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Symmetric Product Graphs

by

Evan Witz

A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of Masters of Science
in Applied and Computational Mathematics

School of Mathematical Sciences

College of Science

Rochester Institute of Technology

May 20, 2015

Abstract

We describe four types of hyperspace graphs; namely, the simultaneous and nonsimultaneous symmetric product graphs, as well as their respective layers. These hyperspace graphs are meant to be analogous to the concepts of hyperspaces in topology, in that they are constructed by taking in another graph as an input in the construction of the hyperspace graph. We establish subgraph relationship between these graphs and establish some properties on the orders and sizes of the graphs, as well as on the degrees of the individual vertices of these graphs. We establish that these graphs are connected (providing that the input graph is connected), and provide a categorization of the graphs G for which the second symmetric product graphs are planar. We investigate the chromatic numbers and hamiltonicity of some of these graph products. We also provide a categorization for the distances between any pair of vertices in the symmetric product graphs. We conclude by discussing a couple of different unanswered questions that could be addressed in the future.

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Contents

1	Introduction	1
2	Definitions	3
3	Subgraphs	6
4	Order, Size and Vertex Degree	11
5	Connectedness	17
6	Planarity	21
7	Colorings	36
8	Distances	39
9	Hamiltonicity	45
10	Conclusion	50

Chapter 1

Introduction

In the field of Topology, there exists a particular type of topological space called a hyperspace. These hyperspaces are constructed by taking in some other topological space as an input, and constructing a new topological space on some family of subsets of the original space. A number of different techniques can be used to construct open sets on this new space, by establishing some notion of how "close" sets are. One particular hyperspace is $\mathcal{F}_k(\mathbb{X})$ [1], which takes in a metric space \mathbb{X} and has a space of

$$\{U \subseteq \mathbb{X} \mid U \text{ is finite, } |U| \leq k\}$$

and a metric defined as

$$d(U, V) = \max \left(\sup_{u \in U} \left(\inf_{v \in V} d(u, v) \right), \sup_{v \in V} \left(\inf_{u \in U} d(u, v) \right) \right),$$

with which a topology can be constructed. (A similar topology can be constructed even if \mathbb{X} is not a *metric* space, but this notion of distances is sufficient for this paper.) By observing graphs as the union of several homeomorphic copies of the unit interval in \mathbb{R}^3 , (i.e, edges, where the intersection of two or more edges at their ends is a vertex), one can construct the topological hyperspace $\mathcal{F}_k(\mathbb{G})$, where \mathbb{G} is an arbitrary graph viewed as a topological space.

This concept of taking in spaces as inputs and constructing new spaces is analogous to the concept of product graphs in graph theory. A product graph is constructed by taking in some number of graphs as input, and constructing a new graph. These product graphs are typically constructed with a vertex set that is some subset of the Cartesian product of the vertex sets of its component graphs,

and adjacencies that are based on the adjacencies of the original graph's vertices. For example, the Cartesian product of two graphs G and H , denoted $G \square H$, is defined as follows:

$$V(G \square H) = \{(u, v) \mid u \in V(G), v \in V(H)\},$$

$$(u_1, v_1)(u_2, v_2) \in E(G \square H) \iff (u_1 = u_2, v_1 v_2 \in E(H)) \vee (u_1 u_2 \in E(G), v_1 = v_2).$$

As the two concepts are very similar, different graph products can be constructed in an effort to mimic different types of topological hyperspaces. [3] As we can construct the hyperspace \mathcal{F}_k of a graph \mathbb{G} when observed as a topological space, it makes sense to try to construct a direct analogy to \mathcal{F}_k in a graph theoretic sense. Hence, we construct two different types of "hyperspace" graph that are meant to mirror the behavior of the topological hyperspace \mathcal{F}_k in two different ways. For some arbitrary graph G (observed in a graph theoretic sense), we construct the two hyperspace graphs $\mathcal{F}_k(G)$ and $\mathcal{S}_k(G)$.

As these graphs are very similar in construction to product graphs, we naturally attempt to answer certain questions about these hyperspace graphs that have been answered about other product graphs. In Chapter 3, we discuss which hyperspace graphs are subgraphs of others. In Chapter 4, we establish some theorems about the orders and sizes of these graphs, as well as the degrees of their vertices. In Chapter 5, we briefly establish that these graphs are connected, provided that the original graph is also connected. In Chapter 6, we give some bounds for the chromatic numbers of these graphs, based on the chromatic number of the original graph. In Chapter 7, we give a categorization for the distances between arbitrary vertices in the hyperspace graphs. In Chapter 8, we show whether or not certain hyperspace graphs are Hamiltonian, or if they contain a Hamiltonian path. Finally, we conclude in Chapter 9 by stating some potential research questions for the future. Of particular note are the questions of uniqueness of these hyperspace graphs (that is, if two hyperspace graphs are isomorphic, must their original graphs also be isomorphic), and how these hyperspace graphs might be used to gain some understanding of the topological space $\mathcal{F}_k(\mathbb{G})$.

Chapter 2

Definitions

Symmetric Product Graph

For all graph theory terminology, we use the vocabulary described in [2].

Let G be a graph. Let $A, B \subseteq V(G)$ be non-empty. We define the k th symmetric product graph of G , denoted by $\mathcal{F}_k(G)$, as follows:

1. $A \in V(\mathcal{F}_k(G)) \iff A \subseteq V(G), 1 \leq |A| \leq k$
2. Let $A, B \subseteq V(G)$ be non-empty. $AB \in E(\mathcal{F}_k(G))$ if $\exists W \subseteq V(G)$ and $ab \in E(G)$ such that the following hold true:
 - (a) $A = W \cup \{a\}$
 - (b) $B = W \cup \{b\}$
 - (c) $A \neq B$.

(In this context, we call ab the *transversal* edge of the adjacency between the vertex sets A and B . Note that this edge is not necessarily unique for a given A and B .)

Consider P_4 , with vertices labeled 1, 2, 3, 4 in order. We can show that the sets $A = \{1, 2\}$ and $B = \{1, 3\}$ are adjacent in $\mathcal{F}_2(P_4)$. Let $W = \{1\}$, $a = 2$ and $b = 3$. Then $A = W \cup \{a\}$ and

$B = W \cup \{b\}$, and $a \neq b$. Thus, $AB \in E(\mathcal{F}_2(G))$. If $|A| \neq |B|$, then we choose our W slightly differently, in that W is now exactly equal to one of A or B . Suppose $A = \{3\}$ and $B = \{3, 4\}$. Then by letting $W = \{3\}$, $a = 3$ and $b = 4$, we have that $A = W \cup \{a\}$, $B = W \cup \{b\}$, and $A \neq B$. Therefore, $AB \in E(\mathcal{F}_2(G))$.

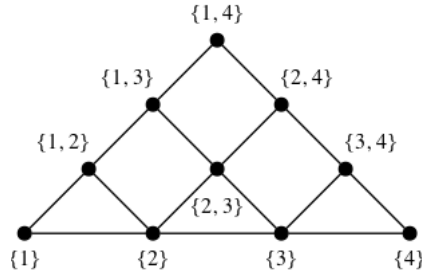


Figure 2.1: $\mathcal{F}_2(P_4)$.

For notational convenience, if G is a graph of order n , we write $\mathcal{F}_n(G)$ as $\mathcal{F}(G)$.

Symmetric Product Layer Graph

Let $V_k = \{A \subseteq V(G) \mid |A| = k\}$. Then we define the k th symmetric product layer graph of G , denoted by $\mathcal{L}_k(G)$, as the vertex-induced subgraph of $\mathcal{F}(G)$ on V_k .

Simultaneous Symmetric Product Graph

Let G be a connected graph. Then we define the k th simultaneous symmetric product graph of G , denoted by $\mathcal{S}_k(G)$, as follows:

1. $A \in V(\mathcal{F}_2(G)) \iff A \subseteq V(G), 1 \leq |A| \leq k$
2. Let $A, B \subseteq V(G)$ be non-empty. Then $AB \in V(G)$ if the following hold true:
 - (a) $\forall a \in A, \exists b \in B$ such that $ab \in E(G)$ or $a = b$.
 - (b) $\forall b \in B, \exists a \in A$ such that $ab \in E(G)$ or $a = b$.
 - (c) $A \neq B$.

For notational convenience, if G is a graph of order n , we write $\mathcal{S}_n(G)$ as $\mathcal{S}(G)$.

Again, we shall consider P_4 , with vertices labeled 1, 2, 3, 4 in order. If $A = \{2\}$, and $B = \{1, 3\}$, then we have that $AB \in E(\mathcal{S}_2(G))$, as $A \neq B$, 2 is adjacent to 1 and therefore $\forall a \in A, \exists b \in B$ such that $ab \in E(G)$, and 1 and 3 are both adjacent to 2 and thus $\forall b \in B, \exists a \in A$ such that $ab \in E(G)$.

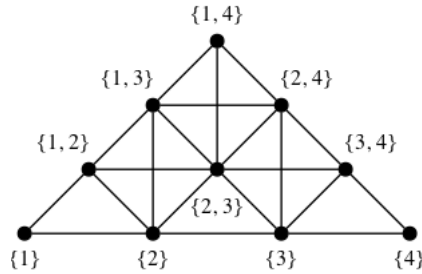


Figure 2.2: $\mathcal{S}_2(P_4)$. Notice that $\mathcal{F}_2(P_4)$ is actually a subgraph of $\mathcal{S}_2(P_4)$ (a result that holds in general, which we will prove later).

Simultaneous Symmetric Product Layer Graph

Let $V_k = \{A \subseteq V(G) \mid |A| = k\}$. Then we define the k th simultaneous symmetric product layer graph of G , denoted by $\mathcal{M}_k(G)$, as the vertex-induced subgraph of $\mathcal{S}(G)$ on V_k .

Chapter 3

Subgraphs

In this chapter, we establish the subgraph relationships of the graph products we defined. The theorems in this chapter are going to be especially useful when we wish to establish the connectedness of these graphs in Chapter 5, the planarity of these graphs in Chapter 6, the chromatic number of these graphs in Chapter 7 and the distances between these graphs in Chapter 8.

Let G be a graph, and let H be a subgraph of G . Let $i, k \in \mathbb{Z}^+$ such that $i \leq k$. In this chapter, we will establish the following relationships between hyperspace graphs.

- $\mathcal{L}_i(G)$ is a subgraph of $\mathcal{F}_k(G)$.
- $\mathcal{M}_i(G)$ is a subgraph of $\mathcal{S}_k(G)$.
- $\mathcal{F}_i(G)$ is a subgraph of $\mathcal{F}_k(G)$.
- $\mathcal{S}_i(G)$ is a subgraph of $\mathcal{S}_k(G)$.
- $\mathcal{L}_k(G)$ is a subgraph of $\mathcal{M}_k(G)$.
- $\mathcal{F}_k(G)$ is a subgraph of $\mathcal{S}_k(G)$.
- $\mathcal{F}_k(H)$ is a subgraph of $\mathcal{F}_k(G)$ (along with similar results for \mathcal{S} , \mathcal{L} and \mathcal{M}).
- If $V(H) \neq V(G)$, then $\mathcal{F}_k(G)$ contains multiple disjoint subgraphs that are isomorphic to $\mathcal{F}_k(G)$ (along with similar results for \mathcal{S} , \mathcal{L} and \mathcal{M}).

Theorem 3.1. *Let G be a graph. Then $\mathcal{L}_i(G)$ is a subgraph of $\mathcal{F}_k(G)$ for $i \leq k$.*

Proof: By definition, $V(\mathcal{L}_i(G)) = \{A \subseteq V(G) \mid |A| = i\}$. Also, $V(\mathcal{F}_k(G)) = \{A \subseteq V(G) \mid |A| \leq k\}$. Since $i \leq k$, it is obvious that $V(\mathcal{L}_i(G)) \subseteq V(\mathcal{F}_k(G))$.

Since the requirement for two vertex sets $A, B \subset V(G)$ to be adjacent in $\mathcal{L}_i(G)$ is exactly the same requirement for those vertex sets to be adjacent in $\mathcal{F}_k(G)$, it is obvious that $AB \in E(\mathcal{L}_i(G)) \rightarrow AB \in E(\mathcal{F}_k(G))$. Hence, we have that $\mathcal{L}_i(G)$ is a subgraph of $\mathcal{F}_k(G)$.

Theorem 3.2. *Let G be a graph. Then $\mathcal{M}_i(G)$ is a subgraph of $\mathcal{S}_k(G)$ for $i \leq k$.*

Proof: By definition, $V(\mathcal{M}_i(G)) = \{A \subseteq V(G) \mid |A| = i\}$. Also, $V(\mathcal{S}_k(G)) = \{A \subseteq V(G) \mid |A| \leq k\}$. Since $i \leq k$, it is obvious that $V(\mathcal{M}_i(G)) \subseteq V(\mathcal{S}_k(G))$.

Since the requirement for two vertex sets $A, B \subset V(G)$ to be adjacent in $\mathcal{M}_i(G)$ is exactly the same requirement for those vertex sets to be adjacent in $\mathcal{S}_k(G)$, it is obvious that $AB \in E(\mathcal{M}_i(G)) \rightarrow AB \in E(\mathcal{S}_k(G))$. Hence, we have that $\mathcal{M}_i(G)$ is a subgraph of $\mathcal{S}_k(G)$.

Theorem 3.3. *Let G be a graph. Then $\mathcal{F}_i(G)$ is a subgraph of $\mathcal{F}_k(G)$ for $i \leq k$.*

Proof: By definition, $V(\mathcal{F}_i(G)) = \{A \subseteq V(G) \mid |A| \leq i\}$. Also, $V(\mathcal{F}_k(G)) = \{A \subseteq V(G) \mid |A| \leq k\}$. Since $i \leq k$, it is obvious that $V(\mathcal{F}_i(G)) \subseteq V(\mathcal{F}_k(G))$.

Since the requirement for two vertex sets $A, B \subset V(G)$ to be adjacent in $\mathcal{F}_i(G)$ is exactly the same requirement for those vertex sets to be adjacent in $\mathcal{F}_k(G)$, it is obvious that $AB \in E(\mathcal{F}_i(G)) \rightarrow AB \in E(\mathcal{F}_k(G))$. Hence, we have that $\mathcal{F}_i(G)$ is a subgraph of $\mathcal{F}_k(G)$.

