2006

Plane Elasticity Using Airy Stress Functions

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A thesis submitted in partial fulfilment of the
requirements for the degree of
MASTERS OF SCIENCE
in
MECHANICAL ENGINEERING

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ABSTRACT

Given a three dimensional solid element in a state of plane stress or plane strain with conservative body forces, the stress components are equal to the appropriate second order partial derivatives of a bi-harmonic function, ϕ , called an Airy Stress Function. It follows that the stress components automatically satisfy the equilibrium conditions.

The function ϕ depends on both the geometry of the body and the loading, which leaves infinite possible stress functions to be developed. As thesis work, I have researched and collected currently existing Airy Stress Functions, made plots of thier stress fields in order to gain a better understanding of how they are developed, and attempted to take the Finite Element Analysis of a real world example -- support structure for a door under the loading of a gas shock -- and compare to results obtained from the use of Airy Stress Functions.

TABLE OF CONTENTS

LIST OF FIGURES

GENERAL OVERVIEW

Current Events

In doing my research, I felt it was a good idea to take some time to talk about what is currently happening with Airy Stress Functions in the field of mechancial engineering.

In 1999 the paper entitled "A New Analytical Solution for Diaphragm Deflection and its Application to Surface-Micromachined Pressure Sensors" was released. In this analytical solution they used the govorning differential equations for the bending of a circular plate

$$\nabla^4 w = \frac{P}{D} + \frac{h}{Dr}\frac{\delta\phi}{\delta r}\frac{\delta^2 w}{\delta r^2} \tag{1}$$

$$\nabla^4 \phi = -\frac{E}{r}\frac{\delta w}{\delta r}\frac{\delta^2 w}{\delta r^2} \tag{2}$$

In which ϕ is the Airy stress function. The assumptions that were made were azimuthal symmetry, clamped boundary conditions, and the amount of stretch, u, at the edge of the diaphragm is zero. After carrying out the differential equation and obtaining equations for both stress and strain, plots were made comparing the analytical results to the output characteristics of a piezoresistive pressure sensor. Both solutions were quite similar. [1]

Fundamental Information Needed for Airy Stress Function Development

In order to develop Airy Stress Functions, some basic mechanical concepts such Stress, Strain, Displacment, and Equilibrium and Compatability are needed. These will be used during the development of our Airy Stress Functions as well as some of the assumptions that need to be made for the use of Airy Stress Functions.

Stress and Strain [3]

There are two types of forces, surface forces and body forces. Surface forces are distributed over a surface, body forces act over the entire body. An example of a surface force is the pressure between two contacting bodies, an example of a body force is gravity. These forces produce stress, which is an internally distributed force within a body. Stresses that act normal to a surface are referred to as normal stresses, and stresses that act tangential to a surface are referred to as shear stresses. Normal stresses cause the element to grow or shrink while shear stresses cause the element to deform and change shape. The dimensionless rate of elongation from normal stress is refered to as normal strain, and is defined by Hooke's Law.

$$\epsilon = \frac{\sigma}{E} \tag{3}$$

E is the Modulus of Elasticity and this equation is only valid for a linear isotropic material in simple tension. An isotropic material is one for which the material properties are independent of direction.

If you strain the element in one direction, it creates strain in all three directions due to the fact you can't create or destroy matter. If you stretch it in one direction, it must shrink in the other two and vice versa. For a linear isotropic material, the strains can be related to each other by Poisson's Ratio, ν, a proportionality constant that relates the contraction to the primary strain. For an element undergoing normal stress in a single direction, the following equations can be used

$$\epsilon_y = \epsilon_z = -\nu\epsilon_x = -\nu\frac{\sigma_x}{E} \tag{4}$$

$$\epsilon_x = \epsilon_z = -\nu\epsilon_y = -\nu\frac{\sigma_y}{E} \tag{5}$$

$$\epsilon_x = \epsilon_y = -\nu\epsilon_z = -\nu\frac{\sigma_z}{E} \tag{6}$$

For a linear, homogeneous, isotropic element undergoing more than one normal stress, linear superposition of Hooke's Law and equations (3) can be used, resulting in the following equations

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)) \tag{7}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)) \tag{8}$$

$$\epsilon_z = \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y)) \tag{9}$$

Using algebra to solve these equations in terms of stress, we find

$$\sigma_x = \frac{E}{(1-2\nu)(1+\nu)}[(1-\nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z)] \tag{10}$$

$$\sigma_y = \frac{E}{(1-2\nu)(1+\nu)}[(1-\nu)\epsilon_y + \nu(\epsilon_x + \epsilon_z)] \tag{11}$$

$$\sigma_z = \frac{E}{(1-2\nu)(1+\nu)}[(1-\nu)\epsilon_z + \nu(\epsilon_x + \epsilon_y)] \tag{12}$$

If we take a problem of plane stress, using x and y as the coordinates for the plane and $\sigma_z = 0$, then equations (4--6) reduce to

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \tag{13}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{14}$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \tag{15}$$

and the equations for stress reduce down to just the $\sigma_x$and σ_y equations of (7--9).

Looking now at the shear stress and shear strain on an element by looking at the effect of τ_{xy} alone, as shown in figure 1

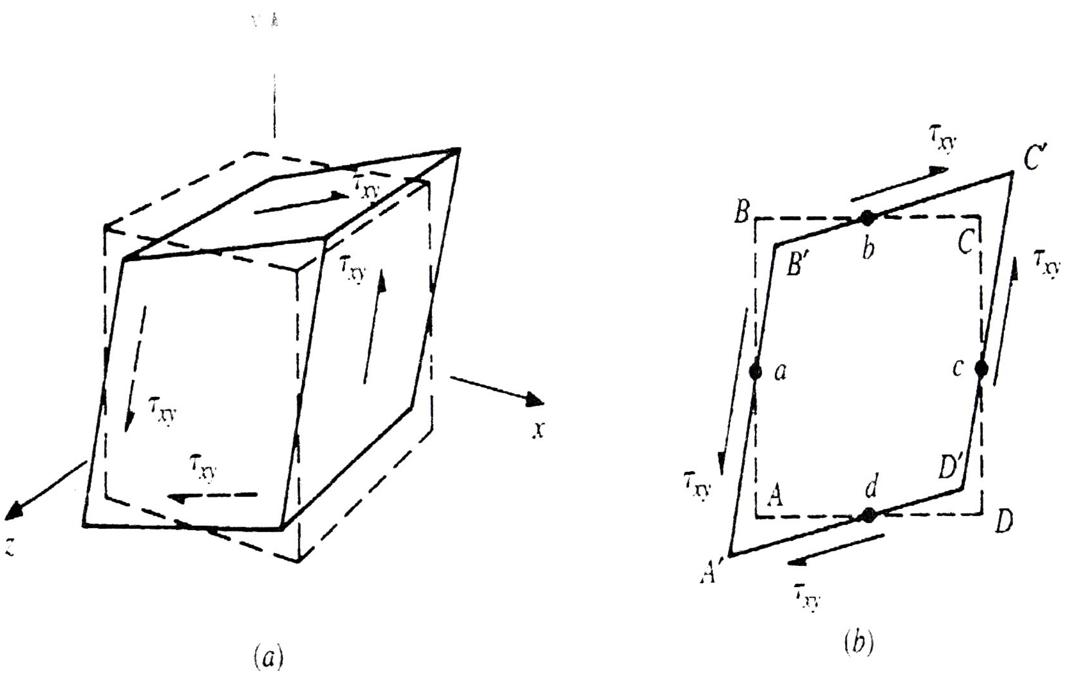


Figure 1: An Element Undergoing Shear Stress and Shear Strain [3]

The shear strain is a measure of the deviation of the stressed element from a rectangular parallelpiped, denoted by γ_{xy}, and defined by

$$\gamma_{xy} = -\Delta\angle BAD = \angle BAD - \angle B'A'D' \tag{16}$$

where γ_{xy} is in dimensionless radians. If the material is linear, homogeneous, and isotropic, we can relate the shear strain to the shear stress directly through the following equation

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{17}$$

where G is the Shear Modulus of Elasticity. For plane problems with x and y defining the coordinate system $\tau_{xz} = \tau_{yz} = 0$, $\gamma_{xz}$and γ_{yz}will both be zero. We can also relate the Shear Modulus, G, to the Modulus of elasticity, E for a linear, homogeneous, isotropic material by

$$G = \frac{E}{2(1+\nu)} \tag{18}$$

Substituting this relationship into equation (17) we get

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \tag{19}$$

We have now defined all of our stresses and strains that occur on an element undergoing a state of plane stress, however we have neglected on important assumption. We are limiting ourselves to small deflection theory, where two dimensional analysis in each of the three planes is valid.

Displacements and Strain Displacement Relationships [3]

The cumulative effect of the strains caused by the varying stresses throughout a structural member cause deflections of the points within the member which allow the deflections to be directly related to the strains

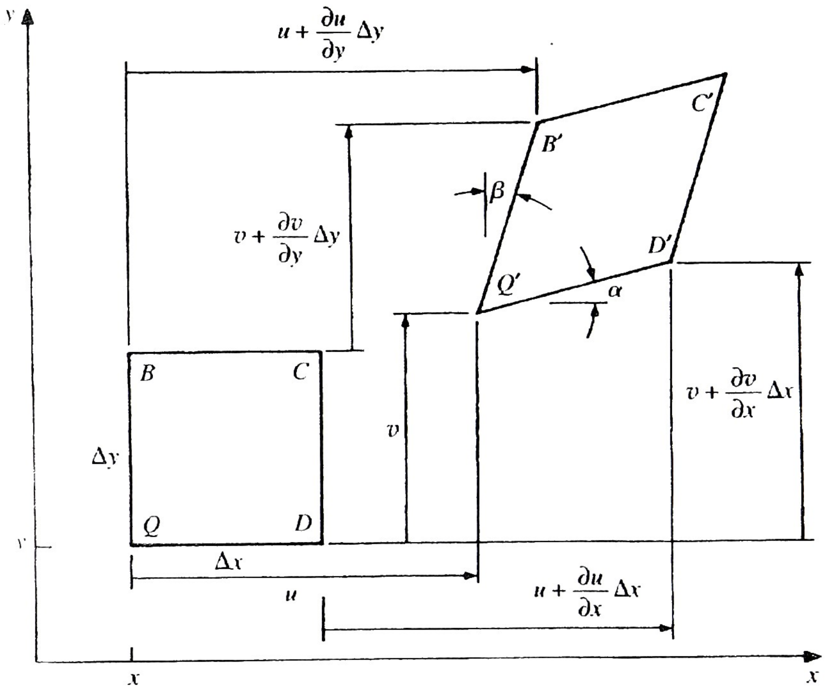


Figure 1.5-1 Rigid body and elastic deflections of an infinitesimal rectangular element.

Figure 2: Rigid Body and Elastic Deflections of an Infinitesimal Element [3]

Consider the point Q in a member where the position of the point Q before loading of the member is located by the coordinates x, y, z with respect to an arbitrary origin as shown in figure 2. An element of infinitesimal dimensions $\Delta x, \Delta y, \Delta z$ originating from the point Q can be constructed where the corners of the initially undeformed element are indicated by QBCD. Stresses, which in turn cause strains, cause point Q to deflect and the element to change geometrically. The deflections of point Q in the x and y directions are denoted by u and v, respectively. The corresponding deflections of points B, C, and D would be identical if the element were rigid and did not rotate and change shape geometrically.

The deflection of point Q can be described by continious functions of x and y. Considering deflections in the x and y plane

$$u = u(x, y) \qquad v = v(x, y) \tag{20}$$

Taking Taylor expansions of the functions u and v about point Q, the deflection of point D in the x direction will be

$$u_D = u + \frac{\partial u}{\partial x}\Delta x + \frac{1\partial^2 u}{2!\partial x^2}(\Delta x)^2 + ... \tag{21}$$

and in the y direction

$$v_D = v + \frac{\partial v}{\partial x}\Delta x + \frac{1\partial^2 v}{2!\partial x^2}(\Delta x)^2 + ... \tag{22}$$

The derivatives are still in terms of x since point D is Δx from Q, and if Δx is taken to be very small, then we can neglect the terms $(\Delta x)^2$ or higher, giving us

$$u_D = u + \frac{\partial u}{\partial x}\Delta x \tag{23}$$

$$v_D = v + \frac{\partial v}{\partial x}\Delta x \tag{24}$$

The deflections of point B can be determined in a similar fashion by taking a Taylor Series expansion for B about point Q, and assuming Δy to be very small, which yields

$$u_B = u + \frac{\partial u}{\partial y}\Delta y \tag{25}$$

$$v_B = v + \frac{\partial v}{\partial y}\Delta y \tag{26}$$

For small deflection theory, the derivative terms are considered small. Thus if $(\frac{\partial v}{\partial x})\Delta x$ is considered small compared with $\Delta x + (\frac{\partial u}{\partial x})\Delta x$, then Q'D' $\approx \Delta x + (\frac{\partial u}{\partial x})\Delta x$ and the rate of elongation of QD is

$$\epsilon_x = \frac{Q'D' - QD}{QD} = \frac{[\Delta x + (\frac{\partial u}{\partial x})\Delta x] - \Delta x}{\Delta x} = \frac{\partial u}{\partial x} \tag{27}$$

And the strain in the y direction of point Q is the rate of elongation of QB,

$$\epsilon_y = \frac{Q'B' - QB}{QB} = \frac{[\Delta y + (\frac{\partial v}{\partial y})\Delta y] - \Delta y}{\Delta y} = \frac{\partial v}{\partial y} \tag{28}$$

The reduction in angle BQD is defined as the shear strain at the point Q and is $\gamma_{xy} = \alpha + \beta$. From figure 2 it can be seen that

$$tan(\alpha) = \frac{(\partial v/\partial x)\Delta x}{\Delta x} = \frac{\partial v}{\partial x} \tag{29}$$

$$tan(\beta) = \frac{(\partial u/\partial y)\Delta y}{\Delta y} = \frac{\partial u}{\partial y} \tag{30}$$

However, if the strains are small, then $\tan(\alpha) \approx \alpha$, and $\tan(\beta) \approx \beta$. We can now represent the shear strain in terms of displacement by

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{31}$$

This definition of shear strain is refered to as the engineering shear strain, which is defined differently than the elasticity shear strain.

Equilibrium & Compatability

We are now going to consider the equilibrium and compatability equations in Cartesian coordinates.

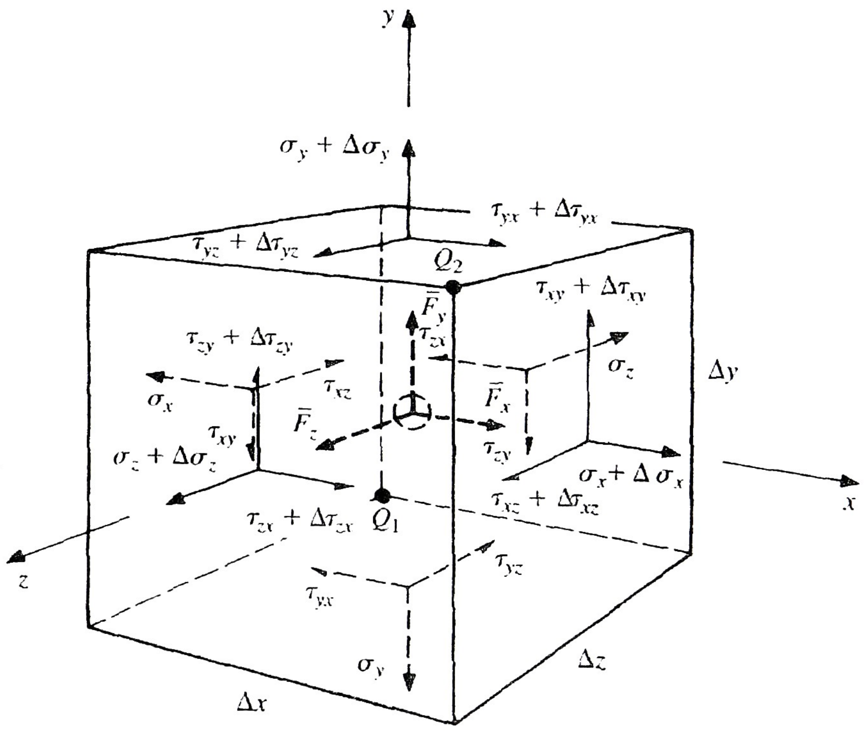


Figure 3: An Element Under Stress[3]

In figure 3, all of the stresses on the element and the body forces F_x, F_y, F_z are shown. In order for equilibrium, the sum of the forces in each direction must be equal to zero. Taking the sum of the forces in the x direction

$$(-\sigma_x + \sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x)\Delta y \Delta z + (-\tau_{xy} + \tau_{xy} + \frac{\partial \tau_{xy}}{\partial y}\Delta y)\Delta x \Delta z +$$

$$(-\tau_{xz} + \tau_{xz} + \frac{\partial \tau_{xz}}{\partial z}\Delta z)\Delta x \Delta y + F_x = 0 \tag{32}$$

Combining terms and dividing by $\Delta x \Delta y \Delta z$, equation (32)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0 \tag{33}$$

The equilibrium equations for the y and z directions can be arrived at in the same fashion

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y = 0 \tag{34}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \tag{35}$$

Taking a state of plane stress, where $\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \tag{36}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \tag{37}$$

If the body is in equilibrium, any segment of the body together with its corresponding internal-force distribution must maintain the segment in static equilibrium. At any given section it is possible to find many stress distributions which will ensure equilibrium. An *acceptable* stress distribution is one which ensures a piecewise-continious-deformation distribution of the body. This is the essential characteristic of *compatability*; i.e. the stress distribution and the resulting deflection distribution must be compatable with the boundary conditions and a continious distribution of deformations so that no "holes" or overlapping of specific points in the body occur [3].

