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An Extension of the Hull White Model for Interest Rate Modeling

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*A Thesis Submitted in Partial Fulfillment of the
Requirements for the Degree of MASTER OF SCIENCE in
APPLIED & COMPUTATIONAL MATHEMATICS*

SCHOOL OF MATHEMATICAL SCIENCES
COLLEGE OF SCIENCE

Rochester Institute of Technology

Rochester, NY

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Abstract

Since the time of the Black-Scholes model published in 1973, the research about mathematical finance models has never stopped. The original Black-Scholes model is for stock and stock derivatives pricing. However, stock derivatives is not the only kind of financial instrument in the market. Fix income derivatives also plays a very important role in the financial market, appealing to many researchers to explore more about their pricing model.

The fundamental theory of Black-Scholes is still employed in the pricing model for fix income derivatives, but there is something else making the research even more complicate: the definition function for the risk neutral interest rate. Like the stock price, part of the risk neutral interest rate also follows Brownian Motion, but still keeps certain term structure as the basic property of interest rate. There are many famous models in history to determine the risk neutral interest rate, but they have some disadvantages in estimating the spot interest rate. In this paper, we will use the historical data to build a spot neutral interest rate estimation model that can give us more accurate information about the imbalance of the fix income derivative prices.

In this research, we use the yield to maturity of the Treasury bonds as our target, and collect the 10 years data of all kinds of Treasury bonds from Jan 3rd, 1994 to Dec 31st. Then we take part of the data which comes from a period when the economy was relatively stable to conduct the data analysis. Then we notice that the change of the interest rate has the shape of its graph as the intersection of two parabolas with opposite directions. Based on this discovery, we build our model and test it with the other part of data from our collection, and our model turns to work well.

To verify the accuracy of the model, we use the built-in model in MATLAB which is based on the similar theory of ours to do a model comparison. The result of the comparison shows that our model works better than the model in MATLAB.

The spot interest rate estimation model in this research gives a new way to describe the properties of interest rate, and also give a more accurate estimation about the future interest rate. The bond, or fix income derivative, pricing model based on this interest rate model should be able to help investors to make better decisions from a new point of view.

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Chapter 1

Research Background

In the financial market, the stock and the bond are the two most important kinds of financial instruments. The stock is an equity instrument. While the bond is a debt instrument. People want to earn money through buying and selling these two financial products. However, the market price fluctuation makes people lose money sometimes: there are risks to investing. Since realizing this, a lot of economists and mathematicians worked for decades to find a risk-free financial investment portfolio of equity products, like stocks, and fixed-income products, like bonds. They even created financial derivatives to hedge risks. An option is such a kind of financial derivatives.

The simplest option gives the holder the right to trade in the future at a previously agreed price but takes away the obligation. To do so, a call option gives the holder the right to buy fixed amount of stock at the agreed price at expiration day. A put option gives the holder the right to sell fixed amount of stock at the agreed price at expiration day [2]. In 1973, the Black-Scholes model was first articulated by Fischer Black and Myron Scholes. Using this model, a risk-free investment portfolio can be built to obtain positive investment return no matter how stock prices would change. In such an investment portfolio, we have positive value of the call option and negative value of stock price, because stocks in the portfolio are borrowed from someone else, we need to buy them back from the market, the money we pay for the stock is the negative part of the value of our portfolio. Therefore, the function of the portfolio value can be given as:

$$\pi = V(S, t) - \Delta S \quad (1.1)$$

where π is the value of the whole portfolio; $V(S, t)$ is the value of the call option at different time and stock prices; Δ is the amount of stock we need to build the risk-free investment portfolio.

Since the stock has random walk property, the change of the stock price, S , satisfies a stochastic differential equation, which is

$$dS = \mu S dt + \sigma S dX \quad (1.2)$$

where X , equals $X(t)$, is a Brownian motion.

The value of the portfolio also has the stochastic property:

$$d\pi = dV - \Delta dS \quad (1.3)$$

By Ito's Lemma [9]

$$d\pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \Delta^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta dS$$

To make the value of a portfolio perfectly predictable, all random terms in this stochastic equation should be canceled out, which means that all dS should be disappeared. This is how we determine the value of Δ . If we choose $\frac{\partial V}{\partial S} = \Delta$, the two terms containing dS in the equation will be canceled out. The randomness is reduced to zero. Then the value of the portfolio is simply a function of time and some known parameters.

According to the economic theory, arbitrage is a process of buying a good in one market at a low price and selling it in another market at a higher price to profit from the price difference. [7] If the stock market is perfect, there are never any arbitrage opportunities. Based on this theory and the time value of money, the value of the portfolio should change at the rate as same as the risk-free interest rate in the market, if and only if the option is priced correctly.

$$d\pi = (V - \Delta S) r dt$$

$$\frac{\partial V}{\partial t} dt + \frac{1}{2} \Delta^2 S^2 \frac{\partial^2 V}{\partial S^2} dt = (V - \Delta S) r dt$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\Delta^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

where r is the risk-free interest rate in the market [2].

This is the Black-Scholes equation. It can be used to price the option, to hedge investment risks, and to find arbitrage opportunities in the market. The creation of the Black-Scholes model is the beginning of mathematical finance innovation. After that, other scholars who worked on mathematical finance found models similar with the Black-Scholes model but built with bonds. The investment portfolio with bonds is quite similar with the one used to build the Black-Scholes model. In this paper, we are trying to build a risk free portfolio of bonds with different maturities, and using this portfolio to find the bond pricing model.

The rest of this paper is organized as follows: The next section of Chapter 1 presents the theory of bond pricing model and popular interest rate models. Then we will introduce the terminology and notation which will be used in our research. In Chapter 2, we will use the historical data of market interest rate to explore the stochastic differential equation of spot interest rate. And we will use MATLAB to run the estimation model, and then compare the estimated interest rate with the market interest rate. Then, we will use the build-in model in MATLAB to conduct a model comparison.

1.1 Literature Review

A bond is a long-term contract under which a borrower agrees to make payments of interest and principal, on specific dates, to the holders of the bond [3, p. 173]. Even though this definition seems quite simple, there are many different kinds of bonds and interest-rate-derivative securities. The one used in building the bond pricing model is the zero-coupon bond. The zero-coupon bond is a contract paying a known fixed amount, the principal, at some given date in the future, the maturity date T [9, p.320]. Clearly, interest rate and maturity date determines the current price of the zero-coupon bond. The modern research about bond pricing model assumes that the interest rate is a function of time, and the short term interest rate, which is also called spot rate, has some kind of mathematical relationship with a term which follows a random walk. Then we need to take two more steps to build the model. First, build a portfolio containing bonds with two different maturity dates. In this way, the value of the investment

portfolio is as following:

$$\pi = V_1(r, t; T_1) - \Delta V_2(r, t; T_2)$$

where π is the value of the portfolio, V_1 is the price of bonds with maturity T_1 , V_2 is the price of bonds with maturity T_2 and Δ is the amount of bonds with maturity T_2 we need to hold for hedging.

Then the change in this portfolio in one time step, dt , is given by Ito's Lemma as [9, p.362]:

$$d\pi = \frac{\partial V_1}{\partial t}dt + \frac{\partial V_1}{\partial r}dr + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2}dt - \Delta\left(\frac{\partial V_2}{\partial t}dt + \frac{\partial V_2}{\partial r}dr + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}dt\right) \quad (1.4)$$

Similarly to how we get the Black-Scholes model, in this portfolio of bonds, we use specific amount of bonds that mature soon and bonds that mature later to eliminate the random term in the function of portfolio value. Therefore we have:

$$\Delta = \frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r} \quad (1.5)$$

Eliminating the random term in $d\pi$, then we have

$$d\pi = \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \left(\frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r}\right)\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\right)\right)dt \quad (1.6)$$

With the non-arbitrage theory, the change of the investment portfolio should be equal to the interest we can get with market risk free spot rate, which means

$$d\pi = r\pi dt \quad (1.7)$$

Therefore we have

$$\left(\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \left(\frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r}\right)\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\right)\right)dt = r\pi dt \quad (1.8)$$

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - \left(\frac{\partial V_1}{\partial r} / \frac{\partial V_2}{\partial r}\right)\left(\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}\right) = r\pi \quad (1.9)$$

With the definition function of π we get

$$\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - (\frac{\partial V_1}{\partial r}/\frac{\partial V_2}{\partial r})(\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2}) = r(V_1 - (\frac{\partial V_1}{\partial r}/\frac{\partial V_2}{\partial r})V_2) \quad (1.10)$$

$$\frac{\frac{\partial V_1}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_1}{\partial r^2} - rV_1}{\frac{\partial V_1}{\partial r}} = \frac{\frac{\partial V_2}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V_2}{\partial r^2} - rV_2}{\frac{\partial V_2}{\partial r}} \quad (1.11)$$

Since V_1 and V_2 are the two bonds we picked randomly with different time to maturity, we can say that the left side of the equation represents a ratio which is independent from the bond's time to maturity. Let $q(t, r)$ denote the common function of this ratio for any bond value V with any maturity date, we can get

$$q(r, t) = \frac{\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} - rV}{\frac{\partial V}{\partial r}} \quad (1.12)$$

There are many ways to represent this stochastic equation with different definition functions of $q(r, t)$. In this way, people can get better financial explanation for the SDE, or make it easier to be solved. An example is given in [9, p.363], the definition function for $q(r, t)$ is

$$q(r, t) = w(r, t)\lambda(r, t) - \mu(r, t) \quad (1.13)$$

Then we can write 1.12 as

$$\frac{\partial V}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 V}{\partial r^2} + (u - \lambda w)\frac{\partial V}{\partial r} - rV = 0 \quad (1.14)$$

This is the stochastic equation of zero-coupon bond price. In this equation, u and w are functions of the spot interest rate, r , and the time, t . Unlike the Black-Scholes model, in which we need to determine the stock price volatility, in the bond pricing model, we also need to determine the risk-neutral spot rate, which means an interest rate model needs to be built. Scholars agree with the bond pricing model above, but for the interest rate model, they have different ideas.

1.1.1 Vasicek Model

Many classic interest rate models have been created to fit the bond pricing model. The Vasicek model is the first dynamic interest rate model, which was articulated in 1977. The Vasicek model has a simple mathematical structure and makes it easy to find the explicit formula for many interest rate derivatives. The Vasicek model is [8]

$$dr = (\eta - \gamma r)dt + \beta dX \quad (1.15)$$

In this stochastic equation, η , γ and β are positive constant parameters, dX is the random term. Vasicek describes the change of interest rate simply as a differential equation plus linearly independent random term, easily resulting in a negative estimated interest rate. Therefore, the Vasicek model is not the best choice to estimate interest rate and to price bonds in reality.

1.1.2 CIR Model

After adjusting certain terms in the Vasicek model, Cox, Ingersoll and Ross introduced an advanced interest rate model, which we call CIR model. In this new model, the random term of the change of interest rate is a function of square root of interest rate with a constant coefficient. The CIR model takes the form [4]:

$$dr = (\eta - \gamma r)dt + \sqrt{\alpha r}dX \quad (1.16)$$

This adjustment keeps the interest rate positive with parameters satisfying certain constraints. Unfortunately, solving the stochastic differential equation of the CIR model is not easy. Even if this complicated equation could be solved, it will possibly result in an inconsistent bond pricing model, which admits arbitrage opportunity, contradicting the assumption of the bond pricing model.

