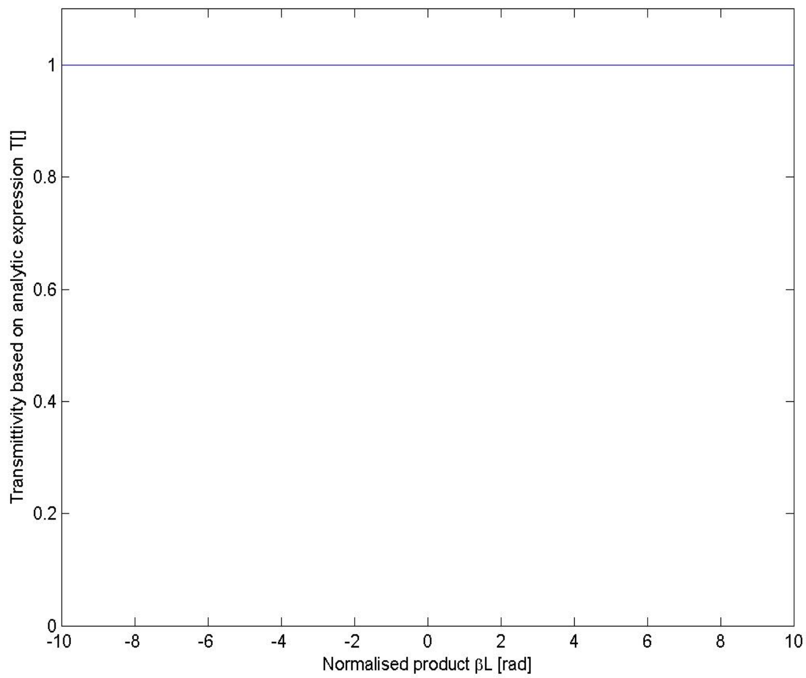

Figure 3.2: Transmittivity plot of a serially coupled DRR under zero loss condition.

The equation derived for transmittivity is used for plotting the graph. We can see that under lossless conditions, the transmittivity is equal to one which indicates that the total power entering and leaving the coupling region are equal.

Transmittivity plot based on numerical expression T_N

We know transmittivity is the product of the transmission coefficient $\tilde{t}$ and its conjugate $\tilde{t}^{\star}$.

$$T_N = \tilde{t} \ \times \ \tilde{t}^{\star}.$$

where,

$$\tilde{t} = \left(\frac{\sqrt{1-\epsilon_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_1} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_2} + e^{i\beta L_2}\ e^{i\beta L_1}}{1 - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta L_1} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2} + \sqrt{1-\epsilon_1}\ e^{i\beta L_2}\ e^{i\beta L_1}} \right).$$

The figure shows the response obtained when $\tilde{t} \times \tilde{t}^{\star}$ is plotted against βL. We can see that the plots from both the analytic and numerical expressions are the same.

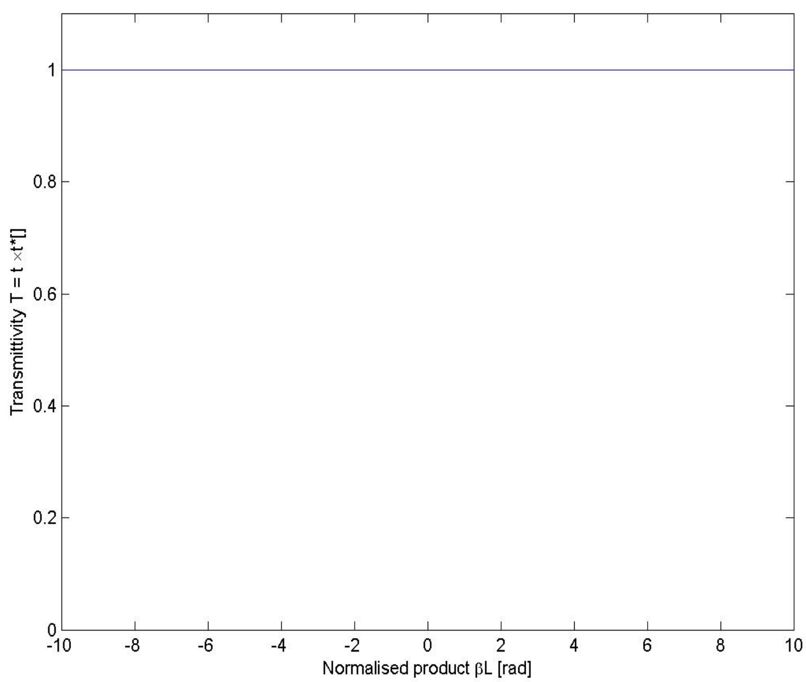

Figure 3.3: Transmittivity plot of a ring resonator where tansmittivity is calculated as $\tilde{t} \times \tilde{t}^{\star}$

Chapter 4

Double Ring Resonator-Transmitted field

The double ring resonator (DRR) consists of two mutually coupled rings coupled to a single bus waveguide. There are three coupling regions in the DRR, one between the rings and the other two between each ring and the bus waveguide. L_1 and L_2 are the circumference of the rings and L_s is the space between the coupling regions ϵ_1 and ϵ_2.

Light propagates in both forward and backward direction as the rings are mutually coupled to each other. There are a total of 24 field points at both ends of the three coupling regions including the input field $\tilde{E}_1$, as shown in Figure (1.1) and (1.2). $\tilde{E}$ represents an electric field in the counter clockwise direction and $\tilde{F}$ represents an electric field in the clockwise direction. $\tilde{F}_2{}'' = 0$ because there is no light entering from the right hand side. The remaining 22 electric fields are represented as linear equations, of which 10 are propagation equations and 12 are coupling equations.

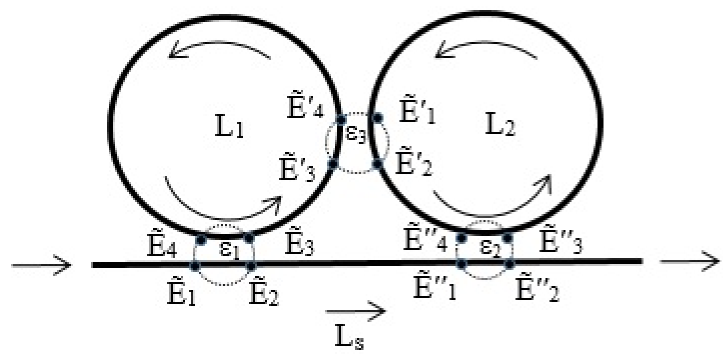

Figure 4.1: Schematic of the DRR, showing electric fields in the counter clockwise direction

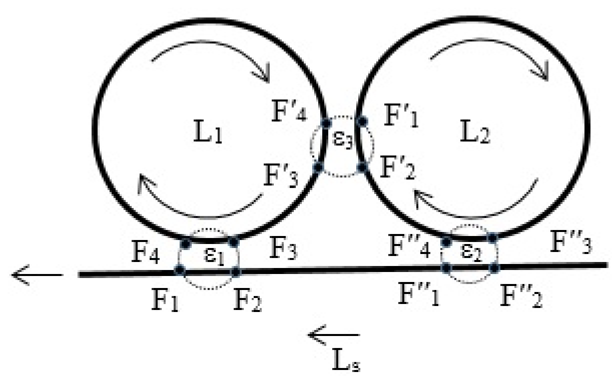

Figure 4.2: Schematic of the DRR, showing electric fields in the clockwise direction

4.1 Output Electric Field

The equations for electric fields are grouped into three :

4.1.1 Only $\tilde{E}$ fields

$$\tilde{E}_3 = i\sqrt{\epsilon_1}\ \tilde{E}_1 + \sqrt{1-\epsilon_1}\ \tilde{E}_4, \tag{4.1}$$

$$\tilde{E}_2 = \tilde{E}_1\ \sqrt{1-\epsilon_1} + \tilde{E}_4\ i\ \sqrt{\epsilon_1}, \tag{4.2}$$

$$\tilde{E}_1{}' = \tilde{E}_3{}''\ e^{i\beta\ 3L_2/4}, \tag{4.3}$$

$$\tilde{E}_3{}' = \tilde{E}_3\ e^{i\beta L_1/4}, \tag{4.4}$$

$$\tilde{E}_4 = \tilde{E}_4{}'\ e^{i\beta\ 3L_1/4}, \tag{4.5}$$

$$\tilde{E}_3{}'' = \tilde{E}_1{}''\ i\sqrt{\epsilon_2} + \sqrt{1-\epsilon_2}\ \tilde{E}_4{}'', \tag{4.6}$$

$$\tilde{E}_4{}'' = e^{i\beta L_2/4}\ \tilde{E}_2{}', \tag{4.7}$$

$$\tilde{E}_1{}'' = \tilde{E}_2\ e^{i\beta L_s}, \tag{4.8}$$

$$\tilde{E}_2{}'' = \tilde{E}_1{}''\ \sqrt{1-\epsilon_2} + i\sqrt{\epsilon_2}\ \tilde{E}_4{}''. \tag{4.9}$$

4.1.2 Only $\tilde{F}$ fields

$$\tilde{F_2}' = \tilde{F_4}'' \; e^{i\beta L_2/4}, \tag{4.10}$$

$$\tilde{F_3}'' = \tilde{F_1}' \; e^{i\beta 3L_2/4}, \tag{4.11}$$

$$\tilde{F_1}'' = i\sqrt{\epsilon_2} \; \tilde{F_3}'', \tag{4.12}$$

$$\tilde{F_4}'' = \sqrt{1-\epsilon_2} \; \tilde{F_3}'', \tag{4.13}$$

$$\tilde{F}_4 = i\sqrt{\epsilon_1} \; \tilde{F}_2 + \sqrt{1-\epsilon_1} \; \tilde{F}_3, \tag{4.14}$$

$$\tilde{F_4}' = e^{i\beta 3L_1/4} \; \tilde{F}_4, \tag{4.15}$$

$$\tilde{F}_3 = e^{i\beta L_1/4} \; \tilde{F_3}', \tag{4.16}$$

$$\tilde{F}_1 = \tilde{F}_3 \; i\sqrt{\epsilon_1} + \sqrt{1-\epsilon_1} \; \tilde{F}_2, \tag{4.17}$$

$$\tilde{F}_2 = e^{i\beta L_s} \; \tilde{F_1}''. \tag{4.18}$$

4.1.3 Mixed $\tilde{E}$ and $\tilde{F}$ fields

$$\tilde{E}_2{}' = \tilde{E}_1{}' \sqrt{1-\epsilon_3} + \tilde{F}_4{}' \, i\sqrt{\epsilon_3}, \tag{4.19}$$

$$\tilde{E}_4{}' = \sqrt{1-\epsilon_3} \, \tilde{E}_3{}' + \tilde{F}_2{}' \, i\sqrt{\epsilon_3}, \tag{4.20}$$

$$\tilde{F}_1{}' = \tilde{F}_2{}' \sqrt{1-\epsilon_3} + \tilde{E}_3{}' \, i \sqrt{\epsilon_3}, \tag{4.21}$$

$$\tilde{F}_3{}' = \tilde{F}_4{}' \sqrt{1-\epsilon_3} + \tilde{E}_1{}' \, i\sqrt{\epsilon_3}, \tag{4.22}$$

4.2 Equation for $\tilde{E}_2{}''$ based on $\tilde{E}_1$ only

The goal of this chapter is to derive an equation for the output electric field $\tilde{E}_2{}''$ based on the input electric field $\tilde{E}_1$ only. The 22 linear equations are solved using the 'Method of Equating Fields' in order to come up with the final equation for $\tilde{E}_2{}''$. The derivation is carried out in several steps yielding equations representing the path followed by the light as it travels along the structure.

From Equation (4.9) we know,

$$\tilde{E}_2{}'' = \tilde{E}_1{}'' \sqrt{1-\epsilon_2} + i\sqrt{\epsilon_2} \, \tilde{E}_4{}''.$$

Combining Equation(4.8) and Equation (4.9), we get

$$\tilde{E_2}'' = \tilde{E_2}\ e^{i\beta L_s}\ \sqrt{1-\epsilon_2} + i\sqrt{\epsilon_2}\ \tilde{E_4}''.$$

Combining the above Equation with Equation (4.2) for $\tilde{E_2}$ and then equation (4.5) for $\tilde{E_4}$, we get

$$\tilde{E_2}'' = e^{i\beta L_s}\ \sqrt{1-\epsilon_2}\ (\tilde{E_1}\ \sqrt{1-\epsilon_1} + \tilde{E_4}\ i\ \sqrt{\epsilon_1}) + i\sqrt{\epsilon_2}\ \tilde{E_4}'',$$

$$\tilde{E_2}'' = e^{i\beta L_s}\ \sqrt{(1-\epsilon_2)\ (1-\epsilon_1)}\ \tilde{E_1} + e^{i\beta L_s}\ \sqrt{1-\epsilon_2}\ i\sqrt{\epsilon_1}\ \tilde{E_4}'\ e^{i\beta 3L_1/4} + i\sqrt{\epsilon_2}\ \tilde{E_4}''. \tag{4.23}$$

And, we know from equation (4.7) that $\tilde{E_4}''$ can be written in terms of $\tilde{E_2}'$, resulting in

$$\tilde{E_2}'' = e^{i\beta L_s}\ \sqrt{(1-\epsilon_2)\ (1-\epsilon_1)}\ \tilde{E_1} + e^{i\beta L_s}\ \sqrt{1-\epsilon_2}\ i\sqrt{\epsilon_1}\ \tilde{E_4}'\ e^{i\beta 3L_1/4} + i\sqrt{\epsilon_2}\ e^{i\beta L_2/4}\ \tilde{E_2}'. \tag{4.24}$$

This particular equation is very important in the analysis of the DRR. The three components of the equation for $\tilde{E_2}''$ represents the three different paths through which light propagates and exits through the DRR in the forward direction.