Looking at the xy plane only, we can define compatability by looking back at our strain equations

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{38}$$

$$\epsilon_y = \frac{\partial v}{\partial y} \tag{39}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{40}$$

Differentiating γ_{xy} with respect to x and then with respect to y yields

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} \tag{41}$$

Noting that

$$\frac{\partial^3 u}{\partial x \partial y^2} = \frac{\partial^2 \epsilon_x}{\partial y^2} \tag{42}$$

$$\frac{\partial^3 v}{\partial x^2 \partial y} = \frac{\partial^2 \epsilon_y}{\partial x^2} \tag{43}$$

we have

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} \tag{44}$$

Equation (44) is called the compatability equation, and neglecting z dependence provides a check on whether a given strain field is compatable in the xy plane [3]

AIRY STRESS FUNCTION DEVELOPMENT

Using the fundamentals, we will develop the general Airy Stress Function in cartesian coordinates. We will then take our Cartesian Development amd transform it into Polar Coordinates. Doing this transformation is easier than going through the whole stress function development -- starting way back at stress, strain, and equilibrium and compatability -- again in polar coordinates.

Cartesian Coordinates

In order to develop the Airy Stress Function for a state of plane stress, we first start by assuming the z surface is stress free, which gives us the following.

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0 \tag{45}$$

Next we look at the Compatability Equation (44)

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} \tag{46}$$

and our strain equations (7--9) and (19)

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \tag{47}$$

$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{48}$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \tag{49}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \tag{50}$$

Plugging our strain equations into the compatability equations and multiplying through by E

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \tag{51}$$

Looking now at our equilibrium equations for a state of plane stress (36) and (37)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0 \tag{52}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \tag{53}$$

taking $\frac{\partial}{\partial x}$ of the first, and $\frac{\partial}{\partial y}$ of the second to get

$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial F_x}{\partial x} = 0 \tag{54}$$

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial F}{\partial y} = 0 \tag{55}$$

Assuming τ_{xy} is continious in x and y, which means that $\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^2 \tau_{xy}}{\partial y \partial x}$, and then adding the two equations together

$$\frac{2\partial^2 \tau_{xy}}{\partial x \partial y} + (\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}) + (\frac{\partial F_x}{\partial x} + \frac{\partial F}{\partial y}) = 0 \tag{56}$$

Plugging (56) into (51)

$$\frac{\partial^2}{\partial y^2}(\sigma_x - \nu\sigma_y) + \frac{\partial^2}{\partial x^2}(\sigma_y - \nu\sigma_x) = -(1+\nu)(\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2}) - (1+\nu)(\frac{\partial F_x}{\partial x} + \frac{\partial F}{\partial y}) \tag{57}$$

Canceling terms

$$\frac{\partial^2 \sigma_x}{\partial^2 y} + \frac{\partial^2 \sigma_y}{\partial^2 x} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - (1+\nu)(\frac{\partial F_x}{\partial x} + \frac{\partial F}{\partial y}) \tag{58}$$

Rearranging the equation results in

$$\frac{\partial^2 \sigma_y}{\partial^2 x} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_x}{\partial^2 y} = -(1+\nu)(\frac{\partial F_x}{\partial x} + \frac{\partial F}{\partial y}) \tag{59}$$

Combining terms

$$\frac{\partial^2}{\partial^2 x}(\sigma_x + \sigma_y) + \frac{\partial^2}{\partial^2 y}(\sigma_x + \sigma_y) = -(1+\nu)(\frac{\partial F_x}{\partial x} + \frac{\partial F}{\partial y}) \tag{60}$$

Using the definition of the Leplacian

$$\nabla^2 = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \tag{61}$$

The equation reduces to

$$\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\nabla(F_x + F_y) \tag{62}$$

Making the assumption that there is a function Φ(x,y), such that the following relationships hold

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2} \tag{63}$$

$$\sigma_y = \frac{\partial^2 \Phi}{\partial x^2} \tag{64}$$

$$\tau_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \tag{65}$$

and using the fact that

$$\sigma_x + \sigma_y = \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial x^2} = \nabla^2 \Phi \tag{66}$$

Equation (62) becomes

$$\nabla^2 \nabla^2 \Phi = -(1+\nu)\nabla(F_x + F_y) \tag{67}$$

This differential equation is called the *biharmonic equation*, and the function Φ(x,y) is called the *Airy Stress Function.* It is developed from our compatability equation, so if the stress function Φ satisfies the equation, then compatability is assured. However, we still need to prove that equilibrium is also satisfied in order to derive stresses from the Airy Stress Function substitution.

In order to prove that equilibrium is still satisfied, we will take our equations for stress in terms of Φ, equations (63--65) and plug them into our plane stress equilibrium equations (36--37) resulting in

$$\frac{\partial^3 \Phi}{\partial y^2 \partial x} - \frac{\partial^3 \Phi}{\partial y^2 \partial x} + F_x = 0 \tag{68}$$

$$\frac{\partial^3 \Phi}{\partial y \partial x^2} - \frac{\partial^3 \Phi}{\partial y \partial x^2} + F_y = 0 \tag{69}$$

If we let the body forces go to zero, $F_x = F_y = 0$, then the above equations are valid and equilibrium is indeed satisfied. [3]

Polar Coordinates

It is sometimes easier to define problems in polar coordinates. In order to do this we will use a transformation of coordinates. Assuming that ϕ is defined in terms of r and θ and applying the chain rule to get the following equations

$$\frac{\delta\phi}{\delta x} = \frac{\delta\phi}{\delta r}\frac{\delta r}{\delta x} + \frac{\delta\phi}{\delta\theta}\frac{\delta\theta}{\delta x} \tag{70}$$

$$\frac{\delta\phi}{\delta y} = \frac{\delta\phi}{\delta r}\frac{\delta r}{\delta y} + \frac{\delta\phi}{\delta\theta}\frac{\delta\theta}{\delta y} \tag{71}$$

Using the following equations to relate cartesian coordinates to polar coordinates

$$r^2 = x^2 + y^2 \tag{72}$$

$$x = r\,cos(\theta) \tag{73}$$

$$y = r\,sin(\theta) \tag{74}$$

$$\theta = \tan^{-1}(\frac{y}{x}) \tag{75}$$

and taking thier derivatives

$$\frac{\delta r}{\delta x} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}(2x) = \frac{x}{r} = cos(\theta) \tag{76}$$

$$\frac{\delta r}{\delta y} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}(2y) = \frac{y}{r} = sin(\theta) \tag{77}$$

$$\frac{\delta\theta}{\delta x} = \frac{1}{1+(\frac{y}{x})^2}(\frac{-y}{x^2}) = -\frac{y}{r^2} = -\frac{sin(\theta)}{r} \tag{78}$$

$$\frac{\delta\theta}{\delta \text{y}} = \frac{1}{1+(\frac{y}{x})^2}(\frac{1}{x}) = \frac{x}{r^2} = \frac{cos(\theta)}{r} \tag{79}$$

Rewriting $\frac{\delta\phi}{\delta x}$and $\frac{\delta\phi}{\delta y}$ using equations (76) through (79) to get

$$\frac{\delta\phi}{\delta x} = \frac{\delta\phi}{\delta r}cos(\theta) - \frac{\delta\phi}{\delta\theta}\frac{\sin(\theta)}{r} \tag{80}$$

$$\frac{\delta\phi}{\delta y} = \frac{\delta\phi}{\delta r}sin(\theta) + \frac{\delta\phi}{\delta\theta}\frac{cos(\theta)}{r} \tag{81}$$

Applying the chain rule again to get the second derivatives

$$\frac{\delta^2\phi}{\delta x^2} = \frac{\delta}{\delta x}(\frac{\delta\phi}{\delta x}) = \frac{\delta}{\delta r}(\frac{\delta\phi}{\delta x})\frac{\delta r}{\delta x} + \frac{\delta}{\delta\theta}(\frac{\delta\phi}{\delta x})\frac{\delta\theta}{\delta x} \tag{82}$$

$$\frac{\delta^2\phi}{\delta y^2} = \frac{\delta}{\delta y}(\frac{\delta\phi}{\delta y}) = \frac{\delta}{\delta r}(\frac{\delta\phi}{\delta y})\frac{\delta r}{\delta y} + \frac{\delta}{\delta\theta}(\frac{\delta\phi}{\delta y})\frac{\delta\theta}{\delta y} \tag{83}$$

$$\frac{\delta^2\phi}{\delta x\delta y} = \frac{\delta}{\delta y}(\frac{\delta\phi}{\delta x}) = \frac{\delta}{\delta r}(\frac{\delta\phi}{\delta y})\frac{\delta r}{\delta x} + \frac{\delta}{\delta\theta}(\frac{\delta\phi}{\delta y})\frac{\delta\theta}{\delta x} \tag{84}$$

subsituting in the values for $\frac{\delta\phi}{\delta x}$ and $\frac{\delta\phi}{\delta y}$

$$\frac{\delta^2\phi}{\delta x^2} = \frac{\delta}{\delta r}(\frac{\delta\phi}{\delta r}cos(\theta) - \frac{\delta\phi}{\delta\theta}\frac{\sin(\theta)}{r})\frac{\delta r}{\delta x} + \frac{\delta}{\delta\theta}(\frac{\delta\phi}{\delta r}cos(\theta) - \frac{\delta\phi}{\delta\theta}\frac{\sin(\theta)}{r})\frac{\delta\theta}{\delta x} \tag{85}$$

$$\frac{\delta^2\phi}{\delta y^2} = \frac{\delta}{\delta r}(\frac{\delta\phi}{\delta r}sin(\theta) + \frac{\delta\phi}{\delta\theta}\frac{cos(\theta)}{r})\frac{\delta r}{\delta y} + \frac{\delta}{\delta\theta}(\frac{\delta\phi}{\delta r}sin(\theta) + \frac{\delta\phi}{\delta\theta}\frac{cos(\theta)}{r})\frac{\delta\theta}{\delta y} \tag{86}$$

$$\frac{\delta^2\phi}{\delta x\delta y} = \frac{\delta}{\delta r}(\frac{\delta\phi}{\delta r}sin(\theta) + \frac{\delta\phi}{\delta\theta}\frac{cos(\theta)}{r})\frac{\delta r}{\delta x} + \frac{\delta}{\delta\theta}(\frac{\delta\phi}{\delta r}sin(\theta) + \frac{\delta\phi}{\delta\theta}\frac{cos(\theta)}{r})\frac{\delta\theta}{\delta x} \tag{87}$$

Expanding out

$$\frac{\delta^2\phi}{\delta x^2} = (\frac{\delta^2\phi}{\delta r^2}cos(\theta) - \frac{\delta^2\phi}{\delta\theta\delta r}\frac{\sin(\theta)}{r} + \frac{\delta\phi}{\delta\theta}\frac{sin(\theta)}{r^2})\frac{\delta r}{\delta x}$$

$$+(\frac{\delta^2\phi}{\delta r\delta\theta}cos(\theta) - \frac{\delta\phi}{\delta r}sin(\theta) - \frac{\delta^2\phi}{\delta\theta^2}\frac{\sin(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{\cos(\theta)}{r})\frac{\delta\theta}{\delta x} \tag{88}$$

$$\frac{\delta^2\phi}{\delta y^2} = (\frac{\delta^2\phi}{\delta r^2}sin(\theta) + \frac{\delta^2\phi}{\delta\theta\delta r}\frac{cos(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{cos(\theta)}{r^2})\frac{\delta r}{\delta y}$$

$$+(\frac{\delta^2\phi}{\delta r\delta\theta}sin(\theta) + \frac{\delta\phi}{\delta r}\cos(\theta) + \frac{\delta^2\phi}{\delta\theta^2}\frac{cos(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{sin(\theta)}{r})\frac{\delta\theta}{\delta y} \tag{89}$$

$$\frac{\delta^2\phi}{\delta x\delta y} = (\frac{\delta^2\phi}{\delta r^2}sin(\theta) + \frac{\delta^2\phi}{\delta r\delta\theta}\frac{cos(\theta)}{r} - \frac{\delta\phi}{\delta r}\frac{cos(\theta)}{r^2})\frac{\delta r}{\delta x}$$

$$+(\frac{\delta^2\phi}{\delta r\delta\theta}sin(\theta) + \frac{\delta\phi}{\delta r}\cos(\theta) + \frac{\delta^2\phi}{\delta\theta^2}\frac{cos(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{sin(\theta)}{r})\frac{\delta\theta}{\delta x} \tag{90}$$

susbtituting in the values for $\frac{\delta r}{\delta x}$, $\frac{\delta\theta}{\delta x}$, $\frac{\delta r}{\delta y}$, and $\frac{\delta\theta}{\delta y}$

$$\frac{\delta^2\phi}{\delta x^2} = (\frac{\delta^2\phi}{\delta r^2}cos(\theta) - \frac{\delta^2\phi}{\delta\theta\delta r}\frac{\sin(\theta)}{r} + \frac{\delta\phi}{\delta\theta}\frac{sin(\theta)}{r^2})cos(\theta)$$

$$- (\frac{\delta^2\phi}{\delta r\delta\theta}cos(\theta) - \frac{\delta\phi}{\delta r}sin(\theta) - \frac{\delta^2\phi}{\delta\theta^2}\frac{\sin(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{\cos(\theta)}{r})\frac{sin(\theta)}{r} \tag{91}$$

$$\frac{\delta^2\phi}{\delta y^2} = (\frac{\delta^2\phi}{\delta r^2}sin(\theta) + \frac{\delta^2\phi}{\delta\theta\delta r}\frac{cos(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{cos(\theta)}{r^2})sin(\theta)$$

$$+ (\frac{\delta^2\phi}{\delta r\delta\theta}sin(\theta) + \frac{\delta\phi}{\delta r}\cos(\theta) + \frac{\delta^2\phi}{\delta\theta^2}\frac{cos(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{sin(\theta)}{r})\frac{cos(\theta)}{r} \tag{92}$$

$$\frac{\delta^2\phi}{\delta x\delta y} = (\frac{\delta^2\phi}{\delta r^2}sin(\theta) + \frac{\delta^2\phi}{\delta r\delta\theta}\frac{cos(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{cos(\theta)}{r^2})cos(\theta)$$

$$- (\frac{\delta^2\phi}{\delta r\delta\theta}sin(\theta) + \frac{\delta\phi}{\delta r}\cos(\theta) + \frac{\delta^2\phi}{\delta\theta^2}\frac{cos(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{sin(\theta)}{r})\frac{sin(\theta)}{r} \tag{93}$$

Multiplying through

$$\frac{\delta^2\phi}{\delta x^2} = \frac{\delta^2\phi}{\delta r^2}cos^2(\theta) - \frac{\delta^2\phi}{\delta\theta\delta r}\frac{\sin(\theta)cos(\theta)}{r} + \frac{\delta\phi}{\delta\theta}\frac{sin(\theta)\cos(\theta)}{r^2}$$

$$- \frac{\delta^2\phi}{\delta r\delta\theta}\frac{sin(\theta)cos(\theta)}{r} + \frac{\delta\phi}{\delta r}\frac{sin^2(\theta)}{r} + \frac{\delta^2\phi}{\delta\theta^2}\frac{\sin^2(\theta)}{r^2} + \frac{\delta\phi}{\delta\theta}\frac{\cos(\theta)sin(\theta)}{r^2} \tag{94}$$

$$\frac{\delta^2\phi}{\delta y^2} = \frac{\delta^2\phi}{\delta r^2}sin^2(\theta) + \frac{\delta^2\phi}{\delta\theta\delta r}\frac{cos(\theta)sin(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{cos(\theta)sin(\theta)}{r^2}$$

$$+ \frac{\delta^2\phi}{\delta r\delta\theta}\frac{cos(\theta)sin(\theta)}{r} + \frac{\delta\phi}{\delta r}\frac{cos^2(\theta)}{r} + \frac{\delta^2\phi}{\delta\theta^2}\frac{cos^2(\theta)}{r^2} - \frac{\delta\phi}{\delta\theta}\frac{sin(\theta)cos(\theta)}{r^2} \tag{95}$$

$$\frac{\delta^2\phi}{\delta x\delta y} = \frac{\delta^2\phi}{\delta r^2}sin(\theta)cos(\theta) + \frac{\delta^2\phi}{\delta r\delta\theta}\frac{cos^2(\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{cos^2(\theta)}{r^2}$$

$$- \frac{\delta^2\phi}{\delta r\delta\theta}\frac{\sin^2(\theta)}{r} - \frac{\delta\phi}{\delta r}\frac{cos(\theta)\sin(\theta)}{r} - \frac{\delta^2\phi}{\delta\theta^2}\frac{sin(\theta)cos(\theta)}{r^2} + \frac{\delta\phi}{\delta\theta}\frac{sin^2(\theta)}{r^2} \tag{96}$$

Using $2\,sin(\theta)cos(\theta) = sin(2\theta)$ and $cos^2(\theta) - sin^2(\theta) = cos(2\theta)$ to reduce the equations to

$$\frac{\delta^2\phi}{\delta x^2} = \frac{\delta^2\phi}{\delta r^2}cos^2(\theta) - \frac{\delta^2\phi}{\delta\theta\delta r}\frac{\sin(2\theta)}{r} + \frac{\delta\phi}{\delta\theta}\frac{sin(2\theta)}{r^2}$$

$$+\frac{\delta\phi}{\delta r}\frac{sin^2(\theta)}{r} + \frac{\delta^2\phi}{\delta\theta^2}\frac{\sin^2(\theta)}{r^2} \tag{97}$$

$$\frac{\delta^2\phi}{\delta y^2} = \frac{\delta^2\phi}{\delta r^2}sin^2(\theta) + \frac{\delta^2\phi}{\delta\theta\delta r}\frac{sin(2\theta)}{r} - \frac{\delta\phi}{\delta\theta}\frac{sin(2\theta)}{r^2}$$

$$+\frac{\delta\phi}{\delta r}\frac{cos^2(\theta)}{r}+\frac{\delta^2\phi}{\delta\theta^2}\frac{cos^2(\theta)}{r^2} \tag{98}$$

$$\frac{\delta^2\phi}{\delta x\delta y}=\frac{\delta^2\phi}{\delta r^2}sin(\theta)cos(\theta)+\frac{\delta^2\phi}{\delta r\delta\theta}\frac{cos(2\theta)}{r}-\frac{\delta\phi}{\delta r}\frac{cos(\theta)\sin(\theta)}{r}$$

$$-\frac{\delta^2\phi}{\delta\theta^2}\frac{sin(\theta)cos(\theta)}{r^2}-\frac{\delta\phi}{\delta\theta}\frac{cos(2\theta)}{r^2} \tag{99}$$