1.1.3 Ho & Lee Model

In 1986, Thomas Ho and Sang Bin Lee developed a new interest rate model, called Ho & Lee model. Different from the Vasicek model and the CIR model, the Ho & Lee model describes interest rate increment as a function of time and the random term. The Ho & Lee model is

$$dr = \eta(t)dt + \beta^{1/2}dX \quad (1.17)$$

where $\eta(t)$ is a function of t , and β is a constant number.

The Ho & Lee model is the first non-arbitrage model of the term structure of interest rates. The key of this model is finding appropriate function of η to obtain the risk neutral interest in the market and to fit the yield curve obtained from market historical data. The Ho and Lee model can successfully estimate the risk-neutral interest rate only if it works with historical data. This technique used to test the model is called yield curve fitting [5].

1.1.4 Hull-White Model

Unlike previous models, in the Hull-White model, all parameters are time dependent, which means that the drift and variable of interest rate r is a function of time t , r itself, and many other variables. With this idea, John Hull and Alan White published an article introducing the Hull-White model as following [6]:

$$dr = [\theta(t) + a(t)(b - r)]dt + \sigma(t)r^\beta dz \quad (1.18)$$

$\theta(t)$ is a time-dependent drift, a is the reversion rate, and σ is the volatility.

All the interest rate models that we have discussed are estimating the spot interest rate and using it to price the bond. However, in reality, it is impossible for us to know the spot interest rate in the future, the data used in the yield curve fitting is just the historical interest rates, which easily biases the estimated risk neutral interest rate in the market. Fortunately, there are many kinds of interest rate derivatives in the financial market. They use the interest rate in the future, which we call the forward interest

rate, to trade in the market. The forward rate curve provides the market estimation of interest rates in the future. If we can use the forward risk neutral interest rate rather than the spot risk neutral interest rate in the bond pricing model, the estimation will be more accurate.

1.1.5 HJM Model

The Heath, Jarrow & Morton model is a forward interest rate model. This model gathers the concepts of all the interest rate models discussed above but with forward interest rates in the market to price bonds with different maturities. Using real market data of interest rate makes the yield curve to be fitted by default in the Heath, Jarrow & Morton model. The HJM model is proved by many scholars to be the most accurate method to estimate the interest rate. However, since more variables are involved, the stochastic equation for the spot interest rate is very hard to be solved. [9]

1.2 Terminology and Notation

In our analysis, let V be the bond value, and let r be the spot interest rate. Because of interest rate's random walk property, we apply the partition $\{t_0, t_1, t_2, t_3, t_4, \dots, t_N\}$ to time interval $[t_0, t_N]$. $T_i = [t_{i-1}, t_i]$ denotes a specific small time interval, $T_{i,j} = [t_{i-j}, t_j]$ and $T^i = [t_0, t_i]$. If the time interval is evenly cut into small time steps, we define a single time step as dt . Then we can represent the whole time interval as $T^N = \bigcup_{i=1}^N T^i = [t_0, t_N]$. Set V_N be the value of bond maturing at t_N . With this notation, if we have a bond with maturity date at t_N , at time t_0 , the time to maturity, τ , is $t_N - t_0$, and we should estimate the spot interest rate over the time interval T^N . At time t_3 , τ is $t_N - t_3$, and we should estimate the spot interest rate over the time interval $T_{3,N}$.

Since part of the change in the spot interest rate is unexpectable and uncertain, the estimation function of the spot interest rate must contain a random term, z , which is a Brownian motion. By the property of Brownian motion, the change of z , which we denote as dz here, is having random value every time.

1.3 Bond Value Estimation Models

In Chapter 1, we talked about how to use bonds with different time to maturity to estimate the bond price. With the understanding of Black-Scholes model, we finally build our bond pricing model and get the stochastic equation of zero-coupon bond price. This bond pricing model is based on three assumptions:

- The spot interest rate used in the model follows a continuous Markov process, or random walk.
- The spot interest rate is a function of time and the random term.
- The market is efficient, means that there is no arbitrage opportunities.
- There is no transaction cost on the underlying.

However, to solve this stochastic equation, the spot interest rate estimation model should be built first. In this paper, we will explore a suitable spot interest rate term structure based on the one-factor dynamic model, which extended the Vasicek model, which is also considered as a kind of Hull-White model. Then we will use our term structure to estimate the spot interest rates in the future. Before we start exploring the spot interest rate estimation model, we should know how the spot interest rate gets used in the whole bond price valuation theory.

To start with a basic one-factor Hull-White interest rate model, we shift the model 1.18 introduced in Section 1.1.4 as

$$dr = [\theta_{(t)} + a_{(t)}(b - r)]dt + \sigma_{(t)}dz \quad (1.19)$$

In this stochastic equation, $\theta_{(t)}$ can be estimated with the forward interest rates in the current market, and $\sigma_{(t)}$ can be estimated as the variance.

With the boundary conditions of the spot interest rate, the price of a bond, that pays off 1 dollar at the maturity day, should be defined by the function [6]:

$$V(t, T) = A(t, T)e^{B(t, T)r} \quad (1.20)$$

where

$$B(t, T) = (1 - e^{-a(T-t)})/a \quad (1.21)$$

$$A(t, T) = \exp\left[\frac{(B(t, T) - T + t)(a\phi - \sigma^2/2)}{a^2} - \frac{\sigma^2 B(t, T)^2}{4a}\right] \quad (1.22)$$

In this model,

$$\phi(t) = a(t)b + \theta(t) - \lambda(t)\sigma(t) \quad (1.23)$$

where λ is a the market price of interest rate risk, which is a function of time [6].

From above, we can see that as long as we get the stochastic equation of the spot interest rate, it will not be hard for us to estimate the value of the bond price, and then evaluate the pricing of the different bonds in the market.

Chapter 2

Spot Interest Rate Estimation Model

In Chapter 1, we introduced how to use bonds with different times to maturity to estimate the bond price. With the understanding of Black-Scholes model, we finally build our bond pricing model and get the stochastic equation of zero-coupon bond price. This bond pricing model is based on three assumptions:

- The spot interest rate used in the model follows a continuous Markov process, or random walk.
- The spot interest rate is a function of time and the random term.
- The market is efficient, means that there is no arbitrage opportunities.
- There is no transaction cost on the underlying.

However, to solve this stochastic equation, the spot interest rate estimation model should be built first. In this paper, we will explore a suitable spot interest rate term structure based on the one-factor dynamic model, Hull White model, and then use the our term structure to estimate the spot interest rates in the future.

2.1 Historical Data Analysis

2.1.1 Data Description

For our empirical investigation, we use two sets of data: Treasury bond data and CPI-U data.

The Treasury bond is always recognized as a typical risk free bond. To find the term structure of the spot interest rate, we use the secondary market rate of the 3-Month Treasury Bill as our target data. The 3 months time to maturity allows us to take this rate as the spot rate for our research. Since the financial crisis in 2007 had a huge influence on the whole financial market, the interest rate data before that time will be a better data source for our research. Besides this, the term structure of interest rate should be able to describe the interest rate movement in the long term. Therefore, a big group of interest rate historical data will make our research more accurate. With this consideration, we choose the secondary market rate of the 3-Month Treasury Bill from January 3rd, 1994 to December 31st, 2003. This data set contains 2501 secondary market rates for all different times in these 10 years. [1]

2.1.2 Data Investigation

With the 10 years data we got, we can first make a graph to show the movement of the interest rate during this period. Figure 2.1 gives us a whole picture about the interest rate from 01/03/1994 to 12/31/2003.

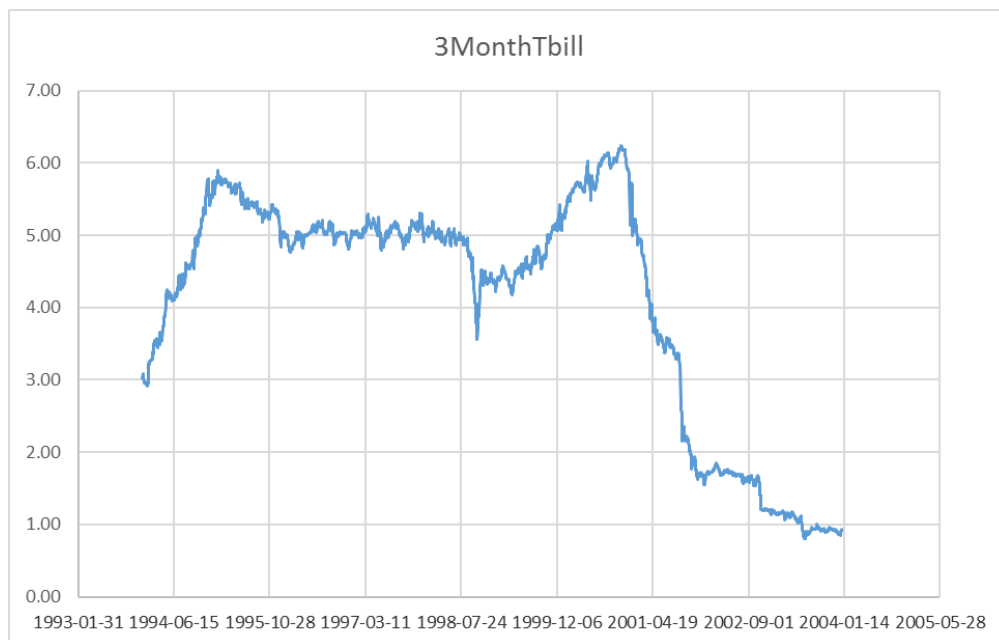


Figure 2.1: 3-Month Treasury Bill YTM

Figure 2.1 gives us an idea about how the interest rate changed, but since the term structure of spot interest rate is a stochastic equation about dr with r , we need another graph to show how much interest rate changed from where it was every time.

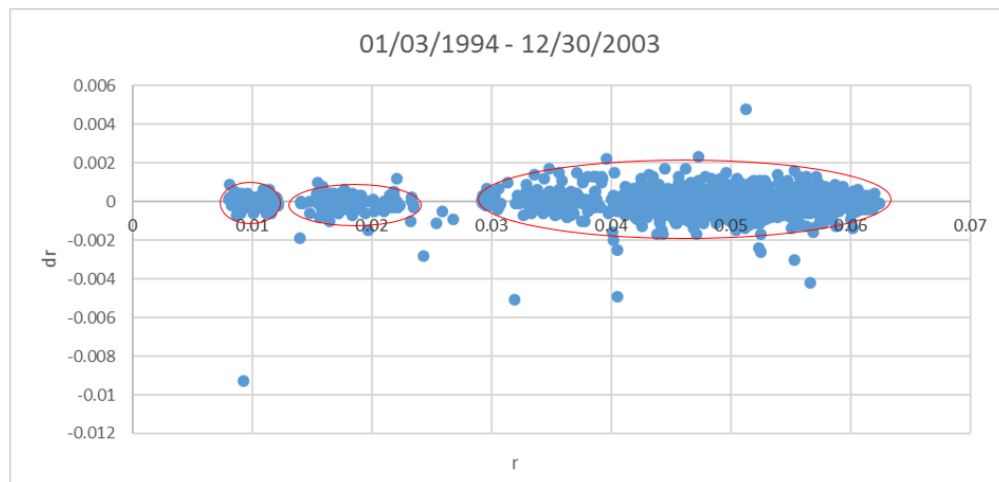


Figure 2.2: dr vs r

In Figure 2.2, dr , at each time point t , is the change of interest rate at time t , which means that the interest rate at the next time point is $r(t+1) = r(t) + dr(t)$.