The first component of the equation, $e^{i\beta L_s}\ \sqrt{(1-\epsilon_2)\ (1-\epsilon_1)}\ \tilde{E_1}$ represents path 1 where the input light entering the structure travels along the bus waveguide after crossing the two coupling regions ϵ_1 and ϵ_2 and exits through the waveguide. This component represents the simplest light path in a DRR structure. Figure 4.3 below shows light path 1 in the forward direction.

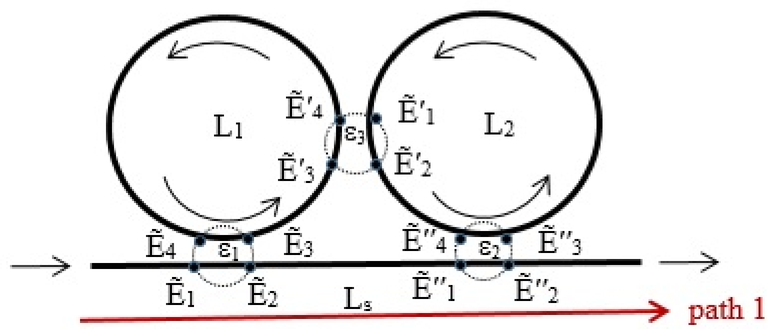

Figure 4.3: Schematic of DRR showing light path 1 in the forward direction

The second component of the equation, $e^{i\beta L_s}\ \sqrt{1-\epsilon_2}\ i\sqrt{\epsilon_1}\ e^{i\beta 3L_1/4}\ \tilde{E}_4{}'$ represents the light leaving the coupling region ϵ_3 between the two rings. The light travels along $\frac{3}{4}$ of the circumference L_1 of the first ring and then enters the two coupling regions ϵ_1 and ϵ_2, between the rings and the waveguide before exiting through the waveguide. Figure 4.5 below shows the light path 2.

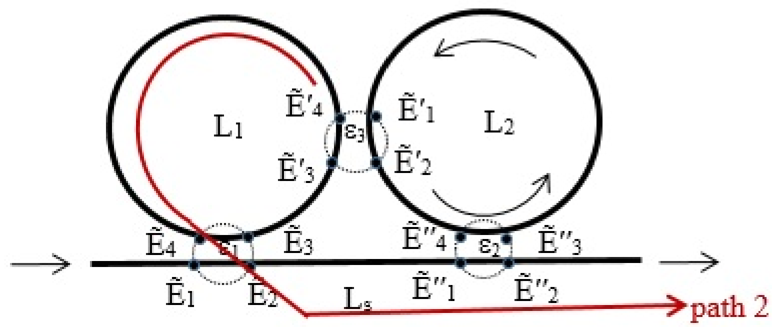

Figure 4.4: Schematic of DRR showing light path 2 in the forward direction

The third component of the equation represents the light leaving the coupling region ϵ_3 between the rings and travels along $\frac{1}{4}$ of the circumference L_2 of the second ring before entering the coupling region ϵ_2 between the second ring and the waveguide and exits through the waveguide. Figure 4.4 below shows the light path 3.

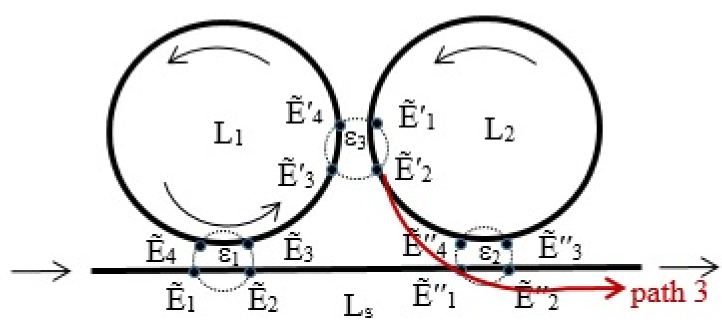

Figure 4.5: DRR, figure showing light path3 in the forward direction

The next step is to solve for $\tilde{E_4}'$ and $\tilde{E_2}'$ in terms of $\tilde{E_1}$. Doing so will yield an equation for $\tilde{E_2}''$ solely in terms of $\tilde{E_1}$, the input field. The three light paths considered above are the only paths that need to be taken into account. All other possible paths (such as light leaving $\tilde{E}_4{}'$ and circulating around ring 1) are accounted for by these three paths.

4.3 Equation for $\tilde{E}_4{}'$ based on $\tilde{E}_1$ only (path 2)

From equation (4.20) we know,

$$\tilde{E}_4{}' = \sqrt{1-\epsilon_3}\ \tilde{E}_3{}' + \ \tilde{F}_2{}'\ i\sqrt{\epsilon_3}.$$

Combining the above equation with Equation (4.4), we get

$$\tilde{E}_4{}' = \sqrt{1-\epsilon_3}\ \tilde{E}_3\ e^{i\beta L_1/4} + \ \tilde{F}_2{}'\ i\sqrt{\epsilon_3}. \tag{4.25}$$

Here the second ring acts like a ring resonator attached to the first ring. All of its light (in the clockwise orientation) comes from the first ring.

4.3.1 Solving for $\tilde{F}_2{}'$ (the ring)

Now we need $\tilde{F}_2{}'$ based on $\tilde{E}_3$. We know from equation (4.10),

$$\tilde{F}_2{}' = \tilde{F}_4{}''\ e^{i\beta L_2/4}.$$

Combining the above equation with Equations (4.13) and (4.11) yields

$$\tilde{F_2}' = \sqrt{1-\epsilon_2}\ \tilde{F_3}''\ e^{i\beta L_2/4},$$

$$\tilde{F_2}' = \sqrt{1-\epsilon_2}\ e^{i\beta L_2/4}\ \tilde{F_1}'\ e^{i\beta 3L_2/4},$$

$$\tilde{F_2}' = \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ \tilde{F_1}'. \tag{4.26}$$

Combining Equations (4.26) and (4.21), yields

$$\tilde{F_2}' = \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ (\tilde{F_2}'\ \sqrt{1-\epsilon_3} + \tilde{E_3}'\ i\ \sqrt{\epsilon_3}),$$

$$= \sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}\ \tilde{F_2}'\ + \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ \tilde{E_3}'\ i\ \sqrt{\epsilon_3},$$

$$\tilde{F_2}'(1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}) = \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ \tilde{E_3}'\ i\ \sqrt{\epsilon_3},$$

$$\tilde{F_2}' = \left[\frac{\sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ i\ \sqrt{\epsilon_3}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}}\right]\ \tilde{E_3}'. \tag{4.27}$$

Combining Equations (4.27) and (4.4) yields,

$$\tilde{F_2}' = \left[\frac{\sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ i\ \sqrt{\epsilon_3}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right]\ \tilde{E_3}\ e^{i\beta L_1/4}. \tag{4.28}$$

Now combining equations (4.28) and (4.24) , we get

$$\tilde{E_4}' = \sqrt{1-\epsilon_3}\ \tilde{E_3}\ e^{i\beta L_1/4} + i\sqrt{\epsilon_3}\ \left[\frac{\sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ i\ \sqrt{\epsilon_3}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right]\ \tilde{E_3}\ e^{i\beta L_1/4}.$$

$$= \frac{\sqrt{1-\epsilon_3}\ e^{i\beta L_1/4}(1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2})\ \tilde{E_3} - (\epsilon_3\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta L_1/4})\ \tilde{E_3}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}},$$

$$= \left[\frac{\sqrt{1-\epsilon_3}\ e^{i\beta L_1/4} - e^{i\beta(L_2+L_1/4)}\ \sqrt{1-\epsilon_2}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right]\ \tilde{E_3},$$

$$\tilde{E_4}' = \left[\frac{\sqrt{1-\epsilon_3} - e^{i\beta L_2}\ \sqrt{1-\epsilon_2}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right]\ e^{i\beta L_1/4}\ \tilde{E_3}.$$

The denominator term $\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}$ is simply the transmission coefficient t_2 after traversing the second ring once. Therefore,

$$\tilde{E_4}' = \left[\frac{\sqrt{1-\epsilon_3} - e^{i\beta L_2}\ \sqrt{1-\epsilon_2}}{1-t_2}\right]\ e^{i\beta L_1/4}\ \tilde{E_3}. \tag{4.29}$$

Now we need $\tilde{E_3}$ in terms of $\tilde{E_1}$.

4.3.2 Solving for $\tilde{E_3}$

From equation (4.1), we have

$$\tilde{E_3} = i\sqrt{\epsilon_1}\ \tilde{E_1} + \sqrt{1-\epsilon_1}\ \tilde{E_4}.$$

Combining equations (4.1) and (4.5), yields

$$\tilde{E_3} = i\sqrt{\epsilon_1}\ \tilde{E_1} + \sqrt{1-\epsilon_1}\ \tilde{E_4}'\ e^{i\beta\ 3L_1/4}. \tag{4.30}$$

From equation (4.29) we have,

$$\tilde{E_4}' = \left[\frac{\sqrt{1-\epsilon_3} - e^{i\beta L_2}\ \sqrt{1-\epsilon_2}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}}\right]\ e^{i\beta L_1/4}\ \tilde{E_3}.$$

Combining equations (4.29) and (4.30) yields,

$$\tilde{E_3} = i\sqrt{\epsilon_1}\ \tilde{E_1} + \sqrt{1-\epsilon_1}\ e^{i\beta\ 3L_1/4}\ \left[\frac{\sqrt{1-\epsilon_3}\ e^{i\beta L_1/4} - e^{i\beta(L_2+L_1/4)}\ \sqrt{1-\epsilon_2}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}}\right]\ \tilde{E_3},$$

$$= i\sqrt{\epsilon_1}\ \tilde{E_1} + \left[\frac{\sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta(L_1+L_2)}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right]\ \tilde{E_3},$$

$$\tilde{E_3}\left(1 - \left[\frac{\sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta(L_1+L_2)}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right]\right) = i\sqrt{\epsilon_1}\ \tilde{E_1},$$

$$\tilde{E_3}\left(\frac{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} + \sqrt{(1-\epsilon_1)\ (1-\epsilon_2)}\ e^{i\beta(L_1+L_2)}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right)$$

$$= i\sqrt{\epsilon_1}\ \tilde{E_1},$$

$$\tilde{E_3} = \left(\frac{i\sqrt{\epsilon_1}(1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2})}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}e^{i\beta L_2} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta(L_1+L_2)}} \right) \tilde{E_1}. \quad (4.31)$$

Combining equations (4.29) and (4.31), yields

$$\tilde{E_4}' = \left[\frac{i\sqrt{\epsilon_1}(\sqrt{1-\epsilon_3}\ e^{i\beta L_1/4} - e^{i\beta(L_2+L_1/4)}\ \sqrt{1-\epsilon_2})}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}e^{i\beta L_2} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta(L_1+L_2)}} \right] \tilde{E_1}.$$

The denominator term $\sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1}$ represents the transmission coefficient t_1 after the light traverses the first ring once. The denominator term $\sqrt{(1-\epsilon_1)(1-\epsilon_2)}\ e^{i\beta(L_1+L_2)}$ represents the transmission coefficient t_X after light traverses a fictitious ring-like structure of length $L_1 + L_2$ and having coupling points ϵ_1 and ϵ_2.

$$\tilde{E_4}' = \left[\frac{i\sqrt{\epsilon_1}(\sqrt{1-\epsilon_3}\ e^{i\beta L_1/4} - e^{i\beta(L_2+L_1/4)}\ \sqrt{1-\epsilon_2})}{1 - t_2 - t_1 + t_X} \right]\ \tilde{E_1}. \tag{4.32}$$

Thus we have $\tilde{E_4}'$ based on $\tilde{E_1}$ only .

Next step is to come up with an equation for $\tilde{E_2}'$ based on $\tilde{E_1}$ only.