Applying equations (97) and (98) to the following equation

$$\nabla^2\phi=\frac{\delta^2\phi}{\delta^2 x}+\frac{\delta^2\phi}{\delta^2 y} \tag{100}$$

resulting in

$$\nabla^2\phi=\frac{\delta^2\phi}{\delta r^2}(cos^2(\theta)+sin^2(\theta))-\frac{\delta^2\phi}{\delta\theta\delta r}\frac{\sin(2\theta)}{r}+\frac{\delta^2\phi}{\delta\theta\delta r}\frac{sin(2\theta)}{r}+$$

$$\frac{\delta\phi}{\delta\theta}\frac{sin(2\theta)}{r^2}-\frac{\delta\phi}{\delta\theta}\frac{sin(2\theta)}{r^2}+\frac{\delta\phi}{\delta r}(\frac{sin^2(\theta)}{r}+\frac{cos^2(\theta)}{r})+\frac{\delta^2\phi}{\delta\theta^2}(\frac{\sin^2(\theta)}{r^2}+\frac{cos^2(\theta)}{r^2}) \tag{101}$$

Canceling terms and reducing

$$\nabla^2\phi=\frac{\delta^2\phi}{\delta r^2}+\frac{\delta\phi}{\delta r}\frac{1}{r}+\frac{\delta^2\phi}{\delta\theta^2}\frac{1}{r^2} \tag{102}$$

Polar Components of Stress in Terms of Airy's Stress Function [4]

Expressions can be obtained which relate the polar stress components $\sigma_{rr}, \sigma_{\theta\theta}$, and $\tau_{r\theta}$ to the cartesian stress components σ_{xx}, σ_{yy} and τ_{xy}

$$\sigma_{rr}=\sigma_{xx}cos^2(\theta)+\sigma_{yy}sin^2(\theta)+\tau_{xy}sin(2\theta) \tag{103}$$

$$\sigma_{\theta\theta}=\sigma_{yy}cos^2(\theta)+\sigma_{xx}sin^2(\theta)-\tau_{xy}sin(2\theta) \tag{104}$$

$$\tau_{r\theta}=(\sigma_{yy}-\sigma_{xx})sin(\theta)cos(\theta)+\tau_{xy}cos(2\theta) \tag{105}$$

If equations (63) to (65) are inserted into the above equations, which is equivelant to setting F_x and F_y equal to zero, then

$$\sigma_{rr}=\frac{\delta^2\phi}{\delta y^2}cos^2(\theta)+\frac{\delta^2\phi}{\delta x^2}sin^2(\theta)-\frac{\delta^2\phi}{\delta x\delta y}sin(2\theta) \tag{106}$$

$$\sigma_{\theta\theta} = \frac{\delta^2\phi}{\delta x^2}cos^2(\theta) + \frac{\delta^2\phi}{\delta y^2}sin^2(\theta) + \frac{\delta^2\phi}{\delta x\delta y}sin(2\theta) \tag{107}$$

$$\tau_{r\theta} = (\frac{\delta^2\phi}{\delta x^2} - \frac{\delta^2\phi}{\delta y^2})sin(\theta)cos(\theta) - \frac{\delta^2\phi}{\delta x\delta y}cos(2\theta) \tag{108}$$

Substituting equations (97) to (99) into the above equations the polar components of stress in terms of Airy Stress function are obtained.

$$\sigma_{rr} = \frac{1}{r}\frac{\delta\phi}{\delta r} + \frac{1}{r^2}\frac{\delta^2\phi}{\delta\theta^2} \tag{109}$$

$$\sigma_{\theta\theta} = \frac{\delta^2\phi}{\delta r^2} \tag{110}$$

$$\tau_{r\theta} = \frac{1}{r^2}\frac{\delta\phi}{\delta\theta} - \frac{1}{r}\frac{\delta^2\phi}{\delta r\delta\theta} \tag{111}$$

Axisymmetric Stresses [5]

Often times, problems in the circular domain are axisymmetric, depending only on r and not on θ. We can therefore simplify the Airy Stress equations to

$$\sigma_{rr} = \frac{1}{r}\frac{\delta\phi}{\delta r} \tag{112}$$

$$\sigma_{\theta\theta} = \frac{\delta^2\phi}{\delta r^2} \tag{113}$$

$$\tau_{r\theta} = 0 \tag{114}$$

$$\nabla^2\phi = \frac{\delta^2\phi}{\delta r^2} + \frac{1}{r}\frac{\delta\phi}{\delta r} = \frac{1}{r}\frac{\delta}{\delta r}[r(\frac{\delta\phi}{\delta r})] \tag{115}$$

$$\nabla^4\phi = [\frac{\delta^2}{\delta r^2} + \frac{1}{r}\frac{\delta}{\delta r}][\frac{\delta^2\phi}{\delta r^2} + \frac{1}{r}\frac{\delta\phi}{\delta r}] \tag{116}$$

$$= \frac{1}{r}\frac{\delta}{\delta r}\left[r\frac{\delta}{\delta r}[\frac{1}{r}\frac{\delta}{\delta r}(r\frac{\delta\phi}{\delta r})]\right] = 0 \tag{117}$$

Integrating ∇^4 in order to get the stresses in terms of constants

$$\frac{\delta}{\delta r}\left[r\frac{\delta}{\delta r}[\frac{1}{r}\frac{\delta}{\delta r}(r\frac{\delta\phi}{\delta r})]\right] = 0 \tag{118}$$

$$r\frac{\delta}{\delta r}[\frac{1}{r}\frac{\delta}{\delta r}(r\frac{\delta\phi}{\delta r})] = C_1 \tag{119}$$

$$\frac{\delta}{\delta r}[\frac{1}{r}\frac{\delta}{\delta r}(r\frac{\delta \phi}{\delta r})] = \frac{C_1}{r} \tag{120}$$

$$\frac{1}{r}\frac{\delta}{\delta r}(r\frac{\delta \phi}{\delta r}) = C_1 ln(r) + C_2 \tag{121}$$

$$\frac{\delta}{\delta r}(r\frac{\delta \phi}{\delta r}) = C_1 ln(r)r + C_2 r \tag{122}$$

$$r\frac{\delta \phi}{\delta r} = C_1(\frac{r^2}{2}ln(r) - \frac{r^2}{2}) + C_2\frac{r^2}{2} + C_3 \tag{123}$$

$$= C_1^{'} r^2 ln(r) + C_2^{'} r^2 + C_3 \tag{124}$$

$$\frac{\delta \phi}{\delta r} = C_1^{'} r ln(r) + C_2^{'} r + \frac{C_3}{r} \tag{125}$$

$$\phi = C_1^{'}(\frac{r^2}{2}ln(r) - \frac{r^2}{2}) + C_1^{'}\frac{r^2}{2} + C_3 ln(r) + C_4 \tag{126}$$

$$\phi = C_1^{''} r^2 ln(r) + C_2^{''} r^2 + C_3 ln(r) + C_4 \tag{127}$$

Substituting into equations (112) through (114), our stresses become

$$\sigma_{rr} = C_1^{''}(1 + 2ln(r)) + 2C_2^{''} + C_3\frac{1}{r^2} \tag{128}$$

$$\sigma_{\theta\theta} = C_1^{''}(3 + 2ln(r)) + 2C_2^{''} - C_3\frac{1}{r^2} \tag{129}$$

$$\tau_{r\theta} = 0 \tag{130}$$

GENERAL CASE STUDIES

Some Airy Stress Functions have been developed in the general form without specific boundary or loading conditions. By developing them first in the general case we give ourselves the ability to quickly solve many specific cases by simply choosing the general function, or combination there of, and apply our known boundary and loading conditions.

Polynomial Functions

Polynomial Functions can be developed in the general case, and are especially useful when it comes to applying the boundary conditions for specific cases as some terms may be determined based on things such as symmetry, the need for an odd or even stress function, or whether a certain stress component must be independent of a certain direction based on the boundary and loading conditions.

First Degree [6]

$$\phi = ax + by \tag{131}$$

the stress field becomes

$$\sigma_{xx} = \sigma_{yy} = \tau_{xy} = 0 \tag{132}$$

This is only useful in order to indicate as stress free field

Second Degree [6]

$$\phi = ax^2 + bxy + cy^2 \tag{133}$$

the stress field becomes

$$\sigma_{xx} = 2c \tag{134}$$

$$\sigma_{yy} = 2a \tag{135}$$

$$\tau_{xy} = -b \tag{136}$$

This gives the case of a uniform stress field over the entire body

Third Degree [6]

$$\phi = ax^3 + bx^2y + cxy^2 + dy^3 \tag{137}$$

the stress field becomes

$$\sigma_{xx} = 2cx + 6dy \tag{138}$$

$$\sigma_{yy} = 6ax + 2by \tag{139}$$

$$\tau_{xy} = -2bx - 2cy \tag{140}$$

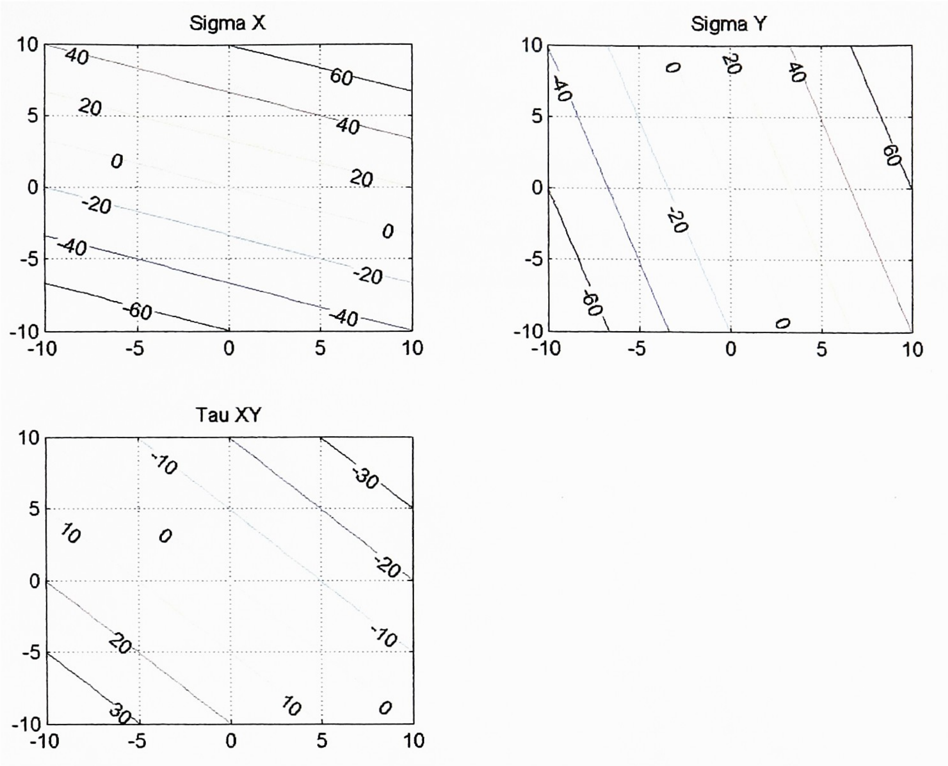


Figure 4: Plot of Linearly Varying Stress Field Over Entire Body

Fourth Degree [6]

$$\phi = ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \tag{141}$$

the stress field becomes

$$\sigma_{xx} = 2cx^2 + 6dxy + 12ey^2 \tag{142}$$

$$\sigma_{yy} = 12ax^2 + 6bxy + 2cy^2 \tag{143}$$

$$\tau_{xy} = -3bx^2 - 4cxy - 3dy^2 \tag{144}$$

In order to ensure compatability, $\nabla^4\phi = 0$, we get the following

$$24a + 8c + 24e = 0 \tag{145}$$

which means that

$$e = -(a + \frac{c}{3}) \tag{146}$$

Substituting back into our stress equations for e, we get

$$\sigma_{xx} = 2cx^2 + 6dxy - 12ay^2 - 4cy^2 \tag{147}$$

$$\sigma_{yy} = 12ax^2 + 6bxy + 2cy^2 \tag{148}$$

$$\tau_{xy} = -3bx^2 - 4cxy - 3dy^2 \tag{149}$$

This yields a stress field that varies according to a second degree polynomial in both $x$and y

When you graph the contour plots, you will end up with either hyperbolas or ellipses depending on whether your constants are negative or posotive and assuming no consants are equal to zero. Some examples are shown below.