From Figure 2.2, there are two observations we can get:

- 1) The interest rate only changed in certain range. The possibility of a big increase or decrease is very small.
- 2) There are three separate "clouds" in the graph: $r = 0.007$ to $r = 0.012$, $r = 0.014$ to $r = 0.025$ and $r = 0.035$ to $r = 0.065$. These three clouds have similar shape, and all make interest rate change in certain range. Based on our investigation, the two separate points represent the things which happened in sudden and had big influence on the whole financial market. The first one bringing interest rate down 16 percent from 0.0319 to 0.0268 is "911" in 2001. The second one is that the way to calculate and use interest rate in court was adjusted in 2002.

To find a more accurate term structure for spot interest rate, we need to find a period of time in which nothing big happened and there is sufficient data our research. For this purpose, we choose the largest "cloud" with the interest rate from 0.035 to 0.065 as our research target. This "cloud" contains interest rate data from 01/03/1994 to 12/29/2000.

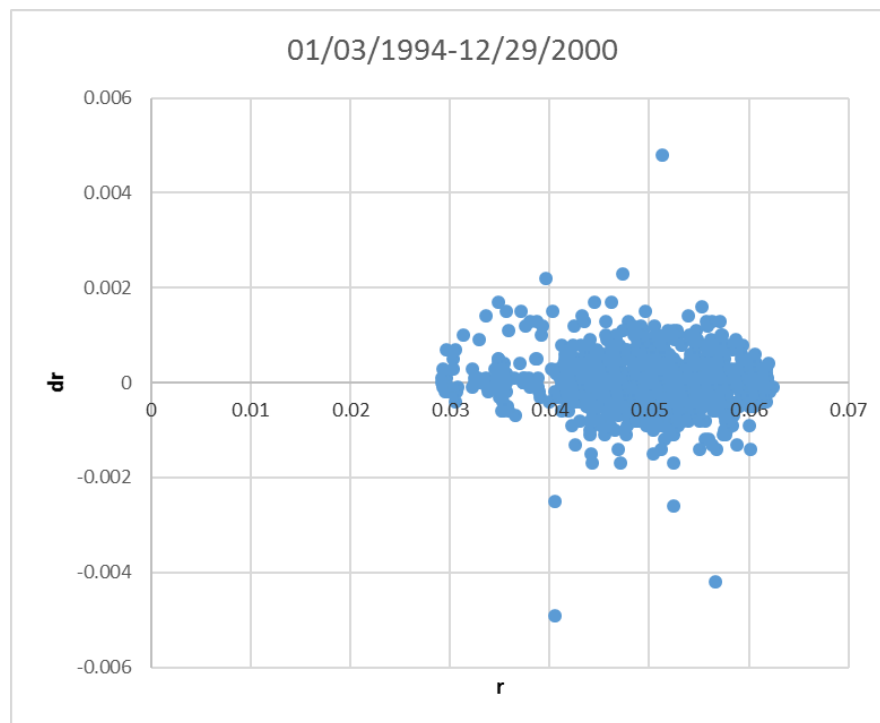


Figure 2.3: dr vs r from 01/03/1994 to 12/30/2003

To make it easier for us to investigate the patterns of how interest rate changes, we focus on the data from 05/03/1994 to 12/29/2000.

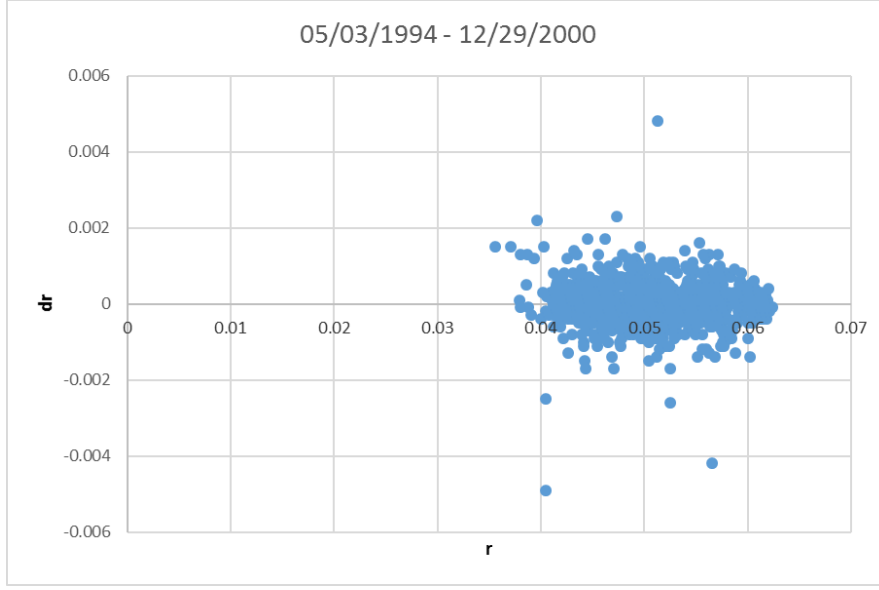


Figure 2.4: dr vs r from 05/03/1994 to 12/29/2000

Figure 2.4 shows the relationship between dr and r for data from 05/03/1994 to 12/29/2000. If we take away the outliers, the shape of the "cloud" is more like the intersection of two parabolas with opposite orientations. Exploring more, we can find that the main part of the dots in the graph have dr in the value range $(-0.0015, 0.0015)$, and the mean interest rate of this group of data is 0.05. Based on these observations, we can make an assumption of our interest rate term structure as that, the dots representing the relationship of dr and r are in the area of the intersection of two parabolas with opposite orientations, which have x axis intersections $(0.04, 0)$ and $(0.06, 0)$, and vertexes $(0.05, 0.0015)$ and $(0.05, -0.0015)$. Solving for the parabolas, we get two functions:

$$dr = -15(r - 0.05)^2 + 0.0015 \quad (2.1)$$

$$dr = 15(r - 0.05)^2 - 0.0015 \quad (2.2)$$

Since our goal is to use one interest rate term structure function to describe all the possible movement of interest rate, and this function has a random term, we can combine the random part of the interest rate term structure with the function above. Based on the previous research about the interest rate term structure, the random term dz follows the Brownian motion and has normal distribution. To keep the random part of dr in the ranges from historical data, we shrink the function by one third, and

therefore we get

$$(-5(r - 0.05)^2 + 0.005)dz \quad (2.3)$$

as the random part of the interest rate term structure function.

From the previous literature review, we can see that most of the researchers tend to assume that the deterministic term is a linear function of r . Here, we can keep using this assumption as a definition of a linear function of r but with some changes:

$$0.01(0.05 - r)dt \quad (2.4)$$

Here, 0.05 is the mean value of r in this data period, and 0.01 is

To summarize, the term structure we build is

$$dr = 0.01(0.05 - r)dt + (-5(r - 0.05)^2 + 0.005)dz \quad (2.5)$$

2.1.3 Stochastic Differential Equation Validation

Since we get term structure function, we can use the starting spot interest rate, the one at 01/03/1994, 0.0402, to run the function in Excel to see if we can get a group of interest rates with the similar shape in the graph. Then we get as many simulations as we want. The following are some graphs we get from generating different random numbers:

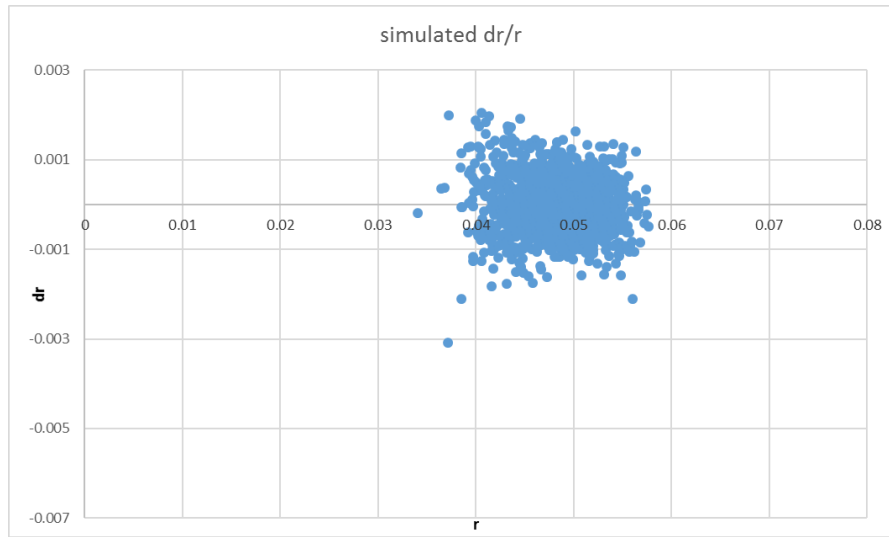


Figure 2.5: Simulation(1) dr vs r from 1994 to 2000

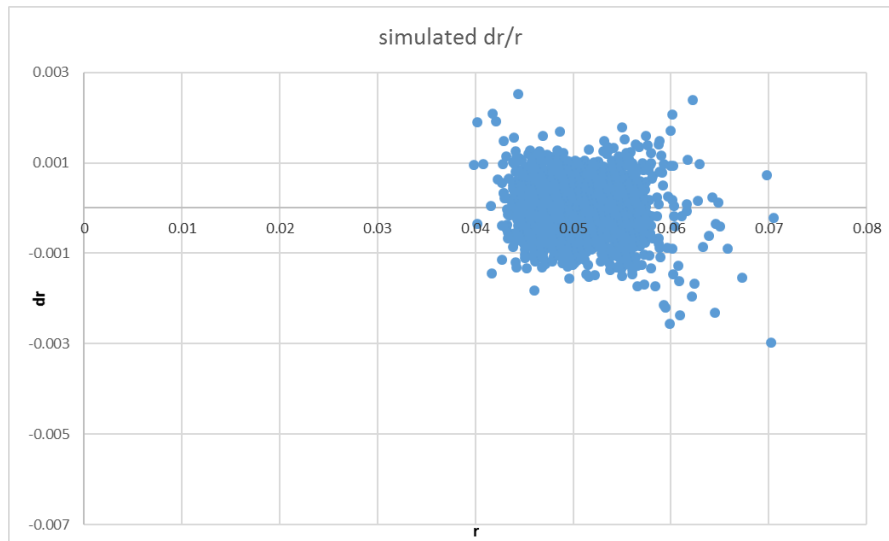


Figure 2.6: Simulation(2) dr vs r from 1994 to 2000

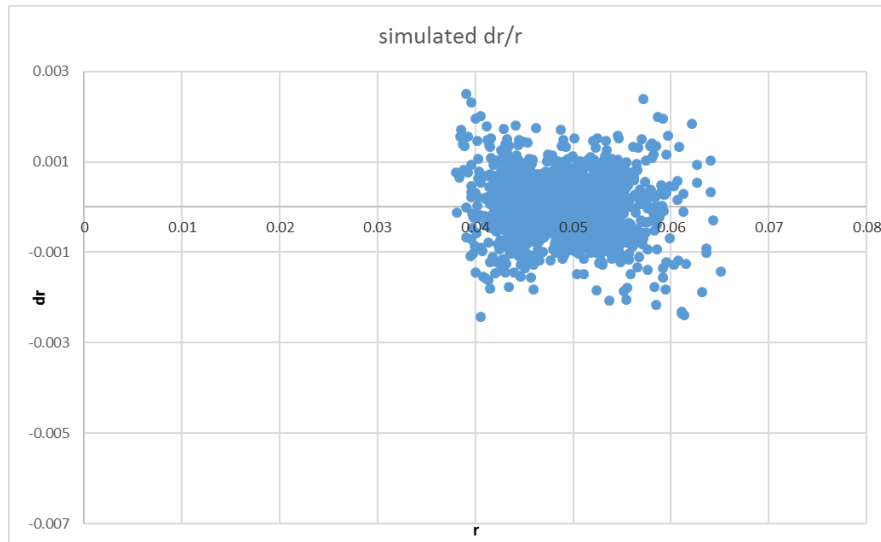


Figure 2.7: Simulation(3) dr vs r from 1994 to 2000

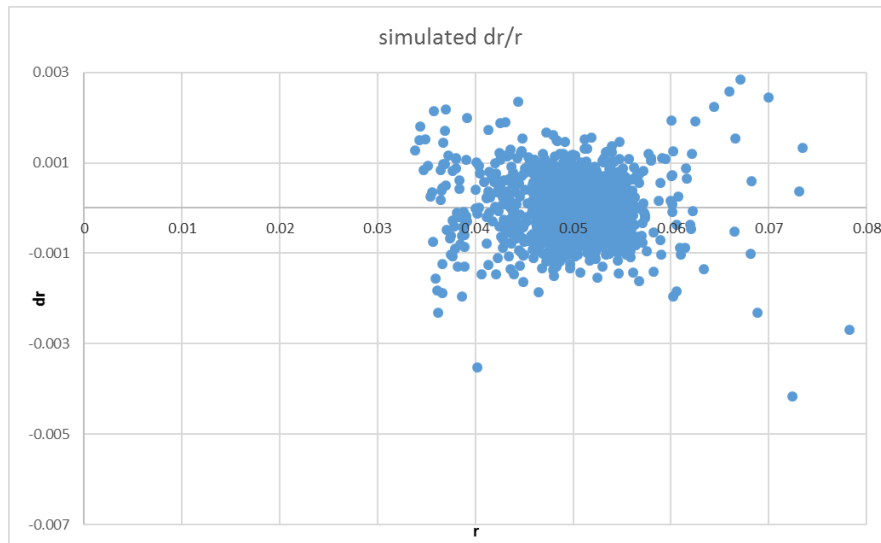


Figure 2.8: Simulation(4) dr vs r from 1994 to 2000

From these graphs, we can see that our estimation function works well. No matter how different these simulated interest rates are from the real ones, we get the big picture of the movement of interest rate now, the structure of the this function is taking us closer to the real interest rate term structure.

2.1.4 Generalization Equation

The function we found above is based on the parameters we got from the historical data. To be able to describe the term structure of interest rate at any time, we need to get the general stochastic differential equation for the spot interest rate. First of all, we need to clarify that, the function which we are trying to get now is describing the change of interest rate in a stable economic environment. Therefore, we are not taking inflation, war or any things with big influence into consideration for now. The change of inflation rate certainly will change the mean value of interest rate, but we need to get other factor into the function to adjust our interest rate simulation function. Even though the environment is stable, the spot interest rate still has some random changes, which are unpredictable but within a certain range. Hence our model contains a deterministic term and a random term.

With this condition clarified, we can get the following information from an efficient market, and the corresponding parameters are showed in the graph:

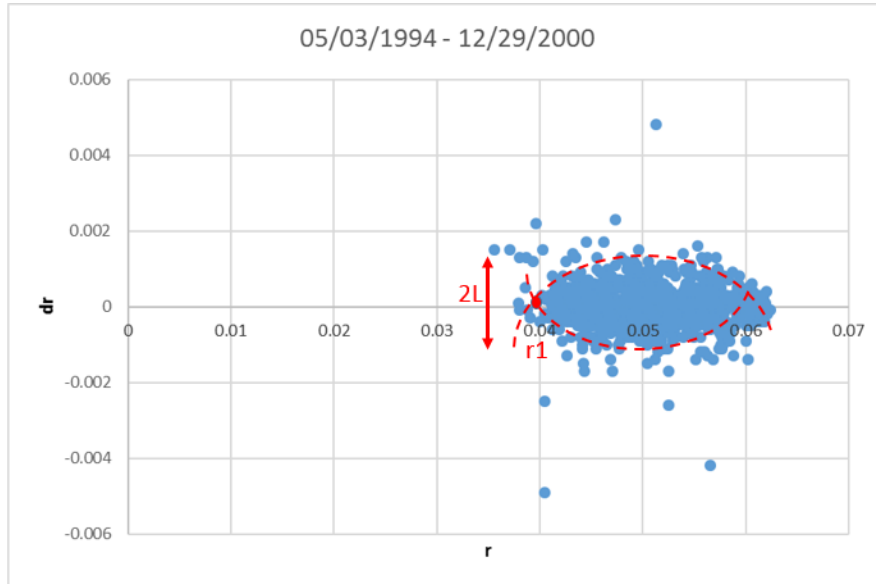


Figure 2.9: dr vs r from 1994 to 2000

- Since the market is stable, we can always use the historical data to get the mean value of spot interest rate, which is called \bar{r} .
- From the historical data, we know when the "cloud" started to show up. The interest rate at that time should be one x axis intersection point of the "cloud",

called r_1 . Then the r value of the other intersection point should be $r_2 = 2\bar{r} - r_1$.

- We can also find the range of dr , as $Max(dr)$ and $Min(dr)$. Let us call the absolute value of these two L . We can get the value of L from the statistical analysis of interest rate's historical data.
- m represents the slope of the first order polynomial function of dr and r . With certain linear analysis tool, we can get the value for m . In our model, m is the speed that the interest rate gets back to the mean value. It is the key parameter to keep the interest rate stable.
- Like we mentioned before, the economic environment we are in is stable. And dz , which is the random term in this stochastic differential equation, follows a Wiener process.

Then we get the generalized function for the change of interest rate:

$$dr = m(\bar{r} - r)dt + (\frac{L}{(r_1 - \bar{r})^2}(r - \bar{r})^2 + L)dz \quad (2.6)$$

2.2 Estimation Model in MATLAB

The previous analysis is based on the historical data. To build a model to estimate the spot interest rate in the future, we will use tools from MATLAB. Since we will not have historical data available to estimate the interest rate in the future, the methodology we use in this section is different.

2.2.1 Methodology

First, to make our estimation more accurate, we will combine the Hull-White spot interest rate model and the model we found in the last section. The Hull-White model uses the forward rate to estimate the deterministic part of the spot interest rate, while gives us available data and makes the estimation more accurate. But for the random part, we will use the model from Equation 2.6, which describes the fluctuation of the current financial market more accurately. The term structure of interest rate we build

in this research is

$$dr = [\theta_{(t)} - mr]dt + (\frac{L}{(r_1 - \bar{r})^2}(r - \bar{r})^2 + L)dz. \quad (2.7)$$

Every parameter is the same as in the model used in Equation 2.6 except $\theta_{(t)}$. Instead of using $m\bar{r}$, which is a constant number, we use a group of values for θ to describe the change of interest rate more accurately. In Hull-White model, the determining function of $\theta_{(t)}$ is

$$\theta_{(t)} = F_t(0, t) + mF(0, t) + \frac{\sigma^2}{2m}(1 - e^{-2mt}) \quad (2.8)$$

where F is the forward interest rate. σ is the volatility of spot interest rate, in other words, the volatility of the whole random part of the spot interest rate. Since the last term in this function is very close to zero, we can ignore it in our calculation. Therefore, we have the calculation function we use in our analysis:

$$\theta_{(t)} = F_t(0, t) + mF(0, t). \quad (2.9)$$

In Hull White model, different yield to maturities with same settle date are used to get the forward interest rate in the market. Because the treasury bonds only have 10 different maturities, we need to use the interpolation technique to get the zero bond YTM and the forward rate at all time points in the time period we are investigating. Since MATLAB has a built-in interpolation function, we can use it directly to get the zero bond YTM from the settle date to the 30-year treasury bond matures.

Once we get the zero bond YTM data, we can use the forward rate calculation function to get the forward rate. The function is

$$F_{t+1} = \frac{e^{-rt}}{e^{-r_{t+1}(t+1)}} - 1 \quad (2.10)$$

Now, we get all the functions we need to get the value for $\theta_{(t)}$. With $\theta_{(t)}$, we can use function $\bar{r} = \frac{\theta_{(t)}}{m}$ to get the floating mean of the spot interest rates. To get more accurate values for m and L , we use statistical analysis.

For m , since it is the speed that r is going back to the mean value, we can use the build-in function in MATLAB to get the slope of the first order polynomial function of dr and r .

L is the range of dr , but to remove the outliers, we omit the 98th percentile of the data. There is also a built-in function in MATLAB help us get this data directly.

Since we define r_1 as the starting point of the cloud, and we assume that our economic environment is stable, which means the spot interest rate we are trying to estimate should be in the cloud, we can take the value of r_1 as the smallest value of historical interest rate.

Now we have all the functions for the parameters we will use in our spot interest rate estimation model in MATLAB.

2.2.2 Spot Interest Rate Estimation Model in MATLAB

If we recall Figure 2.1, of all historical interest rates in our analysis, we can see that the interest rate had big changes at many time points, which means that the economy was not always stable from 1993 to 2004. Since our interest rate estimation model is based on the assumption that the market is in a stable economic environment, we will take the data from 02/27/1996 to 08/27/1998 to test the accuracy of our model. For this data, we use the first half of the data, which is the interest rates from 02/27/1996 to 05/27/1997 as the historical data, to get the value for m , and the range of dr . Then we use all the parameters we get and the YTM's of all kind of Treasury bonds at 05/28/1997 to estimate the spot interest rate, which is the 3 months YTM in our case, from 05/28/1997 to 08/27/1998. Comparing the estimated interest rate with the real interest rates from the second half of the data, which is the interest rates from 05/28/1997 to 08/27/1998, we can see how well our model works.

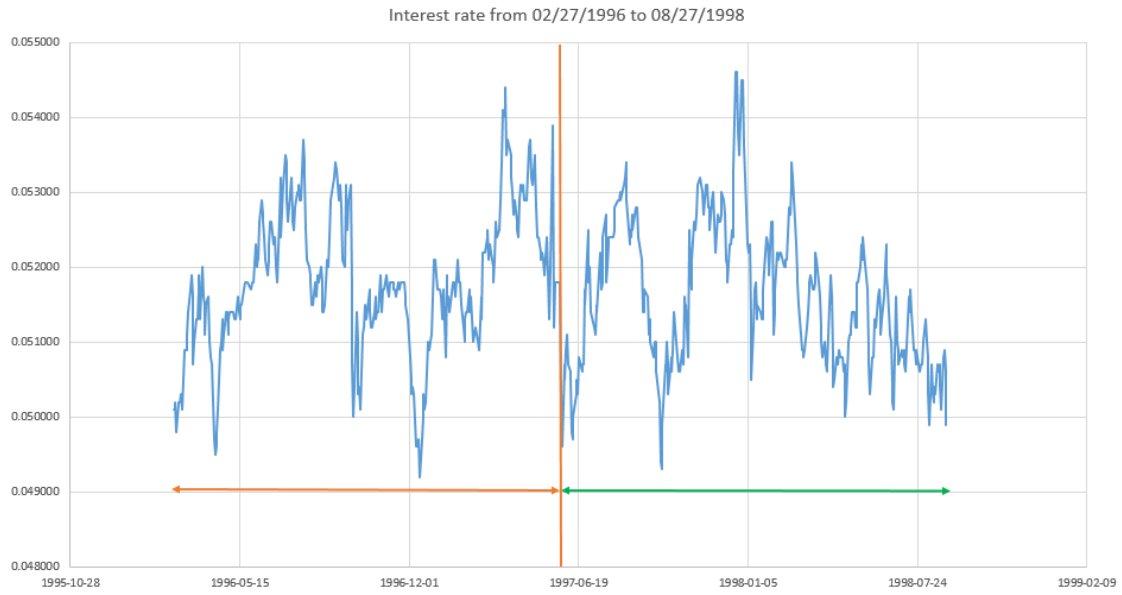


Figure 2.10: Historical estimation data

Since every time we run the model in MATLAB, it generates random numbers to calculate the random term for the interest rate, to make our estimation more accurate, we can use Monte Carlo simulation of the spot interest rate. For example, we can run the model 10 times and see how it works.