4.4 Equation for $\tilde{E_2}'$ based on $\tilde{E_1}$ only (path 3)

From equation (4.19),we know

$$\tilde{E_2}' = \tilde{E_1}'\ \sqrt{1-\epsilon_3} + \tilde{F_4}'\ i\sqrt{\epsilon_3}.$$

Combining Equations (4.19) and (4.3) yields

$$\tilde{E_2}' = \tilde{E_3}''\ e^{i\beta\ 3L_2/4}\ \sqrt{1-\epsilon_3} + \tilde{F_4}'\ i\sqrt{\epsilon_3}. \tag{4.33}$$

4.4.1 Solving for $\tilde{F_4}'$ in terms of $\tilde{E_3}''$ and$\tilde{F_2}$

From Equation(4.15),we know

$$\tilde{F_4}' = e^{i\beta 3L_1/4}\ \tilde{F_4}.$$

Combining the above Equation with Equation (4.14) yields

$$\tilde{F_4}' = e^{i\beta\ 3L_1/4}\ i\sqrt{\epsilon_1}\ \tilde{F_2} + e^{i\beta\ 3L_1/4}\sqrt{1-\epsilon_1}\ \tilde{F_3}. \tag{4.34}$$

Combining Equations (4.34) and (4.16) yields

$$\tilde{F_4}' = e^{i\beta\ 3L_1/4}\ i\sqrt{\epsilon_1}\ \tilde{F_2} + e^{i\beta\ 3L_1/4}\sqrt{1-\epsilon_1}\ e^{i\beta L_1/4}\ \tilde{F_3}'. \tag{4.35}$$

Combining Equations (4.35) and (4.22) yields

$$\tilde{F_4}' = e^{i\beta\ 3L_1/4}\ i\sqrt{\epsilon_1}\ \tilde{F_2} + e^{i\beta L_1}\sqrt{1-\epsilon_1}\ (\tilde{F_4}'\ \sqrt{1-\epsilon_3} + \tilde{E_1}'\ i\sqrt{\epsilon_3}),$$

$$\tilde{F_4}'(1 - e^{i\beta L_1}\sqrt{(1-\epsilon_1)(1-\epsilon_3)}) = e^{i\beta\ 3L_1/4}\ i\sqrt{\epsilon_1}\ \tilde{F_2} + e^{i\beta L_1}\sqrt{1-\epsilon_1}\ i\sqrt{\epsilon_3}\ \tilde{E_1}'. \tag{4.36}$$

From Equation (4.3) we know,

$$\tilde{E_1}' = \tilde{E_3}''\ e^{i\beta\ 3L_2/4}.$$

Thus ,

$$\tilde{F_4}' = \left(\frac{e^{i\beta\ 3L_1/4}\ i\sqrt{\epsilon_1}}{(1 - e^{i\beta L_1}\sqrt{(1-\epsilon_1)(1-\epsilon_3)})} \right)\ \tilde{F_2} + \left(\frac{e^{i\beta L_1}\quad e^{i\beta\ 3L_2/4}\sqrt{1-\epsilon_1}\ i\sqrt{\epsilon_3}}{(1 - e^{i\beta L_1}\sqrt{(1-\epsilon_1)(1-\epsilon_3)})} \right)\ \tilde{E_3}'',$$

$$= \left(\frac{e^{i\beta\ 3L_1/4}\ i\sqrt{\epsilon_1}}{t_1} \right)\ \tilde{F_2} + \left(\frac{e^{i\beta L_1}\quad e^{i\beta\ 3L_2/4}\sqrt{1-\epsilon_1}\ i\sqrt{\epsilon_3}}{t_1} \right)\ \tilde{E_3}''. \qquad (4.37)$$

4.4.2 Solving for $\tilde{F_2}$ in terms of $\tilde{E_1}$

Now we need $\tilde{F_2}$ in terms of $\tilde{E_1}$. From Equation (4.18), we know

$$\tilde{F_2} = e^{i\beta L_s}\ \tilde{F_1}'',$$

$$= e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ \tilde{F_3}'',$$

$$= e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ \tilde{F_1}'\ e^{i\beta 3L_2/4},$$

$$= e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ e^{i\beta 3L_2/4}\ (\tilde{F_2}'\ \sqrt{1-\epsilon_3} + \tilde{E_3}'\ i\ \sqrt{\epsilon_3}),$$

$$= e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ e^{i\beta 3L_2/4}\ \tilde{F_2}'\ \sqrt{1-\epsilon_3} + \tilde{E_3}'\ i\ \sqrt{\epsilon_3} e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ e^{i\beta 3L_2/4},$$

$$= i\sqrt{\epsilon_2}\ \sqrt{1-\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ \tilde{F_2}' -\ \sqrt{\epsilon_2\ \epsilon_3}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\tilde{E_3}'. \qquad (4.38)$$

From Equation (4.27) we have,

$$\tilde{F_2}' = \left[\frac{\sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ i\ \sqrt{\epsilon_3}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right]\ \tilde{E_3}'.$$

Combining the above Equation with Equation (4.38), we get

$$\tilde{F_2} = e^{i\beta L_s}\ i\sqrt{\epsilon_2}\, e^{i\beta 3L_2/4}\ \sqrt{1-\epsilon_3}\ \left[\frac{\sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ i\ \sqrt{\epsilon_3}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right]\ \tilde{E_3}' - \tilde{E_3}'\ \sqrt{\epsilon_2\epsilon_3} e^{i\beta L_s}\ e^{i\beta 3L_2/4},$$

$$= \left[\frac{-\sqrt{\epsilon_2\ \epsilon_3(1-\epsilon_2)(1-\epsilon_3)} e^{i\beta(L_s+7L_2/4)}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right]\ \tilde{E_3}' - \tilde{E_3}'\ \sqrt{\epsilon_2\epsilon_3} e^{i\beta L_s}\ e^{i\beta 3L_2/4},$$

$$= \left(\frac{(-\sqrt{\epsilon_2\epsilon_3(1-\epsilon_2)(1-\epsilon_3)} e^{i\beta(L_s+7L_2/4)}) - \sqrt{\epsilon_2\ \epsilon_3}\ e^{i\beta L_s} e^{i\beta 3L_2/4}(1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)} e^{i\beta L_2})}{1-\sqrt{(1-\epsilon_2)(1-\epsilon_3)}\ e^{i\beta L_2}} \right) \tilde{E_3}',$$

$$= \left[\frac{-\sqrt{\epsilon_2\ \epsilon_3}\ e^{i\beta(L_s+3L_2/4)}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right] \tilde{E_3}'.$$

Combining above Equation with Equation (4.4) we get,

$$\tilde{F_2} = \left[\frac{-\sqrt{\epsilon_2\ \epsilon_3}\ e^{i\beta(L_s+3L_2/4+L_1/4)}}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right] \tilde{E_3}. \tag{4.39}$$

From Equation (4.31) we have,

$$\tilde{E_3} = \left(\frac{i\sqrt{\epsilon_1}(1 - \sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2})}{1 - \sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}e^{i\beta L_2} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta(L_1+L_2)}} \right) \tilde{E_1}.$$

Combining Equations (4.38) and (4.30), we get

$$\tilde{F_2} = \left(\frac{-\sqrt{\epsilon_2\ \epsilon_3}\ e^{i\beta(L_s+3L_2/4+L_1/4)}}{1 - \sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2}} \right)$$

$$\times \left(\frac{i\sqrt{\epsilon_1}(1 - \sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}e^{i\beta L_2})}{1 - \sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}e^{i\beta L_2} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta(L_1+L_2)}} \right) \tilde{E_1},$$

$$= \left(\frac{-i\sqrt{\epsilon_1\ \epsilon_2\ \epsilon_3}e^{i\beta(L_s+3L_2/4+L_1/4)}}{1 - \sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}e^{i\beta L_2} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta(L_1+L_2)}} \right) \tilde{E_1},$$

$$\tilde{F_2} = \left(\frac{-i\sqrt{\epsilon_1\epsilon_2\ \epsilon_3}\ e^{i\beta(L_s+3L_2/4+L_1/4)}}{1 - t_1 - t_2 + t_X} \right) \tilde{E_1}. \quad (4.40)$$

Thus we have $\tilde{F_2}$ based on $\tilde{E_1}$ only.

4.4.3 Solving for $\tilde{F_4}'$ in terms of $\tilde{E_1}$ and $\tilde{E_3}''$

Combining Equations (4.40) and (4.37) yields,

$$\tilde{F_4}' = \left(\frac{e^{i\beta\ 3L_1/4}\ i\sqrt{\epsilon_1}}{1-t_1}\right)\left(\frac{-i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta(L_s+3L_2/4+L_1/4)}}{1-t_1-t_2+t_X}\right)\tilde{E_1}+\left(\frac{e^{i\beta L_1}\quad e^{i\beta\ 3L_2/4}\sqrt{1-\epsilon_1}\ i\sqrt{\epsilon_3}}{1-t_1}\right)\ \tilde{E_3}''.$$

$$= \left(\frac{\epsilon_1\sqrt{\epsilon_2\ \epsilon_3}e^{i\beta(L_s+3L_2/4+L_1)}}{(1-t_1)\ (1-t_1-t_2+t_X)}\right)\tilde{E_1}+\left(\frac{e^{i\beta L_1}\quad e^{i\beta\ 3L_2/4}\sqrt{1-\epsilon_1}\ i\sqrt{\epsilon_3}}{1-t_1}\right)\ \tilde{E_3}''. \tag{4.41}$$

Now we have $\tilde{F_4}'$ based on $\tilde{E_1}$ and $\tilde{E_3}''$.

Combining this equation with the equation for $\tilde{E_2}'$ will yield $\tilde{E_2}'$ based on $\tilde{E_1}$ and $\tilde{E_3}''$.

4.4.4 Solving for $\tilde{E_2}'$ in terms of $\tilde{E_1}$ and $\tilde{E_3}''$

From quation (4.33) , we have

$$\tilde{E_2}' = \tilde{E_3}''\ e^{i\beta\ 3L_2/4}\ \sqrt{1-\epsilon_3} + \tilde{F_4}'\ i\sqrt{\epsilon_3}.$$

Combining Equations (4.33) and (4.41) yields,

$$\tilde{E_2}' = \tilde{E_3}'' e^{i\beta\ 3L_2/4}\sqrt{1-\epsilon_3} + \left(\frac{i\epsilon_1\ \epsilon_3\sqrt{\epsilon_2}e^{i\beta(L_s+3L_2/4+L_1)}}{(1-t_1)\ (1-t_1-t_2+t_X)}\right)\tilde{E_1} - \left(\frac{e^{i\beta(L_1+3L_2/4)}\epsilon_3\ \sqrt{1-\epsilon_1}}{1-t_1}\right)\ \tilde{E_3}'',$$

$$= \left(\frac{i\epsilon_1\ \epsilon_3\sqrt{\epsilon_2}e^{i\beta(L_s+3L_2/4+L_1)}}{(1-t_1)\ (1-t_1-t_2+t_X)}\right)\tilde{E_1} + \left(\frac{e^{i\beta\ 3L_2/4}\ \sqrt{1-\epsilon_3}(a) - e^{i\beta(L_1+3L_2/4)}\epsilon_3\ \sqrt{1-\epsilon_1}}{1-t_1}\right)\ \tilde{E_3}''.$$

Note: $\sqrt{1-\epsilon_3}\ (1-e^{i\beta L_1}\ \sqrt{(1-\epsilon_1)\ (1-\epsilon_3)})e^{i\beta 3L_2/4} - \epsilon_3\ \sqrt{1-\epsilon_1}\ e^{i\beta(L_1+3L_2/4)}$

$$= -(1-\epsilon_3)\ \sqrt{1-\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta 3L_2/4} - \epsilon_3\ \sqrt{1-\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta 3L_2/4} + \sqrt{1-\epsilon_3}\ e^{i\beta 3L_2/4},$$

$$= \sqrt{1-\epsilon_3}\ e^{i\beta 3L_2/4} - \sqrt{1-\epsilon_1}\ e^{i\beta L_1}\ e^{i\beta 3L_2/4}.$$

Thus

$$\tilde{E_2}' = \left(\frac{i\epsilon_1\ \epsilon_3\sqrt{\epsilon_2}e^{i\beta(L_s+3L_2/4+L_1)}}{(1-t_1)\ (1-t_1-t_2+t_X)}\right)\tilde{E_1} + \left(\frac{\sqrt{1-\epsilon_3}\ e^{i\beta 3L_2/4} - \sqrt{1-\epsilon_1}\ e^{i\beta(L_1+3L_2/4)}}{1-t_1}\right)\ \tilde{E_3}''. \tag{4.42}$$

Now we need $\tilde{E_3}''$ in terms of $\tilde{E_1}$ to get the final equation for $\tilde{E_2}'$.