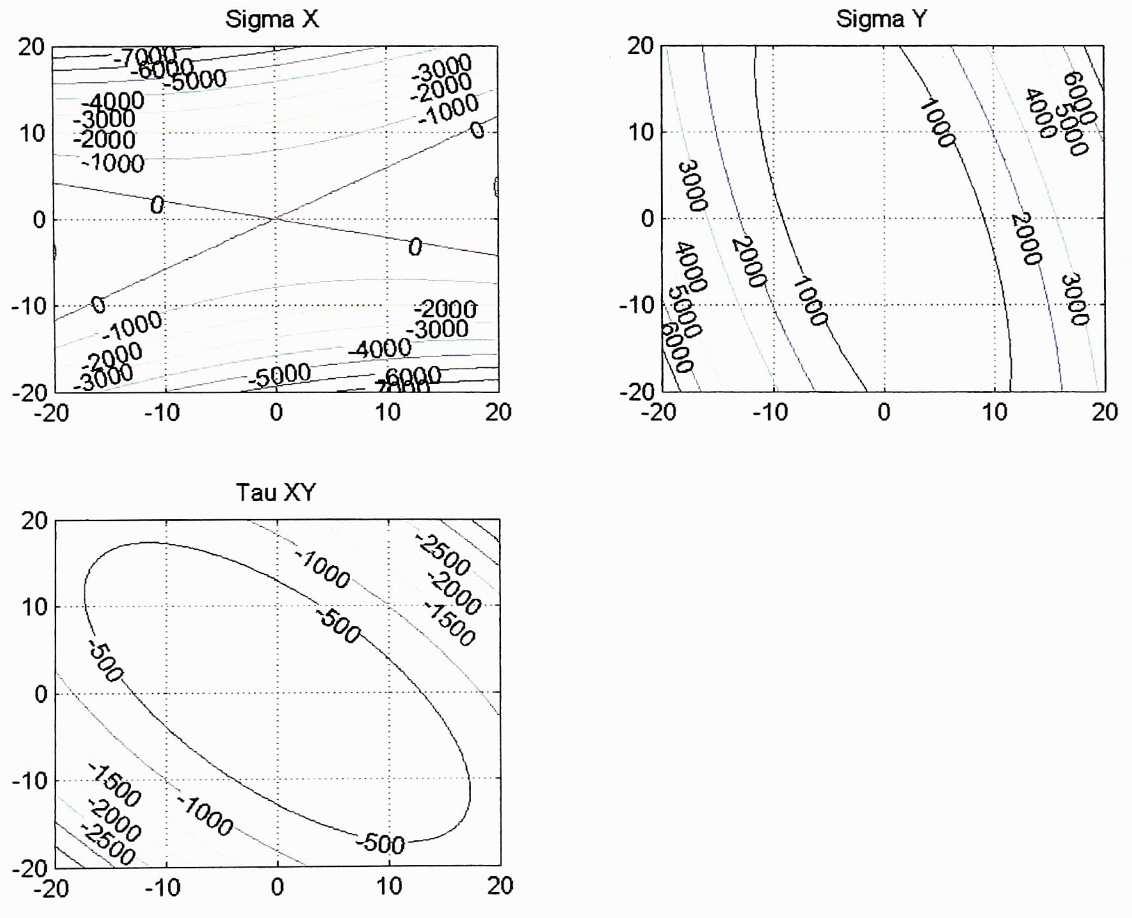


Figure 5: Stress Plot of Fourth Degree Polynomial With All Constants Posotive

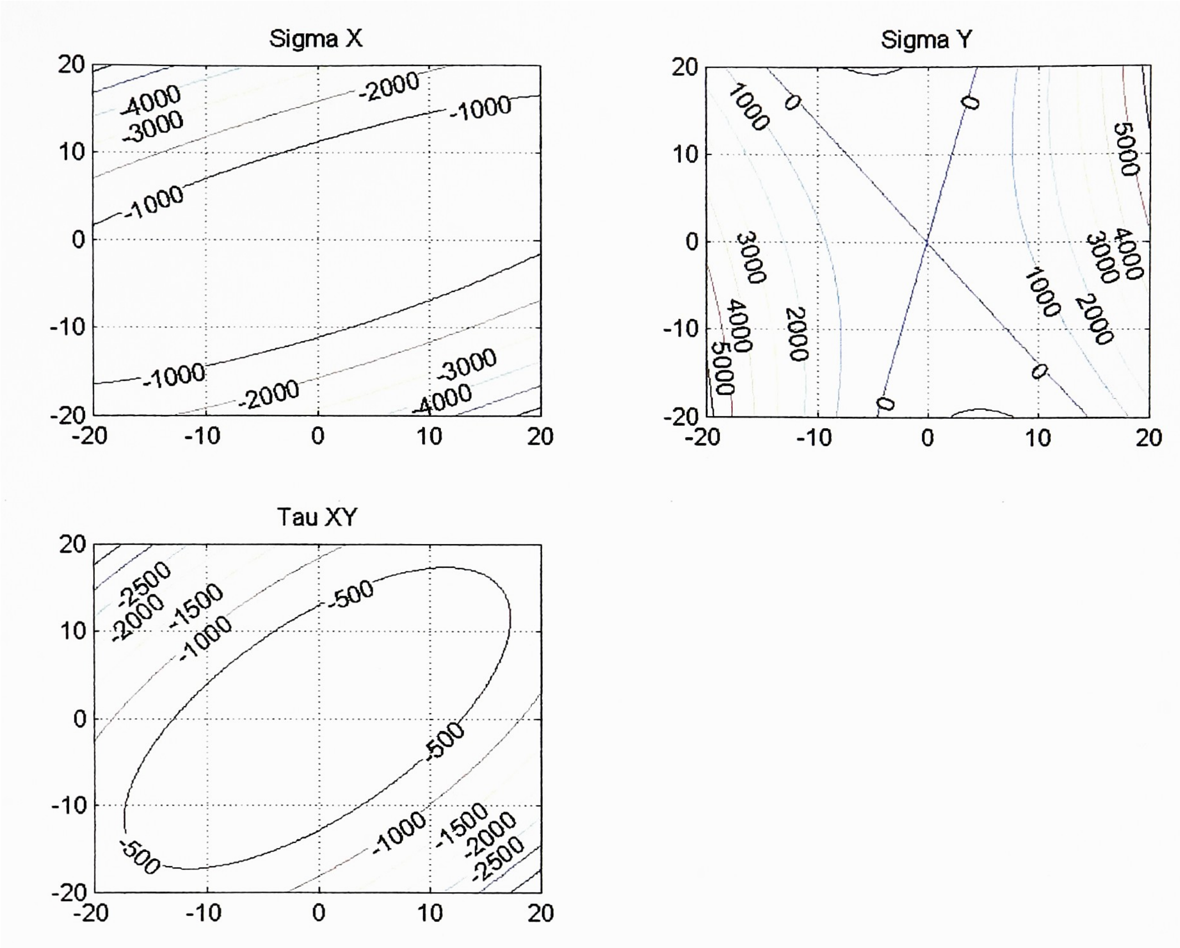


Figure 6: Stress Plot of Fourth Degree Polynomial With Only Constant C Negative

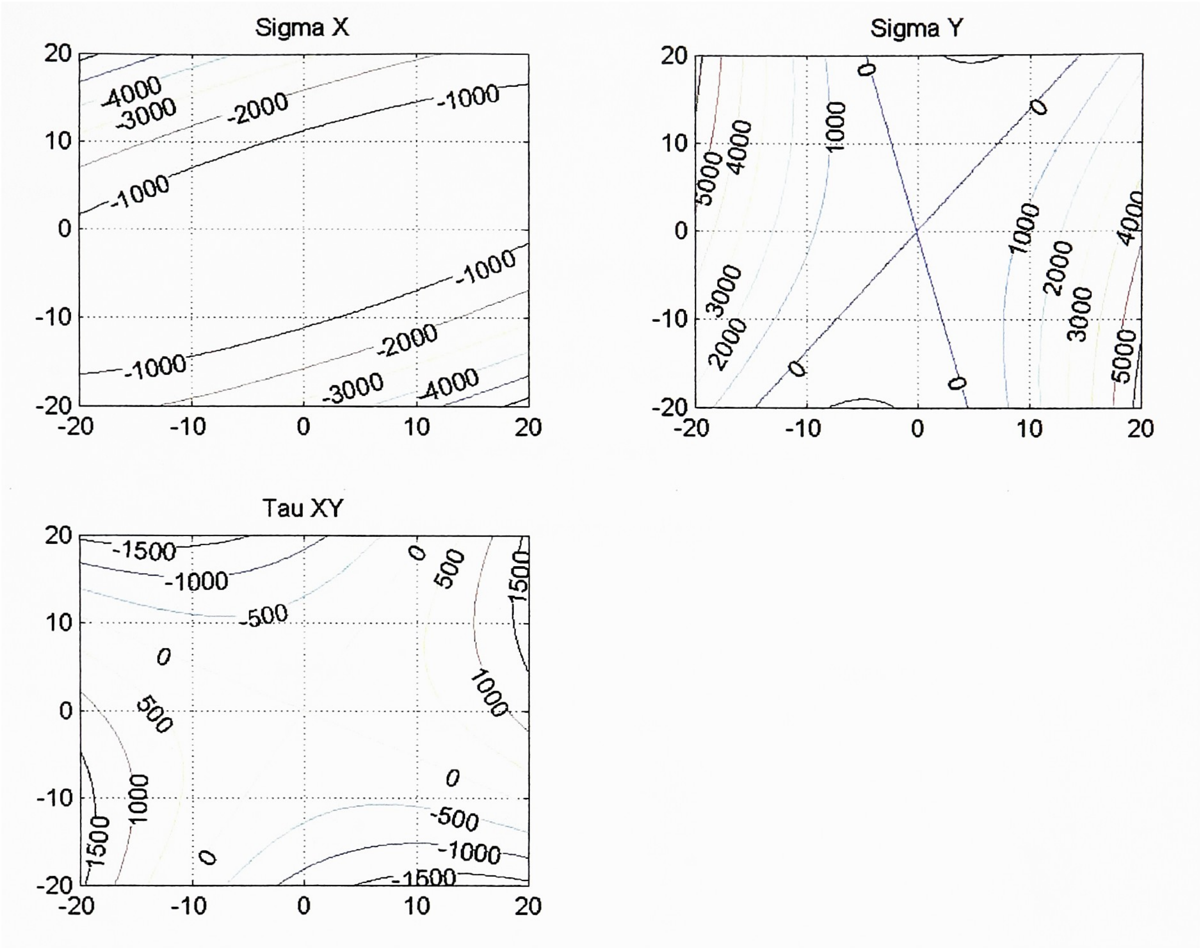


Figure 7: Stress Plot of Fourth Degree Polynomial With Constants B and C Negative

<u>Fifth Degree</u> [6]

$$\phi = ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + exy^4 + fy^5 \tag{150}$$

the stress field becomes

$$\sigma_{xx} = 2cx^3 + 6dx^2y + 12exy^2 + 20fy^3 \tag{151}$$

$$\sigma_{yy} = 20ax^3 + 12bx^2y + 6cxy^2 + 2dy^3 \tag{152}$$

$$\tau_{xy} = -4bx^3 - 6cx^2y - 6dxy^2 - 4ey^3 \tag{153}$$

Checking the compatability equation once again

$$120ax + 24by + 120fy + 24ex + 24cx + 24dy = 0 \tag{154}$$

evaluating the equation for e and f in terms of other constants, we get

$$e = -(5a + c) \tag{155}$$

$$f = -(\frac{b + d}{5}) \tag{156}$$

Substituting back into our stress equations

$$\sigma_{xx} = 2cx^3 + 6dx^2y - 12(5a + c)xy^2 - 4(b + d)y^3 \tag{157}$$

$$\sigma_{yy} = 20ax^3 + 12bx^2y + 6xy^2 + 2dy^3 \tag{158}$$

$$\tau_{xy} = -4bx^3 - 6cx^2y - 6dxy^2 + 4(5a + c)y^3 \tag{159}$$

This yeilds a stress field which is a third degree polynomial in x and y .

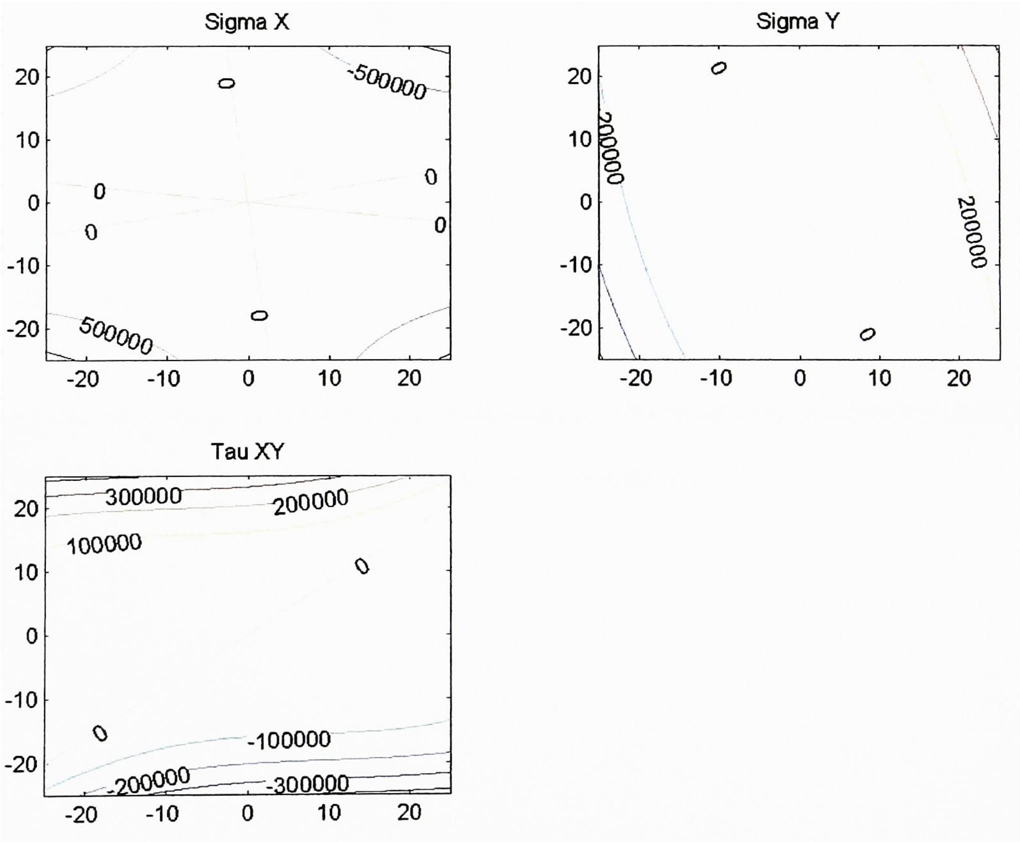


Figure 8: Stress Plot of Fifth Degree Polynomial With All Constants Posotive

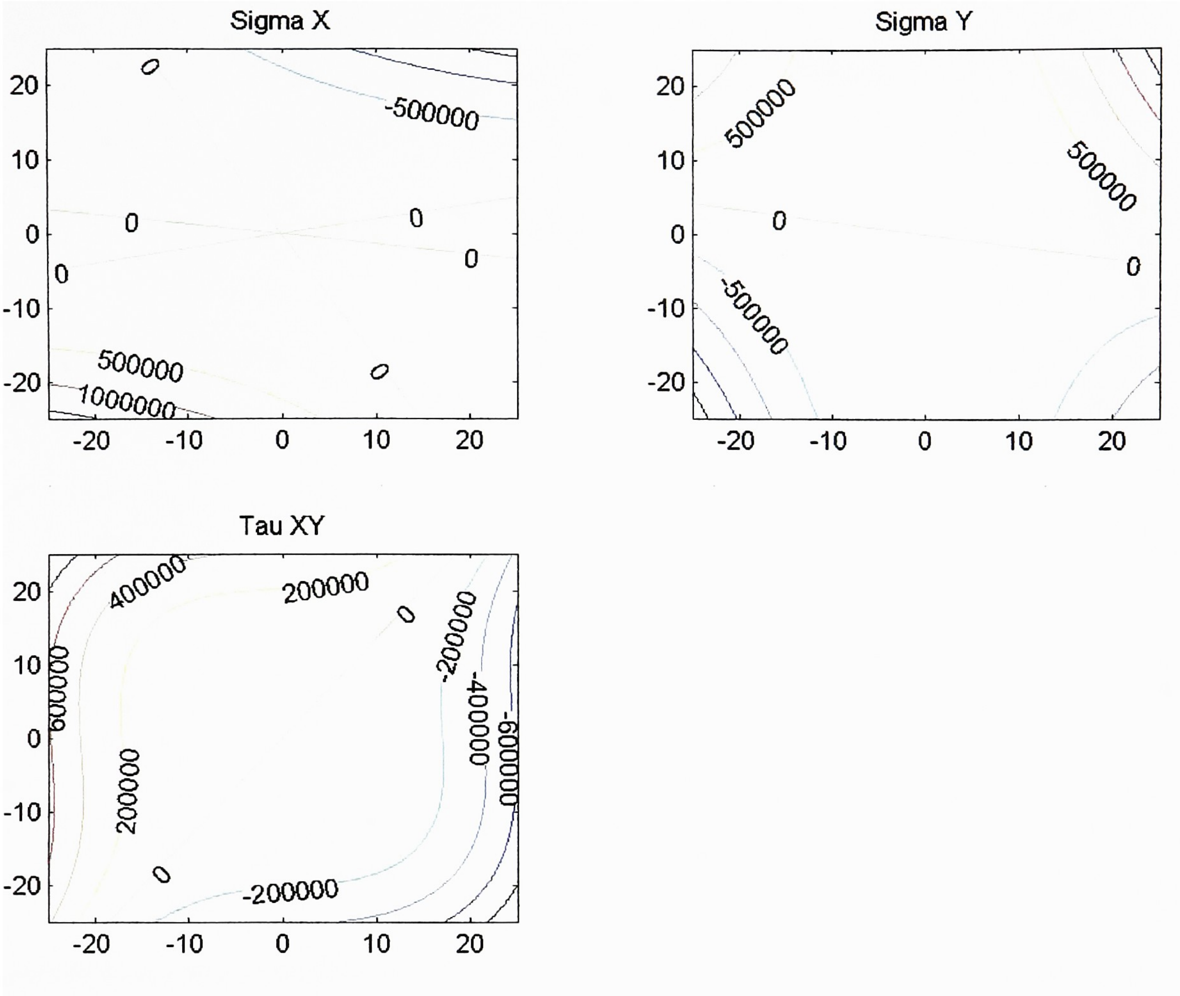


Figure 9: Stress Plot of Fifth Degree Polynomial With Constants A=C=D=1 and B=10

Figure 9 really emphasizes how τ_{xy} varies by a third degree polynomial in x and y

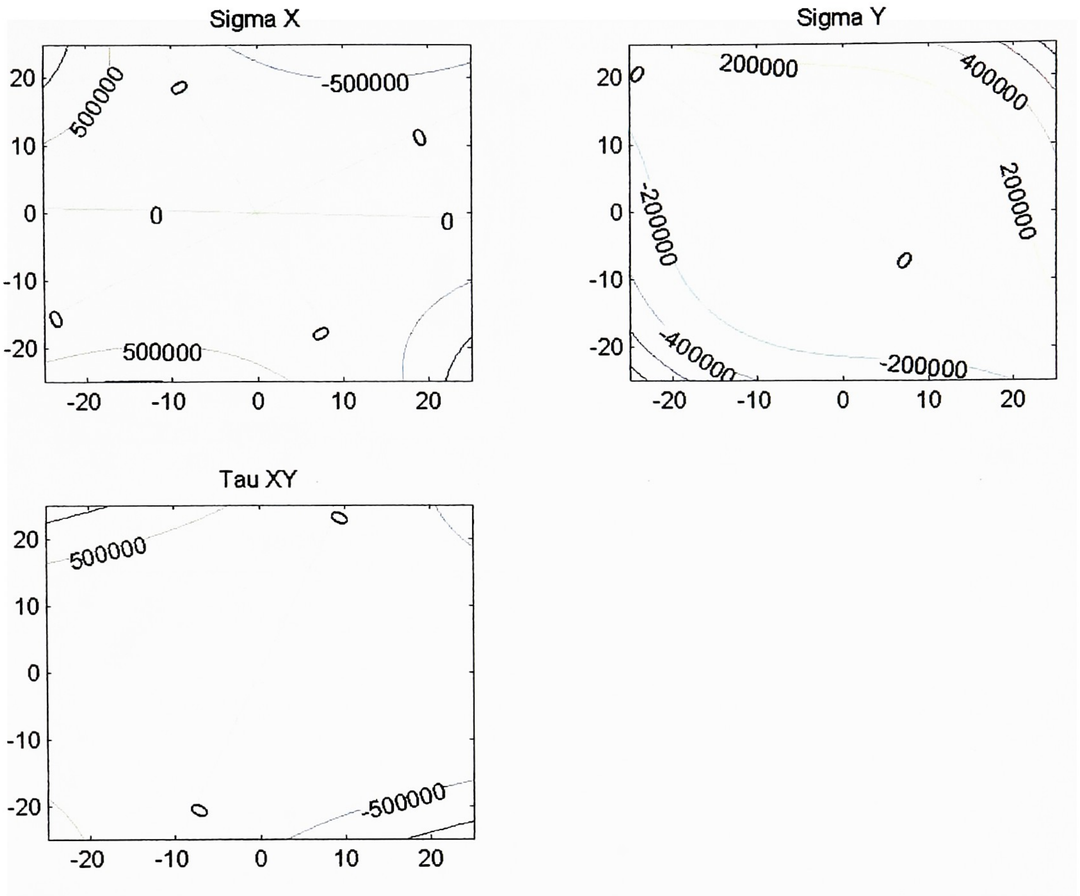


Figure 10: Stress Plot of Fifth Degree Polynomial With Constants A=B=C=1 and D=10

Figure 10 emphasizes the third degree polynomial stress distribution of σ_{yy}

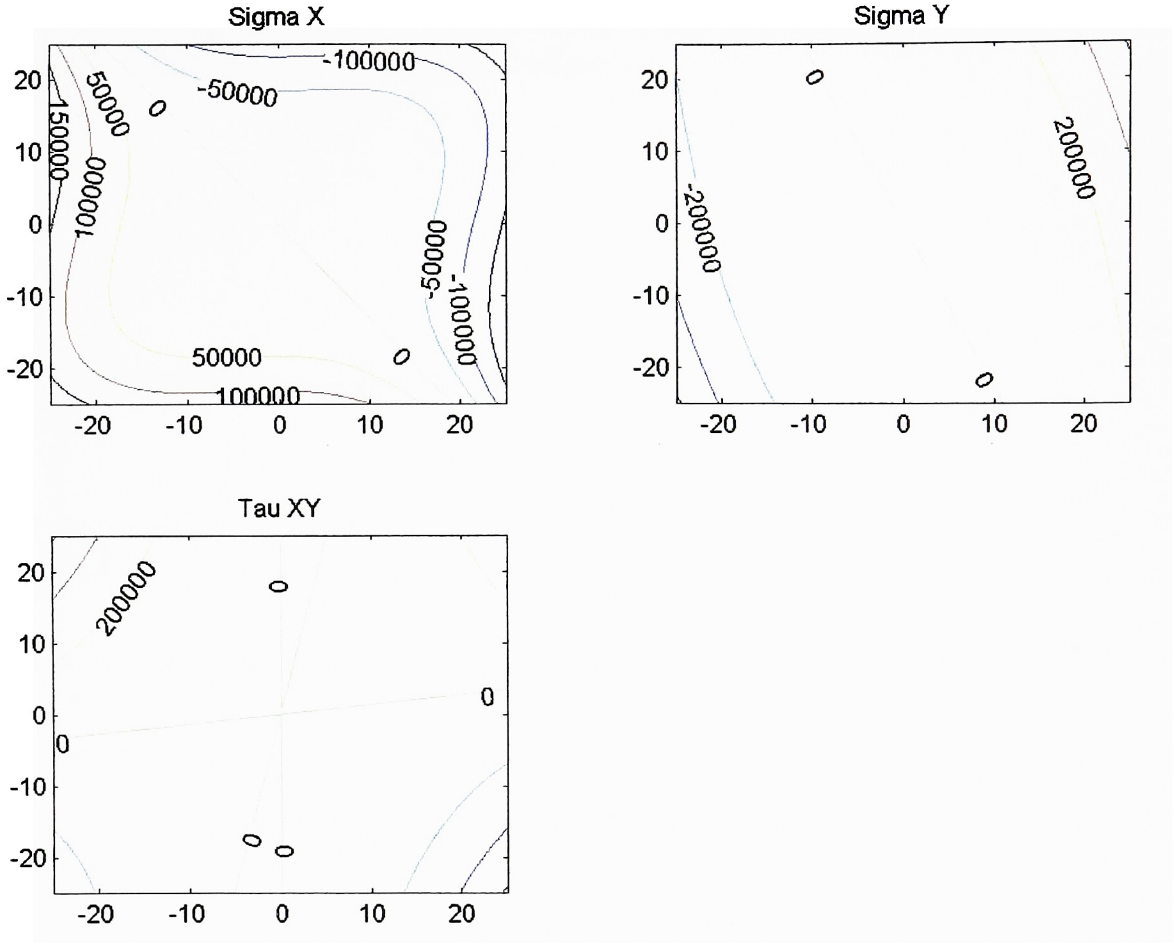


Figure 11: Stress Plot of Fifth Degree Polynomial With Constants A=B=D=1 and C=--5

Figure 11emphasizes the third degree polynomial stress distribution of σ_{xx}

SPECIFIC CASE STUDIES

Some stress functions have been developed for specific boundary conditions, many having come directly from our general case studies. By keeping some things general though -- force, length, etc. -- these functions can also be applied to many different cases even though they are less general.