The first step in our model is using the YTM's of all kinds of Treasury bonds at the settle day, which is 5/28/1997 in our case, to get the zero bond YTM and the forward rate through the next 30 years. Figure 2.11 is the first figure we get from running our model in MATLAB.

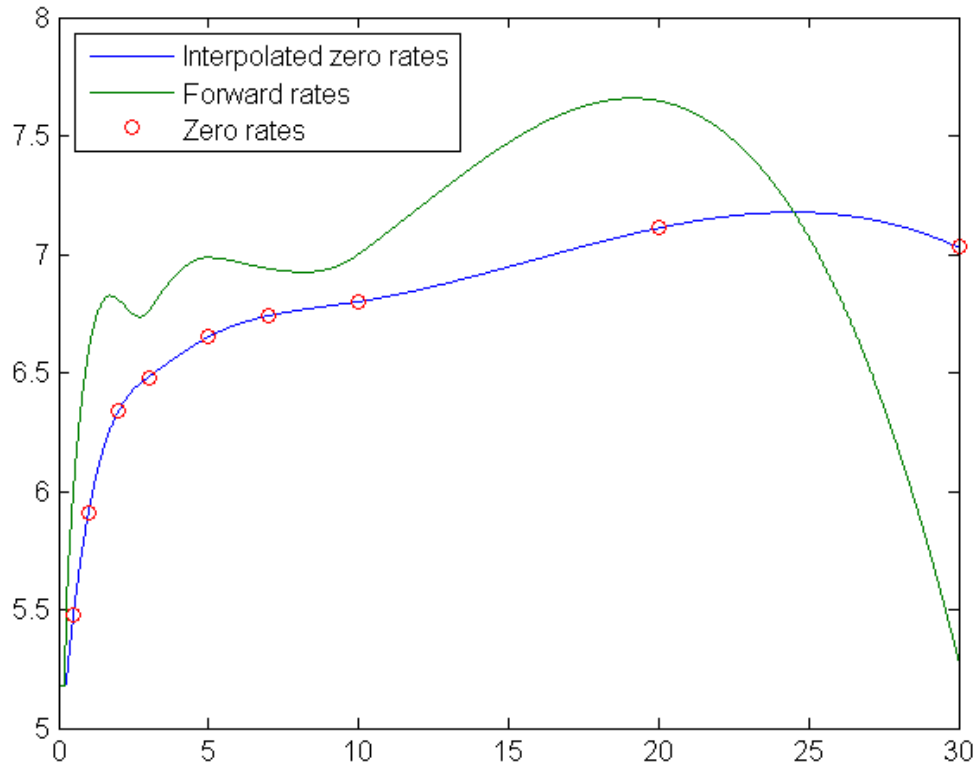


Figure 2.11: Interpolated zero rates and forward rates

From the zero rates and the forward rates, we can get the value for θ and then calculate the change of interest rate, dr , at all time points. Figure 2.12 is showing the "cloud" of 10 trials. In this figure, the x -axis is r , the y -axis is dr .

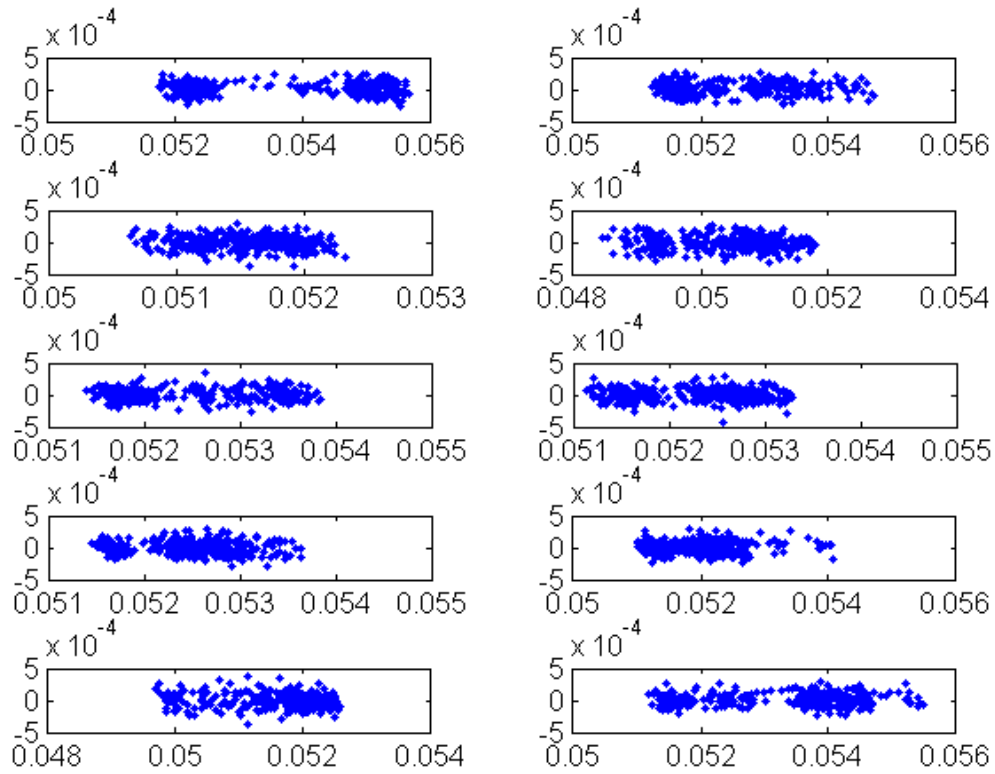


Figure 2.12: dr vs r for 10 trials

From this, we can see if the "cloud" we get from our estimation model is similar with the "cloud" we see from the real interest rate data. In these 10 figures, dr always stays in certain range, and moves around the x -axis randomly, which is consistent with our analysis and the historical data.

With the value of dr , we get the estimated interest rates from 05/28/1997 to 08/27/1998. Figure 2.13 is showing how the 10 interest rate series change over time. The light blue line on the background is the real interest rate from the Treasury bond data from 05/28/1997 to 08/27/1998. Even though the 10 series changes differently over time, the overall direction of change is the same as the real interest rate.

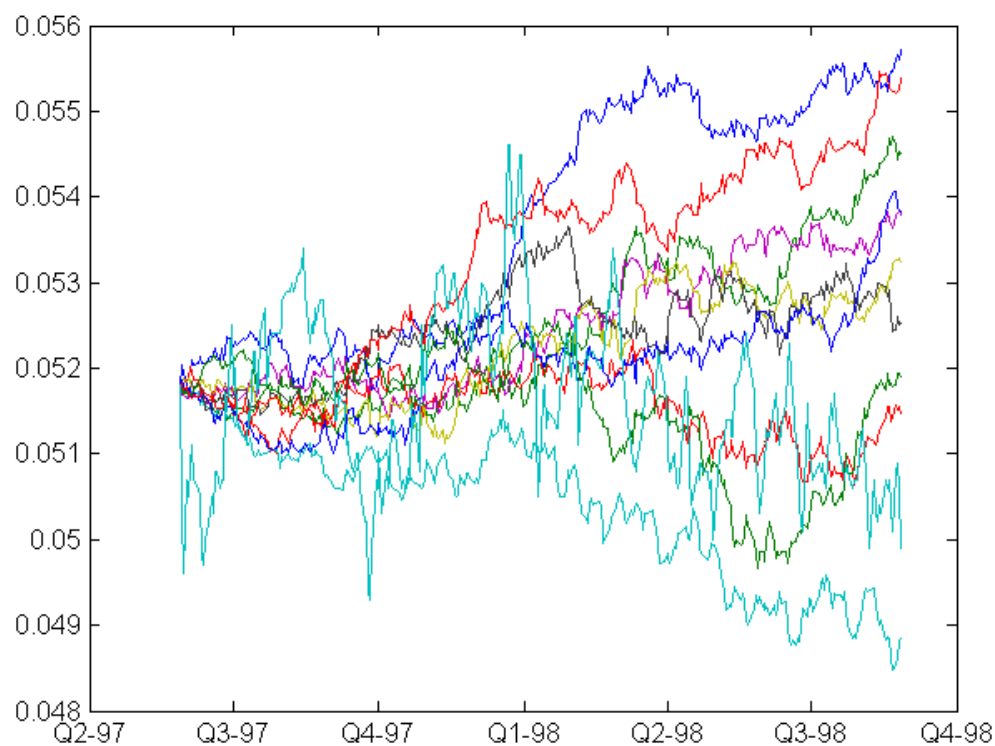


Figure 2.13: Estimated interest rates from 10 trials and the real interest rate

If the figure with 10 interest rate movements is not good enough to tell us how well the model works, we can take the mean value of all 10 estimated interest rates at each time point, and then compare the mean interest rate with the real interest rate, as in Figure 2.14.

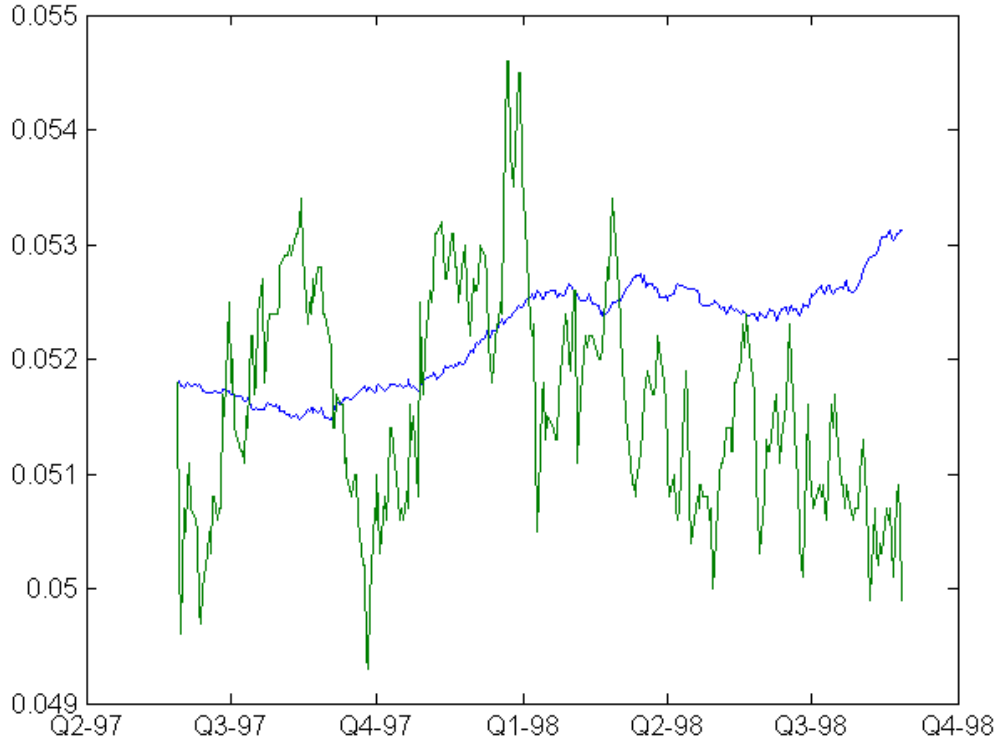


Figure 2.14: Mean of estimated interest rates from 10 trials and the real interest rate

In Figure 2.14, we can see that the mean value of all 10 series of interest rates goes through the movement of the real interest rate, proving that our model can estimate the trend of the changing of the interest rate accurately.