4.4.5 Solving for $\tilde{E_3}''$ in terms of $\tilde{E_2}$ and $\tilde{E_2}'$

From Equation (4.6) , we know

$$\tilde{E_3}'' = \tilde{E_1}''\ i\sqrt{\epsilon_2} + \sqrt{1-\epsilon_2}\ \tilde{E_4}''.$$

Combining the above equation and Equation (1.8) and (1.7), we get

$$\tilde{E_3}'' = e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ \tilde{E_2} + \sqrt{1-\epsilon_2}\ e^{i\beta L_2/4}\ \tilde{E_2}'. \tag{4.43}$$

4.4.6 Solving for $\tilde{E_2}$ in terms of $\tilde{E_1}$

From equation (4.2) we know,

$$\tilde{E_2} = \tilde{E_1}\ \sqrt{1-\epsilon_1} + \tilde{E_4}\ i\ \sqrt{\epsilon_1}.$$

Combining above equation with Equation (4.5), we get

$$\tilde{E_2} = \tilde{E_1}\ \sqrt{1-\epsilon_1} + \tilde{E_4}'\ e^{i\beta 3L_1/4}\ i\ \sqrt{\epsilon_1}. \tag{4.44}$$

From Equation (4.31) we know,

$$\tilde{E_4}' = \left[\frac{i\sqrt{\epsilon_1}(\sqrt{1-\epsilon_3}\ e^{i\beta L_1/4} - e^{i\beta(L_2+L_1/4)}\ \sqrt{1-\epsilon_2})}{1-\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}e^{i\beta L_2} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta(L_1+L_2)}}\right]\ \tilde{E_1},$$

$$= \left[\frac{i\sqrt{\epsilon_1}\sqrt{1-\epsilon_3} - e^{i\beta L_2}\ i\sqrt{\epsilon_1}\sqrt{1-\epsilon_2}}{1-t_1-t_2+t_X}\right]\ e^{i\beta L_1/4}\ \tilde{E_1}.$$

Combining above equation with Equation (4.44), we get

$$\tilde{E_2} = \tilde{E_1}\ \sqrt{1-\epsilon_1} + \left[\frac{i\sqrt{\epsilon_1}\sqrt{1-\epsilon_3} - e^{i\beta L_2}\ i\sqrt{\epsilon_1}\sqrt{1-\epsilon_2}}{1-t_1-t_2+t_X}\right]\ \tilde{E_1}\ e^{i\beta L_1}\ i\ \sqrt{\epsilon_1},$$

$$= \left(\sqrt{1-\epsilon_1} - \epsilon_1\left[\frac{\sqrt{1-\epsilon_3} - e^{i\beta L_2}\sqrt{1-\epsilon_2}}{1-t_1-t_2+t_X}\right]\ e^{i\beta L_1}\right)\ \tilde{E_1}. \quad (4.45)$$

Now combining equations (4.43) and (4.45) , we get

$$\tilde{E_3}'' = \left(\sqrt{1-\epsilon_1} - \ \epsilon_1\left[\frac{\sqrt{1-\epsilon_3} - e^{i\beta L_2}\sqrt{1-\epsilon_2}}{1-t_1-t_2+t_X}\right]\ e^{i\beta L_1}\right)\ e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ \tilde{E_1} + \sqrt{1-\epsilon_2}\ e^{i\beta L_2/4}\ \tilde{E_2}',$$

$$\tilde{E_3}'' = i\sqrt{\epsilon_2}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ \tilde{E_1} - \epsilon_1\ \sqrt{\epsilon_2}\left[\frac{\sqrt{1-\epsilon_3} - e^{i\beta L_2}\sqrt{1-\epsilon_2}}{1-t_1-t_2+t_X}\right] i\ e^{i\beta L_1}\ e^{i\beta L_s}\ \tilde{E_1} + \sqrt{1-\epsilon_2}\ e^{i\beta L_2/4}\ \tilde{E_2}'. \quad (4.46)$$

4.4.7 Solving for $\tilde{E_2}'$ in terms of $\tilde{E}_1$

The combination of equations (4.42) and (4.46) allows us to solve for $\tilde{E_2}'$ solely in terms of $\tilde{E}_1$. Therefore Equation (4.46) can also be written as,

$$\tilde{E_3}'' = \left[\frac{\sqrt{1-\epsilon_1} - e^{i\beta L_2}\sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)} - e^{i\beta L_1}\ \sqrt{1-\epsilon_3)} + e^{i\beta(L_1+L_2)}\ \sqrt{1-\epsilon_2)}}{1 - \sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}e^{i\beta L_2} - \sqrt{(1-\epsilon_1)\ (1-\epsilon_3)}\ e^{i\beta L_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta(L_1+L_2)}} \right] \tilde{E}_1\ e^{i\beta L_s}\ i\sqrt{\epsilon_2} + \sqrt{1-\epsilon_2}\ e^{i\beta L_2/4}\ \tilde{E_2}',$$

$$= \left[\frac{\sqrt{1-\epsilon_1} - e^{i\beta L_2}\sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)} - e^{i\beta L_1}\ \sqrt{1-\epsilon_3)} + e^{i\beta(L_1+L_2)}\ \sqrt{1-\epsilon_2)}}{1 - t_1 - t_2 + t_X} \right] \tilde{E}_1\ e^{i\beta L_s}\ i\sqrt{\epsilon_2} + \sqrt{1-\epsilon_2}\ e^{i\beta L_2/4}\ \tilde{E_2}'.$$

Combining Equation (4.46) with the above equation yields,

$$\tilde{E_2}' = \left(\frac{i\epsilon_1 \ \epsilon_3 \sqrt{\epsilon_2} e^{i\beta(L_s + 3L_2/4 + L_1)}}{(1-t_1)\ (1-t_1-t_2+t_X)} \right) \tilde{E_1}$$

$$+ \left(\frac{\sqrt{1-\epsilon_3}\ e^{i\beta 3L_2/4} - \sqrt{1-\epsilon_1}\ e^{i\beta(L_1+3L_2/4)}}{1-t_1} \right) \times$$

$$\left(\left[\frac{\sqrt{1-\epsilon_1} - e^{i\beta L_2}\ \sqrt{(1-\epsilon_1)(1-\epsilon_2)\ (1-\epsilon_3)} - e^{i\beta L_1}\ \sqrt{1-\epsilon_3} + e^{i\beta(L_2+L_1)} \sqrt{1-\epsilon_2}}{1-t_1-t_2+t_X} \right] \tilde{E_1}\ e^{i\beta L_s}\ i\sqrt{\epsilon_2} + \sqrt{1-\epsilon_2}\ e^{i\beta L_2/4}\ \tilde{E_2}' \right),$$

$$\tilde{E_2}' \left(1 - \left(\frac{\sqrt{(1-\epsilon_2)\ (1-\epsilon_3)}\ e^{i\beta L_2} - \sqrt{(1-\epsilon_1)(1-\epsilon_2)} e^{i\beta(L_1+L_2)}}{1-t_1} \right) \right)$$

$$= \left(\frac{\begin{array}{c}(\sqrt{1-\epsilon_3}\ e^{i\beta 3L_2/4} - \sqrt{1-\epsilon_1}\ e^{i\beta(L_1+3L_2/4)})(e^{i\beta L_s}\ i\sqrt{\epsilon_2}) \times \\ (\sqrt{1-\epsilon_1} - e^{i\beta L_2}\sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)} - e^{i\beta L_1}\sqrt{1-\epsilon_3} + e^{i\beta(L_2+L_1)}\sqrt{1-\epsilon_2}) \\ + i\epsilon_1\ \epsilon_3\sqrt{\epsilon_2} e^{i\beta(L_s+3L_2/4+L_1)}\end{array}}{(1-t_1)\ (1-t_1-t_2+t_X)} \right) \tilde{E_1}.$$

$$\tilde{E_2}'\left(\frac{1-t_1-t_2+t_X}{1-t_1}\right)$$

$$=\left(\frac{\begin{array}{r}(\sqrt{1-\epsilon_3}e^{i\beta 3L_2/4}-\sqrt{1-\epsilon_1}e^{i\beta(L_1+3L_2/4)})(e^{i\beta L_s}i\sqrt{\epsilon_2})\times\\ (\sqrt{1-\epsilon_1}-e^{i\beta L_2}\sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}-e^{i\beta L_1}\sqrt{1-\epsilon_3}+e^{i\beta(L_2+L_1)}\sqrt{1-\epsilon_2})\\ +i\epsilon_1\ \epsilon_3\sqrt{\epsilon_2}e^{i\beta(L_s+3L_2/4+L_1)}\end{array}}{(1-t_1)\ (1-t_1-t_2+t_X)}\right)\tilde{E_1}.$$

Therefore

$$\tilde{E_2}'=$$

$$\left(\frac{\begin{array}{r}(i\sqrt{\epsilon_2(1-\epsilon_3)}\ e^{i\beta(L_s+3L_2/4)}-i\sqrt{\epsilon_2(1-\epsilon_1)}\ e^{i\beta(L_1+3L_2/4+L_s)})\times\\ (\sqrt{1-\epsilon_1}-e^{i\beta L_2}\sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}-e^{i\beta L_1}\sqrt{1-\epsilon_3}+e^{i\beta(L_2+L_1)}\sqrt{1-\epsilon_2})\\ +i\epsilon_1\ \epsilon_3\sqrt{\epsilon_2}e^{i\beta(L_s+3L_2/4+L_1)}\end{array}}{(1-t_1-t_2+t_X)^2}\right)\tilde{E_1}. \tag{4.47}$$

4.5 Combining equations for $\tilde{E_4}'$ and $\tilde{E_2}'$ to obtain output electric field $\tilde{E_2}''$ in terms of $\tilde{E_1}$ only

From Equation (4.24) we have,

$$\tilde{E_2}'' = e^{i\beta L_s} \sqrt{(1-\epsilon_2)\ (1-\epsilon_1)}\ \tilde{E_1} + e^{i\beta L_s} \sqrt{1-\epsilon_2}\, i\sqrt{\epsilon_1}\, \tilde{E_4}'\, e^{i\beta 3L_1/4} + i\sqrt{\epsilon_2}\, e^{i\beta L_2/4}\, \tilde{E_2}'.$$

Now combine Equations (4.24) , (4.32) and (4.47) as the final step to get $\tilde{E_2}''$ in terms of $\tilde{E_1}$. This yields,

$$\tilde{E_2}'' = i\sqrt{\epsilon_2} \left(\frac{\begin{array}{r} (i\sqrt{\epsilon_2(1-\epsilon_3)}\ e^{i\beta(L_s+L_2)} - i\sqrt{\epsilon_2(1-\epsilon_1)}\ e^{i\beta(L_1+L_2+L_s)})\times \\ (\sqrt{1-\epsilon_1} - e^{i\beta L_2}\sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)} - e^{i\beta L_1}\sqrt{1-\epsilon_3} \\ + e^{i\beta(L_2+L_1)}\sqrt{1-\epsilon_2}) \\ + i\epsilon_1\ \epsilon_3\sqrt{\epsilon_2}e^{i\beta(L_s+L_2+L_1)} \end{array}}{(1-t_1-t_2+t_X)^2} \right) \tilde{E_1}$$

$$+ i\sqrt{\epsilon_1(1-\epsilon_2)}\, e^{i\beta(L_1+L_s)} \left[\frac{i\sqrt{\epsilon_1}(\sqrt{1-\epsilon_3} - e^{i\beta L_2}\ \sqrt{1-\epsilon_2})}{1-t_1-t_2+t_X} \right] \tilde{E_1} + e^{i\beta L_s} \sqrt{(1-\epsilon_2)\ (1-\epsilon_1)}\, \tilde{E_1}.$$

Thus

$$\tilde{E_2}'' = \left(\frac{\begin{array}{c}(\epsilon_2\ \sqrt{(1-\epsilon_1)}e^{i\beta(L_s+L_1+L_2)} - \epsilon_2\sqrt{(1-\epsilon_3)}e^{i\beta(L_s+L_2)})\ \times \\ (\sqrt{1-\epsilon_1} - e^{i\beta L_2}\sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)} - e^{i\beta L_1}\sqrt{1-\epsilon_3} + e^{i\beta(L_2+L_1)}\sqrt{1-\epsilon_2}\end{array}}{(1-t_1-t_2+t_X)^2} \right) \tilde{E_1}$$

$$-\frac{\epsilon_1\ \epsilon_3\ \epsilon_2\ e^{i\beta(L_s+L_2+L_1)}}{(1-t_1-t_2+t_X)^2}\tilde{E_1} + \left(\frac{\epsilon_1(1-\epsilon_2)e^{i\beta(L_1+L_2+L_s)} - \epsilon_1\sqrt{(1-\epsilon_2)(1-\epsilon_3)}e^{i\beta(L_1+L_s)}}{1-t_1-t_2+t_X} \right) \tilde{E_1}$$

$$+ e^{i\beta L_s}\ \sqrt{(1-\epsilon_2)\ (1-\epsilon_1)}\ \tilde{E_1}.$$

which can also be written as

$$\tilde{E_2}'' = \left(\frac{N}{(1-t_1-t_2+t_X)^2} \right)\ \tilde{E_1}. \qquad (4.48)$$

where

$$N = (\epsilon_2\ \sqrt{(1-\epsilon_1)}e^{i\beta(L_s+L_1+L_2)} - \epsilon_2\sqrt{(1-\epsilon_3)}e^{i\beta(L_s+L_2)})\times$$
$$(\sqrt{1-\epsilon_1} - e^{i\beta L_2}\sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)} - e^{i\beta L_1}\sqrt{1-\epsilon_3} + e^{i\beta(L_2+L_1)}\sqrt{1-\epsilon_2})$$
$$-\ \epsilon_1\ \epsilon_3\ \epsilon_2\ e^{i\beta(L_s+L_2+L_1)}$$
$$+\ (\epsilon_1(1-\epsilon_2)e^{i\beta(L_1+L_2+L_s)} - \epsilon_1\sqrt{(1-\epsilon_2)(1-\epsilon_3)}e^{i\beta(L_1+L_s)})\ (1-t_1-t_2+t_X)$$
$$+\ e^{i\beta L_s}\ \sqrt{(1-\epsilon_2)\ (1-\epsilon_1)}\ (1-t_1-t_2+t_X)^2.$$

We know

$$\tilde{E_2}'' = \tilde{t}\ \tilde{E}_1$$

Therefore the transmission coefficient $\tilde{t}$ is

$$\tilde{t} = \frac{N}{(1 - t_1 - t_2 + t_X)^2}. \tag{4.49}$$

4.6 Transmittivity plot based on numerical expression

We know transmittivity is the product of the transmission coefficient $\tilde{t}$ and its conjugate $\tilde{t}^{\star}$.

$$T = |\frac{\tilde{E_2}''}{\tilde{E_1}}|^2 = |\tilde{t}|^2 = \tilde{t}\ \times\ \tilde{t}^{\star}. \tag{4.50}$$

Figure 4.6 below shows the response obtained when $\tilde{t}\ \times\ \tilde{t}^{\star}$ is plotted against $\beta L/\pi$ with $\epsilon_1 = \epsilon_2 = 0.1$ and $\epsilon_3 = 0.016$.