Theorem 3.4. *Let G be a graph. Then $\mathcal{S}_i(G)$ is a subgraph of $\mathcal{S}_k(G)$ for $i \leq k$.*

Proof: By definition, $V(\mathcal{S}_i(G)) = \{A \subseteq V(G) \mid |A| \leq i\}$. Also, $V(\mathcal{S}_k(G)) = \{A \subseteq V(G) \mid |A| \leq k\}$. Since $i \leq k$, it is obvious that $V(\mathcal{S}_i(G)) \subseteq V(\mathcal{S}_k(G))$.

Since the requirement for two vertex sets $A, B \subset V(G)$ to be adjacent in $\mathcal{S}_i(G)$ is exactly the same requirement for those vertex sets to be adjacent in $\mathcal{S}_k(G)$, it is obvious that $AB \in E(\mathcal{S}_i(G)) \rightarrow AB \in E(\mathcal{S}_k(G))$. Hence, we have that $\mathcal{S}_i(G)$ is a subgraph of $\mathcal{S}_k(G)$.

Theorem 3.5. *Let G be a graph. Then $\mathcal{L}_k(G)$ is a subgraph of $\mathcal{M}_k(G)$.*

Proof: Let $A \in V(\mathcal{L}_k(G))$. Then $A \subseteq V(G)$ such that $|A| = k$. Therefore, $A \in V(\mathcal{M}_k(G))$.

Let $AB \in E(\mathcal{L}_k(G))$. Then $\exists W \subseteq V(G)$ and an edge $ab \in E(G)$ such that $A = W \cup \{a\}$, $B = W \cup \{b\}$ and $A \neq B$. Let $v \in A$. If $v \neq a$, then $v \in W$ and thus $v \in B$. If $v = a$, then v is adjacent to $b \in B$. Thus, $\forall v \in A$, either $v \in B$, or $v = a$ and thus $ab \in E(G)$. By similar argument, we can get that $\forall v \in B$, either $v \in A$, or $v = b$ and thus $ab \in E(G)$. We know by definition that $A \neq B$. Then by definition, $AB \in E(\mathcal{M}_k(G))$.

Theorem 3.6. *Let G be a graph. Then $\mathcal{F}_k(G)$ is a subgraph of $\mathcal{S}_k(G)$.*

Proof: Let $A \in V(\mathcal{F}_k(G))$. Then $A \subseteq V(G)$ such that $|A| = k$. Therefore, $A \in V(\mathcal{S}_k(G))$.

Let $AB \in E(\mathcal{F}_k(G))$. Then $\exists W \subseteq V(G)$ and an edge $ab \in E(G)$ such that $A = W \cup \{a\}$, $B = W \cup \{b\}$ and $A \neq B$. Let $v \in A$. If $v \neq a$, then $v \in W$ and thus $v \in B$. If $v = a$, then v is adjacent to $b \in B$. Thus, $\forall v \in A$, either $v \in B$, or $v = a$ and thus $ab \in E(G)$. By similar argument, we can get that $\forall v \in B$, either $v \in A$, or $v = b$ and thus $ab \in E(G)$. We know by definition that $A \neq B$. Then by definition, $AB \in E(\mathcal{S}_k(G))$.

Theorem 3.7. *Let G be a graph. Let H be a subgraph of G . Then $\mathcal{F}_k(H)$ is a subgraph of $\mathcal{F}_k(G)$.*

Proof: Suppose $A \in V(\mathcal{F}_k(H))$. Then $A \subset V(H)$ such that $|A| \leq k$, and thus, $A \subset V(G)$ such that $|A| \leq k$. Therefore, $A \in V(\mathcal{F}_k(G))$.

Suppose $A, B \in V(\mathcal{F}_k(H))$ such that $AB \in E(\mathcal{F}_k(H))$. Then there exists a set $W \subset V(H)$ and an edge $ab \in E(H)$ such that $A = W \cup \{a\}$, $B = W \cup \{b\}$ and $A \neq B$. But since H is a subgraph of G , this $W \subset V(G)$ and $ab \in E(G)$ as well, and thus, $AB \in E(\mathcal{F}_k(G))$. Thus, $\mathcal{F}_k(H)$ is a subgraph of $\mathcal{F}_k(G)$.

Using similar arguments, we can prove the following three theorems:

Theorem 3.8. *Let G be a graph. Let H be a subgraph of G . Then $\mathcal{S}_k(H)$ is a subgraph of $\mathcal{S}_k(G)$.*

Theorem 3.9. *Let G be a graph. Let H be a subgraph of G . Then $\mathcal{L}_k(H)$ is a subgraph of $\mathcal{L}_k(G)$.*

Theorem 3.10. *Let G be a graph. Let H be a subgraph of G . Then $\mathcal{M}_k(H)$ is a subgraph of $\mathcal{M}_k(G)$.*

Theorem 3.11. *Let G be an arbitrary graph. Let A be some arbitrary subset of $V(G)$ such that the vertex-induced subgraph of G on A (which we will call H) is connected. Let B be some subset of*

$V(G)\setminus A$. The collection of vertex sets

$$F_B = \{U \cup B \mid U \subseteq A, U \neq \emptyset\}$$

induces a subgraph of $\mathcal{F}(G)$. Call this graph \mathcal{H}_B . Then \mathcal{H}_B contains a subgraph that is graph isomorphic to $\mathcal{F}(H)$, and $|V(\mathcal{F}(H))| = |V(\mathcal{H}_B)|$

Proof: Define the following function:

$$f : V(\mathcal{F}(H)) \rightarrow F_B$$

such that $f(U) = U \cup B$. If $U \in V(\mathcal{F}(H))$, then $U \subseteq A$ and $U \neq \emptyset$, and therefore, $U \cup B \in F_B$, which is our range. Let $V \in F_B$. Then $V = U \cup B$ for some $U \subseteq A, U \neq \emptyset$. Thus, $U \in V(\mathcal{F}(H))$, and thus $f(U) = V$. Hence, our function is onto. Let $U_1, U_2 \in \mathcal{F}(H)$ such that $f(U_1) = f(U_2)$. Then $U_1 \cup B = U_2 \cup B$. Therefore, $U_1 \cup B \setminus B = U_2 \cup B \setminus B$. Since we know that $U_1, U_2 \in \mathcal{F}(H)$, we know that $U_1, U_2 \subseteq A$. Since $B = V(G)\setminus A$, we know A and B are disjoint, and thus, U_1 and U_2 are pairwise disjoint with B . Therefore, we have that $U_1 \cup B \setminus B = U_1$ and $U_2 \cup B \setminus B = U_2$. Then we have that $U_1 = U_2$. Thus, our function is one-one. Since f is one-one and onto, it is a bijection. Hence, $|V(\mathcal{H}_B)| = |V(\mathcal{F}(H))|$.

Let $U_1, U_2 \in \mathcal{F}(H)$ such that U_1 and U_2 are adjacent in $\mathcal{F}(H)$. Then there exists a set $W \subseteq V(\mathcal{F}(H))$ and an edge $u_1 u_2 \in E(H)$ such that $U_1 = W \cup \{u_1\}$ and $U_2 = W \cup \{u_2\}$, with $U_1 \neq U_2$. Let $V_1 = f(U_1) = U_1 \cup B$, and $V_2 = f(U_2) = U_2 \cup B$. Since $U_1 \neq U_2$, we know that $V_1 \neq V_2$ due to the fact that f is a bijection. Let $W^* = W \cup B$. Note that $V_1, V_2 \subseteq V(G)$ are nonempty, and $W^* \subseteq V(G)$. Additionally, $u_1 u_2 \in E(H)$, therefore, $u_1 u_2 \in E(G)$. Notice that $W^* \cup u_1 = W \cup B \cup u_1 = (W \cup u_1) \cup B = U_1 \cup B = V_1$, and $W^* \cup u_2 = W \cup B \cup u_2 = (W \cup u_2) \cup B = U_2 \cup B = V_2$. Then, by definition, V_1 and V_2 are adjacent in $\mathcal{F}(G)$ since $V_1, V_2 \in V(\mathcal{H}_B)$. Therefore, \mathcal{H}_B contains a subgraph that is graph isomorphic to $\mathcal{F}(H)$, and $|V(\mathcal{F}(H))| = |V(\mathcal{H}_B)|$

Using similar arguments, we can show the following:

Theorem 3.12. *Let G be an arbitrary graph. Let A be some arbitrary subset of $V(G)$ such that the vertex-induced subgraph of G on A (which we will call H) is connected. Let B be some subset of $V(G)\setminus A$. The collection of vertex sets*

$$F_B = \{U \cup B \mid U \subseteq A, U \neq \emptyset\}$$

induces a subgraph of $\mathcal{S}(G)$. Call this graph \mathcal{H}_B . Then \mathcal{H}_B contains a subgraph that is graph isomorphic to $\mathcal{S}(H)$, and $|V(\mathcal{S}(H))| = |V(\mathcal{H}_B)|$

Chapter 4

Order, Size and Vertex Degree

Establishing some properties about the orders and sizes of these hyperspace graphs, as well as the degrees of their vertices, could be useful in proving the uniqueness of these hyperspace graphs. That is, if $\mathcal{F}_k(G)$ is isomorphic to $\mathcal{F}_k(H)$, need G be isomorphic to H ? Note that a similar question can be asked of \mathcal{S}_k , as well as \mathcal{L}_k for certain values of k , but for \mathcal{M}_k , there are several graphs for which this is not the case for any $k \neq 1$ (for example, $\mathcal{M}_2(C_4) = \mathcal{M}_2(K_4) = K_6$).

In this chapter, we give values for the following:

- The orders of \mathcal{L}_k , \mathcal{M}_k , \mathcal{F}_k and \mathcal{S}_k ,
- The sizes of \mathcal{L}_k and \mathcal{F}_k .

Additionally, we show that if $\mathcal{F}(G)$ contains a vertex U of degree 2 such that $|V(G)| \geq 3$, then $U = \{u\}$, where u is a singleton of G .

Theorem 4.1. *Suppose G is a graph of order n . Then:*

- The orders of $\mathcal{L}_k(G)$ and $\mathcal{M}_k(G)$ are both $\binom{n}{k}$.
- The orders of $\mathcal{F}_k(G)$ and $\mathcal{S}_k(G)$ are both $\sum_{i=1}^k \binom{n}{i}$.

Proof: This follows directly from the definitions of the vertex sets of our hyperspace graphs.

Theorem 4.2. *Suppose G is a graph of order n and size m . Then the size of $\mathcal{L}_k(G)$ is*

$$\binom{n-2}{k-1}m.$$

Proof: Suppose $AB \in E(\mathcal{L}_k(G))$. Then there exist W and ab as in the definition of our adjacency in $\mathcal{L}_k(G)$. If $a \in W$, then either $b \in W$ and then $A = B$, or $b \notin W$ and $|A| \neq |B|$, and therefore, at least one of A, B is not in $V(\mathcal{L}_k(G))$. Therefore, $a \notin W$. By similar process, we get $b \notin W$. If $|W| < k - 1$, then $|A| = |W \cup \{a\}| < k$, and thus $A \notin V(\mathcal{L}_k(G))$. If $|W| > k - 1$, then $|A| = |W \cup \{a\}| > k$ and thus $A \notin V(\mathcal{L}_k(G))$. Therefore, $|W| = k - 1$.

Suppose there are two vertex sets W_1, W_2 and two edges a_1b_1 and a_2b_2 such that $A = W_1 \cup \{a_1\} = W_2 \cup \{a_2\}$ and $B = W_1 \cup \{b_1\} = W_2 \cup \{b_2\}$. If $a_1 \neq a_2$, then $a_2 \in W_1$. But then $a_2 \in B$, which in turn means that $a_2 \in W_2 \cup \{b_2\}$. If $a_2 \in W_2$, then $|A| = k - 1$, which contradicts our choice of $A \in V(\mathcal{L}_k(G))$. Therefore, $a_2 = b_2$. But then $a_2b_2 \notin E(G)$. This is a contradiction. Therefore, $a_1 = a_2$. Relabel this vertex as a . By similar logic, we get that $b_1 = b_2$. Relabel this vertex as b . Suppose $W_1 \neq W_2$. Since $|W_1| = |W_2| = k - 1$, $\exists w \in W_1 \setminus W_2$. But then if $A = W_1 \cup \{a\} = W_2 \cup \{a\}$, then this means both $w \in A$ (as $w \in W_1$) and $w \notin A$ (as $w \notin W_2$, and $w = a$ would contradict our choice of ab). Therefore, $W_1 = W_2$.

Therefore, each unique edge AB in $\mathcal{L}_k(G)$ corresponds to a unique choice of $k - 1$ vertices to go in W , and a traversal edge ab that is not incident on any vertex in W . Since each of our $k - 1$ vertices must be in both A and B , our choice of edge to traverse must not be incident on any of these $k - 1$ vertices. Hence, we take our choice of edge ab first, and then choose our $k - 1$ vertices from the $n - 2$ vertices that ab is not incident on. The number of ways to make such a selection is given by the formula above.

Theorem 4.3. *Suppose G is a graph of order n and of size m . Then the size of $\mathcal{F}(G)$ is*

$$m2^{n-2} + n2^{n-1} - \sum_{v \in V(G)} 2^{n-d(v)-1},$$

where $d(v)$ is the degree of v in G .

Proof: Without loss of generality, assume $|A| \leq |B|$. We will break this up into two separate problems: counting the number of edges AB where $|A| = |B|$, and counting the number of edges AB where $|A| + 1 = |B|$. Note that these are the only two possibilities for A and B based on the definition for adjacencies in $\mathcal{F}_n(G)$.

Case 1: We wish to count the number of edges AB such that $|A| = |B|$. If $|A| = |B| = k$, then $AB \in E(\mathcal{L}_k(G))$. Then we can count the number of edges of the form given by merely summing the number of edges in each $\mathcal{L}_k(G)$ over $1 \leq k \leq n$. Then the number of edges of this form is

$$\sum_{k=1}^n m \binom{n-2}{k-1} = m2^{n-2}$$

Case 2: We wish to count the number of edges AB such that $|A| + 1 = |B|$. Then there must exist some set W and some edge $ab \in E(G)$ such that $W \cup \{a\} = A$ and $W \cup \{b\} = B$. Since $|B| > |A|$, we must have that $a \in W$, and $b \notin A$. Hence, $B = A \cup \{b\}$. In order to count the number of edges, we shall sum over our choice of b . Hence, the number of edges of this type is

$$\sum_{b \in V(G)} N(b),$$

where $N(b)$ is the number of edges AB such that $B = A \cup \{b\}$.