2.2.3 Estimation different YTM in MATLAB

In Section 1.3, we showed how the estimated spot interest rate get involved into the bond price evaluation model. Now, since we already have a spot interest rate estimation model built in MATLAB, we should be able to estimate prices for bonds with different term to maturities.

Unlike the bond value estimation function we have in Section 1.3, the estimation function we have is a stochastic differential equation of the spot interest rate. It has more terms involved, and more complicate relations between all the deterministic terms, random terms and r . All of these makes it very hard for us to get a clear definition function for the spot interest rate, even harder for the bond price evaluation function.

Even though we cannot get estimation function with stochastic differential equation terms for the bond price, we can still use the traditional theory of interest rate with different term to maturities and the model we built in MATLAB to estimate the bond price at each time point.

Let us use the same notation as we used before, except Y is number of days we define in one year. For the future time interval $T - t$, which is count in days, let's assume that we have n time steps, dt , in it. Then we have

$$V(t, T) = \exp\left[-\left(\sum_1^n r_i\right)\frac{dt}{Y}\right] \quad (2.11)$$

where r_i represents the different spot interest rates in different unit time intervals.

If we want to get the Yield-To-Maturity from the bond price, we can use YTM for Yield-To-Maturity and then combine the definition function of YTM, which is

$$V(t, T) = \exp\left[-YTM(t, T)\left(\frac{T - t}{Y}\right)\right] \quad (2.12)$$

with Function 2.11, we can get

$$YTM(t, T) = \text{mean}(r_i) \quad (2.13)$$

With these two functions, we can calculate the bond price and YTM for all kinds of bonds.

To be more precise, we can run a 100 trial Monte Carlo simulation with our model. The interpolation of zero rates and forward rates is the same since we are using the same YTM's of the same settle date. Since we want to see how the YTM changes with different time to maturities and we need more data, we will just estimate all the spot interest rates after the settle day before April 11th, 2013. Estimating the spot interest rate in such a long time period is not accurate, but it can show us the zero curve we get from our estimation model.

The estimate spot interest rates from 100 trials are in the figure:

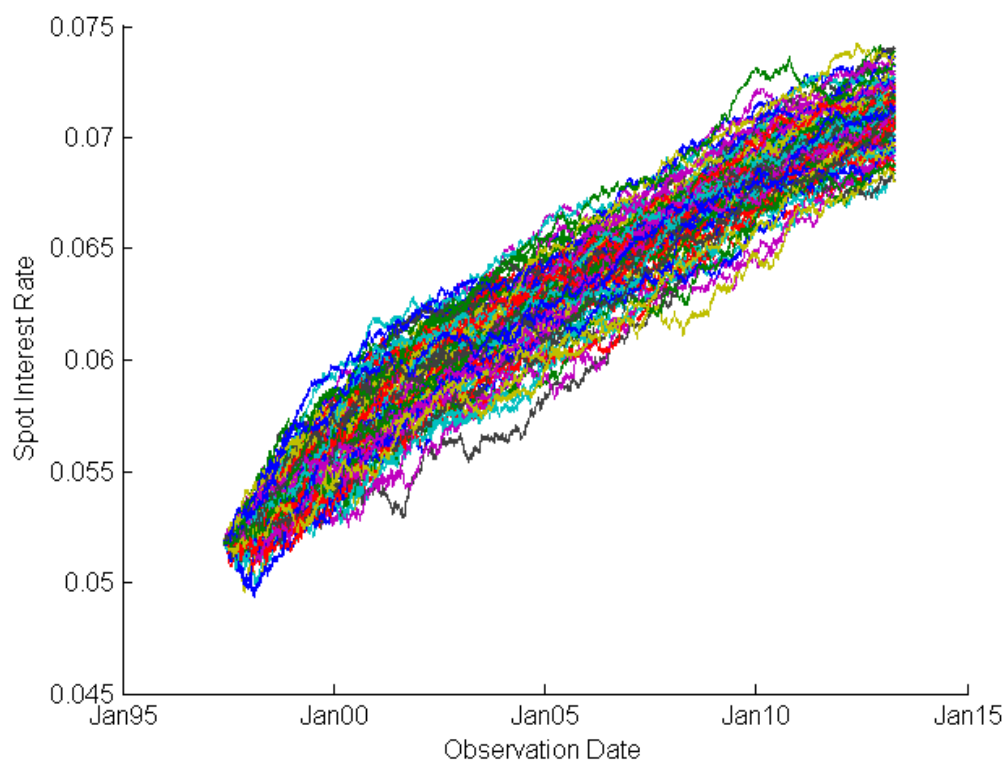


Figure 2.15: Estimated interest rates from 100 trials

Then we can use the mean value of different numbers of spot interest rates to get the YTM for 6month, 1year, 2year, 3year, 5year and 7year Treasury bonds. Taking the 20th trial as an example,

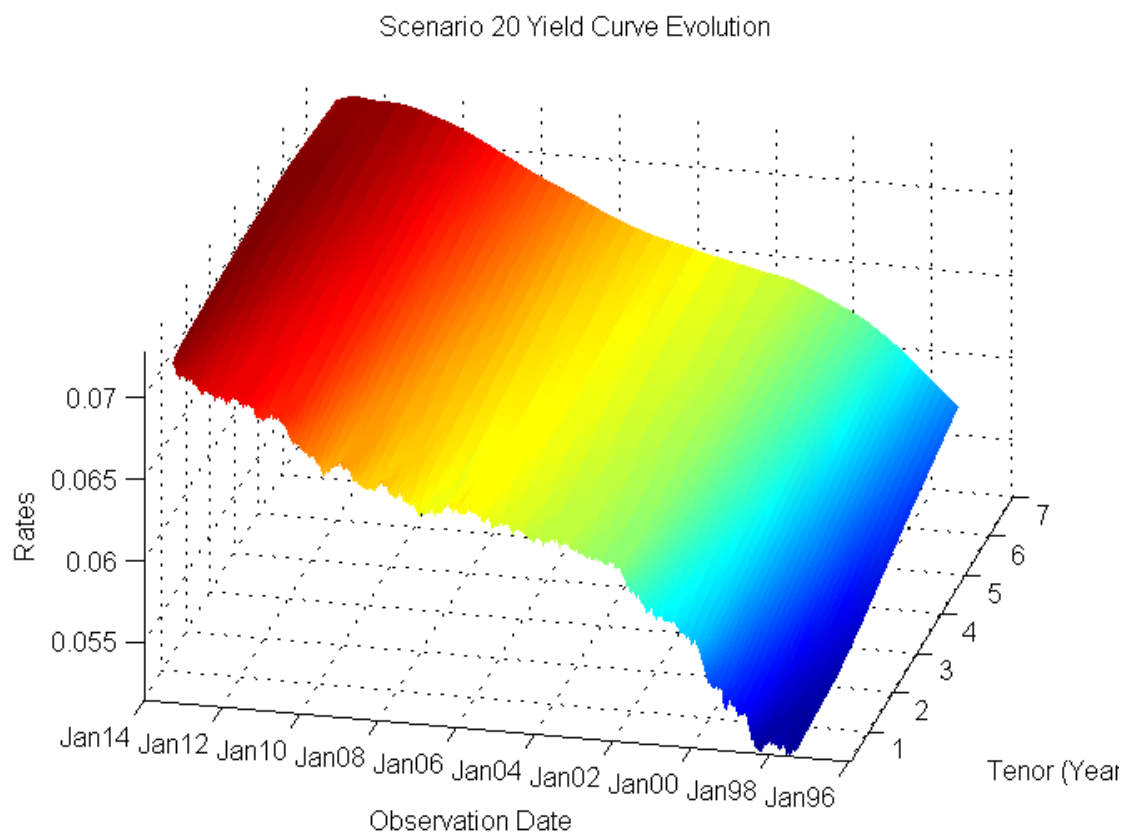


Figure 2.16: Scenario 20 Yield Curve Evolution

In this figure, the "Observation Date" is the dates we estimate the YTM for, "Tenor" is the time to maturities of different bonds. The surface is showing how the YTM changes between different times and different time to maturities. As we can see, the YTM keeps going up with the time going forward, but with some random fluctuations, which is consistent with our theory that the over all trend of the interest rate is going up for a market in a stable economic environment.

If we want to know more about how the term structure is for our model, we can look at this three dimensions graph from another direction,

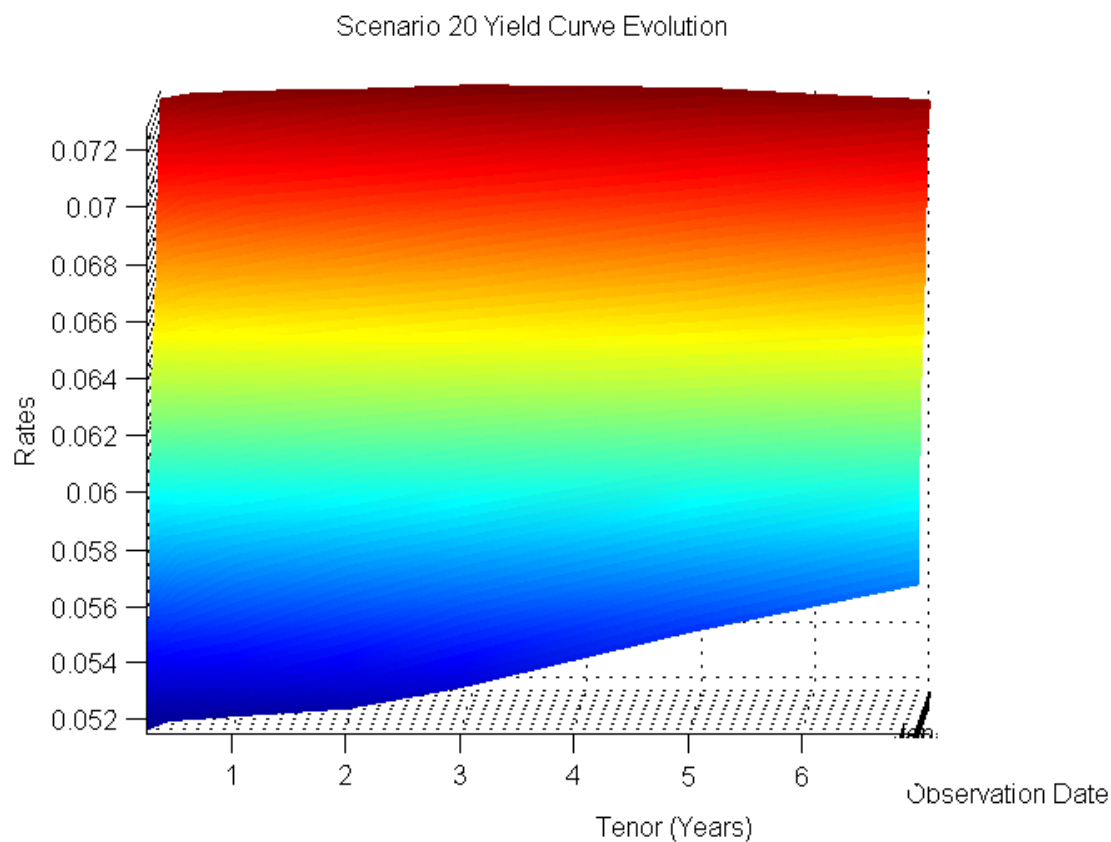


Figure 2.17: Scenario 20 Yield Curve Evolution

The bottom line of the colored part in Figure 2.17 is the zero curve for all different YTM's at some day from our observation dates. It is clear that the zero curve is going up with longer time to maturity, in our case, the tenor. We can run the model in MATLAB one more time to get a figure with the zero curves from some trials, therefore we get a better idea about how our model works with estimating different YTM's.