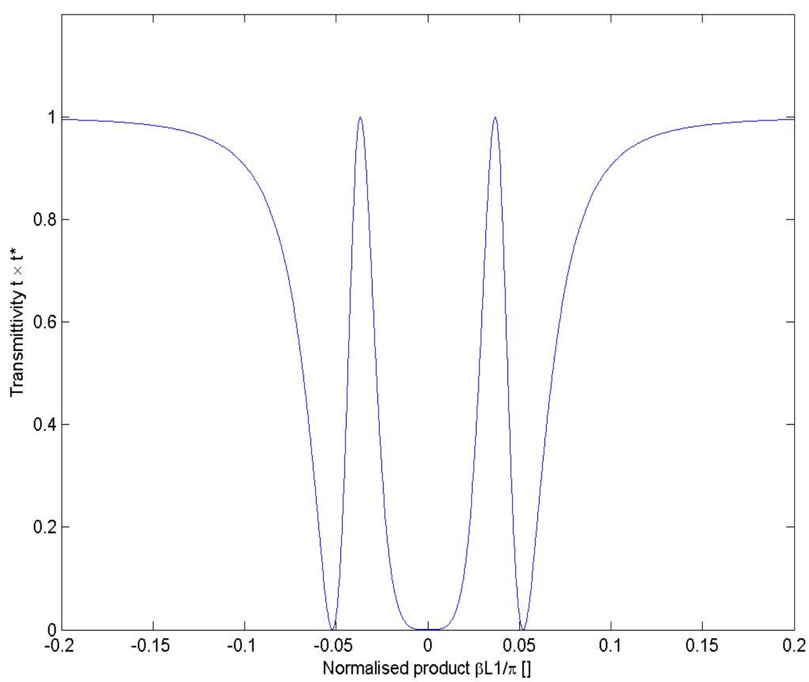

Figure 4.6: Transmittivity plot of a DRR where transmittivity is calculated as $\tilde{t} \times \tilde{t}^{\star}$

Chapter 5

Double Ring Resonator-Reflected Field

The goal of this chapter is to derive an equation for the reflected field $\tilde{F}_1$ based on $\tilde{E}_1$ only.

5.1 $\tilde{F}_1$ based on $\tilde{F}_3{}'$ and $\tilde{F}_1{}'$

From Equation (4.17) we know

$$\tilde{F}_1 = \tilde{F}_3 \; i\sqrt{\epsilon_1} + \sqrt{1-\epsilon_1}\; \tilde{F}_2.$$

Combining above equation with Equation (4.16) , yields

$$\tilde{F}_1 = \tilde{F_3}' \; e^{i\beta L_1/4} \; i\sqrt{\epsilon_1} + \sqrt{1-\epsilon_1} \; \tilde{F}_2. \tag{5.1}$$

From equation (4.18),

$$\tilde{F}_2 = e^{i\beta L_s} \; \tilde{F_1}'',$$

Combining above equation with Equation (4.12), yields

$$\tilde{F}_2 = e^{i\beta L_s} \; i\sqrt{\epsilon_2} \; \tilde{F_3}'',$$

Combining above equation with equation (4.11), yields

$$\tilde{F}_2 = e^{i\beta L_s} \; i\sqrt{\epsilon_2} \; e^{i\beta 3L_2/4} \; \tilde{F_1}'. \tag{5.2}$$

Combining Equations (5.1) and (5.2), yields

$$\tilde{F}_1 = e^{i\beta L_1/4} \; i\sqrt{\epsilon_1} \; \tilde{F_3}' + \; i\sqrt{\epsilon_2}\sqrt{1-\epsilon_1} \; e^{i\beta L_s} \; e^{i\beta 3L_2/4} \; \tilde{F_1}'. \tag{5.3}$$

The right hand side of the above equation constitutes two paths over which light can reach $\tilde{F}_1$. Figure 5.1 shows light path 1 and 2 in the backward direction.

In the following pages, $\tilde{F_3}'$ and $\tilde{F_1}'$ are derived solely in terms of $\tilde{E}_1$.

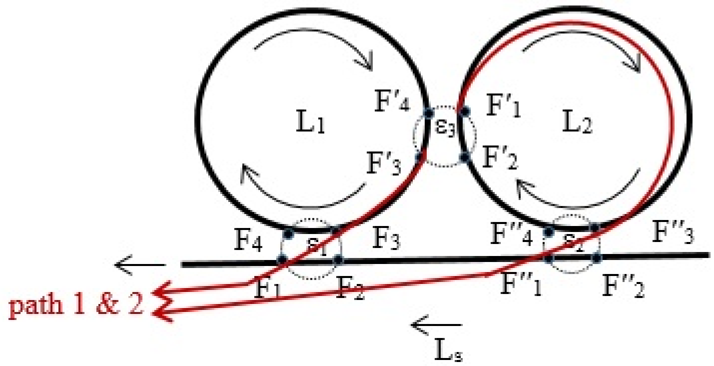

Figure 5.1: Schematic of DRR showing light path 1 and 2 in the backward direction

5.1.1 Equation for $\tilde{F_3}'$ based on $\tilde{F_1}'$, $\tilde{E_1}$ and $\tilde{E_4}$

Equation (4.22) states

$$\tilde{F_3}' = \tilde{F_4}' \sqrt{1-\epsilon_3} + \tilde{E_1}' \, i\sqrt{\epsilon_3},$$

From Equation (4.3), we know

$$\tilde{E_1}' = \tilde{E_3}'' \, e^{i\beta \; 3L_2/4},$$

Combining the above equation with Equation (4.6) yields

$$\tilde{E_1}' = (\tilde{E_1}'' \, i\sqrt{\epsilon_2} + \sqrt{1-\epsilon_2} \, \tilde{E_4}'') \, e^{i\beta \; 3L_2/4},$$

$$\tilde{E_1}' = i\sqrt{\epsilon_2} \;\; e^{i\beta \; 3L_2/4} \, \tilde{E_1}'' + \sqrt{1-\epsilon_2} \, e^{i\beta \; 3L_2/4} \, \tilde{E_4}'',$$

Combining the above equation with Equations (4.8) and (4.7) yields

$$\tilde{E_1}' = e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ \ e^{i\beta\ 3L_2/4}\ \tilde{E_2} + e^{i\beta L_2}\ \sqrt{1-\epsilon_2}\ \tilde{E_2}',$$

Combining the above equation with Equations (4.2) and (4.19) yields

$$\tilde{E_1}' = (\tilde{E_1}\sqrt{1-\epsilon_1}+\tilde{E_4}\, i\,\sqrt{\epsilon_1})\, e^{i\beta L_s}\, i\sqrt{\epsilon_2}\ \ e^{i\beta\ 3L_2/4}+e^{i\beta L_2}\,(\tilde{E_1}'\sqrt{1-\epsilon_3}+\tilde{F_4}'\, i\sqrt{\epsilon_3})\sqrt{1-\epsilon_2},$$

$$= \tilde{E_1}\sqrt{1-\epsilon_1}\, e^{i\beta L_s}\, i\sqrt{\epsilon_2}\ \ e^{i\beta\ 3L_2/4}-\tilde{E_4}\ \sqrt{\epsilon_1\epsilon_2}\, e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}+\tilde{E_1}'\sqrt{1-\epsilon_3}\sqrt{1-\epsilon_2}\, e^{i\beta L_2}$$
$$+\tilde{F_4}'\ i\sqrt{\epsilon_3}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2},$$

$$\tilde{E_1}'(1-\sqrt{1-\epsilon_3}\sqrt{1-\epsilon_2}\, e^{i\beta L_2}) = \tilde{E_1}\sqrt{1-\epsilon_1}\, e^{i\beta L_s}\, i\sqrt{\epsilon_2}\ \ e^{i\beta\ 3L_2/4}-\tilde{E_4}\ \sqrt{\epsilon_1\epsilon_2}\, e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}$$
$$+\tilde{F_4}'\ i\sqrt{\epsilon_3}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2},$$

$$\tilde{E_1}' = \tilde{E_1}\ \left(\frac{\sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ e^{i\beta\ 3L_2/4}}{1-\ \sqrt{1-\epsilon_3}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}}\right)-\tilde{E_4}\ \left(\frac{\sqrt{\epsilon_1\epsilon_2}\ e^{i\beta L_s}\ \ e^{i\beta\ 3L_2/4}}{1-\ \sqrt{1-\epsilon_3}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}}\right)$$

$$+\tilde{F_4}'\ \left(\frac{i\sqrt{\epsilon_3}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}}{1-\ \sqrt{1-\epsilon_3}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}}\right).$$

Note that, as introduced earlier this chapter, the transmission coefficient for ring 2 is $t_2 = e^{i\beta L_2} \sqrt{1-\epsilon_3}\,\sqrt{1-\epsilon_2}$. Using t_2, $\tilde{E}_1{}'$ can also be written as

$$\tilde{E}_1{}' = \tilde{E}_1 \left(\frac{\sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ i\sqrt{\epsilon_2}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right) - \tilde{E}_4 \left(\frac{\sqrt{\epsilon_1\epsilon_2}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right)$$

$$+ \tilde{F}_4{}' \left(\frac{i\sqrt{\epsilon_3}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}}{1-t_2} \right). \quad (5.4)$$

Combining Equations (5.4) and (4.22) yields

$$\tilde{F}_3{}' = \tilde{F}_4{}' \sqrt{1-\epsilon_3} - \tilde{E}_1 \left(\frac{\sqrt{\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right) - \tilde{E}_4 \left(\frac{i\ \sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right)$$

$$- \tilde{F}_4{}' \left(\frac{\epsilon_3\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}}{1-t_2} \right),$$

$$\tilde{F}_3{}' = \tilde{F}_4{}' \left(\sqrt{1-\epsilon_3} - \left(\frac{\epsilon_3\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}}{1-t_2} \right) \right) - \tilde{E}_1 \left(\frac{\sqrt{\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right)$$

$$- \tilde{E}_4 \left(\frac{i\ \sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right),$$

$$= \tilde{F_4}' \left(\frac{\sqrt{1-\epsilon_3}(1-t_2) - \epsilon_3\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}}{1-t_2} \right) - \tilde{E_1} \left(\frac{\sqrt{\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right)$$

$$- \tilde{E_4} \left(\frac{i\ \sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right),$$

As stated above $t_2 = e^{i\beta L_2}\ \sqrt{1-\epsilon_3}\ \sqrt{1-\epsilon_2}$. Therefore the equation becomes

$$\tilde{F_3}' = \tilde{F_4}' \left(\frac{\sqrt{1-\epsilon_3} - \sqrt{1-\epsilon_2}\ e^{i\beta L_2}}{1-t_2} \right) - \tilde{E_1} \left(\frac{\sqrt{\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right)$$

$$- \tilde{E_4} \left(\frac{i\ \sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2} \right). \quad (5.5)$$

From Equation (4.15)we know,

$$\tilde{F_4}' = e^{i\beta 3L_1/4}\ \tilde{F_4}.$$

Combining above equation with Equation (4.14) yields

$$\tilde{F_4}' = e^{i\beta 3L_1/4}\ (i\sqrt{\epsilon_1}\ \tilde{F_2} + \sqrt{1-\epsilon_1}\ \tilde{F_3}),$$

$$= e^{i\beta 3L_1/4}\ i\sqrt{\epsilon_1}\ \tilde{F_2} + e^{i\beta 3L_1/4}\ \sqrt{1-\epsilon_1}\ \tilde{F_3}.$$

Combining above equation with Equation (4.16) yields

$$\tilde{F_4}' = e^{i\beta 3L_1/4}\ i\sqrt{\epsilon_1}\ \tilde{F_2} + e^{i\beta 3L_1/4}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_1/4}\ \tilde{F_3}'. \quad (5.6)$$