To count $N(b)$, we must count the number of choices that we have for A . Notice that each valid choice for A corresponds to a unique adjacency, as if $A_1 \neq A_2$, then the two edges A_1B and A_2B cannot be the same. For a set A to be a valid choice, there must exist some $a \in A$ such that $ab \in E(G)$. In order to better count these number of choices, we further subdivide our cases based on the number of vertices in A . Hence,

$$N(b) = \sum_{k=1}^{n-1} N_A(b, k),$$

where $N_A(b, k)$ is the number of valid choices for A of size k given a vertex b .

$N_A(b, k)$ is just equal to the total number ways to choose k vertices to be in A minus the number of ways to choose k vertices to be in A such that none of them are neighbors of b . Therefore,

$$N_A(b, k) = \binom{n-1}{k} - \binom{n-d(b)-1}{k}.$$

Thus,

$$N(b) = \sum_{k=1}^{n-1} \left(\binom{n-1}{k} - \binom{n-d(b)-1}{k} \right) = 2^{n-1} - 2^{n-d(b)-1},$$

and therefore, the total number of edges of the type AB where $|A| + 1 = |B|$ is just

$$\sum_{b \in V(G)} 2^{n-1} - 2^{n-d(b)-1} = n2^{n-1} - \sum_{b \in V(G)} 2^{n-d(b)-1}.$$

Hence, the total number of edges in $\mathcal{F}(G)$ is just

$$m2^{n-2} + n2^{n-1} - \sum_{v \in V(G)} 2^{n-d(v)-1}.$$

Theorem 4.4. *Let G be an arbitrary graph of order n . Then $\mathcal{M}_k(G)$ is a complete graph of order $\binom{n}{k}$ if and only if $\delta(G) \geq n - k$.*

Proof: Suppose $\delta(G) \geq n - k$. Let A, B be arbitrary vertex sets of size k . Let $v \in A$. Then since $d(v) \geq n - k$, there are at most $k - 1$ vertices $u \in G$ such that v is both not adjacent to u and that $u \neq v$. Thus, by the pidgeonhole principle, there must be at least one vertex u in B such that v is adjacent to u or $u = v$. This holds for any arbitrary vertex in A , and by symmetry, hold for any arbitrary vertex in B with respect to A . Then by the definition of the adjacencies in $\mathcal{M}_k(G)$, $AB \in E(\mathcal{M}_k(G))$.

Conversely, suppose that there exists a vertex $v \in V(G)$ such that $d(v) \leq n - k - 1$. Then, there are at least k vertices in $u \in G$ such that v is both not adjacent to u and $v \neq u$. Let A be an arbitrary vertex set of G containing v , and let B be an arbitrary vertex set containing k vertices of the type u as described above. Then since v is not adjacent to any vertex in B , $AB \notin E(\mathcal{M}_k(G))$, and thus $\mathcal{M}_k(G)$ is not complete.

Example: Consider the graph $G = K_n - e$, where $e = u_1u_2$. Then $\mathcal{S}(G)$ can be constructed as follows:

The vertex set $\{u_1\}$ is adjacent to every vertex that does not contain u_2 . The vertex set $\{u_2\}$ is adjacent to every vertex that does not contain u_1 . Every vertex in $V(\mathcal{S}(G)) \setminus \{\{u_1\}, \{u_2\}\}$ is adjacent to every other vertex in $V(\mathcal{S}(G))$. Therefore, $\mathcal{S}(G)$ can be identified, as it is $K_{2^n - 3}$, along with two additionally vertices, which are connected to all but $2^{n-1} - 1$ of the original vertices, of which there are 2^{n-2} that they have in common.

Theorem 4.5. *Let G be a connected graph such that $|V(G)| \geq 3$. Then A is a vertex of degree 2 in $\mathcal{F}(G)$ if and only if A is singleton containing a vertex of degree 1 in G .*

Proof: Suppose A is a singleton $\{a\}$, where $d(a) = 1$. Let B be adjacent to A . If $W = \{a\}$, then our edge is ab , where b is the only vertex adjacent to a ; otherwise, A would have two vertices. If $W = \emptyset$, then our edge must also be ab , where b is the only vertex adjacent to a ; otherwise, A could not be one of our two sets. Hence, A is only adjacent to two sets: $\{b\}$, and $\{a, b\}$. Hence, our forward proof holds. We shall now show that the backwards proof also holds: if A is either a singleton, but of a vertex of degree 2 or more, or if A is not a singleton, then A has at least 3 neighbors.

Suppose A is a singleton $\{a\}$ where $d(a) \geq 2$. Let b_1, b_2 be adjacent to a . Then the following sets are adjacent to A :

- $\{b_1\}$
- $\{b_2\}$
- $\{a, b_1\}$
- $\{a, b_2\}$.

Thus, A is not a vertex set of degree 2.

Now suppose A contains two unique vertices a_1 and a_2 . We need to show that there are three vertex sets adjacent to A . This argument needs to be broken into two cases.

Case 1: a_1 is not adjacent to a_2 . There must exist some vertex b_1 adjacent to a_1 and some vertex b_2 adjacent to a_2 . We must subdivide into further cases:

Subcase 1: $b_1 \in A$. Then following sets are adjacent to A :

- $A \setminus \{a_1\}$ ($W = A \setminus \{a_1\}$, choice of edge $a_1 b_1$)
- $A \setminus \{b_1\}$ ($W = A \setminus \{b_1\}$, choice of edge $a_1 b_1$)
- $A \cup \{b_2\} \setminus \{a_2\}$ ($W = A \setminus \{a_2, b_2\}$, choice of edge $a_2 b_2$)

Note that even if $b_1 = b_2$, none of these sets are the same. Therefore, A is not a vertex set of degree 2.

Subcase 2: $b_1 \notin A$. Then the following sets are adjacent to A :

- $A \cup \{b_1\}$ ($W = A$, choice of edge $a_1 b_1$)
- $A \cup \{b_1\} \setminus \{a_1\}$ ($W = A \setminus \{a_1\}$, choice of edge $a_1 b_1$)
- $A \cup \{b_2\} \setminus \{a_2\}$ ($W = A \setminus \{a_2, b_2\}$, choice of edge $a_2 b_2$)

Again, note that even if $b_1 = b_2$, none of these sets are the same. Therefore, A is not a vertex set of degree 2.

Case 2: a_1 is adjacent to a_2 . Since $|V(G)| \geq 3$ and G is connected, there must exist some other vertex b adjacent to either a_1 or a_2 . Without loss of generality, assume it is adjacent to a_1 . Then we divide into two subcases:

Subcase 1: $b \in A$. Then following sets are adjacent to A :

- $A \setminus \{a_1\}$ ($W = A \setminus \{a_1\}$, choice of edge a_1b)
- $A \setminus \{b\}$ ($W = A \setminus \{b\}$, choice of edge a_1b)
- $A \setminus \{a_2\}$ ($W = A \setminus \{a_2\}$, choice of edge a_1a_2)

Note that even if $b_1 = b_2$, none of these sets are the same. Therefore, A is not a vertex set of degree 2.

Subcase 2: $b \notin A$. Then the following sets are adjacent to A :

- $A \cup \{b\}$ ($W = A$, choice of edge a_1b_1)
- $A \cup \{b\} \setminus \{a_1\}$ ($W = A \setminus \{a_1\}$, choice of edge a_1b)
- $A \cup \setminus \{a_2\}$ ($W = A \setminus \{a_2\}$, choice of edge a_1a_2)

Hence, the backward proof is true as well. Therefore, the theorem holds.

Theorem 4.6. *For graphs such that $|V(G)| \geq 3$, a vertex of degree 2 in $S(G)$ corresponds to a singleton containing a vertex of degree 1 in G .*

Proof: By similar logic to the last proof, we have that $A = \{a\}$ for some degree one vertex a is only adjacent to the sets $\{b\}$ and $\{a, b\}$, where b is adjacent to a . Similarly, since $\mathcal{F}(G)$ is a subgraph of $\mathcal{S}(G)$, we have that any vertex A not of the form $\{a\}$ for some vertex of degree 1 is going to be adjacent to the same vertices as outlined in the backward portion of the last proof. Hence, the theorem holds.

Chapter 5

Connectedness

For each of our proofs in this section, we are going to assume that G is a connected graph. If this is the case, then we have that each of our hyperspace graphs are connected.

Theorem 5.1. $\mathcal{L}_k(G)$ is a connected graph.

First, we prove that for any vertex set W such that $|W| = n-1$, there exists a path between $A = W \cup \{a\}$ and $B = W \cup \{b\}$ in $\mathcal{L}_k(G)$ for any $a, b \notin W$.

Since G is connected, we know there exists a path between a and b . Label the vertices of this path u_1, u_2, \dots, u_l , with $a = u_1$ and $b = u_l$. Let $i_0 = l$. Let $I = \{i | u_i \in A\}$. We know this set is non-empty, as $a = u_1$. Let i_1 be the largest number of I , i_2 be the second largest, and so on until i_r (where $i_r = 1$). Note $r \leq l$. For all $0 \leq j \leq r$, define the set U_j as $(A \cup \{b\}) \setminus \{u_{i_j}\}$ (see the figure below for an explanation). Note that in particular, $U_0 = A$, as and $A \cup \{b\} \setminus \{a\} = W \cup \{b\} = B$, as $u_{i_r} = a$ and $A \cup \{b\} \setminus \{a\} = W \cup \{b\} = B$. We then construct the following path in $\mathcal{L}_k(G)$:

$$U_j, U_j \setminus \{u_{i_{j+1}}\} \cup \{u_{i_{j+1}+1}\}, U_j \setminus \{u_{i_{j+1}}\} \cup \{u_{i_{j+1}+2}\}, \dots, U_j \setminus \{u_{i_{j+1}}\} \cup \{u_{i_j-1}\}, U_j \setminus \{u_{i_{j+1}}\} \cup \{u_{i_j}\} = U_{j+1}.$$

This path connects the vertices U_j and U_{j+1} . Then by adjoining each of the paths between each U_j and U_{j+1} , we get a walk between the vertices $U_0 = A$ and $U_r = B$. Hence, there is a path between the vertices A and B in $\mathcal{L}_k(G)$.

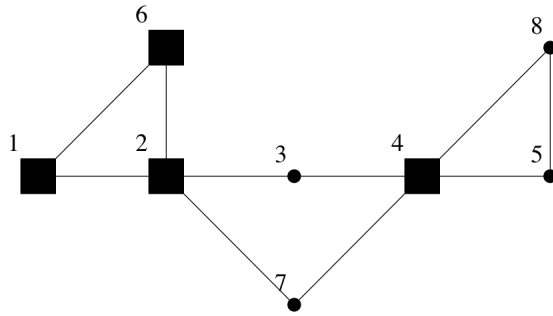


Figure 5.1: An example of the connectedness between two sets A and B, where A and B differ by one vertex. We wish to show that set A (pictured above by the square vertices) and set B (pictured below) are connected.

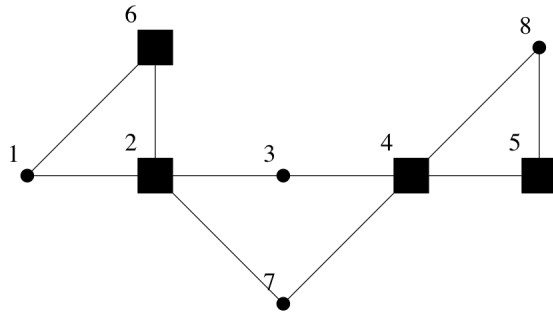


Figure 5.2: The set B. Notice how we have drawn the graph in a way such that a path between vertex 1 and vertex 5 (the two vertices by which A and B differ) are directly in the center of the graph.

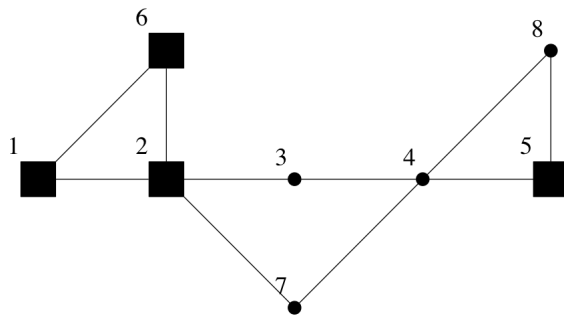


Figure 5.3: By our notation in the proof above, $i_0 = 5, i_1 = 4, i_2 = 2$ and $i_3 = 1$, with $r = 3$. We start by moving $v_{i_1} = v_4$ over to the position of $v_{i_0} = 5$.

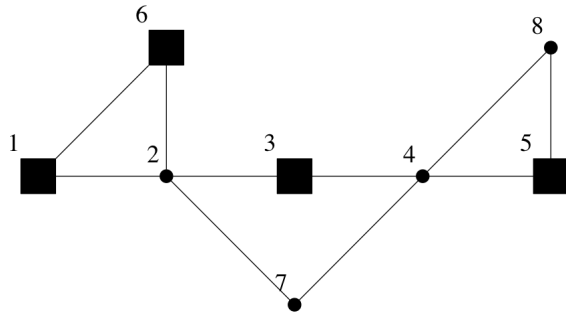


Figure 5.4: We then start moving $v_{i_2} = v_2$ towards $v_{i_1} = v_4$. The two vertices are not adjacent in our choice of path this time, so this will take two steps.

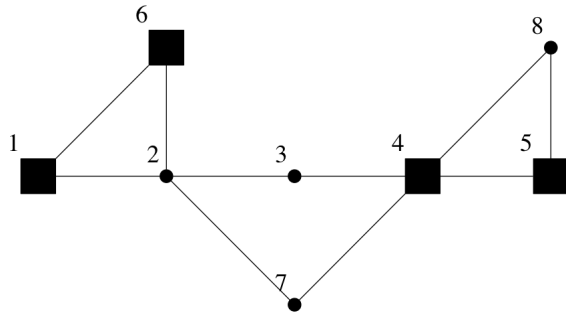


Figure 5.5: Continuing the movement of the last step.