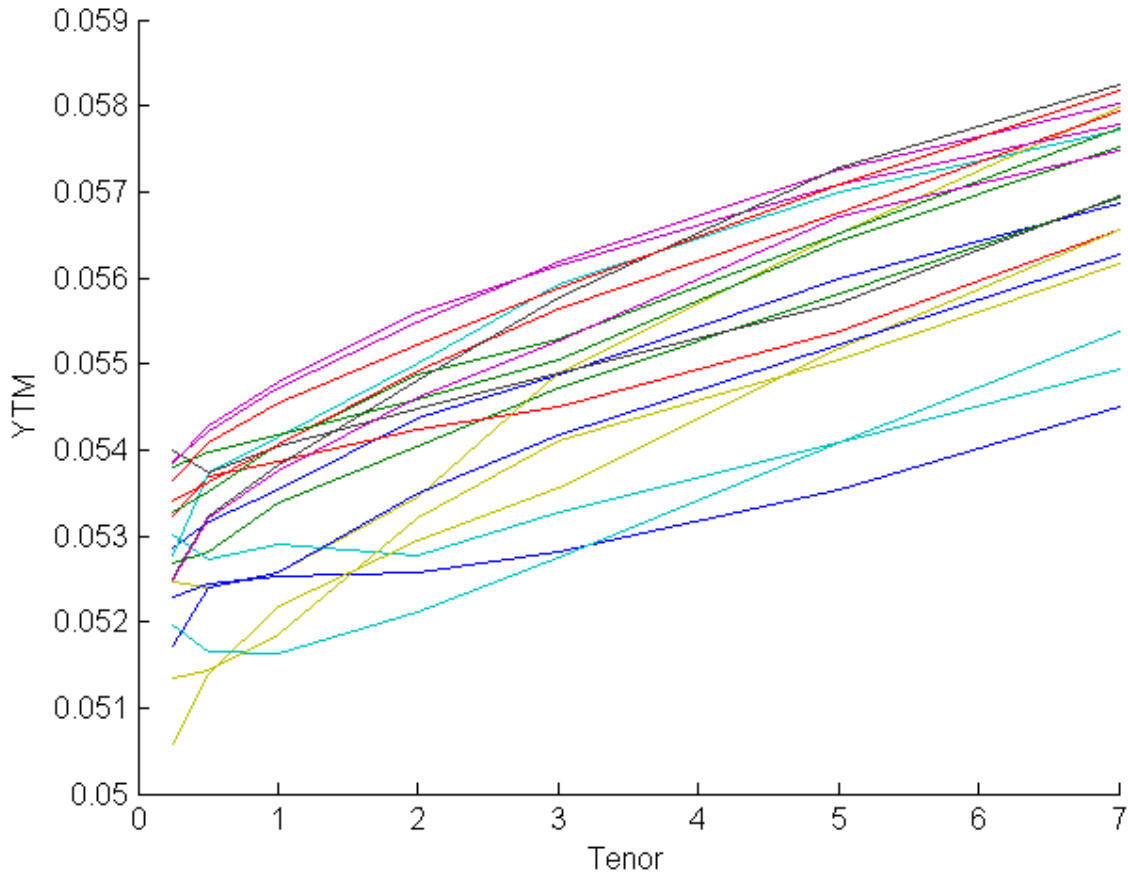


Figure 2.18: 20 out of 100-trials Zero Curves

In this figure, we take the zero curves from 20 trials with the settle date June 24th, 1997. Here we can see that, even though every zero curve moves in different path, the over all trend of the moving is the same. This result proves that our model can estimate the YTM with properties from interest rate term structure theory.

Here, we just estimated the different YTMs. However, since the U.S Treasury displays the value of the Treasury bonds with the YTM calculated based on the current market price, we just need to use the YTM to represent the value of the bond. If we do need to bond price, we can use Equation 2.12 to get it.

With our model in MATLAB, people can use any historical data to get the value for the parameters, and then, with the market YTMs for different Treasury bonds, they can estimated any zero curves they want. Our fundamental theory for this interest rate estimation model is based on many assumption about the current financial market. If the market is not as same as what we described in our theory, the estimation could

be biased. If the market is stable like we said, we need to make sure our estimation is accurate within certain range, which we will discuss in the next section.

2.2.4 Interest Rate Estimation Model Evaluation

From the estimation result we get from our model, we can pick the first 400 YTM for the 3-year Treasury bond coming from all 100 trials. The 400 YTM data covers the time period from May 28th, 1997 to December 31st, 1998. If we plot the 100 estimate paths of the 3-year bond YTM and the real data of the 3-year YTM from the market, we will get the following figure:

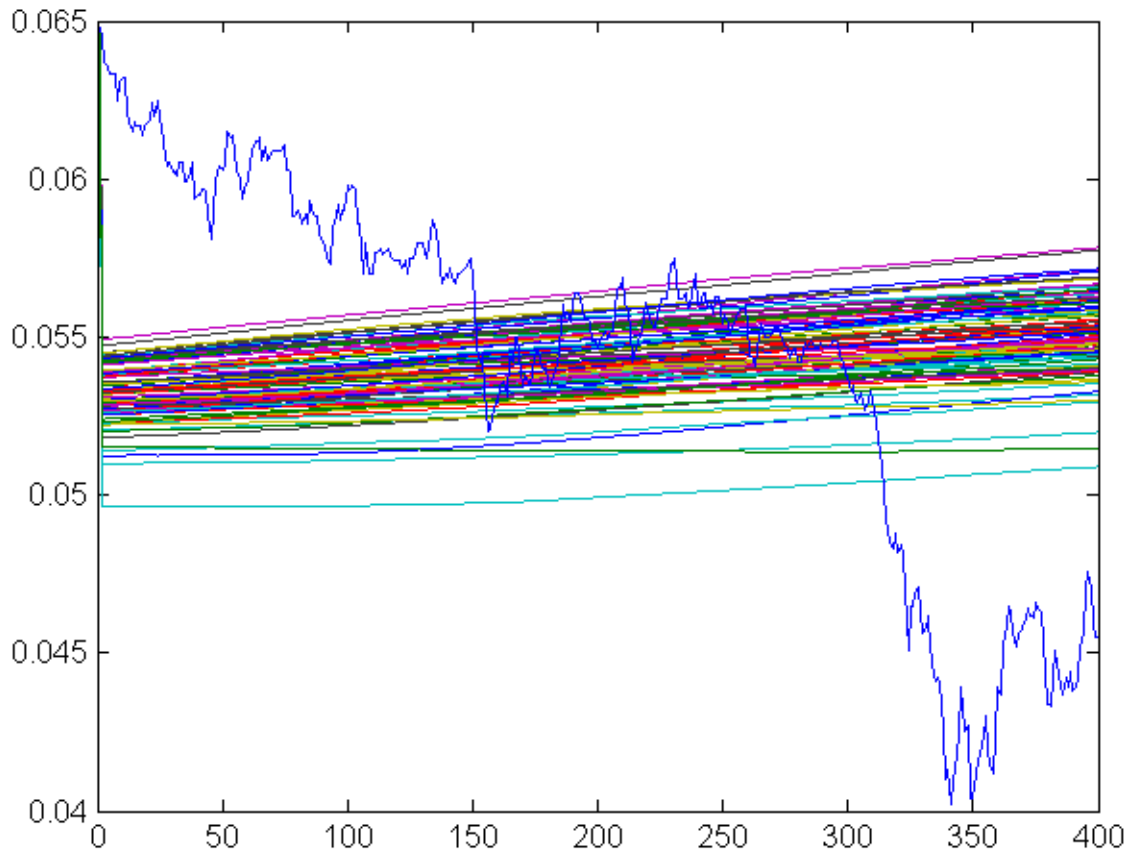


Figure 2.19: 3yr YTM for 100 trials

In this figure, the x axis is the time steps going forward, the y axis is the value of YTM. The line clearly going down is the YTM movement from the real market data. From the figure, we can see that our estimate YTM's are changing "too" smoothly, while the real YTM has more random fluctuation. However, since our 3-yr YTM is the average value

of all spot interest rates in the maturity period, it will turn to be more smooth than the real data. But we can also see that, the range of the estimate YTM's covers most part of the range of the real YTM's. From this point of view, our model is providing an accurate estimation of the range of the YTM in the future, not exactly the value of YTM at every single time point in the future.

2.3 Model Comparison

To measure how well our model is working, we need to find another model to do the model comparison. Since we are using the one factor Hull-White model, we can use the use the build-in Hull-White Single Factor Model in MATLAB as our comparison. The algorithm of the Hull-White Single Factor Model from MATLAB will be not discussed in details here, but we use the parameters we got from our model and the market YTM's from the same settle day to estimate the YTM's of the 3-year Treasury bond for the same time period.

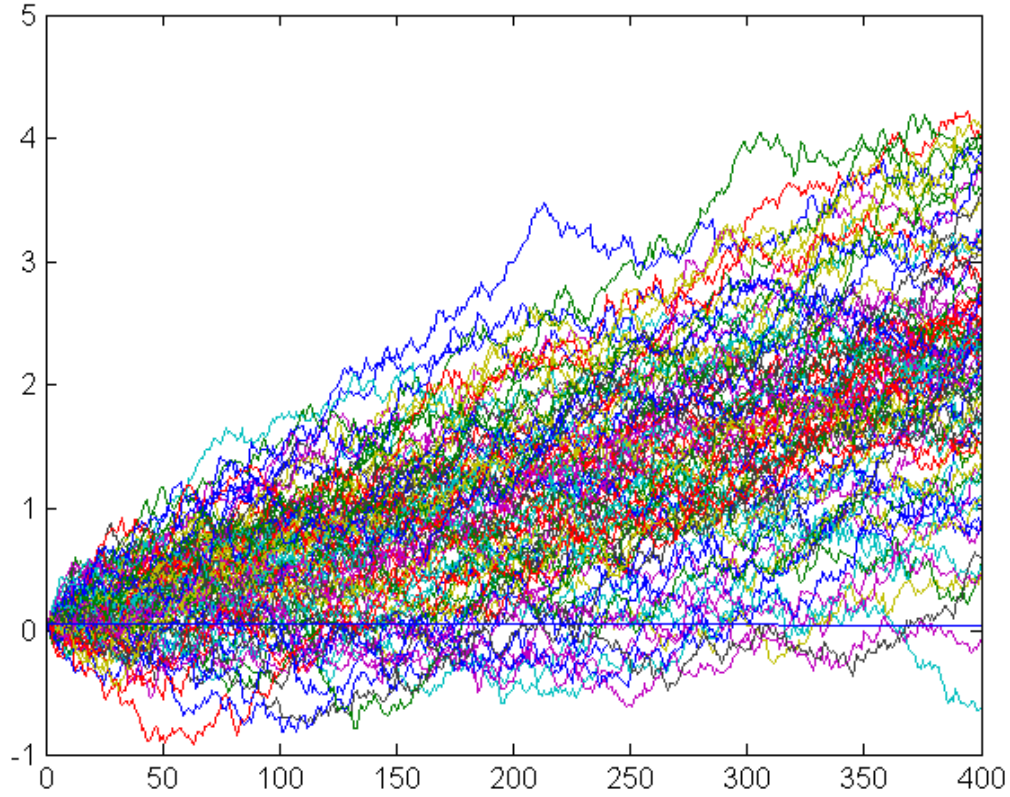


Figure 2.20: 3yr YTM for 100 trials from the HW MATLAB

Figure 2.20 is the about all 100 YTM paths and the path of the real market data, which is almost like a flat line a little bit above the x axis. The reason for what it looks like, is that this simulation produces very big YTM, like 4, and very small YTM, like -1. The range of the estimate YTM is very different from the real one. Actually, for the one factor spot interest rate estimation models, the estimate interest rate is very possible to be negative. But, because of the special parameter of the random term in our model, the estimate interest rate is kept to be positive all the time, which is an advantage of our model over other one factor models.