Combining Equations (5.5) and (5.6) yields,

$$\tilde{F_3}' = (e^{i\beta 3L_1/4}\, i\sqrt{\epsilon_1}\, \tilde{F}_2 + e^{i\beta 3L_1/4}\, \sqrt{1-\epsilon_1}\, e^{i\beta L_1/4}\, \tilde{F_3}') \left(\frac{\sqrt{1-\epsilon_3} - \sqrt{1-\epsilon_2}\, e^{i\beta L_2}}{1-t_2} \right)$$

$$- \tilde{E}_1 \left(\frac{\sqrt{\epsilon_2\epsilon_3}\, \sqrt{1-\epsilon_1}\, e^{i\beta L_s}\, e^{i\beta\, 3L_2/4}}{1-t_2} \right)$$

$$- \tilde{E}_4 \left(\frac{i\, \sqrt{\epsilon_1\epsilon_2\epsilon_3}\, e^{i\beta L_s}\, e^{i\beta\, 3L_2/4}}{1-t_2} \right),$$

$$\tilde{F_3}' = \tilde{F}_2 \left(\frac{i\sqrt{\epsilon_1}\sqrt{1-\epsilon_3}\, e^{i\beta 3L_1/4} - i\sqrt{\epsilon_1}\, \sqrt{1-\epsilon_2}\, e^{i\beta L_2}\, e^{i\beta 3L_1/4}}{1-t_2} \right)$$

$$+ \tilde{F_3}' \left(\frac{t_1 - t_X}{1-t_2} \right) - \tilde{E}_1 \left(\frac{\sqrt{\epsilon_2\epsilon_3}\, \sqrt{1-\epsilon_1}\, e^{i\beta L_s}\, e^{i\beta\, 3L_2/4}}{1-t_2} \right) - \tilde{E}_4 \left(\frac{i\, \sqrt{\epsilon_1\epsilon_2\epsilon_3}\, e^{i\beta L_s}\, e^{i\beta\, 3L_2/4}}{1-t_2} \right),$$

$$\tilde{F_3}'\left(1-\left(\frac{t_1-t_X}{1-t_2}\right)\right) = \tilde{F_2}\left(\frac{i\sqrt{\epsilon_1}\sqrt{1-\epsilon_3}\ e^{i\beta 3L_1/4}\ -i\sqrt{\epsilon_1}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta 3L_1/4}}{1-t_2}\right)$$

$$-\tilde{E_1}\left(\frac{\sqrt{\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2}\right)-\tilde{E_4}\left(\frac{i\ \sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2}\right),$$

$$\tilde{F_3}'\left(\frac{1-t_2-t_1-t_X}{1-t_2}\right) = \tilde{F_2}\left(\frac{i\sqrt{\epsilon_1}\sqrt{1-\epsilon_3}\ e^{i\beta 3L_1/4}\ -i\sqrt{\epsilon_1}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta 3L_1/4}}{1-t_2}\right)$$

$$-\tilde{E_1}\left(\frac{\sqrt{\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2}\right)-\tilde{E_4}\left(\frac{i\ \sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_2}\right).$$

Therefore,

$$\tilde{F_3}' = \tilde{F_2}\left(\frac{i\sqrt{\epsilon_1}\sqrt{1-\epsilon_3}\ e^{i\beta 3L_1/4}\ -i\sqrt{\epsilon_1}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta 3L_1/4}}{1-t_1-t_2+t_X}\right)$$

$$-\tilde{E_1}\left(\frac{\sqrt{\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_1-t_2+t_X}\right)-\tilde{E_4}\left(\frac{i\ \sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_1-t_2+t_X}\right). \tag{5.7}$$

As seen above, $F_2 = i\sqrt{\epsilon_2}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}$. Therefore the equation becomes

$$\tilde{F_3}' = F_1' \left(\frac{-\sqrt{\epsilon_1\ \epsilon_2}\sqrt{1-\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta 3L_1/4}\ + \sqrt{\epsilon_1 \epsilon_2}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta 3L_1/4}}{1 - t_1 - t_2 + t_X} \right)$$

$$- \tilde{E_1} \left(\frac{\sqrt{\epsilon_2 \epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1 - t_1 - t_2 + t_X} \right) - \tilde{E_4} \left(\frac{i\ \sqrt{\epsilon_1 \epsilon_2 \epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1 - t_1 - t_2 + t_X} \right). \tag{5.8}$$

Combining the above equation with the equation for $\tilde{F_1}$ yields $\tilde{F_1}$ in terms of F_1', $\tilde{E_1}$ and $\tilde{E_4}$.

5.2 Solving for $\tilde{F_1}$ in terms of F_1', $\tilde{E_1}$ and $\tilde{E_4}$

From Equation (5.3), we know

$$\tilde{F_1} = e^{i\beta L_1/4}\ i\sqrt{\epsilon_1}\ \tilde{F_3}' +\ i\sqrt{\epsilon_2}\sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ \tilde{F_1}'.$$

Combining above equation with Equation (5.8) yields,

$$\tilde{F}_1 = e^{i\beta L_1/4}\, i\sqrt{\epsilon_1} \left(F_1' \left(\frac{-\sqrt{\epsilon_1\ \epsilon_2}\sqrt{1-\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta 3L_1/4} \\ +\sqrt{\epsilon_1\epsilon_2}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta 3L_1/4}}{1-t_1-t_2+t_X} \right)\right)$$

$$-e^{i\beta L_1/4}\, i\sqrt{\epsilon_1} \left(\tilde{E}_1 \left(\frac{\sqrt{\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_1-t_2+t_X} \right) + \tilde{E}_4 \left(\frac{i\ \sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}}{1-t_1-t_2+t_X} \right)\right)$$

$$+\ i\sqrt{\epsilon_2}\sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ \tilde{F_1}',$$

$$\tilde{F}_1 = F_1' \left(\frac{-i\ \epsilon_1\sqrt{\ \epsilon_2}\sqrt{1-\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1} \\ +i\ \epsilon_1\sqrt{\epsilon_2}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1}}{1-t_1-t_2+t_X} \right)$$

$$+\tilde{E}_1 \left(\frac{-i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right) + \tilde{E}_4 \left(\frac{\epsilon_1\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right)$$

$$+\ i\sqrt{\epsilon_2}\sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ \tilde{F_1}',$$

$$\tilde{F}_1 = F_1' \left(\frac{\begin{array}{l} -i\ \epsilon_1\sqrt{\ \epsilon_2}\sqrt{1-\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1} \\ \quad +i\ \epsilon_1\sqrt{\epsilon_2}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1} \end{array}}{1-t_1-t_2+t_X} + \ i\sqrt{\epsilon_2}\sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4} \right)$$

$$+\tilde{E}_1 \left(\frac{-i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right) + \tilde{E}_4 \left(\frac{\epsilon_1\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right).$$

$$= F_1' \left(\frac{\begin{array}{l} -i\ \epsilon_1\sqrt{\ \epsilon_2}\sqrt{1-\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1} \\ \quad +i\ \epsilon_1\sqrt{\epsilon_2}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1} \\ \qquad +\ i\sqrt{\epsilon_2}\sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}(1-t_1-t_2+t_X) \end{array}}{1-t_1-t_2+t_X} \right)$$

$$+\tilde{E}_1 \left(\frac{-i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right) + \tilde{E}_4 \left(\frac{\epsilon_1\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right). \tag{5.9}$$

5.3 Solving for F_1' in terms of $\tilde{E}_1$ and $\tilde{E}_4$

From Equation (4.21) we know,

$$F_1' = \tilde{F}_2{}'\ \sqrt{1-\epsilon_3} + \tilde{E}_3{}'\ i\ \sqrt{\epsilon_3}.$$

Combining the above equation with Equations (4.10) and (4.4) yields

$$F_1' = \tilde{F}_4{}'' \; e^{i\beta L_2/4} \; \sqrt{1-\epsilon_3} + \tilde{E}_3 \; e^{i\beta L_1/4} \; i \; \sqrt{\epsilon_3}$$

Combining the above equation with Equations (4.13) and (4.1) yields

$$F_1' = \sqrt{1-\epsilon_2} \; \tilde{F}_3{}'' \; e^{i\beta L_2/4} \; \sqrt{1-\epsilon_3} + (i\sqrt{\epsilon_1} \; \tilde{E}_1 + \sqrt{1-\epsilon_1} \; \tilde{E}_4) \; e^{i\beta L_1/4} \; i \; \sqrt{\epsilon_3},$$

$$F_1' = \sqrt{(1-\epsilon_2)(1-\epsilon_3)} \; \tilde{F}_3{}'' \; e^{i\beta L_2/4} - \sqrt{\epsilon_1 \; \epsilon_3} \; \tilde{E}_1 \; e^{i\beta L_1/4} + i \; \sqrt{\epsilon_3}\sqrt{1-\epsilon_1} \; \tilde{E}_4 \; e^{i\beta L_1/4}.$$

Combining the above equation with Equation (4.11) yields

$$F_1' = \sqrt{(1-\epsilon_2)(1-\epsilon_3)} \; e^{i\beta L_2} \; \tilde{F}_1{}' - \sqrt{\epsilon_1 \; \epsilon_3} \; \tilde{E}_1 \; e^{i\beta L_1/4} + i \; \sqrt{\epsilon_3}\sqrt{1-\epsilon_1} \; \tilde{E}_4 \; e^{i\beta L_1/4},$$

$$F_1' \left(1 - \sqrt{(1-\epsilon_2)(1-\epsilon_3)} \; e^{i\beta L_2} \right) = -\sqrt{\epsilon_1 \; \epsilon_3} \; \tilde{E}_1 \; e^{i\beta L_1/4} + i \; \sqrt{\epsilon_3}\sqrt{1-\epsilon_1} \; \tilde{E}_4 \; e^{i\beta L_1/4},$$

$$F_1' = \left(\frac{-\sqrt{\epsilon_1 \; \epsilon_3} \; e^{i\beta L_1/4}}{1 - \sqrt{(1-\epsilon_2)(1-\epsilon_3)} \; e^{i\beta L_2}} \right) \; \tilde{E}_1 + \left(\frac{i \; \sqrt{\epsilon_3}\sqrt{1-\epsilon_1} \; e^{i\beta L_1/4}}{1 - \sqrt{(1-\epsilon_2)(1-\epsilon_3)} \; e^{i\beta L_2}} \right) \; \tilde{E}_4,$$

$$= \left(\frac{-\sqrt{\epsilon_1 \; \epsilon_3} \; e^{i\beta L_1/4}}{1-t_2} \right) \; \tilde{E}_1 + \left(\frac{i \; \sqrt{\epsilon_3}\sqrt{1-\epsilon_1} \; e^{i\beta L_1/4}}{1-t_2} \right) \; \tilde{E}_4. \quad (5.10)$$

5.4 $\tilde{F}_1$ in terms of $\tilde{E}_1$ and $\tilde{E}_4$

Combining equations for $\tilde{F}_1$ and F_1' yields $\tilde{F}_1$ in terms of $\tilde{E}_1$ and $\tilde{E}_4$. From Equation (5.9) , we know

$$\tilde{F}_1 = F_1' \left(\frac{\begin{array}{l} -i\ \epsilon_1\sqrt{\epsilon_2}\sqrt{1-\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1} \\ \quad +i\ \epsilon_1\sqrt{\epsilon_2}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1} \\ \qquad +\ i\sqrt{\epsilon_2}\sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}(1-t_1-t_2+t_X) \end{array}}{1-t_1-t_2+t_X} \right)$$

$$+\tilde{E}_1 \left(\frac{-i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right) + \tilde{E}_4 \left(\frac{\epsilon_1\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right).$$

Combining the above equation and Equation (5.10) yields

$$\tilde{F_1} = \left(\left(\frac{-\sqrt{\epsilon_1\ \epsilon_3}\ e^{i\beta L_1/4}}{1-t_2} \right)\ \tilde{E_1} + \left(\frac{i\ \sqrt{\epsilon_3}\sqrt{1-\epsilon_1}\ e^{i\beta L_1/4}}{1-t_2} \right)\ \tilde{E_4} \right) \times$$

$$\left(\frac{\begin{array}{l} -i\ \epsilon_1\sqrt{\ \epsilon_2}\sqrt{1-\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1} \\ \quad +i\ \epsilon_1\sqrt{\epsilon_2}\ \sqrt{1-\epsilon_2}\ e^{i\beta L_2}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\ e^{i\beta L_1} \\ \qquad +\ i\sqrt{\epsilon_2}\sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}(1-t_1-t_2+t_X) \end{array}}{1-t_1-t_2+t_X} \right)$$

$$+\tilde{E_1}\ \left(\frac{-i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right) + \tilde{E_4}\ \left(\frac{\epsilon_1\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right),$$

$$= \left(\left(\frac{-\sqrt{\epsilon_1\ \epsilon_3}\ e^{i\beta L_1/4}}{1-t_2} \right)\ \tilde{E_1} + \left(\frac{i\ \sqrt{\epsilon_3}\sqrt{1-\epsilon_1}\ e^{i\beta L_1/4}}{1-t_2} \right)\ \tilde{E_4} \right) \times$$

$$\left(\frac{\begin{array}{l} i\sqrt{\epsilon_2}\sqrt{1-\epsilon_1}e^{i\beta L_s}e^{i\beta L_2/4} - i\sqrt{\epsilon_2}\sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}e^{i\beta L_2}e^{i\beta 3L_2/4}e^{i\beta L_s} \\ \qquad - i\sqrt{\epsilon_2}\sqrt{1-\epsilon_3}e^{i\beta L_1}e^{i\beta L_s}e^{i\beta 3L_2/4} + i\sqrt{\epsilon_2}\sqrt{1-\epsilon_2}e^{i\beta L_1}e^{i\beta L_2}e^{i\beta 3L_2/4}e^{i\beta L_s} \end{array}}{1-t_1-t_2+t_X} \right)$$

$$+\tilde{E_1}\ \left(\frac{-i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right) + \tilde{E_4}\ \left(\frac{\epsilon_1\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right).$$