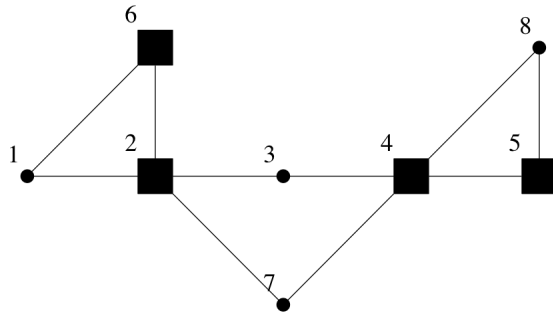


Figure 5.6: Finally, we move $v_{i_3} = v_1$ to $v_{i_2} = v_2$. We now exactly match the set B as pictured above. Hence, A and B are connected.

Let $A = \{a_1, a_2, \dots, a_n\}$ and $B = \{b_1, b_2, \dots, b_n\}$ such that $A \neq B$. Relabel these vertices so that any common vertices A and B share occur at the beginning of the set and with the same index; i.e., relabel the vertices of A and B such that $A = \{b_1, b_2, \dots, b_r, a_{r+1}, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$. For $i \geq r$, $V_i = \{b_1, b_2, \dots, b_i, a_{i+1}, \dots, a_k\}$. Note in particular that $V_r = A$ and $V_k = B$. Let $W_i = \{b_1, b_2, \dots, b_{i-1}, a_{i+1}, \dots, a_k\}$. Then $V_{i-1} = W_i \cup \{a_i\}$ and $V_i = W_{\{i\}} \cup \{b_i\}$. Since we know by construction that $a_i \neq b_i$ for $i \geq r + 1$, we know that our previous statement holds for V_{i-1} and V_i when $i \geq r + 1$. Hence, for $i \geq r$, there exists a path between V_i and V_{i+1} . Then by combining all of these paths for $r \leq i \leq n$, we obtain a walk between A and B . Since this is true for arbitrary vertex sets, we have that $\mathcal{L}_k(G)$ is connected.

Theorem 5.2. $\mathcal{M}_k(G)$ is connected.

As $\mathcal{L}_k(G)$ is a subgraph of $\mathcal{M}_k(G)$ with the same number of vertices, it is obvious that $\mathcal{M}_k(G)$ is connected.

Theorem 5.3. $\mathcal{F}_k(G)$ is connected. ($V(G) \geq n$).

Since each $\mathcal{L}_i(G)$ is connected, we merely need to prove that each $\mathcal{L}_i(G)$ has at least one connection to $\mathcal{L}_{i+1}(G)$ for all $i \leq k - 1$, and we have by construction that $\mathcal{F}_k(G)$ is connected. Take an arbitrary vertex set A of G of size i , with $i \leq n - 1$. Then A is a proper subset of $V(G)$. Let u be a vertex in $V(G) \setminus A$ and v be a vertex in A . Then since G is connected, there exists a path v_1, v_2, \dots, v_r such that $v_1 = u$ and $v_r = v$. Since $u \notin A$ yet $v \in A$, there must exist a vertex v_j such that $v_j \notin A$ yet $v_{j+1} \in A$. Then by selecting our set $W = A$ and our edge as $v_{j+1}v_j$, we have that $B = A + \{v_j\}$ is adjacent to A in $\mathcal{F}_k(G)$. Since $A \in V(\mathcal{L}_i(G))$ and $B \in V(\mathcal{L}_{i+1}(G))$, we have a connection between $\mathcal{L}_i(G)$ and $\mathcal{L}_{i+1}(G)$ in $\mathcal{F}_k(G)$. Hence, we have that $\mathcal{F}_n(G)$ is connected.

Theorem 5.4. $\mathcal{S}_k(G)$ is connected.

Since $\mathcal{F}_n(G)$ is a subgraph of $\mathcal{S}_k(G)$ with the same number of vertices, it is obvious that $\mathcal{S}_k(G)$ is connected.

Chapter 6

Planarity

In previous works, there have been efforts to categorize which graphs \mathbb{G} (observed as topological spaces) are such that $\mathcal{F}_k(G)$ is embeddable in \mathcal{R}_n for particular values of n . This is very similar in concept to establishing which graphs G (observed in a graph theoretic sense) are such that $\mathcal{F}_k(G)$ (or $\mathcal{S}_k(G)$) are planar. Hence, in this chapter, we provide a categorization for which graphs G are such that $\mathcal{F}_2(G)$ and $\mathcal{S}_2(G)$ are planar. Note that as $\mathcal{F}_2(G)$ is a subgraph of $\mathcal{F}_k(G)$ for $k \geq 2$, the set of graphs for which $\mathcal{F}_k(G)$ is planar will be a subset of the set of graphs for which $\mathcal{F}_2(G)$ is planar for $k \geq 2$ (and the same holds for $\mathcal{S}_k(G)$ as well).

Planarity of $\mathcal{F}_2(G)$

Theorem 6.1. *$\mathcal{F}_2(P_n)$ is planar for all $n \in \mathbb{N}$.*

To draw $\mathcal{F}_2(P_n)$ in a planar way, we shall specify locations to place each of the vertices, and then argue that if each edge is drawn as the line segment connecting the two vertices it is incident on, that none of the edges will intersect.

Label the vertices of P_n with the numbers 1 through n , such that $\forall 2 \leq i \leq n-1$, i is adjacent to $i-1$ and $i+1$. Then draw the vertices of $\mathcal{F}_2(P_n)$ on the xy -coordinate plane as follows:

- If $A = \{i\}$, then draw A at the point $(i, 0)$.

- If $A = \{i, j\}$ with $i < j$, then draw A at the point $(i, j - i)$.

Suppose that $AB \in E(\mathcal{F}_2(G))$. Then $\exists W \subseteq V(P_n)$ and an edge $ab \in E(P_n)$ such that $A = W \cup \{a\}$ and $B = W \cup \{b\}$. Since $A \neq B$, at least one of a and b is not in W . Then if $|W| \geq 2$, we have that at least one of A and B is of order at least 3. This contradicts the fact that $A, B \in V(\mathcal{F}_2(G))$. Therefore, $|W| \leq 1$. We shall now divide the edges AB into three categories, based on our possible choices of W and ab .

- $W = \emptyset$. In this case, both A and B are singletons, with $a = i$ and $b = i + 1$. Then the edge AB connects the points $(i, 0)$ and $(i + 1, 0)$.
- $W = \{i\}$, $a = j$, $b = j + 1$ with $j \geq i$. If $j = i$, then $A = \{i\}$ and $B = \{i, i + 1\}$, and the edge AB connects the points $(i, 0)$ and $(i, 1)$. If $j > i$, then $A = \{i, j\}$ and $B = \{i, j + 1\}$, and the edge AB connects the points $(i, j - i)$ and $(i, j - i + 1)$.
- $W = \{i\}$, $a = j$, $b = j + 1$ with $j < i$ (equivalently, $j + 1 \leq i$). If $j + 1 = i$, then $A = \{i\}$ and $B = \{i - 1, j\}$, and the edge AB connects the points $(i, 0)$ and $(i - 1, 1)$. If $j + 1 < i$, then $A = \{j + 1, i\}$ and $B = \{j, i\}$, and the edge AB connects the points $(j + 1, i - j - 1)$ and $(j, i - j)$.

Notice that the first two types of edges form a subset of the unit grid, whereas the third type of edges connect points of the form (i, j) to $(i + 1, j - 1)$.

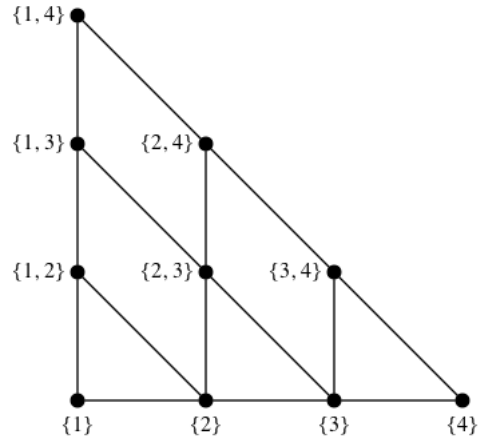


Figure 6.1: A plane drawing of $\mathcal{F}_2(P_4)$. Note that $\mathcal{F}_2(P_n)$ can be drawn in a plane way through similar construction.

As shown above, when drawn this way, $\mathcal{F}_2(P_4)$ is plane. This holds in general for P_n when drawn in this manner. Hence, $\mathcal{F}_2(P_n)$ is planar.

Theorem 6.2. $\mathcal{F}_2(C_n)$ is planar if and only if $n \leq 3$.

Proof: We will start by showing that $\mathcal{F}_2(C_3)$ is planar, then showing that $\mathcal{F}_2(C_4)$ is non-planar by constructing a $K_{3,3}$ subdivision of that graph. From there, we will show a general method to construct a $K_{3,3}$ subdivision of $\mathcal{F}_2(C_n)$ for $n \geq 4$.

Figure 6.2 gives a plane drawing of $\mathcal{F}_2(C_3)$, hence it is planar.

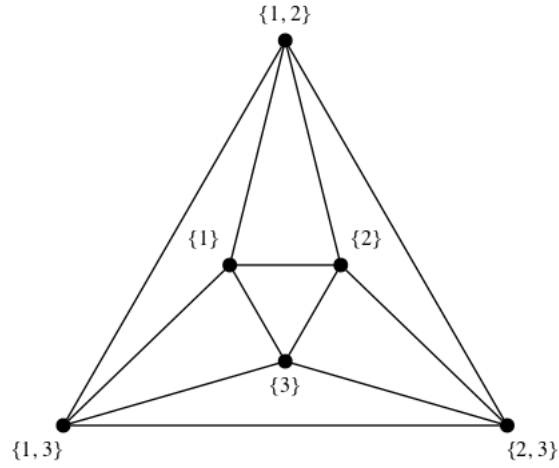


Figure 6.2: A plane drawing of $\mathcal{F}_2(C_3)$.

Since we have a plane drawing of $\mathcal{F}_2(C_4)$, it must be planar.

To prove: $\mathcal{F}_2(C_4)$ is non-planar. Label the vertices of C_4 with the numbers 1 through 4 such that i is adjacent to $i + 1$ for $i < 4$, and 1 and 4 are adjacent. Figure 6.3 shows that $\mathcal{F}_2(C_4)$ has a $K_{3,3}$ subdivision.

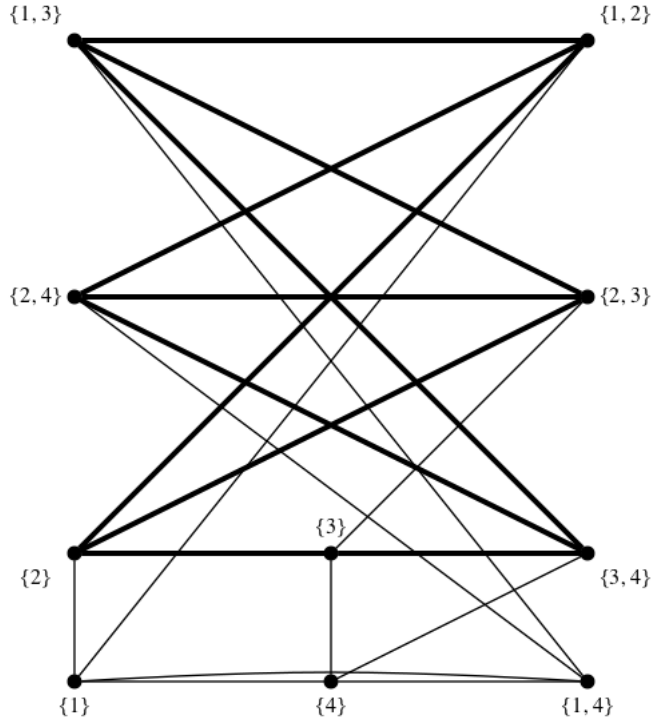


Figure 6.3: A drawing of $\mathcal{F}_2(C_4)$. The $K_{3,3}$ subdivision has bold edges.

Since $\mathcal{F}_2(C_4)$ contains a $K_{3,3}$ subdivision, it is non-planar.

To prove: $\mathcal{F}_2(C_n)$ is non-planar for $n > 4$. Label the vertices of C_n with the numbers 1 through n such that i is adjacent to $i + 1$ for $i < n$, and 1 and n are adjacent. We then construct a $K_{3,3}$ subdivision of this graph much in the same way that we constructed it for C_4 . One partition contains the vertices $\{1, 3\}$, $\{2, 4\}$ and $\{2\}$, and the other partition contains the vertices $\{1, 2\}$, $\{2, 3\}$ and $\{3, 4\}$. The paths that we have between the vertices are almost exactly the same. The only difference in the construction of our paths for C_n in general is that $\{1, 3\}$ is now no longer adjacent to $\{3, 4\}$ and $\{2, 4\}$ is no longer adjacent to $\{1, 2\}$. Rather, we take the path between $\{1, 3\}$ and $\{3, 4\}$ to be

$$\{1, 3\}, \{3, n\}, \{3, n - 1\}, \dots, \{3, 5\}, \{3, 4\}$$

and the path between $\{2, 4\}$ and $\{1, 2\}$ to be

$$\{2, 4\}, \{2, 5\}, \dots, \{2, n\}, \{1, 2\}.$$

Thus, $\mathcal{F}_2(C_n)$ has a $K_{3,3}$ subdivision for $n > 4$, and thus, $\mathcal{F}_2(C_n)$ is non-planar.

Theorem 6.3. $\mathcal{F}_2(S_n)$ is planar if and only if $n \leq 3$.

Proof: By Theorem 3.7, since S_n is a subgraph of $S_{n'}$ for $n \leq n'$, it is sufficient to prove the following statements:

1. $\mathcal{F}_2(S_3)$ is planar.
2. $\mathcal{F}_2(S_4)$ is nonplanar.