To make this comparison more clear, the term "Mean Square Error", or "MSE" has to be introduced here. MSE is the mean value of the all the squares of the difference between the real data and the estimate data.

$$MSE = \frac{1}{n} \sum_{i=1}^n (\hat{Y}_i - Y_i)^2 \quad (2.14)$$

In this function, n is the number of data we have, \hat{Y}_i is the real data, Y_i is the estimate data.

It is a method to measure how well the estimation model is doing. The smaller the MSE is, the better the estimation model is.

In our case, for every trial, we take the square of the difference between the real YTM and the estimate YTM, then take the mean of all 400 numbers. For the result vector from estimate YTM from our model, we name it as ThreeyrMSE. For the result vector from real YTM from MATLAB build-in model, we can name it as ThreeyrMSEML. Then, we can run a code in MATLAB to calculate the percentage of elements in ThreeyrMSE smaller than the corresponding numbers in ThreeyrMSEML, and we call this percentage the "Success rate" of our model. The result of the "Success rate" is 1, meaning that our model perfectly beats the MATLAB build-in model.

Chapter 3

Conclusion

In this paper, we introduced the use of stochastic differential equation in research about modern financial market, especially the Black-Scholes model, and the extended use in bond pricing models. Then, we reviewed and compared different models scholars built to estimate the spot risk neutral interest rate. In this process, we found that they all have different advantages and disadvantages, but none of them can perfectly describe movement of the real spot interest rate in the market. Therefore, we decide to explore a more accurate estimation model in our way.

To be not impacted by the financial crisis in 2007, we choose to work on the real Treasury bond data from January 3rd, 1994 to December 31st, 2003, the 10 years data. From this data, we found that the change of the spot interest rate has a shape like a "cloud". The random change of the interest rate stays in the intersection of two parabolas with opposite directions. We built our model based on this discovery and the fundamental theory about the interest rate's stochastic differential equation.

Then, we built the model in MATLAB. With the value of the parameters from the historical data, and the real market data from the starting time point, we got a estimation result from 100 trials. The result gave us a reasonable estimate of the range of the future interest rates and YTM's. And the model comparison also proved the success of our model.

Even though our model can give a relatively accurate estimation of the range of the future interest rate, a model with one factor is still not enough. As we can see from the historical data, we have three separate clouds, which means the property of movement

of interest rate has been changed with the change of the economic environment. Macro economic indicators, like CPI, should be introduced into the interest rate estimation model.

Appendix A

MATLAB Code

A.1 Estimating Function: XLF1

```
%Settle(date)
%YTM's on the Settle date (numbers)
%Tenor(term to maturities)
%and the HW model to get the estimated YTM at different time and many
%trials

function EstiZeroRates = XLF1( ZeroRates,Tenor, m, nPeriod, Range, npath, numinyear, r
%ZeroRates are YTM's on the Settle date (numbers)
%Tenor(term to maturities)
%Settle is the date of all YTM's, or current date
%m is ALPHA, which we will get from the historical data
%nPeriod is the number of dt we want to estimate
nTenors = numel(Tenor);
dt = 1/numinyear;
nsteps = nPeriod;
seedrate = ZeroRates(1,1);
ratesdata(:,1)=(0:dt:Tenor(end,1))';
ratesdata(:,2)= interp1(Tenor,ZeroRates,ratesdata(:,1),'spline',seedrate);
ratesdata(:,4)=exp(-ratesdata(:,1).*ratesdata(:,2)); %discount factor
```

```

ratesdata(2:end,3)=(ratesdata(1:end-1,4)./ratesdata(2:end,4)-1)/dt; %forwards rate
ratesdata(1,3)= seedrate;
Fwd=ratesdata(1:(nPeriod+1),3);
dF=Fwd(2:(nPeriod+1),:)-Fwd(1:nPeriod,:);
theta = dF+m*Fwd(1:nsteps); %+(sigma^2/(2*a))*(1-exp(-2*a*t)); (ignored)
Rmean = theta / m;
Termstructure = (0:dt:(dt*(nsteps-1)))';
estinsteps = nsteps;
nEstiTenor = numel(EstiTenor);
EstiZeroRates = zeros(nPeriod+1,nEstiTenor,npath);
EstiZeroRates(1,:,:) = repmat((ZeroRates(1:(nEstiTenor))))',[1 1 npath]);
deter = zeros(estinsteps - 1,1);
dz = zeros(estinsteps - 1,1);
dr = zeros(estinsteps - 1,1);
for j = 1:npath,
    randterm = randn(estinsteps,1);
    randterm(abs(randterm) > 1) = [];
    while length(randterm) < (estinsteps )
        randterm = [randterm; randn((estinsteps) - length(randterm),1)];
        randterm(abs(randterm) > 1) = [];
    end
    for k = 1,
        for i = 2:(estinsteps + 1),
            dz(i-1,1) = (((Range / ((r1 - Rmean((i-1),1))^2))*(EstiZeroRates(i-1, k,j)
            EstiZeroRates(i, k, j) = EstiZeroRates(i-1, k,j) + (theta(i-1,1) - m * Est
            deter(i-1,1) =(theta(i-1,1) - m * EstiZeroRates(i-1, k,j))* dt;
            dr(i-1,1) = (theta(i-1,1) - m * EstiZeroRates(i-1, k,j))* dt + dz(i-1,1);
        end
    end
end

end

for j = 1:npath,
    for i = 2:(estinsteps - (EstiTenor(end)*numinyear - 1)),
        for k = 2:nEstiTenor,

```



```

        EstiZeroRates(i, k, j) = mean(EstiZeroRates( i:(i + EstiTenor(k)*numinyear
end
end
end
end

```

A.2 Full Script

```

zeroratefile = 'data1994to2003.xlsx';
zerorateHis = dataset('XLSFile',zeroratefile,'Sheet','His02 27 1996 - 05 27 1997');
zerorateEsti = dataset('XLSFile',zeroratefile,'Sheet','Esti05 28 1997 - 08 27 1998');
zeroyrReal = dataset('XLSFile',zeroratefile,'Sheet','Historical Data');

HisThreemonthTbill = struct(...    %NOTE HERE: the date column can not be used as a ma
    'Time',[],...
    'Rate',[]);

EstiThreemonthTbill = struct(...    %NOTE HERE: the date column can not be used as a m
    'Time',[],...
    'Rate',[]);

ThreemonthTbill = struct(...    %NOTE HERE: the date column can not be used as a matri
    'Time',[],...
    'Rate',[],...
    'CPI',[]);

Datedata = struct( 'Date', []);

HisThreemonthTbill.Time = zerorateHis.date;
HisThreemonthTbill.Rate = zerorateHis.ThreeMonthTbill_1;

EstiThreemonthTbill.Time = zerorateEsti.date;
EstiThreemonthTbill.Rate = zerorateEsti.ThreeMonthTbill_1;

```

```
Datedata.Date = zerorateEsti.date;
```

```
% Get the time frame of the data we want to simulate, numeric the date data
% first
%In our analysis, we use trading dates, assuming that there are 252 trading days in on
numDate = datenum(Datedata.Date);
numsteps=252;
%Determine dt
dt=1/numsteps;
%Settle date is the start day, the start of our estimation
Settle = '28-May-1997';
%Get all the Tbill YTM at the settle date, use the maturity dates and YTM rates to bui
Tenor = [1/4 1/2 1 2 3 5 7 10 20 30]';
ZeroRates = [5.18 5.48 5.91 6.34 6.48 6.65 6.74 6.80 7.11 7.03]'/100;
EstiTenor = [1/4 1/2 1 2 3 5 7]';
rdrlinear = polyfit(HisThreemonthTbill.Rate(1:(end - 1),1), diff(HisThreemonthTbill.Ra
m = -rdrlinear(1,1);
Range = prctile(abs(diff(HisThreemonthTbill.Rate)), 98);
npath = 100;
r1 = HisThreemonthTbill.Rate(1,1);
numinyear = 252;
nPeriod = 252*30;
EstiZeroRates = XLF1( ZeroRates,Tenor, m, nPeriod, Range, npath, numinyear, r1, EstiTe
xaxisdates = [1:1:(nPeriod + - 252*EstiTenor(end))] + busdate(Settle);
i = 20;
figure;
surf(EstiTenor(1:end),xaxisdates, EstiZeroRates(2:(end - 252*EstiTenor(end)),:,i))
axis tight
datetick('y','mmmyy');
shading interp
xlabel('Tenor (Years)');
```

```

ylabel('Observation Date');
zlabel('Rates');
set(gca,'View',[-49 32]);
title(sprintf('Scenario %d Yield Curve Evolution\n',i));

figure('Name', '100-trials Spot Interest Rate Simulation');
xlabel('Observation Date');
ylabel('Spot Interest Rate');
hold on;
    plot(xaxisdates, squeeze(EstiZeroRates(2:(end - 252*EstiTenor(end)),1,:)))
datetick('x','mmmyy');
%axis tight

figure('Name','20 out of 100-trials Zero Curves');
xlabel('Tenor');
ylabel('YTM');
hold on;
    plot(EstiTenor(1:end), squeeze(EstiZeroRates(300,:,1:20)))

Threeyear = zeros(400,100);
for i = 1:100,
    Threeyear(:,i) = EstiZeroRates(1:400,5,i);
end

% use the matlab buildin model to estimate the interest rate
Sigma = svd(HisThreemonthTbill.Rate);
Alpha = m;
ZeroDates = datemnth(Settle,Tenor*12);
Compounding = 2;
Basis = 0;
RateSpec = intenvset('StartDates', Settle,'EndDates', ZeroDates,...
    'Rates', ZeroRates,'Compounding',Compounding,'Basis',Basis);
hw1 = HullWhite1F(RateSpec,Alpha,Sigma);

```

```

scenarios = hw1.simTermStructs(nPeriod,...
    'nTrials',npath,...
    'deltaTime',dt);
HWThreeyear = zeros(400,100);
for i = 1:100,
    HWThreeyear(:,i) = scenarios(1:400,5,i);
end
realthreeyr = zeroYrReal.real;
MSE = zeros(100,1);
MSEML = zeros(100,1);
for i = 1:100,
    MSE(i,1) = sum((Threeyear(:,i) - realthreeyr).^2)/numel(realthreeyr);
    MSEML(i,1) = sum((HWThreeyear(:,i) - realthreeyr).^2)/numel(realthreeyr);
end

successRate = fprintf('%2.3g%%\n',100*nnz(MSE < MSEML)/numel(MSE));

figure('Name','3yr YTM for 100 trials');
plot(Threeyear);
hold on;
plot(realthreeyr);

```

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