Cantilever Beam Loaded at End [7]

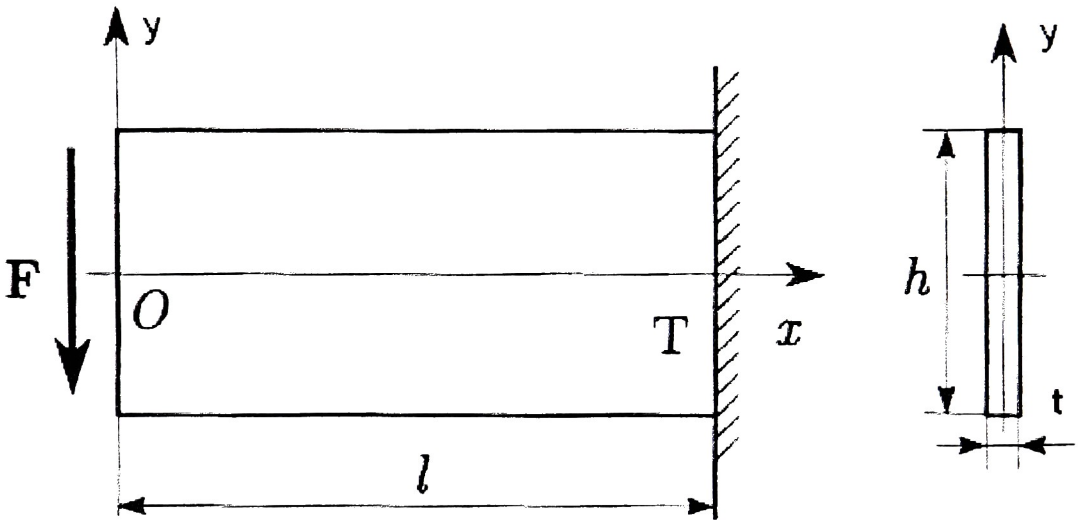


Figure 12: Cantilever Beam Loaded at End [7]

$$\phi = bxy + dxy^3 \tag{160}$$

where b and d are constants.

$$\frac{\partial^2 \phi}{\partial^2 y} = \sigma_{xx} = 6dxy \tag{161}$$

$$\frac{\partial^2 \phi}{\partial^2 x} = \sigma_{yy} = 0 \tag{162}$$

$$\frac{\partial^2 \phi}{\partial^3 y} = \tau_{xy} = \; - b - 3dy^2 \tag{163}$$

Evaluating the stess vector at the top and bottom of the beam.

$$\vec{\sigma}_{top} = \tau_{xy}\hat{e}_x + \sigma_{yy}\hat{e}_y \tag{164}$$

$$\vec{\sigma}_{bottom} = -\tau_{xy}\hat{e}_x - \sigma_{yy}\hat{e}_y \tag{165}$$

from these two equations and the fact that upper and lower parts of the beam are stress free, it follows that

$$\tau_{xy_{top}} = -b - \frac{3}{4}dh^2 = 0 \tag{166}$$

$$\sigma_{yy_{top}} = 0 \tag{167}$$

$$\tau_{xy_{bottom}} = b + \frac{3}{4}dh^2 = 0 \tag{168}$$

$$\sigma_{yy_{bottom}} = 0 \tag{169}$$

solving for b in terms of d

$$b = -\frac{3dh^2}{4} \tag{170}$$

Evaluating the stress vector at the free end of the beam

$$\vec{\sigma}_{free\,end} = \sigma_{xx}\hat{e}_x - \tau_{xy}\hat{e}_y \tag{171}$$

Evaluating the stress vector at x=0

$$\sigma_{free\,end} = (b + 3dy^2)\hat{e}_y \tag{172}$$

therefore

$$t\int_{-\frac{h}{2}}^{\frac{h}{2}} (b + 3dy^2)dy = -F \tag{173}$$

$$t(\frac{bh}{2} + \frac{3dh^3}{24} + \frac{bh}{2} + \frac{3dh^3}{24}) = -F \tag{174}$$

$$t(bh + \frac{dh^3}{4}) = -F \tag{175}$$

subbing in equation (170) and evaluating the constants b and d

$$t(-\frac{3dh^3}{4} + \frac{dh^3}{4}) = -F \tag{176}$$

$$\frac{tdh^3}{2} = F \tag{177}$$

$$d = \frac{2F}{th^3} \tag{178}$$

$$b = -(\frac{3h^2}{4})(\frac{2F}{th^3}) \tag{179}$$

$$b = -\frac{3F}{2th} \tag{180}$$

Using the equation for the axial moment of inertia of the cross section

$$I = \frac{th^3}{12} \tag{181}$$

and the solved constants to evaluate the stresses

$$\sigma_{xx} = 6dxy \tag{182}$$

$$\sigma_{xx} = \frac{12F}{th^3}xy \tag{183}$$

$$\sigma_{xx} = \frac{F}{I}xy \tag{184}$$

$$\tau_{xy} = -b - 3dy^2 \tag{185}$$

$$\tau_{xy} = \frac{3F}{2th} - \frac{6Fy^2}{th^3} \tag{186}$$

$$\tau_{xy} = \frac{F}{2}(\frac{3}{th} - \frac{12y^2}{th^3}) \tag{187}$$

$$\tau_{xy} = \frac{F}{2I}(\frac{h^2}{4} - y^2) \tag{188}$$

$$\sigma_{yy} = 0 \tag{189}$$

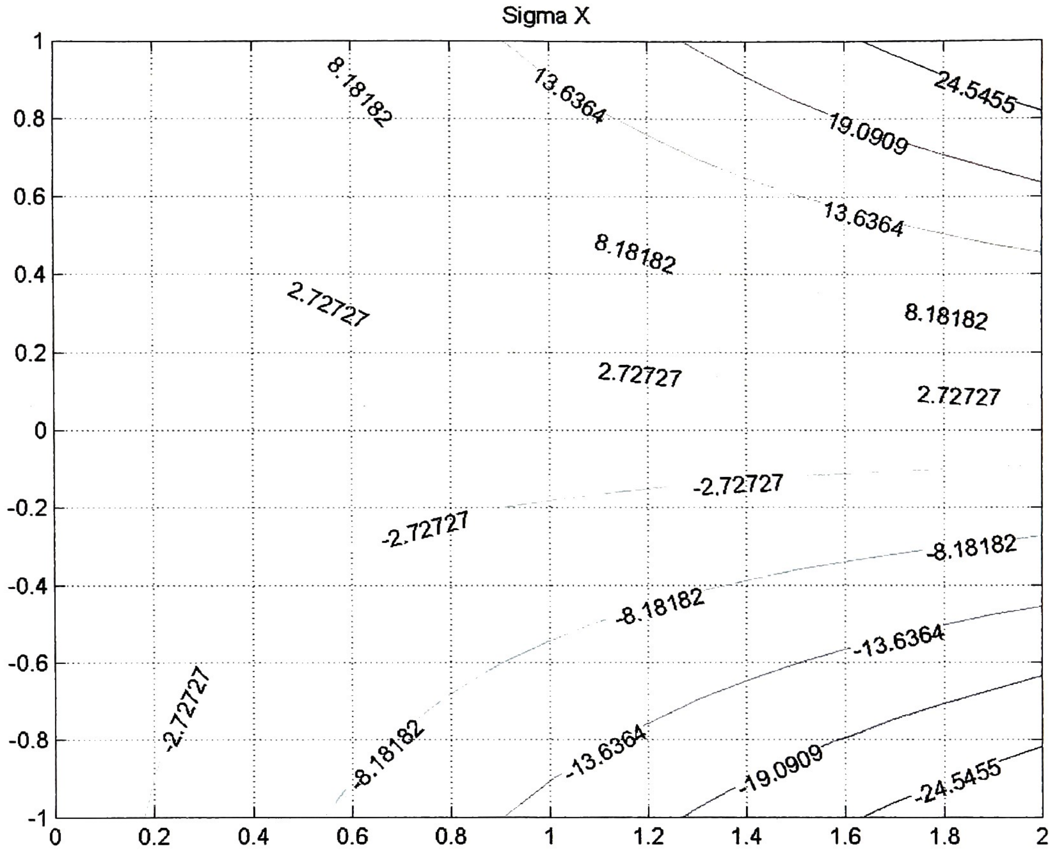


Figure 13: Stress Plot of σ_{zz} for Cantilever Beam Loaded at End with F=10, t=1, h=2, L=2

Examining the contour plot of σ_{xx}, figure 13, it makes sense that the stresses are symmetric about the line y=0, which is the centroidal axis of a square beam. It also makes sense that the beam is in tension above the line y=0 and in compression below the line y=0 as indicated on the contour plot where positive numbers indicate tension and negative numbers indicate compression.

It also makes sense that σ_{xx} increases as you increase your moment arm and get closer to the end of the beam, as well as when you move further from the centroidal axis. Which is reflected in our σ_{xx} equation as well as our contour plots.

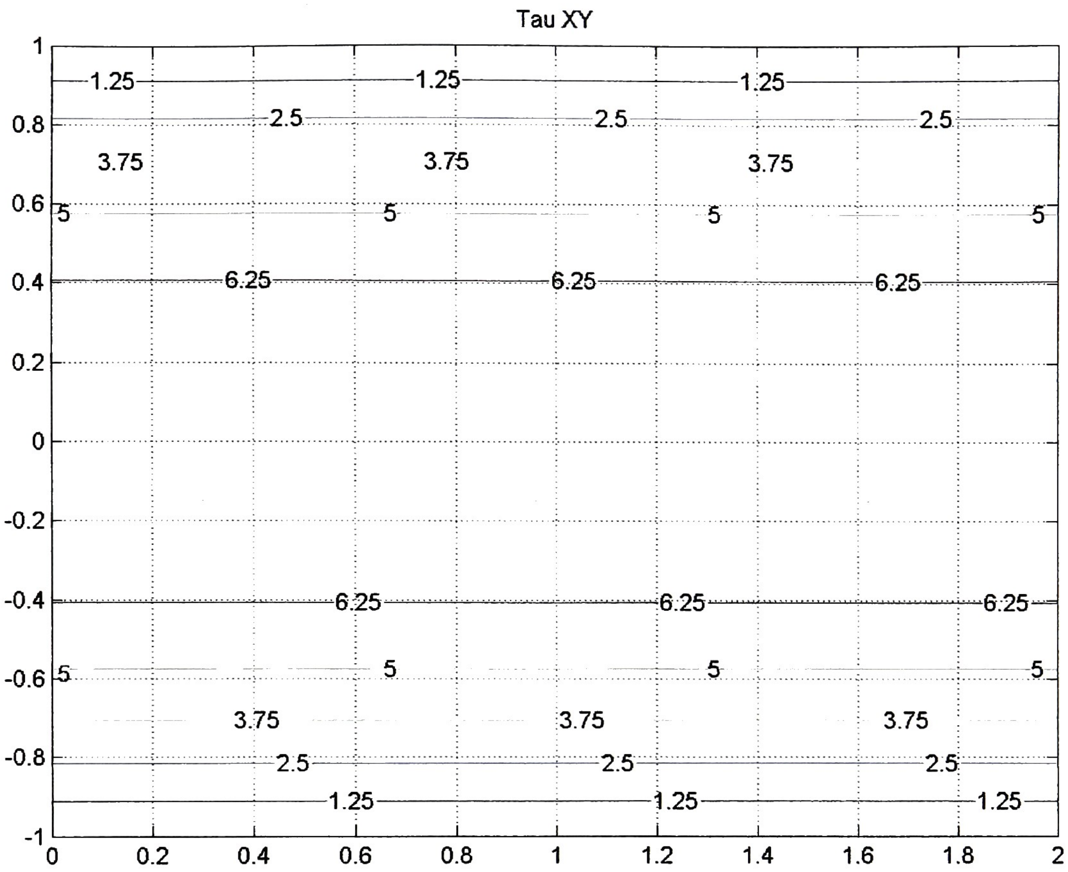


Figure14: Stress Plot of τ_{xy} for Cantilever Beam Loaded at End with F=10, t=1, h=2, L=2

The contour plot of τ_{xy}, figure 14, also makes sense since it is constant as you move across the beam in the x direction, and dependent only on the distance y away from the centroidal axis y=0. The shear stress decreases to zero at the top and bottom of the beam which fits our initial conditions where$\tau_{xy} = 0$ at y=+/-$\frac{h}{2}$ The maximum value for shear stress occurs at the centroidal axis y=0 and is equal to $\frac{3F}{2th}$

.

Curved Beam Under Action of Couples [7]

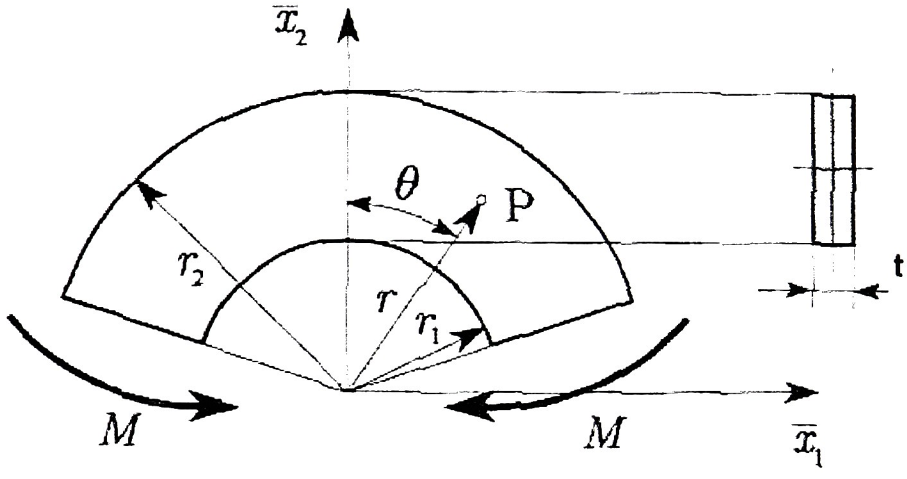


Figure 15: Curved Beam Under the Action of Couples [7]

Using the following stress function

$$\phi = c_1 ln(r) + c_2 r^2 + c_3 r^2 ln(r) + c_4 \tag{190}$$

and using axial symmetry we have

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} \tag{191}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \tag{192}$$

$$\sigma_{r\theta} = 0 \tag{193}$$

using the axial symmetry and our stress function ϕ, the stress components become

$$\sigma_{rr} = \frac{c_1}{r^2} + 2c_2 + 2c_3 ln(r) + c_3 \tag{194}$$

$$\sigma_{\theta\theta} = -\frac{c_1}{r^2} + 2c_2 + 2c_3 ln(r) + 3c_3 \tag{195}$$

Applying the following boundary conditions

$$\sigma_{rr} = \sigma_{r\theta} = 0 \text{ at } r=r_1 \tag{196}$$

$$\sigma_{rr} = \sigma_{r\theta} = 0 \text{ at } r=r_2 \tag{197}$$

$$t\int_{r_1}^{r_2} \sigma_{\theta\theta} dr = 0 \tag{198}$$

$$t\int_{r_1}^{r_2}\sigma_{\theta\theta}rdr = M \tag{199}$$

Applying boundary conditions (196) and (197) to the stress component equations (194) and (195) we get

$$\frac{c_1}{r_1^2} + 2c_2 + 2c_3 ln(r_1) + c_3 = 0 \tag{200}$$

$$\frac{c_1}{r_2^2} + 2c_2 + 2c_3 ln(r_2) + c_3 = 0 \tag{201}$$

Using equation (195) in boundary condition (198) and dividing the t out since it is equal to 0

$$\int_{r_1}^{r_2}\sigma_{\theta\theta}dr = [c_3(1+2ln(r_2)) + 2c_2 + \frac{c_1}{r_2^2}]r_2 - [c_3(1+2ln(r_1)) + 2c_2 + \frac{c_1}{r_1^2}]r_1 = 0 \tag{202}$$

Similarly, applying equation (195) to boundary condition (199)

$$\int_{r_1}^{r_2}\sigma_{\theta\theta}rdr = \ -c_1 ln(\frac{r_2}{r_1}) + c_2(r_2^2 - r_1^2) + c_3[r_2^2(1+ln(r_2) - r_1^2(1+ln(r_1)] = \frac{M}{t} \tag{203}$$