If the first statement is true, then we have that $\mathcal{F}_2(S_n)$ is a subgraph of $\mathcal{F}_2(S_3)$ for $n \leq 3$, and therefore $\mathcal{F}_2(S_n)$ must also be planar for $n \leq 3$. If the second statement is true, we have that $\mathcal{F}_2(S_4)$ is a subgraph of $\mathcal{F}_2(S_n)$ for $n \geq 4$, and therefore $\mathcal{F}_2(S_n)$ is also nonplanar for $n \geq 4$.

To show that $\mathcal{F}_2(S_3)$ is planar, note that by Theorem 3.6, $\mathcal{F}_2(S_3)$ is a subgraph of $\mathcal{S}_2(S_3)$. In figure 6.11, we have a plane drawing of $\mathcal{S}_2(S_3)$. Therefore, $\mathcal{S}_2(S_3)$ is planar, and thus, $\mathcal{F}_2(S_3)$ is planar.

To prove: $\mathcal{S}_2(S_4)$ is non-planar. Below is a drawing of $\mathcal{S}_2(S_4)$ that illustrates a K_5 subdivision.

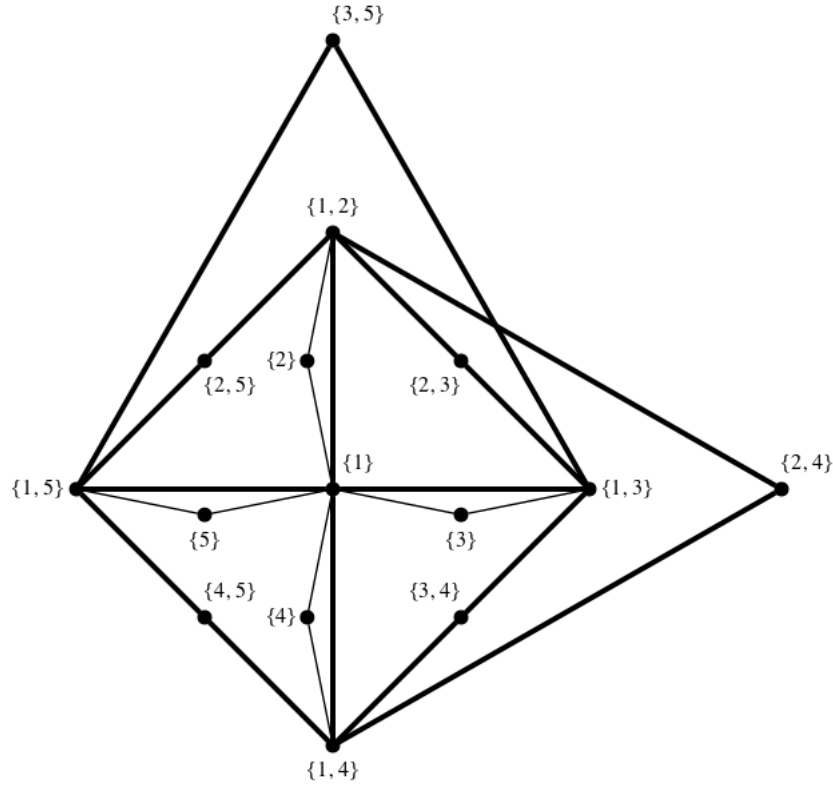


Figure 6.4: A drawing of $\mathcal{F}_2(S_4)$. The K_5 subdivision has bold edges.

Note that the bolded edges above form a K_5 subdivision, with endpoints $\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}$ and $\{1, 5\}$. Hence, $\mathcal{F}_2(S_4)$ is non-planar.

Theorem 6.4. *Let G be the graph in Figure 6.5. Then $\mathcal{F}_2(G)$ is planar.*

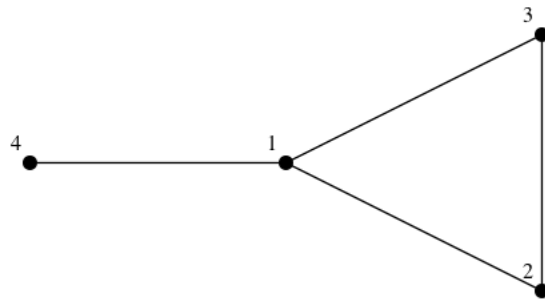


Figure 6.5: The graph G .

Below is a plane drawing of $\mathcal{F}_2(G)$.

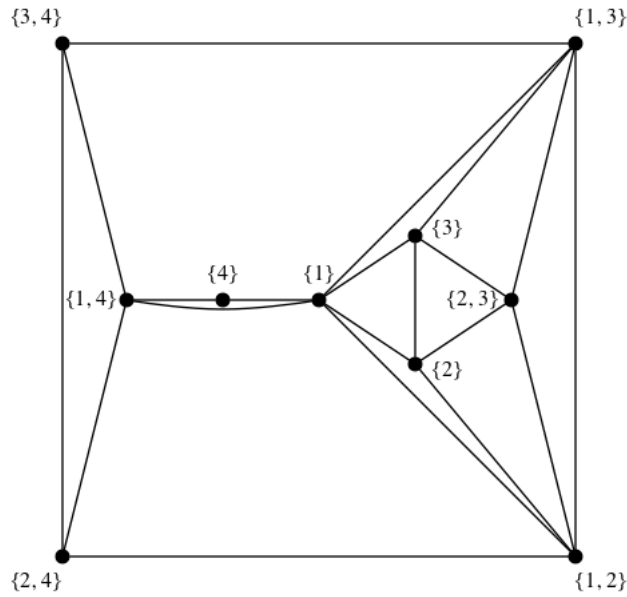


Figure 6.6: A drawing of $\mathcal{F}_2(G)$.

Since we have a plane drawing of $\mathcal{F}_2(G)$, $\mathcal{F}_2(G)$ is planar.

Theorem 6.5. *Let G be the graph in Figure 6.7. Then $\mathcal{F}_2(G)$ is non-planar.*

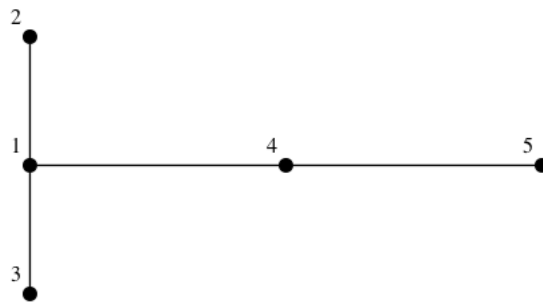


Figure 6.7: The graph G .

Below is a drawing of $\mathcal{F}_2(G)$. To show that it is non-planar, we show that it has a $K_{3,3}$ subdivision.

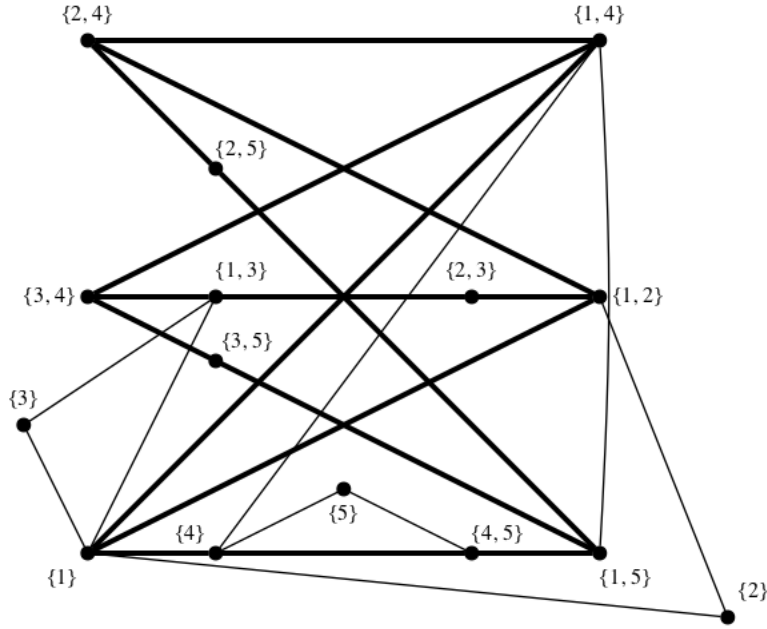


Figure 6.8: A drawing of $\mathcal{F}_2(G)$. The $K_{3,3}$ subdivision has bold edges.

Since $\mathcal{F}_2(G)$ has a $K_{3,3}$ subdivision, $\mathcal{F}_2(G)$ is non-planar.

Theorem 6.6. *Let G be a connected graph. Then $\mathcal{F}_2(G)$ is planar if and only if G is one of the following:*

- P_n for some $n \in \mathbb{N}$.
- C_3 .
- S_3 .
- The graph G pictured in figure 6.5.

We already have that if G is one of the above graphs, then $\mathcal{F}_2(G)$ is planar. Thus, we need to prove that if G is a graph such that $\mathcal{F}_2(G)$ is planar, then G is necessarily one of the above graphs. Let G be a graph such that $\mathcal{F}_2(G)$ is planar. If there is a vertex of degree 4 or more in G , then S_4 is a subgraph of G , which by Theorem 3.7 means that $\mathcal{F}_2(S_4)$ is a subgraph of $\mathcal{F}_2(G)$. Then by Theorem 6.3, since

$\mathcal{F}_2(S_4)$ is non-planar, $\mathcal{F}_2(G)$ is also non-planar. Hence, the maximum degree that a vertex in G can have is 3.

Suppose the maximum degree of a vertex in G is 3. Label this vertex 1, and its neighbors 2, 3 and 4. Suppose that $|V(G)| \geq 5$. Then since G is connected and the degree of 1 is 3, there must exist some vertex (label it 5) that is adjacent to at least one of 2, 3 and 4. Without loss of generality, suppose it is adjacent to 4. Then the graph H pictured in Figure 6.7 is a subgraph of G , which by Theorem 3.7 means that $\mathcal{F}_2(H)$ is a subgraph of $\mathcal{F}_2(G)$. Then by Theorem 6.5, since $\mathcal{F}_2(H)$ is non-planar, $\mathcal{F}_2(G)$ is also non-planar. This is a contradiction. Thus, if the maximum degree of a vertex in G is 3, then G must have exactly 4 vertices. In this case, we have 4 possibilities:

- $G = S_3$. This is one of the cases we stated in the theorem.
- G is the graph pictured in Figure 6.5 This is another one of the cases we stated in the theorem.
- G is K_4 less one edge. Label the vertices of this graph with the numbers 1 through 4 such that 2 is not adjacent to 4. Then the path 1, 2, 3, 4 is a cycle in G , and thus C_4 is a subgraph of G . Then by Theorem 3.7, $\mathcal{F}_2(C_4)$ is a subgraph of $\mathcal{F}_2(G)$, and then by Theorem 6.2, $\mathcal{F}_2(C_4)$ is non-planar, and thus $\mathcal{F}_2(G)$ is also non-planar.
- G is K_4 . We use similar logic to the previous case to show that $\mathcal{F}_2(G)$ is non-planar.

Suppose now that the maximum degree of a vertex in G is at most 2. Then G is either P_n for some $n \in \mathbb{N}$ (which is one of the cases stated in this theorem) or G is C_n for some $n \in \mathbb{N}$. If G is C_n for $n \geq 4$, then by Theorem 6.1, $\mathcal{F}_2(G)$ is non-planar. Then if G is C_n for some $n \in \mathbb{N}$, n must be 3, and thus $G = C_3$ (which is the last case outlined in the theorem). Therefore, our theorem holds.

Planarity of $\mathcal{S}_2(G)$

Theorem 6.7. $\mathcal{S}_2(P_n)$ is planar if and only if $n \leq 4$.

By Theorem 3.8, since P_n is a subgraph of $P_{n'}$ for $n \leq n'$, it is sufficient to prove the following statements:

1. $\mathcal{S}_2(P_4)$ is planar.
2. $\mathcal{S}_2(P_5)$ is nonplanar.

If the first statement is true, then we have that $\mathcal{S}_2(P_n)$ is a subgraph of $\mathcal{S}_2(P_4)$ for $n \leq 4$, and therefore $\mathcal{S}_2(P_n)$ must also be planar for $n \leq 4$. If the second statement is true, we have that $\mathcal{S}_2(P_5)$ is a subgraph of $\mathcal{S}_2(P_n)$ for $n \geq 5$, and therefore $\mathcal{S}_2(P_n)$ is also nonplanar for $n \geq 5$.

To prove: $\mathcal{S}_2(P_4)$ is planar. To prove this, we must merely draw $\mathcal{S}_2(P_4)$ in a plane way.

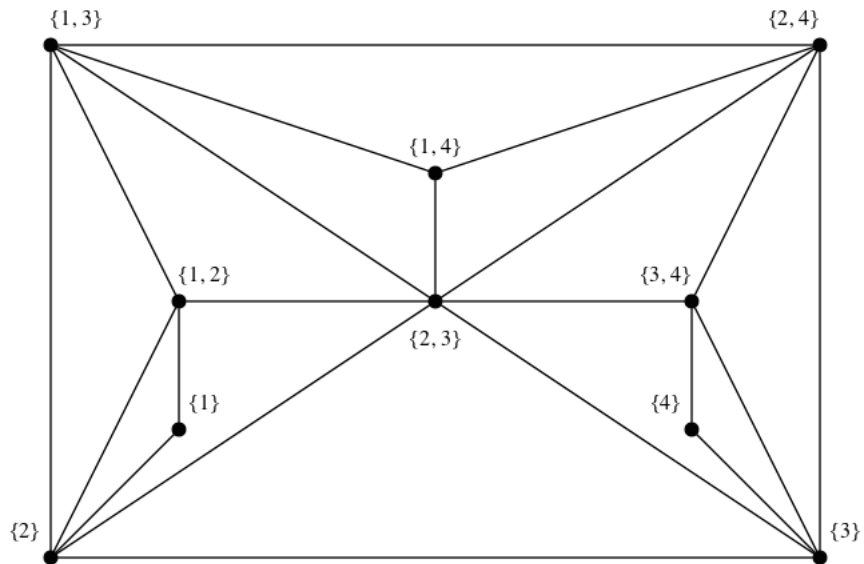


Figure 6.9: A plane drawing of $\mathcal{S}_2(P_4)$

Since we have a plane drawing of $\mathcal{S}_2(P_4)$, it must be planar.