Therefore

$$\tilde{F_1} = \left(\frac{\begin{array}{l}\big(i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta L_1/4}\ e^{i\beta 3L_2/4}\big)\ \times \\ \quad (-\sqrt{1-\epsilon_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}e^{i\beta L_2} + \sqrt{1-\epsilon_3}e^{i\beta L_1} - \sqrt{1-\epsilon_2}e^{i\beta L_1}e^{i\beta L_2})\end{array}}{(1-t_2)(1-t_1-t_2+t_X)} \right)\ \tilde{E_1}$$

$$+ \left(\frac{\begin{array}{l}\big(\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_1/4}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\big)\ \times \\ \quad (-1+\epsilon_1 + \sqrt{(1-\epsilon_2)(1-\epsilon_3)}e^{i\beta L_2} - \epsilon_1\ \sqrt{(1-\epsilon_2)(1-\epsilon_3)}\ e^{i\beta L_2} \\ \qquad + \sqrt{(1-\epsilon_1)(1-\epsilon_3)}e^{i\beta L_1} - \sqrt{(1-\epsilon_1)(1-\epsilon_2)}e^{i\beta L_1}e^{i\beta L_2})\end{array}}{(1-t_2)(1-t_1-t_2+t_X)} \right)\ \tilde{E_4}$$

$$+\tilde{E_1}\ \left(\frac{-i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right) + \tilde{E_4}\ \left(\frac{\epsilon_1\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}}{1-t_1-t_2+t_X} \right),$$

$$= \left(\frac{\begin{array}{l}\big(i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta L_1/4}\ e^{i\beta 3L_2/4}\big)\times \\ \quad (-\sqrt{1-\epsilon_1} + \sqrt{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}e^{i\beta L_2} + \sqrt{1-\epsilon_3}e^{i\beta L_1} - \sqrt{1-\epsilon_2}e^{i\beta L_1}e^{i\beta L_2}) \\ \qquad - i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ \sqrt{1-\epsilon_1}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}(1-\sqrt{(1-\epsilon_3)(1-\epsilon_2)}e^{i\beta L_2})\end{array}}{(1-t_2)(1-t_1-t_2+t_X)} \right) \tilde{E_1}$$

$$+ \left(\frac{\begin{array}{l}\big(\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_1/4}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\big)\ \times (-1+\epsilon_1+t_2-\epsilon_1\ t_2+t_1-t_X) \\ \quad +\ \epsilon_1\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta\ 3L_2/4}\ e^{i\beta L_1/4}(1-\sqrt{(1-\epsilon_3)(1-\epsilon_2)}e^{i\beta L_2})\end{array}}{(1-t_2)(1-t_1-t_2+t_X)} \right)\ \tilde{E_4}.$$

Therefore ,

$$\tilde{F_1} = \left(\frac{i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta L_1/4}\ e^{i\beta 3L_2/4}(-2\sqrt{1-\epsilon_1}(1-t_2) + e^{i\beta L_1}(\sqrt{1-\epsilon_3} - \sqrt{1-\epsilon_2}e^{i\beta L_2}))}{(1-t_2)(1-t_1-t_2+t_X)} \right)\ \tilde{E_1}$$

$$+ \left(\frac{\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_1/4}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}(2\epsilon_1 - 2t_2\ \epsilon_1 - 1 + t_2 + t_1 - t_X)}{(1-t_2)(1-t_1-t_2+t_X)} \right)\ \tilde{E_4}. \tag{5.11}$$

5.4.1 Solving for $\tilde{F_1}$ in terms of $\tilde{E_1}$ only

From Equation (4.32) we have $\tilde{E_4}$ based on $\tilde{E_1}$.

$$\tilde{E_4} = \tilde{E_1} \left(\frac{e^{i\beta L_1}\ i\sqrt{\epsilon_1(1-\epsilon_3)} - e^{i\beta(L_1+L_2)}\ i\sqrt{\epsilon_1(1-\epsilon_2)}}{1-t_1-t_2+t_X} \right).$$

Combining the above equation with Equation (5.11) yields,

$$\tilde{F_1} = \left(\frac{i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta L_1/4}\ e^{i\beta 3L_2/4}(-2\sqrt{1-\epsilon_1}(1-t_2) + e^{i\beta L_1}(\sqrt{1-\epsilon_3} - \sqrt{1-\epsilon_2}e^{i\beta L_2}))}{(1-t_2)(1-t_1-t_2+t_X)} \right) \tilde{E_1}$$

$$+ \left(\frac{\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_1/4}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}(2\epsilon_1 - 2t_2\ \epsilon_1 - 1 + t_2 + t_1 - t_X)}{(1-t_2)(1-t_1-t_2+t_X)} \right) \times$$

$$\left(\frac{e^{i\beta L_1}\ i\sqrt{\epsilon_1(1-\epsilon_3)} - e^{i\beta(L_1+L_2)}\ i\sqrt{\epsilon_1(1-\epsilon_2)}}{1-t_1-t_2+t_X} \right) \tilde{E_1},$$

$$= \left(\frac{i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta L_1/4}\ e^{i\beta 3L_2/4}(-2\sqrt{1-\epsilon_1}(1-t_2) + e^{i\beta L_1}(\sqrt{1-\epsilon_3} - \sqrt{1-\epsilon_2}e^{i\beta L_2}))}{(1-t_2)(1-t_1-t_2+t_X)} \right) \tilde{E_1}$$

$$+ \left(\frac{\begin{array}{r}\sqrt{\epsilon_2\epsilon_3}\ e^{i\beta L_1/4}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}(2\epsilon_1 - 2\ t_2\ \epsilon_1 - 1 + t_2 + t_1 - t_X)\times \\ (e^{i\beta L_1}\ i\sqrt{\epsilon_1(1-\epsilon_3)} - e^{i\beta(L_1+L_2)}\ i\sqrt{\epsilon_1(1-\epsilon_2)})\end{array}}{(1-t_2)(1-t_1-t_2+t_X)^2} \right) \tilde{E_1}.$$

$$= \left(\frac{i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_s}\ e^{i\beta L_1/4}\ e^{i\beta 3L_2/4}(-2\sqrt{1-\epsilon_1}(1-t_2) + e^{i\beta L_1}(\sqrt{1-\epsilon_3} - \sqrt{1-\epsilon_2}e^{i\beta L_2}))}{(1-t_2)(1-t_1-t_2+t_X)} \right) \tilde{E_1}$$

$$+ \left(\frac{\begin{array}{r}i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_1/4}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}(2\epsilon_1 - 2t_2\ \epsilon_1 - (1-t_1-t_2+t_X))\times \\ e^{i\beta L_1}(\sqrt{(1-\epsilon_3)} - e^{i\beta L_2}\sqrt{(1-\epsilon_2)})\end{array}}{(1-t_2)(1-t_1-t_2+t_X)^2} \right) \tilde{E_1}.$$

$$= i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_1/4}\ e^{i\beta L_s}\ e^{i\beta 3L_2/4}\times$$

$$\left(\frac{\begin{array}{l}(1-t_1-t_2+t_X)(-2\sqrt{1-\epsilon_1}(1-t_2)+e^{i\beta L_1}(\sqrt{1-\epsilon_3}-\sqrt{1-\epsilon_2}e^{i\beta L_2})\\ \quad +e^{i\beta L_1}(\sqrt{(1-\epsilon_3)}-e^{i\beta L_2}\sqrt{(1-\epsilon_2)}(2\epsilon_1-2t_2\ \epsilon_1-(1-t_1-t_2+t_X))\end{array}}{(1-t_2)(1-t_1-t_2+t_X)^2} \right)\ \tilde{E}_1,$$

$$= i\sqrt{\epsilon_1\epsilon_2\epsilon_3}e^{i\beta L_1/4}e^{i\beta L_s}e^{i\beta 3L_2/4}\left(\frac{\begin{array}{l}-2\sqrt{1-\epsilon_1}(1-t_2)(1-t_1-t_2+t_X)\\ \quad +e^{i\beta L_1}(\sqrt{(1-\epsilon_3)}-e^{i\beta L_2}\sqrt{(1-\epsilon_2)}(2\epsilon_1-2\ t_2\epsilon_1)\end{array}}{(1-t_2)(1-t_1-t_2+t_X)^2} \right)\tilde{E}_1.$$

Thus the final equation for $\tilde{F}_1$ becomes,

$$\tilde{F}_1 = i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_1/4}e^{i\beta L_s}e^{i\beta 3L_2/4}\left(\frac{\begin{array}{l}-2\sqrt{1-\epsilon_1}(1-t_1-t_2+t_X)\\ \quad +2\epsilon_1\ e^{i\beta L_1}(\sqrt{(1-\epsilon_3)}-e^{i\beta L_2}\sqrt{(1-\epsilon_2)}\end{array}}{(1-t_1-t_2+t_X)^2} \right)\tilde{E}_1. \tag{5.12}$$

We know

$$\tilde{F}_1 = \tilde{r}\ \tilde{E}_1$$

Therefore the reflection coefficient $\tilde{r}$ is

$$\tilde{r} = i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\ e^{i\beta L_1/4}e^{i\beta L_s}e^{i\beta 3L_2/4}\left(\frac{\begin{array}{l}-2\sqrt{1-\epsilon_1}(1-t_1-t_2+t_X)\\ \quad +2\epsilon_1\ e^{i\beta L_1}(\sqrt{(1-\epsilon_3)}-e^{i\beta L_2}\sqrt{(1-\epsilon_2)}\end{array}}{(1-t_1-t_2+t_X)^2}\right). \tag{5.13}$$

5.5 Reflectivity plot based on numerical expression

We know reflectivity R is the product of the reflection coefficient $\tilde{r}$ and its conjugate $\tilde{r}^\star$.

$$R = |\frac{\tilde{F}_1}{\tilde{E}_1}|^2 = |\tilde{r}|^2 = \tilde{r}\ \times\ \tilde{r}^\star. \tag{5.14}$$

Figure 5.2 shows the response obtained when $\tilde{r}\ \times\ \tilde{r}^\star$ is plotted against $\beta L/\pi$ with $\epsilon_1 = \epsilon_2 = 0.1$ and $\epsilon_3 = 0.016$.

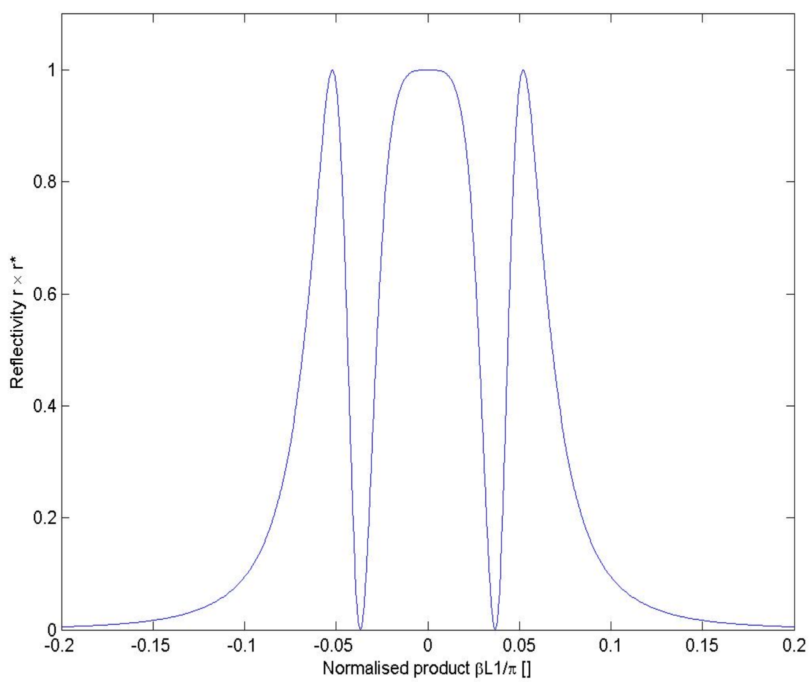

Figure 5.2: Reflectivity plot of a DRR where reflectivity R is calculated as $\tilde{r} \times \tilde{r}^{\star}$

Chapter 6

Sanity Checks

The goal of this chapter is to establish the correctness of the derived equations for the transmitted and reflected electric field of the DRR . This can be done by several methods. One method is by substituting ϵ_1, ϵ_2 or $\epsilon_3 = 0$ in the equations and checking if the results match the corresponding equations for simpler structures. Another method is by checking if the plots of the obtained results match known results in journal papers. Th final sanity check would be checking if power is conserved by adding T and R. The different sanity checks are performed on both transmission and reflection in the following sections.

6.1 Sanity check for transmission equation

This section performs the sanity check on the transmitted field equation (4.48). The correctness of the derived equation for the transmitted electric field of the DRR can be checked by substituting ϵ_1 and $\epsilon_3 = 0$ orϵ_2 and $\epsilon_3 = 0$ in the equation (4.48) and then comparing the resulting equation with the equation for the output electric field of a single ring resonator.