Comparing equations, we find that when (200) and (201) are satisfied, equation (202) is also satisfied. Carrying out the algebra and solving for the constants

$$c_1 = \frac{4M}{K} r_1^2 r_2^2 ln(\frac{r_2}{r_1}) \tag{204}$$

$$c_2 = \ -\frac{M}{K}[r_2^2 - r_1^2 + 2(r_2^2 ln(r_2) - r_1^2 ln(r_1))] \tag{205}$$

$$c_3 = \frac{2M}{K}(r_2^2 - r_1^2) \tag{206}$$

where

$$K = [(r_2^2 - r_1^2)^2 - 4r_1^2 r_2^2 (ln(\frac{r_2}{r_1}))^2]t \tag{207}$$

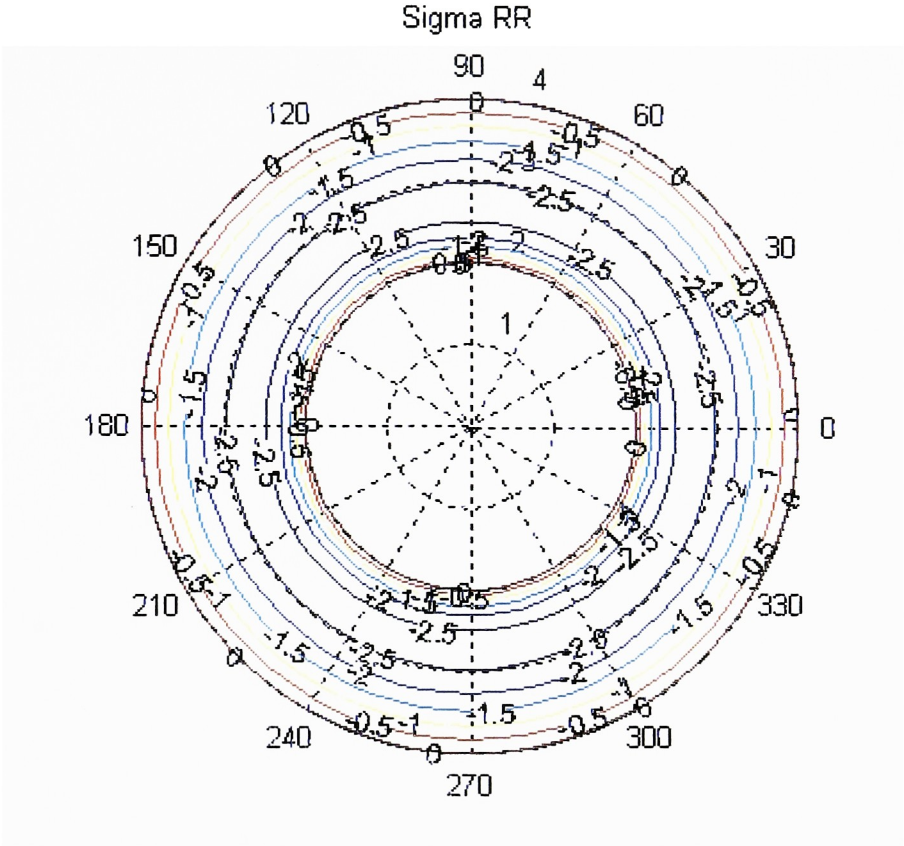


Figure 16: Stress Plot of σ_{rr} for Curved Beam Under Action of Couples with r1=2, r2=4, t=1, M=10

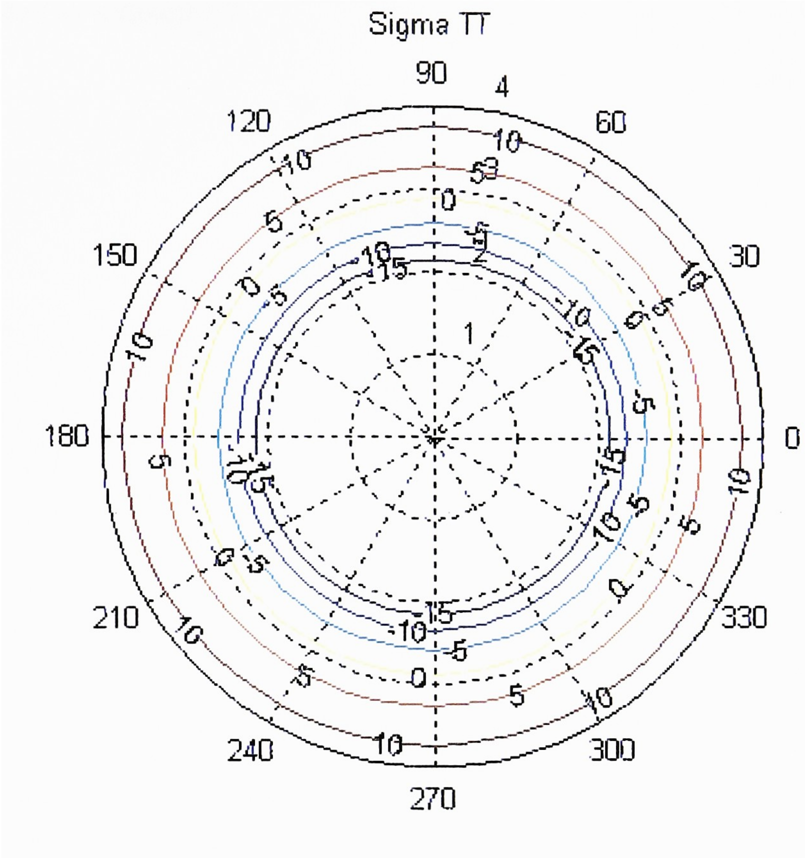


Figure 17: Stress Plot of $\sigma_{\theta\theta}$ for Curved Beam Under Action of Coupleswith r1=2, r2=4, t=1, M=10

This makes sense since as shown in figure 16 our σ_{rr} goes from zero at the inner radius to zero at the outer radius, coinciding with our boundary conditions. Our contour plot of $\sigma_{\theta\theta}$, figure 17, also makes sense since, similar to a beam in bending, portions towards the inside radius are in compression and portions towards the outer radius are in tension. The dividing radius between tension and compression is exactly like the centroidal axis of a beam in bending.

Elastic Disc Loaded by a Couple [7]

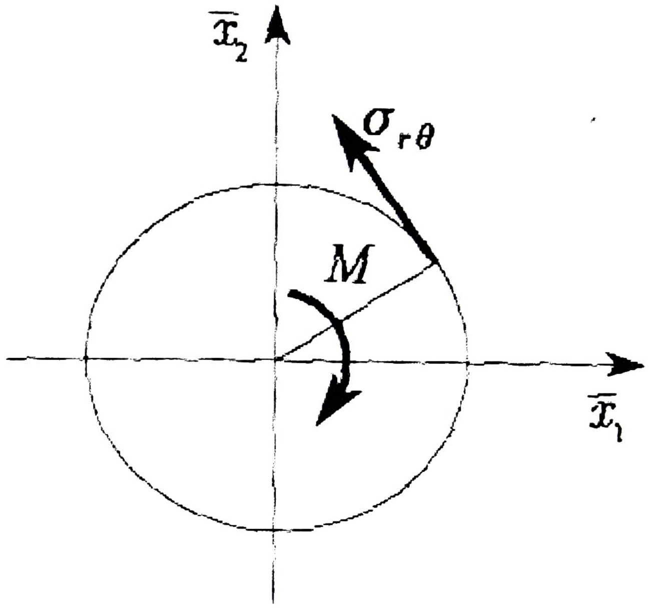


Figure 18: Elastic Disc Loaded By Couple [7]

The tangential stress shown in figure 18 is applied at the outer edge, so that

$$\sigma_{rr} = 0 \tag{208}$$

Assuming a stress fuction

$$\phi = c_1\theta \tag{209}$$

and letting $t =$ thickness, we get a stress field of

$$\sigma_{rr} = 0 \tag{210}$$

$$\sigma_{\theta\theta} = 0 \tag{211}$$

$$\sigma_{r\theta} = \frac{c_1}{r^2} \tag{212}$$

In order to determine the constant, $c_1$we use the global equilibrium condition

$$M = [\sigma_{r\theta}2\pi r]rt = 2\pi rc_1 \tag{213}$$

in order to get

$$c_1 = \frac{M}{2\pi t} \tag{214}$$

giving a stress field of

$$\sigma_{rr} = 0 \tag{215}$$

$$\sigma_{\theta\theta} = 0 \tag{216}$$

$$\sigma_{r\theta} = \frac{M}{2\pi t r^2} \tag{217}$$

Taking r from .05 to .2, M=5 and t=1 we get figure 19

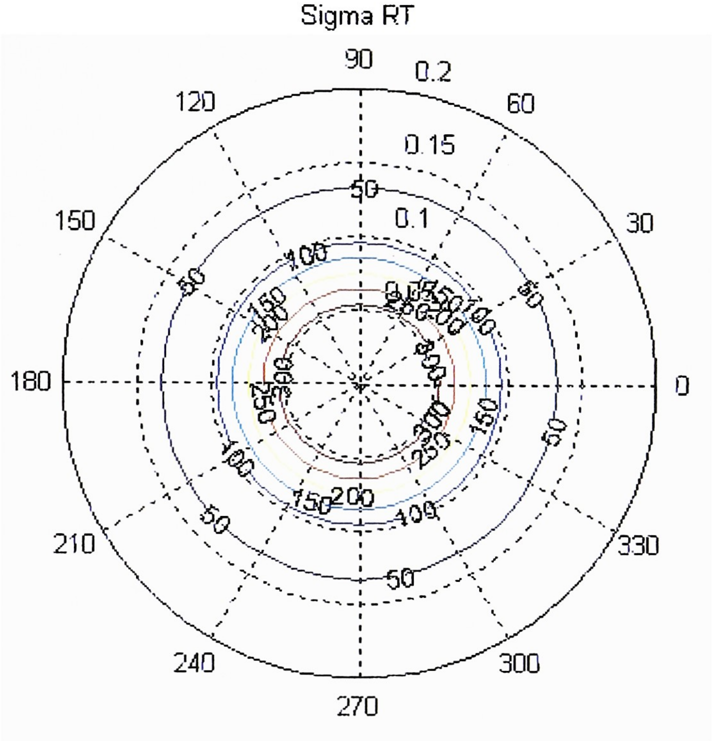


Figure 19: Stress Plot of $\sigma_{r\theta}$ for Elastic Disc Loaded by Couples

When generating the countour plot for $\sigma_{r\theta}$, figure 19, the reason for choosing such small values for r is the fact that we have an r^2in the denominator of our $\sigma_{r\theta}$ equation which makes $\sigma_{r\theta}$ trend to zero really fast as r gets larger. This makes it hard to get a good feel for our contour plot at larger values of r. Examining smaller values of r we can in fact see the trend towards zero as r gets larger.

Simply Supported Uniformly Loaded Beam [4]

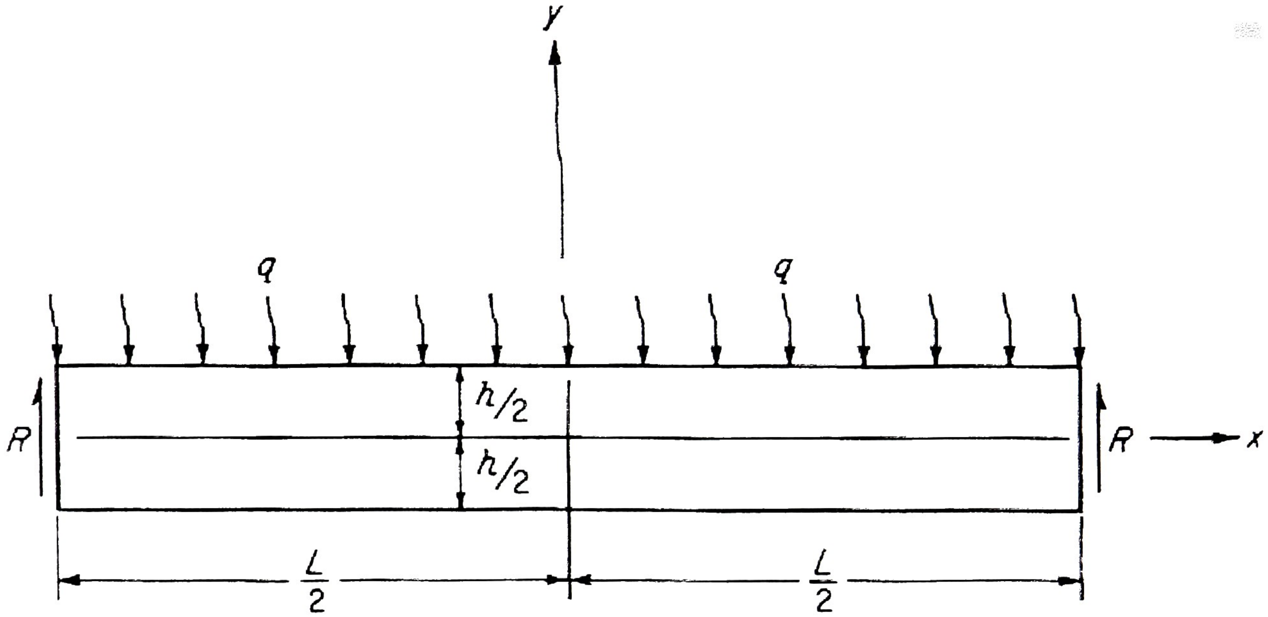


Figure 20: Simply Supported Uniformly Loaded Beam [4]

Employing the following boundary conditions

at $y = \frac{h}{2}$ we have

$$\sigma_{yy} = -q \tag{218}$$

$$\tau_{xy} = 0 \tag{219}$$

at $y = -\frac{h}{2}$ we have

$$\sigma_{yy} = 0 \tag{220}$$

$$\tau_{xy} = 0 \tag{221}$$

at $x = \pm\frac{L}{2}$ we have

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} dy = R = \frac{qL}{2} \tag{222}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} dy = 0 \tag{223}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} y dy = 0 \tag{224}$$

Since the bending moment is maximum at $x = 0$ and decreases with change in the positive or negative x-directions. This can only be possible with a stress function that contains even functions of x

Another key thing to note is that σ_{yy} varies from zero at $y = -\frac{h}{2}$ to a maximum value of $-q$ at $y = \frac{h}{2}$ which means that the stress function must contain odd functions of y .

In order to satisfy both of these conditions on our stress function, we try a polynomial function that is even in x and odd in y.

$$\Phi = ax^2 + bx^2y + cy^3 + dx^4 + ex^4y + fxy^3 + gy^5 \tag{225}$$

Since we are ignoring body forces on the beam, the stress function must also satisfy $\nabla^4\Phi = 0$, which helps us reduce the constants to

$$d = 0 \tag{226}$$

$$g = -\frac{(e+f)}{5} \tag{227}$$

Evaluating the cartesian stress components

$$\sigma_{xx} = 6cy + 6fx^2y + 20gy^3 \tag{228}$$

$$\sigma_{yy} = 2a + 2by + 12dx^2 + 12ex^2y + 2fy^3 \tag{229}$$

$$\tau_{xy} = -2bx - 4ex^3 - 6fxy^2 \tag{230}$$

plugging in d and g

$$\sigma_{xx} = 6cy + 6fx^2y - 4(e+f)y^3 \tag{231}$$

$$\sigma_{yy} = 2a + 2by + 12ex^2y + 2fy^3 \tag{232}$$

$$\tau_{xy} = -2bx - 4ex^3 - 6fxy^2 \tag{233}$$

Examination of boundary conditions (218--221) shows that σ_{yy} must be independent of x since it is consistant valued as you move in the x-direction. This means $e = 0$, which gives

$$\sigma_{xx} = 6cy + 6fx^2y - 4fy^3 \tag{234}$$

$$\sigma_{yy} = 2a + 2by + 2fy^3 \tag{235}$$

$$\tau_{xy} = -2bx - 6fxy^2 \tag{236}$$

Evaluating boundary condition (220) and subbing in our condition on e

$$0 = 2a + 2b(-\frac{h}{2}) + 2f(-\frac{h}{2})^3 \tag{237}$$

$$0 = a - \frac{bh}{2} - \frac{fh^3}{8} \tag{238}$$

Evaluating boundary condition (218)

$$-q = 2a + 2b(\frac{h}{2}) + 2f(\frac{h}{2})^3 \tag{239}$$

$$-\frac{q}{2} = a + \frac{bh}{2} + \frac{fh^3}{8} \tag{240}$$

adding equations (238) and (240) together and evaluating for a

$$a = -\frac{q}{4} \tag{241}$$

From boundary condtions (219) and (221) we get

$$0 = -2x(b + 3f(\pm\frac{h}{2})^2) \tag{242}$$

either $-2x$ or $(b + 3f(\pm\frac{h}{2})^2)$ must equal zero, so

$$b = -\frac{3}{4}fh^2 \tag{243}$$

subbing equations (241) and (243) into equation (238)

$$-\frac{q}{4} - (\frac{h}{2})(-\frac{3}{4})fh^2 - \frac{fh^3}{8} = 0 \tag{244}$$

$$-\frac{q}{4} = -\frac{3fh^3}{8} + \frac{fh^3}{8} \tag{245}$$

$$f = \frac{q}{h^3} \tag{246}$$

using f in equation (243) to evaluate b

$$b = -\frac{3q}{4h} \tag{247}$$

With knowing the values of a, b, and c one can check to see that equations (222) and (223) are satisfied. We can use equation (224) to solve for the remaining constant c

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} (6cy^2 + 6(\frac{q}{h^3})(\frac{L}{2})^2y^2 - 4(\frac{q}{h^3})y^4)dy = 0 \tag{248}$$

$$(2cy^3 + (\frac{1}{2})(\frac{q}{h^3})(L^2)y^3 - (\frac{4}{5})(\frac{q}{h^3})y^5)\Big|_{\frac{-h}{2}}^{\frac{h}{2}} = 0 \tag{249}$$

solving equation (249) gives

$$\frac{ch^3}{2} + \frac{qL^2}{8} - \frac{qh^2}{20} = 0 \tag{250}$$

using the moment of inertia of a unit-width beam

$$I = \frac{h^3}{12} \tag{251}$$

to get

$$6cI = \frac{qh^2}{20} - \frac{qL^2}{8} \tag{252}$$

$$c = \frac{q}{6I}(\frac{h^2}{20} - \frac{L^2}{8}) \tag{253}$$

$$c = \frac{q}{240I}(2h^2 - 5L^2) \tag{254}$$

Now that we have all of our constants evaluated in terms of known conditions, we can sub them into equations (234--236), and rearrange terms to get our final equations for the cartesian components of stress

$$\sigma_{xx} = \frac{q}{8I}(4x^2 - L^2)y + \frac{q}{60I}(3h^2y - 20y^3) \tag{255}$$

$$\sigma_{yy} = \frac{q}{24I}(-h^3 + 4y^3 - 3h^2y) \tag{256}$$

$$\tau_{xy} = \frac{qx}{8I}(h^2 - 4y^2) \tag{257}$$

The conventional strength of materials solution for this problem, namely, that $\sigma_{xx} = My/I$, gives

$$\sigma_{xx} = \frac{q}{8I}(4x^2 - L^2)y \tag{258}$$

which is identical to the first term in equation (255). The second term in equation (255) is a correction term for the strength of materials solution. In the strength of materials solution it is assumed that plane sections remain plane sections after bending. This is not exactly true, and as a consequence the solution obtained lacks the correction term shown in our solution. It is clear that the correction term is small when L>>h and the strength of materials solution will be sufficiently accurate.