To prove: $\mathcal{S}_2(P_5)$ is nonplanar.

Below is a drawing of $\mathcal{S}_2(P_5)$. To show that it is non-planar, we show that it has a K_5 subdivision.

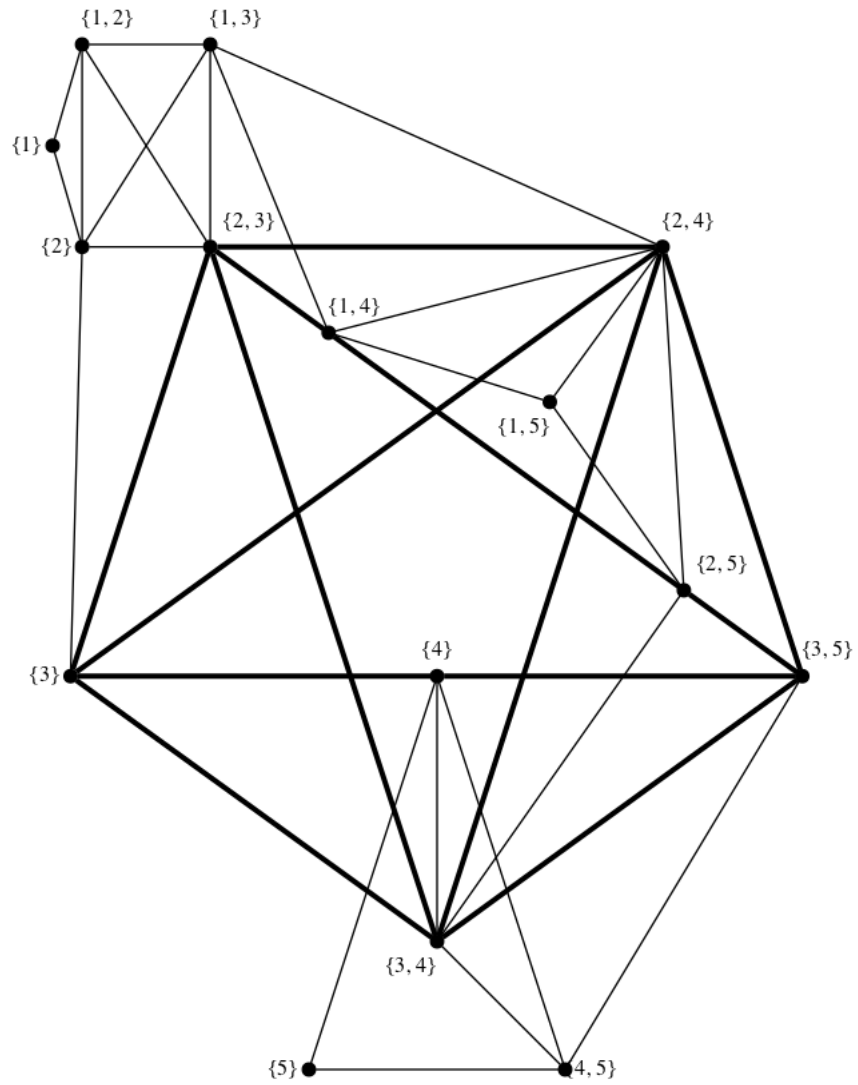


Figure 6.10: A drawing of $\mathcal{S}_2(P_5)$. The K_5 subdivision has bold edges.

Consider \mathcal{H} to be the subgraph of $\mathcal{S}_2(P_5)$ induced by the vertex set $\{\{3\}, \{2, 3\}, \{3, 4\}, \{2, 4\}, \{3, 5\}\}$. This is isomorphic to K_5 minus two edges incident on the same vertex (corresponding to $\{3, 5\}$). Consider the paths: $\{3, 5\}, \{2, 5\}, \{1, 4\}, \{2, 3\}$ and $\{3, 5\}, \{4\}, \{3\}$. These paths are disjoint from each

other, as well as \mathcal{H} (less the endpoints of the paths). Additionally, these paths join $\{3, 5\}$ with $\{2, 3\}$ and $\{3\}$ respectively. Thus, $\mathcal{F}_2(P_5)$ has a K_5 subdivision. Therefore, by Kuratowski's theorem, $\mathcal{F}_2(P_5)$ is nonplanar.

Theorem 6.8. $\mathcal{S}_2(C_n)$ is nonplanar for $n \geq 3$.

Proof: $\mathcal{S}_2(C_3) = K_6$. Thus, $\mathcal{F}_2(C_3)$ is nonplanar.

For C_4 , label the vertices of C_4 with the numbers 1 through 4 such that 1 is not adjacent to 3 and 2 is not adjacent to 4. The subgraph of $\mathcal{F}_2(C_4)$ induced by the vertex set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}, \{1, 3\}\}$ is isomorphic to K_5 . Thus, $\mathcal{F}_2(C_4)$ is nonplanar.

For $n \geq 5$, as P_5 is a subgraph of C_n , by Theorem 3.8, we have that $\mathcal{S}_2(P_5)$ is a subgraph of $\mathcal{S}_2(C_n)$. Since $\mathcal{S}_2(P_5)$ is nonplanar, $\mathcal{S}_2(C_n)$ is nonplanar for $n \geq 5$.

Theorem 6.9. $\mathcal{S}_2(S_n)$ is planar if and only if $n \leq 3$.

By Theorem 3.8, since S_n is a subgraph of $S_{n'}$ for $n \leq n'$, it is sufficient to prove the following statements:

1. $\mathcal{S}_2(S_3)$ is planar.
2. $\mathcal{S}_2(S_4)$ is nonplanar.

If the first statement is true, then we have that $\mathcal{S}_2(S_n)$ is a subgraph of $\mathcal{S}_2(S_3)$ for $n \leq 3$, and therefore $\mathcal{S}_2(S_n)$ must also be planar for $n \leq 3$. If the second statement is true, we have that $\mathcal{S}_2(S_4)$ is a subgraph of $\mathcal{S}_2(S_n)$ for $n \geq 4$, and therefore $\mathcal{S}_2(S_n)$ is also nonplanar for $n \geq 4$.

To prove: $\mathcal{S}_2(S_3)$ is planar. To prove this, we must merely draw $\mathcal{S}_2(S_3)$ in a plane way.

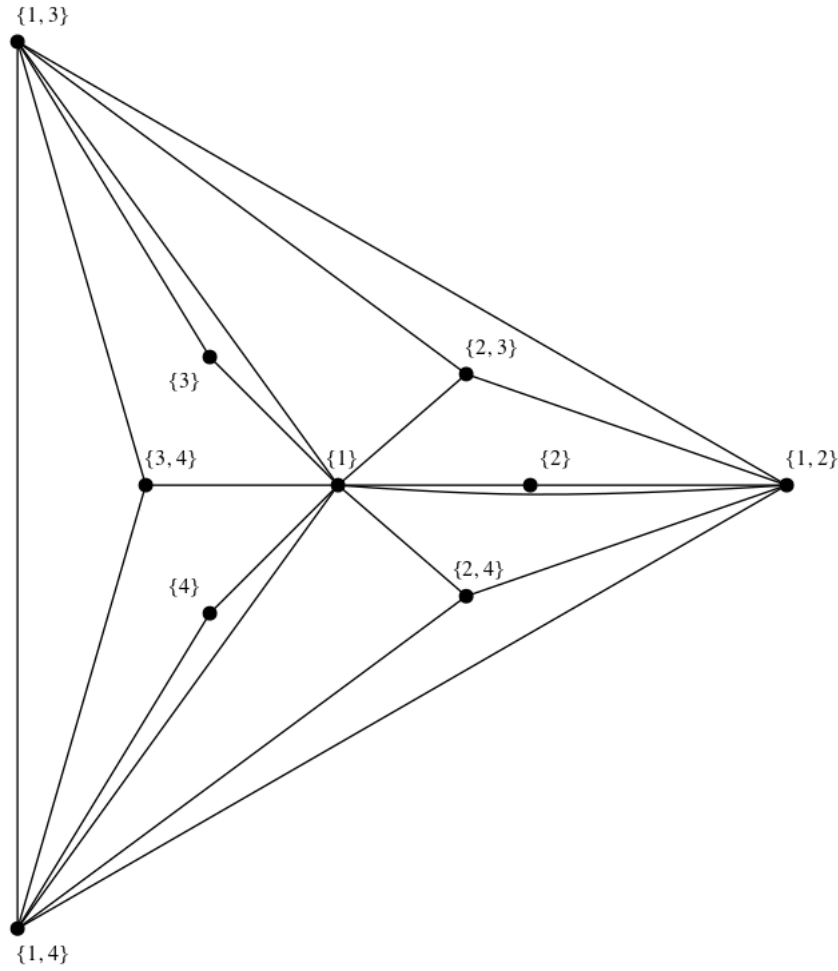


Figure 6.11: A plane drawing of $\mathcal{S}_2(S_3)$

Since we have a plane drawing of $\mathcal{S}_2(S_3)$, it must be planar.

To prove: $\mathcal{S}_2(S_4)$ is non-planar. Note that by Theorem 3.6, $\mathcal{F}_2(S_4)$ is a subgraph of $\mathcal{S}_2(S_4)$. By Theorem 6.3, we have that $\mathcal{F}_2(S_4)$ is non-planar. Therefore, $\mathcal{S}_2(S_4)$ is non-planar.

Theorem 6.10. *Let G be the graph in Figure 6.12. Then $\mathcal{S}_2(G)$ is non-planar.*

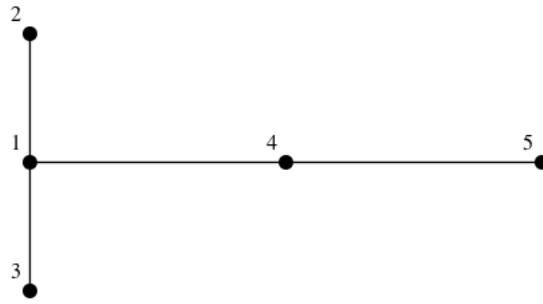


Figure 6.12: The graph G .

To prove: $\mathcal{S}_2(G)$ is non-planar. Note that by Theorem 3.6, $\mathcal{F}_2(G)$ is a subgraph of $\mathcal{S}_2(S_4)$. By Theorem 6.5, we have that $\mathcal{F}_2(G)$ is non-planar. Therefore, $\mathcal{S}_2(G)$ is non-planar.

Theorem 6.11. *Let G be a connected graph. Then $\mathcal{S}_2(G)$ is planar if and only if one of the following is true:*

- $G = P_n$ for $n \leq 4$.
- $G = S_3$.

Proof: We already have by previous theorems that if G is one of the above graphs, then $\mathcal{S}_2(G)$ is planar. Hence, we need to prove that if $\mathcal{S}_2(G)$ is planar, then G is one of the above graphs. Let G be an arbitrary connected graph such that $\mathcal{S}_2(G)$ is planar. If G is not a tree, then it must have a cycle. However, if this were the case, then there would be some n such that C_n is a subgraph of G , and thus $\mathcal{S}_2(C_n)$ would be a subgraph of $\mathcal{S}_2(G)$. Since $\mathcal{S}_2(C_n)$ is non-planar for $n \geq 3$, this cannot be the case, as $\mathcal{S}_2(G)$ is planar. Then G must be a tree. If G has a vertex of degree 4 or more, then S_4 would be a subgraph of G , and thus $\mathcal{S}_2(S_4)$ would be a subgraph of $\mathcal{S}_2(G)$. As $\mathcal{S}_2(S_4)$ is non-planar, this cannot be the case. Hence, G 's maximum degree is at most 3. If G 's maximum degree is exactly 3, then G has a vertex v of degree 3. If G has more than four vertices, however, then one of the vertices that v is adjacent to must also be adjacent to another vertex. This would mean that the graph given in Figure 6.12 would be a subgraph of G , and then we arrive at a contradiction, as \mathcal{S}_2 of that graph is non-planar. Thus, if G 's maximum degree is 3, it must be S_3 . If G 's maximum degree is 2 or less, then since G is a tree, it must be a path. Since G is planar, we have by Theorem 6.7 that it must be P_n with $n \leq 4$. Therefore, our theorem holds.

Chapter 7

Colorings

In this chapter, we establish a couple theorems re the chromatic numbers of our hyperspace graphs, as for a given graph product, it is a common question to ask if the chromatic number of a product graph can be obtained from the chromatic numbers of their component graphs. These theorems (as well as the concepts behind them) are later used in Chapter 9, when attempting to establish whether or not certain hyperspace graph contain Hamiltonian paths.

Theorem 7.1. $\mathcal{L}_i(G)$ is at worst k -colorable for all $1 \leq i \leq n$, where k is the chromatic number of G .

Proof: By definition, an edge AB exists in $L_i(G)$ if and only if there exists some set W and some edge ab such that $A = W \cup \{a\}$ and $B = W \cup \{b\}$, with $A \neq B$. Since $|A| = |B| = i$, this means that $|W| = i - 1$ and $a, b \notin W$. Color each vertex of G with a number between 0 and $k - 1$. Let the coloring of an arbitrary vertex v be c_v . Once this is done, color each vertex of $L_i(G)$ such that its number is equal to the sum of the numbers of the vertices that are in the set corresponding to that vertex, taken mod k . Suppose $AB \in L_i(G)$. Then there exist a set W and an edge ab as described above. Then we have that the color of A is given by

$$\left(\sum_{v \in W} c_v \right) + c_a,$$

and that the color of B is given by

$$\left(\sum_{v \in W} c_v \right) + c_b.$$

If these two numbers were the same taken mod k , then if we were to set these two equations equal to

each other mod k and subtract the large sums over the vertices in W from both sides, we would get that $c_a = c_b \pmod k$. Since we know that a and b are adjacent, they do not have the same number mod k . This is a contradiction; therefore, B 's number must be different than A 's number, mod k . Hence, we have a k -coloring of $L_i(G)$.

Remark: G being k -chromatic does not imply that $\mathcal{L}_i(G)$ is k -chromatic. In particular, notice that $\mathcal{L}_2(K_4) = K_{2,2,2}$ is 3-chromatic, whereas K_4 is 4-chromatic.