6.1.1 $\epsilon_2 = \epsilon_3 = 0$

When $\epsilon_2 = \epsilon_3 = 0$, Equation (4.48) becomes,

$$\tilde{E_2}'' = \left(\frac{\begin{array}{r}(\epsilon_1 e^{i\beta(L_1+L_2+L_s)} - \epsilon_1 e^{i\beta(L_1+L_s)})(1-t_1-t_2+t_X) \\ +\,(e^{i\beta L_s}\ \sqrt{(1-\epsilon_1})(1-t_1-t_2+t_X)^2\end{array}}{(1-t_1-t_2+t_X)^2} \right) \tilde{E_1},$$

$$= \left(\frac{-\epsilon_1\ e^{i\beta(L_1+L_s)}(1-e^{i\beta L_2}) + (e^{i\beta L_s}\ \sqrt{1-\epsilon_1})(1-t_1-t_2+t_X)}{1-t_1-t_2+t_X} \right) \tilde{E_1}.$$

When $\epsilon_2 = \epsilon_3 = 0$,

$$1-t_1-t_2+t_X = 1-e^{i\beta L_2} - \sqrt{1-\epsilon_1}\ e^{i\beta L_1} + \sqrt{1-\epsilon_1}\ e^{i\beta(L_1+L_2)},$$

$$= (1-e^{i\beta L_2})(1-\sqrt{1-\epsilon_1}\ e^{i\beta L_1}).$$

Therefore,

$$\tilde{E_2}'' = \left[\frac{-\epsilon_1\ e^{i\beta(L_1+L_s)}(1-e^{i\beta L_2}) + e^{i\beta L_s}\ \sqrt{1-\epsilon_1}\ (1-e^{i\beta L_2})(1-\sqrt{1-\epsilon_1}\ e^{i\beta L_1})}{(1-e^{i\beta L_2})(1-\sqrt{1-\epsilon_1}\ e^{i\beta L_1})}\right]\tilde{E_1}.$$

Note that all L_2 terms drop out, as makes physical sense:

$$\tilde{E_2}'' = \left[\frac{-\epsilon_1\ e^{i\beta(L_1+L_s)} + e^{i\beta L_s}\ \sqrt{1-\epsilon_1}\ (1-\sqrt{1-\epsilon_1}\ e^{i\beta L_1})}{(1-\sqrt{1-\epsilon_1}\ e^{i\beta L_1})}\right]\tilde{E_1},$$

$$= \left[\frac{-\epsilon_1\ e^{i\beta(L_1+L_s)} + e^{i\beta L_s}\ \sqrt{1-\epsilon_1} - e^{i\beta L_s}\ (1-\epsilon_1)\ e^{i\beta L_1}}{(1-\sqrt{1-\epsilon_1}\ e^{i\beta L_1})}\right]\tilde{E_1},$$

$$= \left[\frac{e^{i\beta L_s}\ (\sqrt{1-\epsilon_1} - e^{i\beta L_1})}{(1-\sqrt{1-\epsilon_1}\ e^{i\beta L_1})}\right]\tilde{E_1},$$

$$= \left[\frac{1-\sqrt{1-\epsilon_1}\ e^{-i\beta L_1}}{1-\sqrt{1-\epsilon_1}\ e^{i\beta L_1}}\right] e^{i\pi}\ e^{i\beta L_1}\ e^{i\beta L_s}.$$

which is equal to the output electric field of a single ring resonator multiplied by a factor $e^{i\beta L_s}$ [10]. This is the expected result.

6.1.2 $\epsilon_1 = \epsilon_3 = 0$

When $\epsilon_2 = \epsilon_3 = 0$,

$$1 - t_1 - t_2 + t_X = (1 - \sqrt{(1-\epsilon_2)}e^{i\beta L_2} + \sqrt{(1-\epsilon_2)}\ e^{i\beta(L_1+L_2)},$$

$$= (1 - e^{i\beta L_1})\ (1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2}).$$

Therefore Equation (4.48) becomes,

$$\tilde{E_2}'' = \left(\frac{\begin{array}{r}(\epsilon_2 e^{i\beta(L_1+L_2+L_s)} - \epsilon_2 e^{i\beta(L_1+L_s)})(1 - e^{i\beta L_2}\sqrt{1-\epsilon_2} - e^{i\beta L_1} + e^{i\beta(L_1+L_2)}\sqrt{1-\epsilon_2} \\ + (e^{i\beta L_s}\ \sqrt{(1-\epsilon_2})((1 - e^{i\beta L_1})\ (1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2}))^2\end{array}}{((1 - e^{i\beta L_1})\ (1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2}))^2} \right) \tilde{E}_1,$$

$$= \left(\frac{\begin{array}{r}-\epsilon_2\ e^{i\beta(L_2+L_s)}(1 - e^{i\beta L_1})(1 - e^{i\beta L_1})\ (1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2}) \\ + (e^{i\beta L_s}\ \sqrt{(1-\epsilon_2})((1 - e^{i\beta L_1})\ (1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2}))^2\end{array}}{((1 - e^{i\beta L_1})\ (1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2}))^2} \right) \tilde{E}_1,$$

$$= \left(\frac{-\epsilon_2\ e^{i\beta(L_2+L_s)} + e^{i\beta L_s}\ \sqrt{(1-\epsilon_2}\ (1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2})}{1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2}} \right) \tilde{E}_1,$$

$$= \left(\frac{-\epsilon_2\ e^{i\beta(L_2+L_s)} + e^{i\beta L_s}\ \sqrt{(1-\epsilon_2} - e^{i\beta(L_s+L_2)} + \epsilon_2\ e^{i\beta(L_s+L_2)}}{1 - \sqrt{1-\epsilon_2}\ e^{i\beta L_2}} \right) \tilde{E}_1,$$

$$= \left(\frac{e^{i\beta L_s} \; (\sqrt{(1-\epsilon_2} - \; e^{i\beta L_2})}{1-\sqrt{1-\epsilon_2} \; e^{i\beta L_2}} \right) \tilde{E}_1,$$

$$= \left[\frac{1-\sqrt{1-\epsilon_2} \; e^{-i\beta L_2}}{1-\sqrt{1-\epsilon_2} \; e^{i\beta L_2}} \right] e^{i\pi} \; e^{i\beta L_2} \; e^{i\beta L_s}.$$

which is equal to the output electric field of a single ring resonator multiplied by a factor $e^{i\beta L_s}$ [10]. This is the expected result.

6.2 Sanity check for reflection equation

This section performs the sanity check on the reflected field equation (5.34). The correctness of the derived equation for the reflected electric field of the DRR can be checked by substituting ϵ_1, ϵ_2 or $\epsilon_3 = 0$ in the equation (5.34). The reflected electric field should be equal to zero when either one of these values is zero .

Another way to establish the correctness of the equation is by matching with known results from previous work in journal papers.

6.2.1 ϵ_1, ϵ_2 or $\epsilon_3 = 0$

From Equation (5.12), we know

$$\tilde{F_1} = i\sqrt{\epsilon_1\epsilon_2\epsilon_3}\, e^{i\beta L_1/4} e^{i\beta L_s} e^{i\beta 3L_2/4} \left(\frac{\begin{array}{c} -2\sqrt{1-\epsilon_1}(1-t_1-t_2+t_X) \\ +\, 2\epsilon_1\; e^{i\beta L_1}(\sqrt{(1-\epsilon_3)} - e^{i\beta L_2}\sqrt{(1-\epsilon_2)} \end{array}}{(1-t_1-t_2+t_X)^2} \right) \tilde{E_1}.$$

As the derived equation for the reflected electric field $\tilde{F_1}$ contains a multiplicative factor $i\sqrt{\epsilon_1\epsilon_2\epsilon_3}e^{i\beta L_1/4}e^{i\beta L_s}e^{i\beta 3L_2/4}$, when either ϵ_1, ϵ_2 or $\epsilon_3 = 0$, the equation becomes zero which is the expected result as there will be no reflected electric field when there is no coupling between the rings.

6.2.2 Matching known results

Several parameters were set to match the reflectivity response in a previously published work on DRR [4] and a comparison is performed. The coupling coefficients ϵ_1 and ϵ_2 were set at 0.1 .The coupling between the rings ϵ_3 was set at different values of 0.05, 0.016 and 0.0028. The reflection response thus obtained is given in Figure 6.1.

The reflection profiles obtained at the different values of ϵ_3 were found to match perfectly with the reflectivity graph given in (fig.3) in a previously reported study[4]. The highest value of coupling ($\epsilon_3 = 0.05$) generates a four peak reflection profile while the lowest value ($\epsilon_3 = 0.0028$) generates a two peak profile. The number of zeros in the reflection profiles of each coupling value also matches with the corresponding values in (fig.3). We can also see that the FSR corresponds to 2π in both cases.

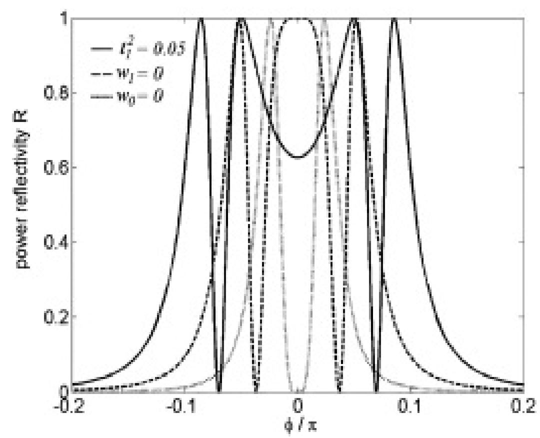

Fig. 3. Reflectivity profiles for $t^2 = 0.1$ and three different values of $t_1^2 =$ 0.05, 0.016($w_1 = 0$), and 0.0028($w_0 = 0$). Note that the FSR corresponds to $\Delta\phi = 2\pi$.

Figure 6.1: fig 3 , Chremmos and Uzunoglu (2005).

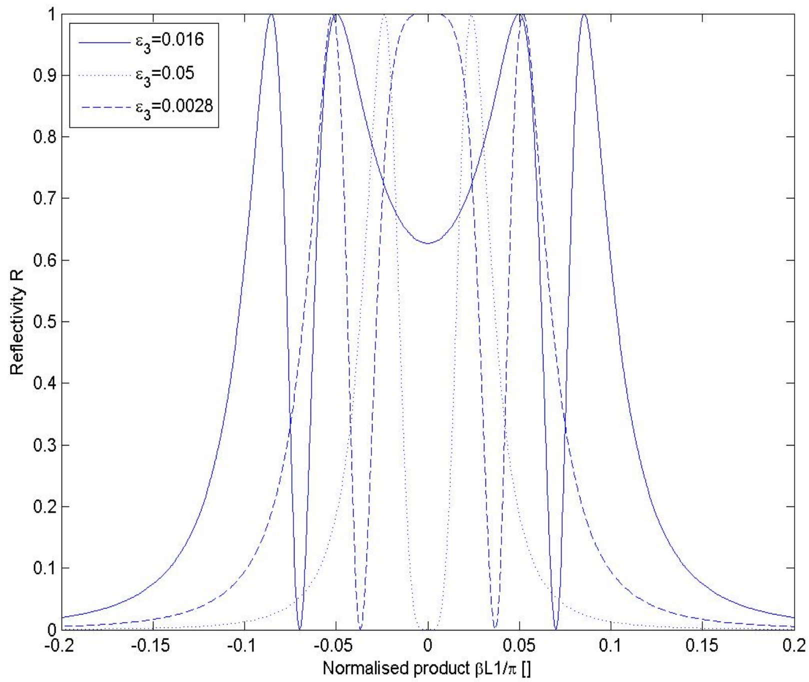

Figure 6.2: Reflectivity plot at $\epsilon_1 = \epsilon_2 = 0.1$ and $\epsilon_3 = 0.05, 0.016, 0.0028$.

6.3 Conservation of power $|\tilde{r}|^2 + |\tilde{t}|^2 = 1$

The conservation of power is checked by adding the equations for reflectivity $|\tilde{r}|^2$ and transmittivity $|\tilde{t}|^2$ in Matlab. Figure 6.3 shows that the result is one which proves that power is conserved.

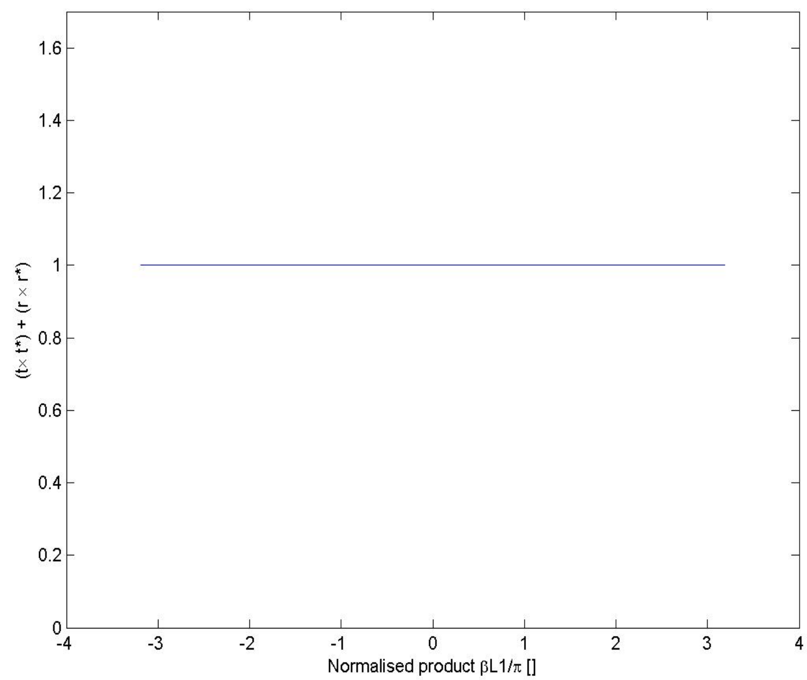

Figure 6.3: Conservation of power plot of DRR based on $(\tilde{t} \times \tilde{t}^{\star}) + (\tilde{r} \times \tilde{r}^{\star})$.

Chapter 7

Conclusion

This thesis performed a detailed analysis of a parallel coupled double ring resonator (DRR) using the method of equating fields (MEF). An analysis was first performed on simpler structures like a single ring resonator and uncoupled double ring resonator in the early chapters in order to develop several important limiting cases for comparison. In the later chapters, derivations for the transmitted field and reflected field of the DRR were carried out using MEF by solving a linear system of 22 equations describing the field points. We obtained generalized transmission and reflection coefficients of the DRR. The results obtained may be very useful in DRR design as the circumference of the rings as well as the coupling coefficients can be varied.

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