The countour plot of σ_{xx}, figure 21, shows how the upper portion of the beam is in compression with negative numbers, and the lower portion of the beam is in tension with posotive numbers. It also shows how the values of stress are symmetric about the centroidal axis y=0, where σ_{xx} is equal to zero. Since the beam is uniformly loaded, the stresses are also symmetric about the line x=0. It can also be seen that increasing the moment arm taken about the end of the beam by moving towards the center of the beam means that to achieve the same amount of stress you need not stray as far from the centroidal axis y=0.

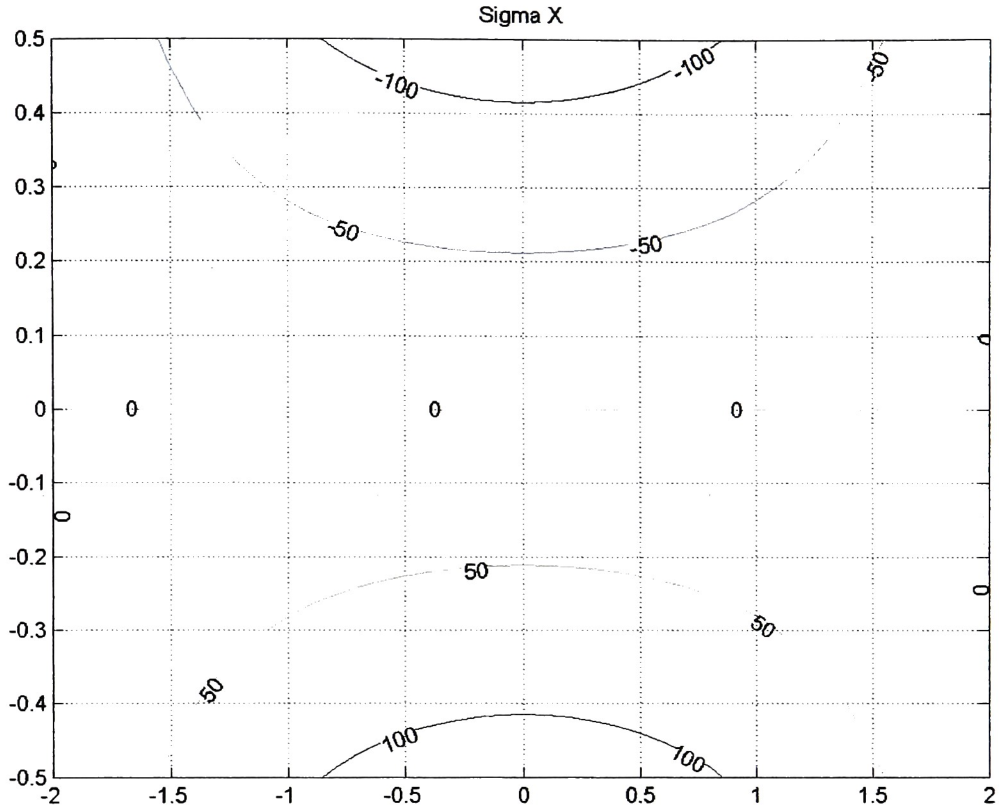


Figure 21: Stress Plot of σ_{xx} for Simply Supported Uniformly Loaded Beam

The contour plot of σ_{yy}, figure 22, shows how σ_{yy} goes from zero to at -h/2 to Q at h/2 for a beam of uniform thickness equal to one. This agrees completely with the boundary conditions of our problem.

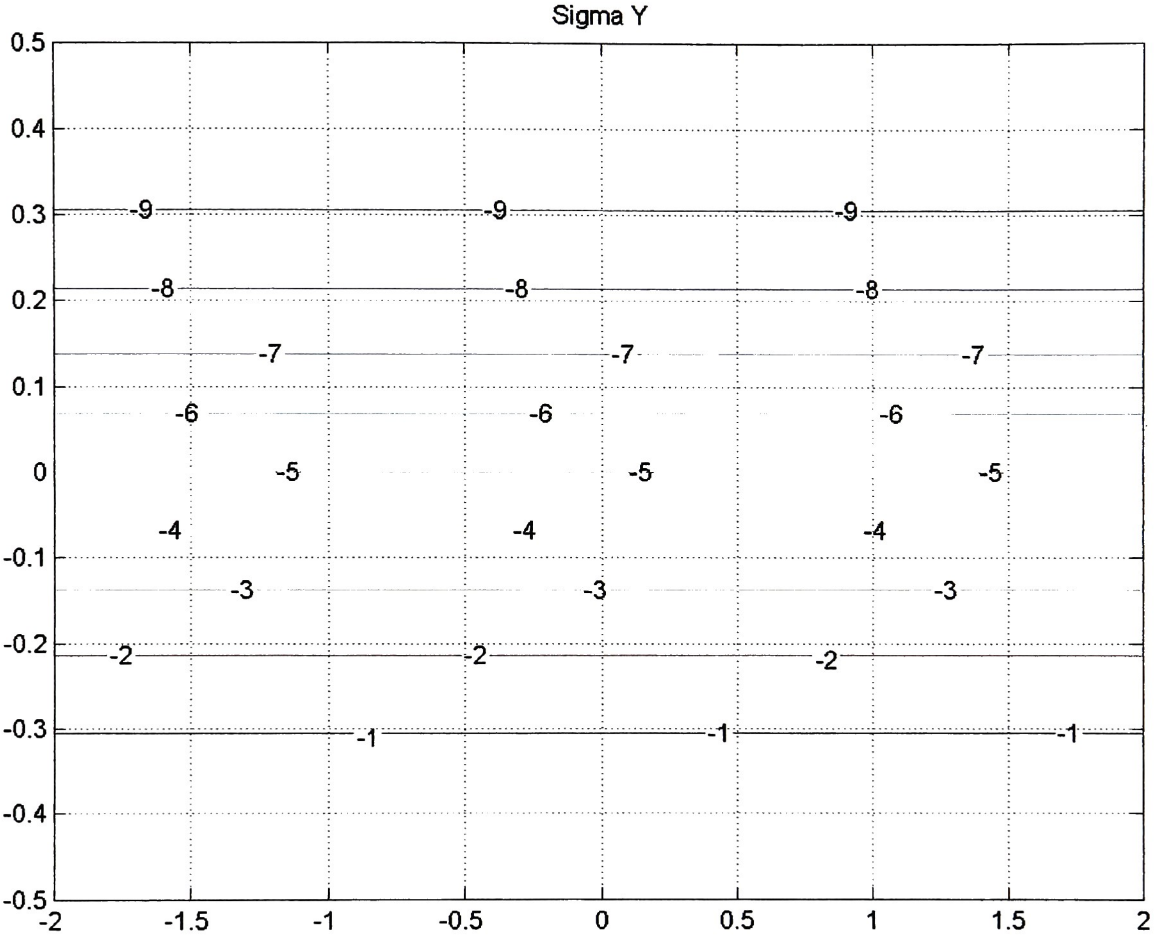


Figure 22: Stress Plot of σ_{yy} for Simply Supported Uniformly Loaded Beam

The contour plot of τ_{xy}, figure 23, shows how the shear stress is symmetric about the centroidal axis y=0 It also shows it is symmetric in magnitude about the line x=0 varying only between posotive and negative shear stress. Since all of our y terms in our τ_{xy} equation are squared and the largest the y value can be is +/-$\frac{h}{2}$, that means that$h^2 - 4y^2$ will always be posotive and our x values determine whether we have posotive or negative shear stress. Examining our equation, we come to the conclusion that values of x<0 indicate negative shear stress, and values of x>0 indicate posotive values of shear stress.

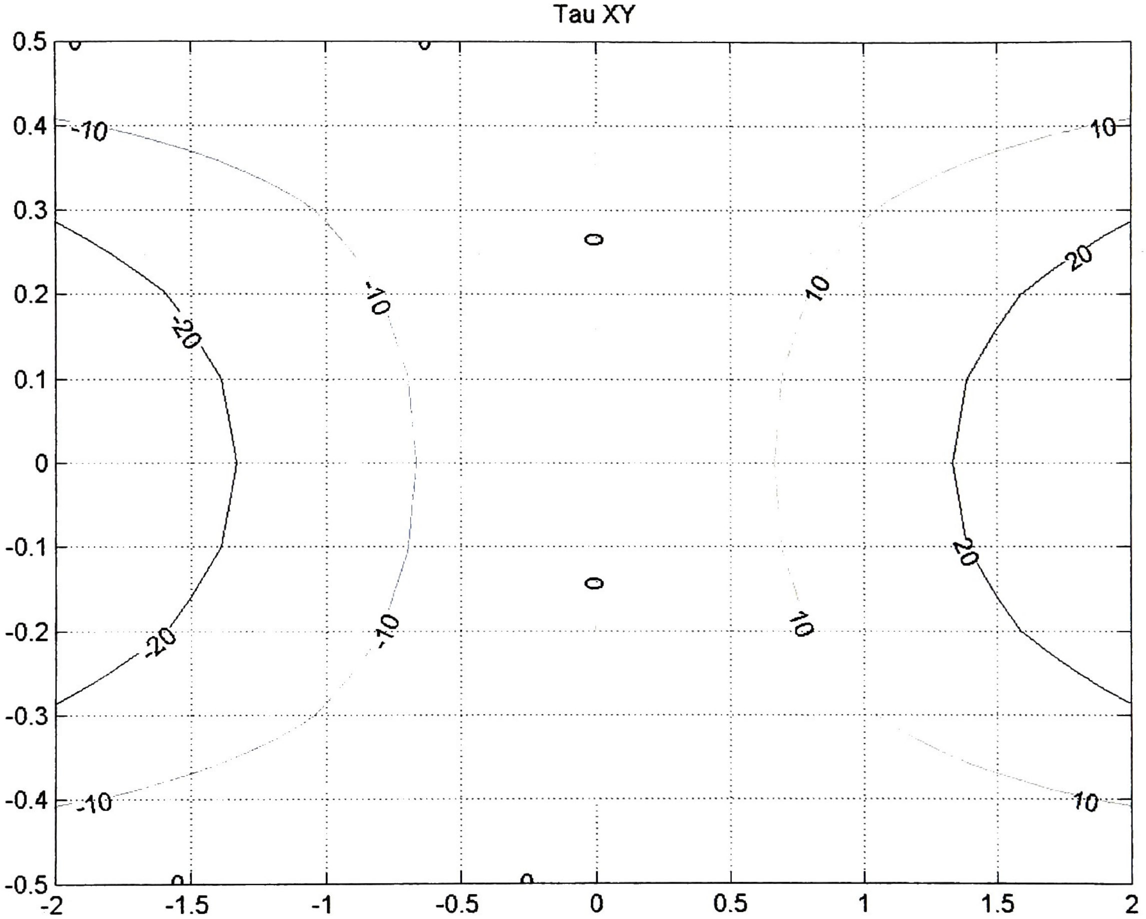


Figure 23: Stress Plot of τ_{xy} for Simply Supported Uniformly Loaded Beam

Infinite Plate with Circular Hole Subjected to Uniaxial Tension [4]

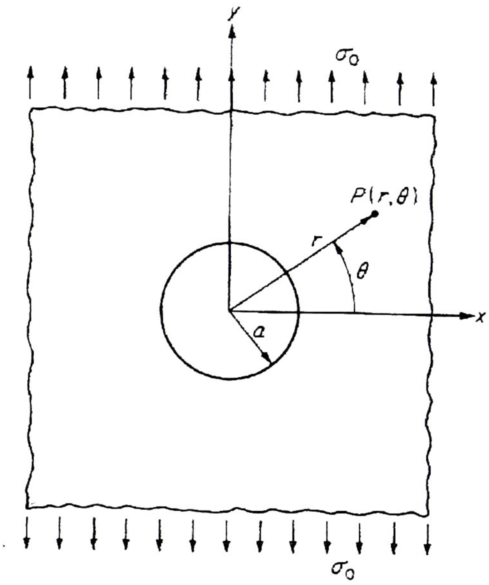


Figure 24: Infinite Plate With Circular Hole Subjected to Uniaxial Tension [4]

at $r = a$ we have

$$\sigma_{rr} = \tau_{r\theta} = 0 \tag{259}$$

at $r = \infty$

$$\sigma_{yy} = \sigma_o \tag{260}$$

$$\sigma_{xx} = \tau_{xy} = 0 \tag{261}$$

In order to solve this problem, we must use the linear superposition of two stress functions. The first function must be chosen in order to satisfy the boundary conditions at $r = \infty$. The second function must be chosen to have an associated stress that cancels the stress on the boundary of the hole without influencing the stress at $r = \infty$.

For the uniaxial tension we choose a second degree polynomial of

$$\Phi = ax^2 + bxy + cy^2 \tag{262}$$

and since we only want tension in the y direction we apply boundary condition (260)

$$\Phi = ax^2 = \frac{\sigma_o x^2}{2} \tag{263}$$

this stress function satisfies the conditions of a plate without a hole in it, since the stresses in a plate without a hole are given by equations (259) and (261) .

If a hole of radius a is cut into the plate, the stresses σ_{rr}, $\sigma_{r\theta}$, and $\tau_{r\theta}$ on the boundary of the imaginary hole can be computed from the following equations

$$\sigma^I_{rr} = \sigma_o \sin^2(\theta) = \frac{\sigma_o}{2}(1\text{-}\cos(2\theta)) \tag{264}$$

$$\sigma^I_{\theta\theta} = \sigma_o \cos^2(\theta) = \frac{\sigma_o}{2}(1 + \cos(2\theta)) \tag{265}$$

$$\tau^I_{r\theta} = \sigma_o \cos(\theta)\sin(\theta) = \frac{\sigma_o}{2}\sin(2\theta) \tag{266}$$

since in the origonal problem, the boundary condtions at $r = a$ are given by equation (259), the boundary condtions to be satisfied by the stresses associated with the second stress function are

at $r = a$

$$\sigma_{rr} = -\sigma_o \sin^2(\theta) = -\frac{\sigma_o}{2}(1\text{-}\cos(2\theta)) \tag{267}$$

$$\tau_{r\theta} = -\sigma_o \cos(\theta)\sin(\theta) = -\frac{\sigma_o}{2}\sin(2\theta) \tag{268}$$

at $r = \infty$

$$\sigma_{rr} = \sigma_{\theta\theta} = \tau_{r\theta} = 0 \tag{269}$$

From equations (267) through (269) it becomes apparent that the stresses $\sigma_{rr}$and $\tau_{r\theta}$ are functions of $\sin(2\theta)$ and $\cos(2\theta)$, suggesting the following

$$\phi^{(n)} = (a_n r^n + b_n r^{-n} + c_n r^{2+n} + d_n r^{2-n})cos(n\theta) \ with \ n \geq 2 \tag{270}$$

as a possible stress function since if we set n=2 it will yield stresses in termes of $\cos(2\theta)$ and $\sin(2\theta)$ as follows

$$\sigma_{rr} = (a_2(-2) - b_2(6)r^{-4} + d_2(-4)r^{-2})cos(2\theta) \tag{271}$$

$$\tau_{r\theta} = (a_2(2) - b_2(6)r^{-4} + c_2(6)r^2 + d_2(-2)r^{-2})sin(2\theta) \tag{272}$$

$$\sigma_{\theta\theta} = (a_2(2) + b_2(6)r^{-4} + c_2(10)r^2)cos(2\theta) \tag{273}$$

Further inspection of the stress function $\phi^{(2)}$ shows that it can only satisfy the boundary conditions at $\tau_{r\theta}$ due to the fact that all the terms in σ_{rr} are multiplied by $\cos(2\theta)$ making it impossible to get the $-\frac{\sigma_0}{2}$ portion of the boundary condition [see eq. (267)]. In order to satisfy the boundary conditions on σ_{rr} we need a stress function which when added to the first function $\phi^{(2)}$ will not influence $\tau_{r\theta}$ while at the same time allow us to satisfy the boundary conditions of σ_{rr}.Looking at the stress function $\phi^{(0)}$

$$\phi^{(0)} = a_0 + b_0 ln(r) + c_0 r^2 + d_0 r^2 ln(r) \tag{274}$$

which has the following stresses

$$\sigma_{rr} = \frac{b_0}{r^2} + 2c_0 + d_0(1 + 2ln(r)) \tag{275}$$

$$\sigma_{\theta\theta} = -\frac{b_0}{r^2} + 2c_0 + d_0(3 + 2ln(r)) \tag{276}$$

$$\tau_{r\theta} = 0 \tag{277}$$

which will have no effect on $\tau_{r\theta}$ while at the same time help us satisfy the boundary conditions for σ_{rr}. We must now test the stress function $\phi^{(0)} + \phi^{(2)}$ to see if it is applicable.