Theorem 7.2. $\mathcal{F}(G)$'s chromatic number is $k + 1$, where k is the chromatic number of G .

Proof: By definition, an edge AB exists in $\mathcal{F}(G)$ if and only if there exists some set W and some edge ab such that $A = W \cup \{a\}$ and $B = W \cup \{b\}$, with $A \neq B$. Color each vertex of G with a number between 1 and k . Let the coloring of an arbitrary vertex v be c_v . Once this is done, color each vertex of $F(G)$ such that its number is equal to the sum of the numbers of the vertices that are in the set corresponding to that vertex, taken mod $k + 1$. Suppose $AB \in F(G)$. Then there exist a set W and an edge ab as described above. Since $A \neq B$, we have that at least one of $a, b \notin W$. We shall split this into three cases.

1. $a, b \notin W$. Then we have that the color of A is given by

$$\left(\sum_{v \in W} c_v \right) + c_a,$$

and that the color of B is given by

$$\left(\sum_{v \in W} c_v \right) + c_b.$$

If these two numbers were the same taken mod $k + 1$, then if we were to set these two equations equal to each other mod $k + 1$ and subtract the large sums over the vertices in W from both sides, we would get that $c_a = c_b \pmod{k + 1}$. Since we know that a and b are adjacent, they do not have the same number mod k . This is a contradiction; therefore, B 's number must be different than A 's number, mod $k + 1$.

2. $a \notin W, b \in W$. Then we have that the color of A is given by

$$\left(\sum_{v \in W} c_v \right) + c_a,$$

and that the color of B is given by

$$\left(\sum_{v \in W} c_v \right).$$

If these two numbers were the same taken mod k , then if we were to set these two equations equal to each other mod k and subtract the large sums over the vertices in W from both sides, we would get that $c_a = 0 \pmod{k+1}$. Since we know that c_a is between 1 and $k \pmod{k+1}$, it cannot be equal to $0 \pmod{k+1}$. This is a contradiction; therefore, B 's number must be different than A 's number, mod $k+1$.

3. $a \in W, b \notin W$. By similar argument to the last case, we can show that B 's label must be different than A 's label, mod $k+1$.

Then for every edge $AB \in \mathcal{F}(G)$, their colors must be different. Hence, we have a $k+1$ -coloring of $\mathcal{F}(G)$.

Additionally, as $\mathcal{F}(G)$ contains an embedded copy of G , namely $\mathcal{L}_{n-1}(G)$, along with an additional vertex that is adjacent to every vertex of this embedded copy. Hence, $\mathcal{F}(G)$ is at best $k+1$ colorable. Therefore, $\mathcal{F}(G)$'s chromatic number must be $k+1$.

Chapter 8

Distances

In this chapter, we ultimately establish a categorization for the distances between two arbitrary vertex sets U and V in $\mathcal{F}_k(G)$ and $\mathcal{S}_k(G)$. While these theorems may not be the best for getting the distance between two vertex sets from a computational standpoint, they may be useful in proofs for future research questions.

Theorem 8.1. *Define the distance between a vertex u and a set of vertices V as follows:*

$$d(u, V) = \min_{v \in V} d(u, v).$$

Take two sets $A, B \in \mathcal{S}(G)$. Then the distance between the vertex sets A and B in $\mathcal{S}(G)$ is equal to

$$\max \left\{ \max_{a \in A} d(a, B), \max_{b \in B} d(b, A) \right\}.$$

Proof: Let the value specified in the above theorem be called d_S . By definition, we know that either there exists some vertex $a \in A$ such that $d(a, B) = d_S$ or that there exists some vertex $b \in B$ such that $d(b, A) = d_S$. Without loss of generality, assume that the former case is true.

Suppose that the true distance between A and B in $\mathcal{S}(G)$ is less than d_S . Call this value k .

Then there exists a sequence of sets of vertices U_1, U_2, \dots, U_{k+1} such that $U_1 = A$, $U_{k+1} = B$, and the sets U_i and U_{i+1} are adjacent in $\mathcal{S}(G)$ for all $i \leq k$.

Let $u_1 = a$. By definition, we know that there must exist some vertex $u_2 \in U_2$ such that either $u_1 = u_2$ or $u_1 u_2 \in E(G)$.

Through iterative construction, we can build a sequence of vertices u_i such that $u_i \in U_i$, and either $u_i = u_{i+1}$ or $u_i u_{i+1} \in E(G)$ for all $i \leq k$.

This sequence of vertices $\{u_i\}_{i=1}^{k+1}$ readily induces a path of at most length k between a and some vertex $u_{k+1} \in B$, once duplicate vertices and loops are eliminated. Then we have that $d(a, B) \leq k$.

But we already stated that $d(a, B) = d_S > k$. This is a contradiction. Hence, $d(A, B) \geq d_S$.

To show that $d(A, B) \leq d_S$, we construct a path of length d_S between A and B .

Label the vertices in A as $\{a_1, a_2, \dots, a_r\}$.

Since every vertex in $d(a_i, B) \leq d_S$, we know that for every vertex $a \in A$, there must exist some vertex $b \in B$ such that $d(a, b) \leq d_S$. Then for every vertex $a_i \in A$, there must exist a path $u_{i,1}, u_{i,2}, \dots, u_{i,d_i+1}$ such that $u_{i,1} = a_i$, $u_{i,d_i+1} \in B$, and $d_i \leq d_S$.

Define the following sets

$$U_j = \bigcup_{1 \leq i \leq r} u_{i,j}$$

for all $j \leq d_S + 1$, where $u_{i,j}$ is defined as we did before for $j \leq d_i + 1$, and $u_{i,j} = u_{i,d_i+1}$ for all $j > d_i + 1$. Notice that every vertex in each U_j has at least one vertex in U_{j-1} and U_{j+1} that it is either the same as or adjacent to; namely, $u_{i,j}$ is either the same as or adjacent to both $u_{i,j-1}$ and $u_{i,j+1}$. Lastly, notice that by our definitions, $U_1 = A$ and $U_{d_S} \subset B$.

In a similar method, define the sequence of sets V_j in a similar method, but with $V_1 = B$ and $V_{d_S} \subset A$.

Now let $W_j = U_j \cup V_{d_S+1-(j-1)}$. Then every vertex in each W_j will have some vertex in W_{j-1} and W_{j+1} that it is adjacent to (namely, if the vertex was originally in U_j , its corresponding vertices in U_{j+1} and U_{j-1} , and if the vertex was originally in $V_{d_S+1-(j-1)}$, its corresponding vertices). Then we have a sequence of vertex sets such that $W_1 = A$, $W_{d_S+1} = B$, and as long as $W_j \neq W_{j+1}$, $W_j W_{j+1} \in E(G)$. However, if some $W_j = W_{j+1}$, we would be able to construct a path of distance less than d_S between A and B by simply omitting the repeated vertices and removing any loops in the sequence. Since we already determined this to be impossible, each W_j must be different than W_{j+1} .

Therefore, $d(A, B) \leq d_S$, and thus, $d(A, B) = d_S$.

Theorem 8.2. *Let G be an arbitrary tree of order n . Then $\text{diam}(\mathcal{F}(G)) = n - 1$.*

Proof: We can partition $\mathcal{F}(G)$ into sets of vertices, where each of our partitions contains all of the vertex sets of a fixed size. Since we know that the partition containing all vertex sets of size k is adjacent to at most two other partitions, namely the one containing all vertex sets of size $k + 1$ and the one containing all vertex sets of size $k - 1$, we know that $d(V(G), \{v\})$ is at least $k - 1$ for every vertex $v \in V(G)$. Therefore, $\text{diam}(\mathcal{F}(G)) \geq n - 1$.

We now need to prove that $d(U, V) \leq n - 1$ for any two vertex sets $U, V \subset V(G)$. We do this by structural induction on the number of vertices of our original graph n .

Base: $n = 1$. In this case, our tree is just a single vertex, and $\mathcal{F}(G)$ is also merely a single vertex. A single vertex has diameter $0 = 1 - 1 = n - 1$, so the theorem holds for the base case.

Inductive step: Suppose that for all trees G of size k , $\mathcal{F}(G)$ has diameter $k - 1$ or less.

Let G be an arbitrary tree of size $k + 1$. Let $U, V \subset V(G)$. We need to prove that $d(U, V) \leq (k + 1) - 1 = k$.

Since G is a tree of order 2 or more, we know that G has a leaf. Let v be some arbitrary leaf of G .

Case 1: $v \notin U, v \notin V$. In this case, U and V are both vertex sets of the induced graph H that we get by removing the leaf v from G . Since this graph is obtained by removing a leaf from a tree of size $k + 1$, we know that H is a tree of size k . Then by inductive hypothesis, we know that $d_{\mathcal{F}(H)}(U, V) \leq k - 1$. But since H is a subgraph of G , we know that $\mathcal{F}(H)$ is a subgraph of $\mathcal{F}(G)$, and thus, $d_{\mathcal{F}(G)}(U, V) \leq k - 1 \leq k$, and we have the theorem hold.

Case 2: $v \in U, v \notin V$. Let u be the vertex that v is adjacent to in G . Then U is adjacent to the set $W = U - \{v\} \cup \{u\}$ in $\mathcal{F}(G)$. Since W and V both do not contain v , we know by Case 1 that $d(W, V) \leq k - 1$. Therefore, $d(U, V) \leq (k - 1) + 1 = k$, and the theorem holds.

Case 3: $v \notin U, v \in V$. See case 2.

Case 4: $v \in U, v \in V$. This case requires its own subcases.

Subcase 1: $U = \{v\}$ or $V = \{v\}$. Since we know that H is a tree, it has at least two leaves. Select a new leaf of the tree G . Since this new leaf is not in at least one of U and V , we can use Case 1, 2 or 3 to show that $d(U, V) \leq k$.

Subcase 2: $U \neq \{v\}$ and $V \neq \{v\}$. Consider the sets $U^* = U \setminus \{v\}$ and $V^* = V \setminus \{v\}$. Note that both of

these sets are nonempty. Let H be the induced subgraph of G obtained by removing v . Note that H is a tree. Then we have that $d_{\mathcal{F}(H)}(U^*, V^*) \leq k - 1$. Let $d = d_{\mathcal{F}(H)}(U^*, V^*) \leq k - 1$. Therefore, there exist a sequence of subsets of $V(H)$, denoted $\{W_i^*\}_{i=0}^d$, such that $W_0^* = U^*$, $W_d^* = V^*$, and each W_i^* is adjacent to W_{i+1}^* . Therefore, $W_i^* \setminus W_{i+1}^*$ contains at most one vertex, and vice-versa. If $v_i \in W_i^* \setminus W_{i+1}^*$ then there exists a $v \in W_{i+1}^*$ such that $v_1 v \in E(G)$, and vice-versa. Lastly, the two sets are not equal.

Now let W_i be defined as $W_i^* \cup \{v\}$. $W_i \setminus W_{i+1}$ still contains at most one vertex, since both contain v as a vertex and thus is not in $W_i \setminus W_{i+1}$. We similarly have that $W_{i+1} \setminus W_i$ contains at most one vertex. If $v_i \in W_i^{star} \setminus W_{i+1}^*$ then there exists a $v \in W_{i+1}^*$ such that $v_1 v \in E(G)$, and vice-versa. Lastly, we still have that $W_i \neq W_{i+1}$, as neither of their respective W^* sets contained j as a vertex, and therefore W_i and W_{i+1} must differ on the same vertex that their respective W^* sets differed on. Therefore, we have that W_i is adjacent to W_{i+1} in G . Additionally, note that $U = W_0$ and $V = W_d$. Then we have that $\{W_i\}_{i=0}^d$ is a walk from U to V of length $d \leq k - 1 \leq k$ in $\mathcal{F}(G)$. Therefore, $d_{\mathcal{F}(G)}(U, V)$ is at most $d \leq k$, and the theorem holds.

Then by inductive hypothesis, we have that $d(U, V) \leq n - 1$ for all trees of size n . Therefore, $\text{diam}(\mathcal{F}(G)) \leq n - 1$, and thus, $\text{diam}(\mathcal{F}(G)) = n - 1$.

Theorem 8.3. *Let G be an arbitrary connected graph of size n . Then $\text{diam}(\mathcal{F}(G)) = n - 1$.*

Proof: By similar argument used in the last proof, we can get that $\text{diam}(\mathcal{F}(G)) \geq n - 1$.

Let H be a spanning tree of G . Then H is a subgraph of G , and thus $\mathcal{F}(H)$ is a subgraph of $\mathcal{F}(G)$. Then by properties of the diameter of graphs, $\text{diam}(\mathcal{F}(G)) \leq \text{diam}(\mathcal{F}(H)) = n - 1$. Thus, $\text{diam}(\mathcal{F}(G)) \leq n - 1$, and thus, $\text{diam}(\mathcal{F}(G)) = n - 1$.

Theorem 8.4. *Let G be an arbitrary graph of size n . Let $U, V \subset V(G)$. If there exists a partition of the vertices of G such that the induced subgraph of G on each of those partitions is connected, each set of the partition contains at least one vertex in U and at least one vertex in V or no vertex in either U or V , and the partition has k sets, then $d(U, V) \leq n - k$.*

Proof: Suppose such a partition exists. Call the sets of this partition P_1, P_2, \dots, P_k . Let H_i be the induced subgraph of G on P_i . Let n_i be the order of H_i . Then $\sum_{i=1}^k n_i = n$. For all $0 \leq i \leq k$, let

$$U_i = \left(V \cap \bigcup_{j=1}^i P_j \right) \cup \left(U \cap \bigcup_{j=i+1}^k P_j \right).$$

Note that $U_0 = U$, $U_k = V$, and that for all $0 \leq i \leq k-1$, $U_i \setminus P_{i+1} = U_{i+1} \setminus P_{i+1}$; in other words, the only vertices in which U_i and U_{i+1} can possibly differ by are those vertices that are in P_{i+1} . Additionally, if P_{i+1} contains no vertices in either U or V , then $U_i = U_{i+1}$.

To prove: $d(U_i, U_{i+1}) \leq n_i - 1$. Trivially, if P_{i+1} contains no vertices in either U or V , then $U_i = U_{i+1}$, and thus $d(U_i, U_{i+1}) = 0 = 1 - 1 \leq n_i - 1$. Suppose $P_{i+1} \cap U \neq \emptyset$ and $P_{i+1} \cap V \neq \emptyset$. Let $W = U_i \setminus P_{i+1} = U_{i+1} \setminus P_{i+1}$. Consider the family of vertex sets

$$F = \{W \cup A \mid A \subset P_{i+1}, A \neq \emptyset\}.$$

Then by previous theorem, we have that the induced subgraph of $\mathcal{F}(G)$ on F is isomorphic to $\mathcal{F}(H_{i+1})$, and both U_i and U_{i+1} are in F . Then by the prior theorem, we have that

$$d_F(U_i, U_{i+1}) \leq |V(H_{i+1})| - 1 = n_i - 1.$$

Then since the induced subgraph of $\mathcal{F}(G)$ on F is a subgraph of $\mathcal{F}(G)$, we have that

$$d_{\mathcal{F}(G)}(U_i, U_{i+1}) \leq d_F(U_i, U_{i+1}) \leq n_{i+1} - 1.$$

Then $d(U, V) \leq \sum_{i=0}^{k-1} d(U_i, U_{i+1}) \leq \sum_{i=0}^{k-1} n_{i+1} - 1 = n - k$. Therefore, the theorem holds.

Theorem 8.5. *Let G be an arbitrary graph of size n . Let $U, V \subset V(G)$. Let P be the largest partition of the vertices of G such that the induced subgraph of G on each of those partitions is connected, each set of the partition contains at least one vertex in U and at least one vertex in V OR no vertex in either U or V , and the partition has k sets, then $d(U, V) = n - k$.*

Proof: We know that there exists a trivial partition of the vertices of G that satisfies the criterion above; namely, $P^* = \{V(G)\}$. Since the partition can have at most n non-empty sets, there must be a largest element of the family of all partitions that satisfy the above criterion above. Let P be such a partition, and let k be its size. By the previous proof, we have that $d(U, V) \leq n - k$.

Suppose that $d(U, V) \leq n - k - 1$. Then there exists a collection of vertices $\{U_i\}_{i=0}^{n-k-1}$ such that $U_0 = U$, $U_{n-k-1} = V$ and U_i is adjacent to U_{i+1} for $0 \leq i \leq n - k - 2$. Let e_i be the transversal edge between U_i and U_{i+1} (review chapter 1 for the definition of a transversal edge). Consider the graph H , where $V(H) = V(G)$ and $E(H) = \{e_i \mid 0 \leq i \leq n - k - 2\}$. Then H has at least $k + 1$ components, as it has n vertices and only $n - k - 1$ edges. We shall show that each of the components of H has at least one vertex contained in both U and V , or no vertex in either U or V .

Suppose U contains a vertex in some arbitrary component of H . We will show that if U_i contains some vertex in this component, then U_{i+1} also contains some vertex in that component, and by an inductive argument, as $U_0 = U$ contains some vertex in that component, we have that $U_{n-k-1} = V$ contains some vertex in that component.

Suppose U_i contains some vertex u in a component of H . Then $U_i = W \cup \{u\}$ and $U_{i+1} = W \cup \{v\}$ for some $W \subset V(G)$ and $v \in V(G)$. But then uv is a transversal edge, and thus $uv \in E(H)$. Then v is in the same component of u , and thus U_{i+1} contains some vertex in the same component.

By similar argument (working backwards), we end up getting that if V contains some vertex in some arbitrary component of H , then U also contains some vertex in that same component of H . Then each component of H either has some vertex in both U or V , or no vertex in either U or V . Let $P' = \{A \mid A \text{ is the set of vertices of some component of } H\}$. Then P' has at least $k + 1$ sets, is a partition of $V(G)$, and satisfies the criterion given in the statement of the theorem of this proof. But this contradicts our choice of P . Therefore, $d(U, V) > n - k - 1$, and therefore, $d(U, V) = n - k$.

Chapter 9

Hamiltonicity

Whenever a new category a graph is created, it is a common graph theoretic question to ask whether or not the graph is Hamiltonian. However, since the categorization of Hamiltonian graphs in general is an open problem, we have chosen to focus on a specific category of hyperspace graphs for this area: specifically, we are looking at $\mathcal{F}_k(P_n)$. To start, we first examine $\mathcal{F}_2(P_n)$.

Theorem 9.1. $\forall n \in \mathbb{N}$, $\mathcal{F}_2(P_n)$ is Hamiltonian.

The Hamiltonian path of $\mathcal{F}_2(P_n)$ is readily observed by looking at P_n . First, start with the vertex set $\{1\}$. Then, iterate through the vertex sets $\{1, 2\}$, $\{1, 3\}$, ..., $\{1, n\}$. Next, move to the vertex set $\{2, n\}$, and then go through the vertex sets $\{2, n-1\}$, ..., $\{2, 4\}$. Next, move to the vertex set $\{3, 4\}$, and start over, incrementing the second vertex to n , increasing the first vertex by one, decrementing the second vertex to the vertex that is a distance of two to the right of the first vertex, and then incrementing the first vertex by one once more. Once all of these steps have been completed, the path should currently end at $\{n-1, n\}$, and should cover all of the vertices in $\mathcal{F}_2(P_n)$ excluding the singletons and every set of the form $\{i, i+1\}$ for i even (with the possible exception of $\{n-1, n\}$, if n is odd). Continue this path by moving to the vertex set $\{n\}$. Here is an image of the path we have taken thus far:

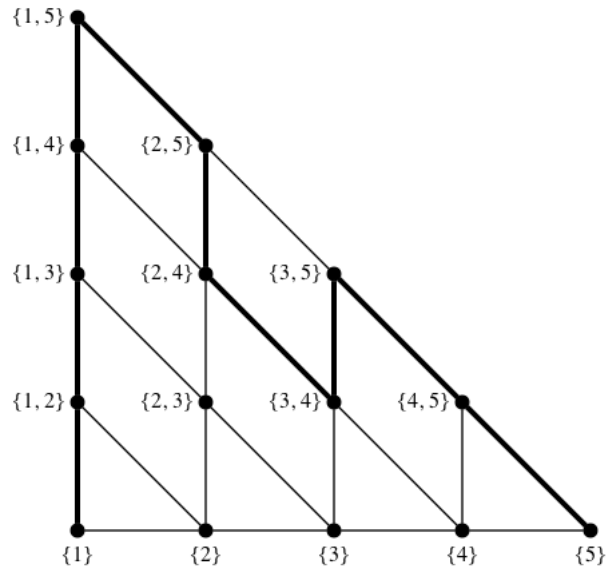


Figure 9.1: A Hamiltonian cycle of $\mathcal{F}_2(P_4)$.

We complete this cycle by closely following the path $\{n\}, \{n-1\}, \dots, \{1\}$; however, for each set of the form $\{i, i+1\}$ that has not been visited, we will instead take the edges $\{i+1\}, \{i, i+1\}, \{i\}$ instead of the edge $\{i+1\}, \{i\}$. See Figure 9.2.

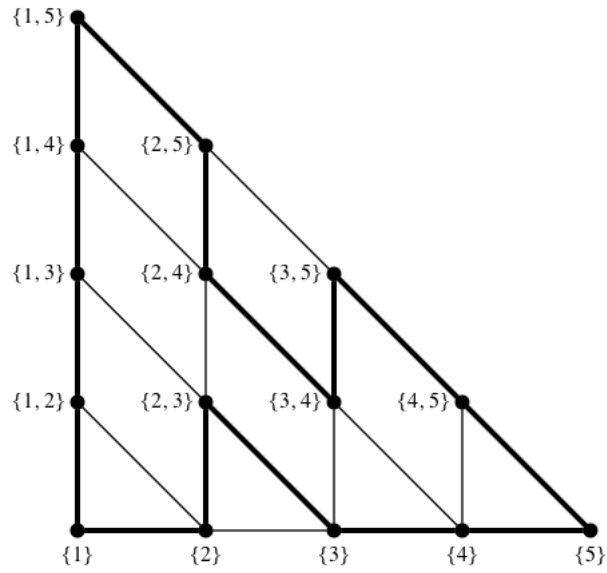


Figure 9.2: The first part of a Hamiltonian cycle of $\mathcal{F}_2(P_5)$.

Similarly, for n even, the Hamiltonian cycle is shown in Figure 9.3.

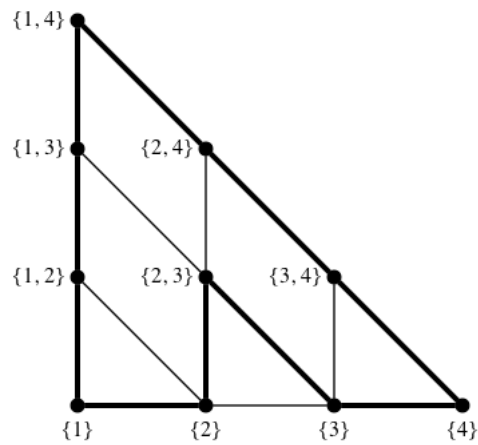


Figure 9.3: A Hamiltonian cycle of $\mathcal{F}_2(P_5)$.

Hence, we have that $\mathcal{F}_2(P_n)$ is Hamiltonian $\forall n$.

The question then arises: is $\mathcal{F}_3(P_n)$ Hamiltonian? To start, we checked to see if $\mathcal{L}_3(P_n)$ has a Hamiltonian path, to see if we could, in some form, "attach" this Hamiltonian path to the Hamiltonian cycle. The reason why we chose to attack the problem this way was to possibly implement some iterative method to show that $\mathcal{F}_k(P_n)$ is Hamiltonian in general.

To start, we found that $\mathcal{L}_3(P_4)$ has a Hamiltonian path (rather trivially, as we have that $\mathcal{L}_3(P_4) \simeq \mathcal{L}_1(P_4) \simeq P_4$). Next, we attempted to see if $\mathcal{L}_3(P_5)$ has a Hamiltonian path. Note that this graph is actually bipartite, as the original graph P_5 is also bipartite. Then we immediately have that this graph does not contain a Hamiltonian path; in one of its partitions, there are 6 vertices (specifically, the vertex sets whose constituent vertices sum to an even number) and in its other partition, there are 4 vertices (the vertex sets whose constituent vertices sum to an odd number). Since this graph is bipartite, if it did have a Hamiltonian path, it would have to alternate between vertices between its first partition and vertices in its second partition. However, since this number of vertices differs by more than one, this is impossible. We can generalize this statement to the following:

Theorem 9.2. $\mathcal{L}_k(P_n)$ does NOT contain a Hamiltonian path for any n such that $2 \leq k \leq n - 2$ and $n \geq 5$ is odd.

(Credit to Brendan Shah for this generating functions proof)

Let $j = \frac{n-1}{2} \geq 2$. Since P_n is bipartite, by theorem 7.1, we know that $\mathcal{L}_k(G)$ is bipartite. Label the vertices of P_n with the numbers 0 and 1 such that the two vertices of degree 1 are both labeled with a 1, and no two adjacent vertices are labeled with the same number. Then $\mathcal{L}_k(G)$ is bipartite, with the vertices in the first partition consisting of those vertex sets whose constituent vertices sum to 0 mod 2, and the vertices in the second partition consisting of those vertex sets whose constituent vertices sum to 1 mod 2. We wish to show that the number of vertices in these sets differ by more than 1. Then our problem can be stated as follows:

Given the string 1010...101, of length n select some odd number of characters k from the string. What is the difference between the number of ways we can choose an even number of 1s and the number of ways we choose an odd number of 1s?

The number of elements in the first partition is given by

$$\sum_{i \text{ odd}} \binom{j+1}{i} \binom{j}{k-i},$$

whereas the number of elements in the second partition is exactly the same, but with i even. Hence, the difference in size between these two partitions is given by

$$\sum_{i=0}^k (-1)^i \binom{j+1}{i} \binom{j}{k-i}.$$

We can create a generating function for this number on k by writing it as

$$\sum_{k \geq 0} \left(\sum_{i=0}^k (-1)^i \binom{j+1}{i} \binom{j}{k-i} \right) x^k,$$

which is the product of $\sum_{i \geq 0} \binom{j+1}{i} (-x)^i$ and $\sum_{i \geq 0} \binom{j}{i} x^i$. These are respectively equal to $(1-x)^{j+1}$ and $(1+x)^j$. Therefore, their product is given by $(1-x)(1-x^2)^j$, whose k th coefficient can be obtained by the binomial theorem. This coefficient is $(-1)^{\frac{k+1}{2}} \binom{j}{\frac{k-1}{2}}$ for k odd and $(-1)^{\frac{k}{2}} \binom{j}{\frac{k}{2}}$ for k even. Since $j \geq 2$ and $\frac{k-1}{2}$ and $\frac{k}{2}$ are both bounded between 1 and j by our restrictions, we know that both of these coefficients are greater than j , which is in turn greater than 2. Hence, the number of elements in these partition differs by at least 2, and thus, $\mathcal{L}_k(G)$ is not bipartite.

Chapter 10

Conclusion

While we have answered a number of graph theoretic questions about these hyperspace graphs, there are still many more directions that this research could be taken in the future. There are still more ways to examine these graphs in a graph theoretic sense, and the connections between these graphs and the topological space $\mathcal{F}_k(\mathbb{G})$ (if any meaningful connection can be made) are still open. In particular, we have the following questions:

- If $\mathcal{F}_k(G)$ is isomorphic to $\mathcal{F}_k(H)$, is it necessary that G is isomorphic to H ?
 - What about $\mathcal{S}_k(G)$?
 - How about $L_k(G)$ (for $k \leq n - 1$)?
 - It is not necessarily the case for $\mathcal{M}_k(G)$. For example, $\mathcal{M}_2(C_4)$ is graph isomorphic to $\mathcal{M}_2(K_4)$.
- How do the cliques of the $\mathcal{F}_k(G)$ relate to the dimension of the hyperspace $\mathcal{F}_k(\mathbb{G})$? (similarly for $\mathcal{S}_k(G)$, etc.)
- If there is a simplicial map from G to H , is there a simplicial map from $\mathcal{F}_k(G)$ to $\mathcal{F}_k(H)$? (similarly for $\mathcal{S}_k(G)$, etc.)

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