$$\phi^{(0)} + \phi^{(2)} = a_0 + b_0 ln(r) + c_0 r^2 + d_0 r^2 ln(r)$$

$$(a_2 r^2 + b_2 r^{-2} + c_2 r^4 + d_2) cos(2\theta) \tag{278}$$

$$\sigma_{rr} = \frac{b_0}{r^2} + 2c_0 + d_0(1 + 2ln(r))$$

$$-(2a_2 + 6b_2 r^{-4} + 4d_2 r^{-2}) cos(2\theta) \tag{279}$$

$$\sigma_{\theta\theta} = -\frac{b_0}{r^2} + 2c_0 + d_0(3 + 2ln(r))$$

$$(2a_2 + 6b_2 r^{-4} + 10c_2 r^2) cos(2\theta) \tag{280}$$

$$\tau_{r\theta} = (2a_2 - 6b_2 r^{-4} + 6c_2 r^2 - 2d_2 r^{-2}) sin(2\theta) \tag{281}$$

Since $\sigma_{rr} = \sigma_{\theta\theta} = \tau_{r\theta} = 0$ as r $\rightarrow \infty$ we know

$$c_0 = d_0 = a_2 = c_2 = 0 \tag{282}$$

reducing our equations to

$$\sigma_{rr} = \frac{1}{r^2}[b_0 - (6b_2 r^{-2} + 4d_2) cos(2\theta)] \tag{283}$$

$$\sigma_{\theta\theta} = \frac{1}{r^2}[-b_0 + 6b_2 r^{-2} cos(2\theta)] \tag{284}$$

$$\tau_{r\theta} = -\frac{1}{r^2}[(6b_2 r^{-2} + 2d_2) sin(2\theta)] \tag{285}$$

From the boundary conditions at r=a we get

$$\tau_{r\theta} = -\frac{1}{a^2}[(\frac{6b_2}{a^2} + 2d_2)sin(2\theta)] = -\frac{\sigma_o}{2}sin(2\theta) \tag{286}$$

$$\sigma_{rr} = \frac{1}{a^2}[b_0 - (\frac{6b_2}{a^2} + 4d_2)cos(2\theta)] = -\frac{\sigma_o}{2}(1 - cos(2\theta)) \tag{287}$$

Evaluating these two equations to get our constant values of

$$b_o = -\frac{\sigma_o a^2}{2} \tag{288}$$

$$b_2 = \frac{\sigma_o a^4}{4} \tag{289}$$

$$d_2 = -\frac{\sigma_o a^2}{2} \tag{290}$$

subbing these values into equations (283) through (285)

$$\sigma_{rr}^{II} = -\frac{\sigma_o a^2}{2r^2}[1 + (\frac{3a^2}{r^2} - 4)cos(2\theta)] \tag{291}$$

$$\sigma_{\theta\theta}^{II} = \frac{\sigma_o a^2}{2r^2}(1 + \frac{3a^2}{r^2}cos(2\theta)) \tag{292}$$

$$\tau_{r\theta}^{II} = -\frac{\sigma_o a^2}{2r^2}[(\frac{3a^2}{r^2} - 2)sin(2\theta)] \tag{293}$$

The solution for the origonal problem is obtained by superposition as follows

$$\sigma_{rr} = \sigma_{rr}^{I} + \sigma_{rr}^{II} = \frac{\sigma_o}{2}\left\{(1 - \frac{a^2}{r^2})[1 + (\frac{3a^2}{r^2} - 1)cos(2\theta)]\right\} \tag{294}$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta}^{I} + \sigma_{\theta\theta}^{II} = \frac{\sigma_o}{2}\left\{(1 + \frac{a^2}{r^2}) + (1 + \frac{3a^2}{r^2})cos(2\theta)\right\} \tag{295}$$

$$\tau_{r\theta} = \tau_{r\theta}^{I} + \tau_{r\theta}^{II} = \frac{\sigma_o}{2}\left\{(1 + \frac{3a^2}{r^2})(1 - \frac{a^2}{r^2})sin(2\theta)\right\} \tag{296}$$

The above equations give the stress at any point in the body as defined in r and θ. In order to define the stesses along the x-axis, we can set θ=0 and r=x to get .

$$\sigma_{rr} = \sigma_{xx} = \frac{\sigma_o}{2}(1 - \frac{a^2}{x^2})\frac{3a^2}{x^2} \tag{297}$$

$$\sigma_{\theta\theta} = \sigma_{yy} = \frac{\sigma_o}{2}(2 + \frac{a^2}{x^2} + \frac{3a^2}{x^4}) \tag{298}$$

$$\tau_{r\theta} = \tau_{xy} = 0 \tag{299}$$

The distribution of σ_{xx}/σ_o and σ_{yy}/σ_o is plotted in figure 25 as a function of position along the x-axis. Examining the graph closer, it is apparent that an infinite plate under uniaxial tension with a hole in it increases the σ_{yy} stress by a factor of three. This factor is often called the stress concentration factor.

80 ELEMENTARY ELASTICITY

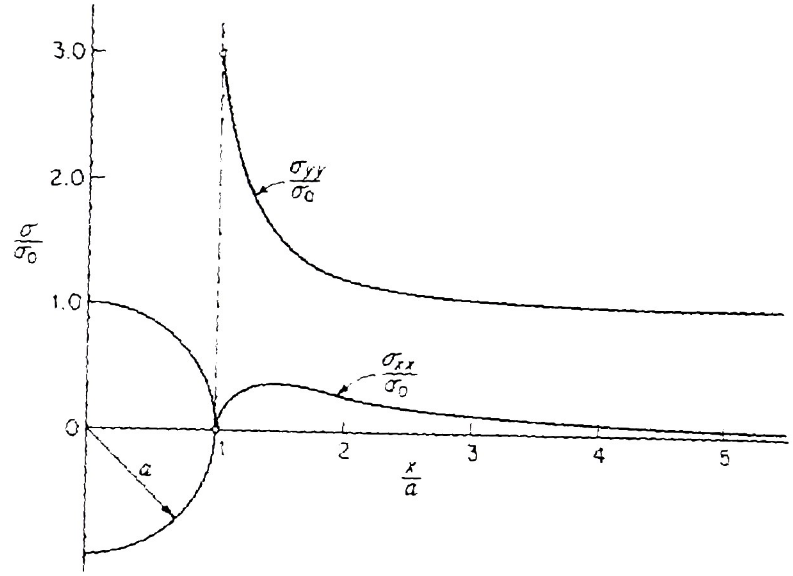


Figure 3.7 Distribution of σ_{xx}/σ_0 and σ_{yy}/σ_0 along the x axis.

Figure 25: Distribution of σ_{xx}/σ_o and σ_{yy}/σ_o along the x-axis [4]

In a similar manner, the stresses along the y axis can be determined by setting $\theta=\frac{\pi}{2}$ and r=y as follows

$$\sigma_{rr} = \sigma_{yy} = \frac{\sigma_o}{2}(2 - \frac{5a^2}{y^2} + \frac{3a^2}{y^4}) \tag{300}$$

$$\sigma_{\theta\theta} = \sigma_{xx} = \frac{\sigma_o}{2}(\frac{a^2}{y^2} - \frac{3a^2}{y^4}) \tag{301}$$

$$\tau_{r\theta} = \tau_{xy} = 0 \tag{302}$$

A distribution of σ_{xx}/σ_o and σ_{yy}/σ_o is plotted as a function of position along the y-axis in figure 26. It can be noted that σ_x/σ_o = -1 at the boundary of the hole; thus the influence of the hole not only produces a stress concentration, but in this case also produces a change in the sign of the stresses

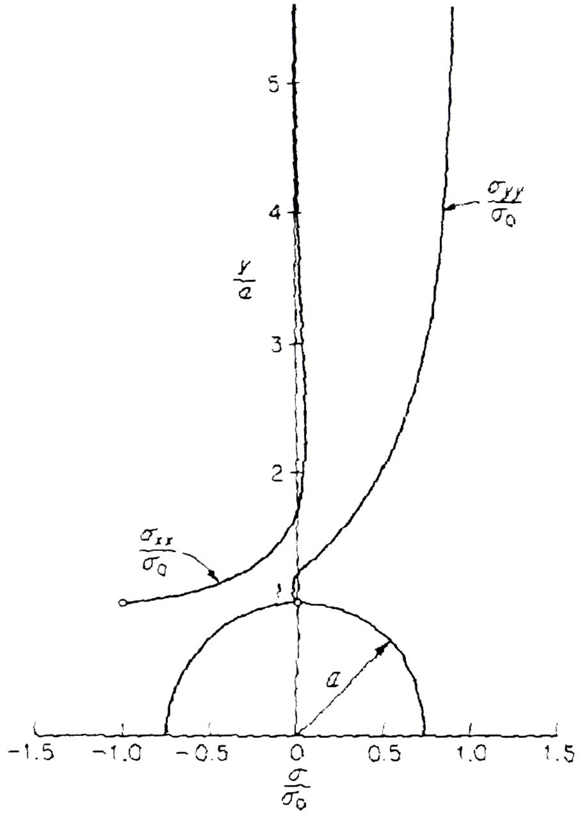


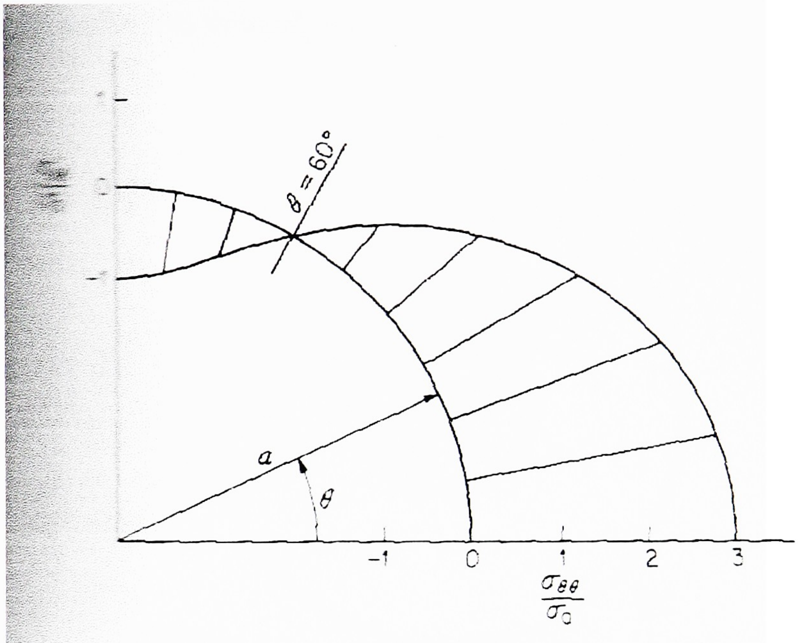
Figure 26: Distribution of σ_{xx}/σ_o and σ_{yy}/σ_o along the y-axis [4]

The distribution of $\sigma_{\theta\theta}$ about the boundary of the hole is found by setting r=a

$$\sigma_{rr} = \tau_{r\theta} = 0 \tag{303}$$

$$\sigma_{\theta\theta} = \sigma_o(1 + 2cos(2\theta)) \tag{304}$$

The distribution of $\sigma_{\theta\theta}/\sigma_o$ about the boundary of the hole is shown in figure 27. The maximum $\sigma_{\theta\theta}/\sigma_o$ occurs at the x-axis ($\sigma_{\theta\theta}/\sigma_o = 3$), and the minimum occurs at the y-axis ($\sigma_{\theta\theta}/\sigma_o =$ -1). At the point defined by θ=60°on the boundary of the hole, all stresses are zero. This type of point is commonly referred to as a singular point.



3.9 Distribution of $\sigma_{\theta\theta}/\sigma_0$ about the boundary of the hole.

Figure 27: Distribution of $\sigma_{\theta\theta}/\sigma_o$ About the Boundary of the Hole [4]

Thick Walled Cylinder With to Internal and External Pressures [5]

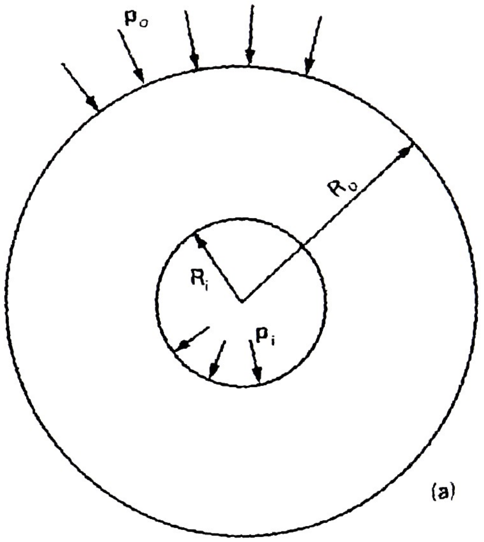


Figure 28: Thick Walled Cylinder With Internal and External Pressures [5]

A thick walled cylinder subjected to internal and external pressures is a case of axisymmetrical loading so we can use a simplified version of the comaptability equation reducing it to

$$\nabla^4\Phi = [\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}][\frac{\delta^2\phi}{\delta r^2} + \frac{1}{r}\frac{\delta\phi}{\delta r}] \tag{305}$$

which must be equal to zero since there are no body forces present. Rewriting this equation into the form

$$\frac{1}{r}\left[r\left(\frac{1}{r}[r\frac{\delta\phi}{\delta r}]\frac{\delta}{\delta r}\right)\frac{\delta}{\delta r}\right]\frac{\delta}{\delta r} \tag{306}$$

Setting this equation equal to zero and working the integration through it can be reduced to the following equation in terms of constants

$$\phi = c_1 r^2 ln(r) + c_2 r^2 + c_3 ln(r) + c_4 \tag{307}$$

Using the equations that relate ϕ to stress we get the following stresses.

$$\sigma_{rr} = c_1(1 + 2ln(r)) + 2c_2 + c_3\frac{1}{r^2} \tag{308}$$

$$\sigma_{\theta\theta} = c_1(3 + 2ln(r)) + 2c_2 - c_3\frac{1}{r^2} \tag{309}$$

$$\sigma_{r\theta} = 0 \tag{310}$$

Applying the following boundary conditions

$$\sigma_{rr}(R_o) = -p_o \tag{311}$$

$$\sigma_{rr}(R_i) = -p_i \tag{312}$$

We get two out of the three necessary equations to determine the constants

$$-p_i = c_1(1 + 2ln(\mathrm{R}_i)) + 2c_2 + c_3\frac{1}{R_i^2} \tag{313}$$

$$-p_o = c_1(1 + 2ln(\mathrm{R}_o)) + 2c_2 + c_3\frac{1}{R_o^2} \tag{314}$$

In order to get a third equation we must look at the displacement expressions for axisymmetrical problems, mainly the displacement u_θ.

$$u_\theta = \frac{1}{E}[4c_1r\theta + c_4cos(\theta) - c_5sin(\theta) + c_6r] \tag{315}$$

At first glance it seems to add more constants, but since the expression for u_θ is multi-valued in theta and our example which has $u_\theta(0) = u_\theta(2\pi) = u_\theta(4\pi)$, etc. is not multi-valued in theta, we know that c_1 must be equal to zero which leaves us with two equations and two unkowns

$$-p_i = 2c_2 + c_3\frac{1}{R_i^2} \tag{316}$$

$$-p_o = 2c_2 + c_3\frac{1}{R_o^2} \tag{317}$$

Solving these two equations we get the following

$$c_2 = \frac{p_iR_i^2 - p_oR_0^2}{2(R_0^2 - R_i^2)} \tag{318}$$

$$c_3 = \frac{R_i^2R_0^2(p_o - p_i)}{(R_0^2 - R_i^2)} \tag{319}$$

Subbing these into equations (308) and (309) gives the following stress distribitions

$$\sigma_{rr} = \frac{p_iR_i^2 - p_oR_0^2}{(R_0^2 - R_i^2)} + \frac{R_i^2R_0^2(p_o - p_i)}{r^2(R_0^2 - R_i^2)} \tag{320}$$

$$\sigma_{\theta\theta} = \frac{p_iR_i^2 - p_oR_0^2}{(R_0^2 - R_i^2)} - \frac{R_i^2R_0^2(p_o - p_i)}{r^2(R_0^2 - R_i^2)} \tag{321}$$

Case 1: External Pressure Equal to Zero. Setting p_o=0

$$\sigma_{rr} = \frac{R_i^2 p_i}{R_o^2 - R_i^2}(1 - \frac{R_o^2}{r^2}) \tag{322}$$

$$\sigma_{\theta\theta} = \frac{p_i R_i^2}{(R_0^2 - R_i^2)}(1 + \frac{R_o^2}{r^2}) \tag{323}$$

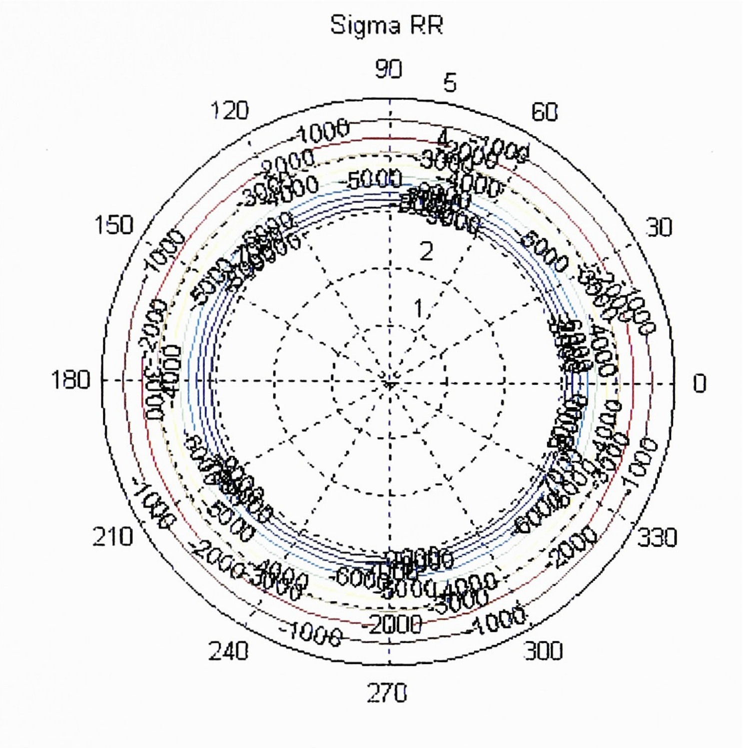


Figure 29: Stress Plot of σ_{rr} with External Pressure Equal to Zero with $R_i = 3$, $R_o = 5$, $pi = 10000$

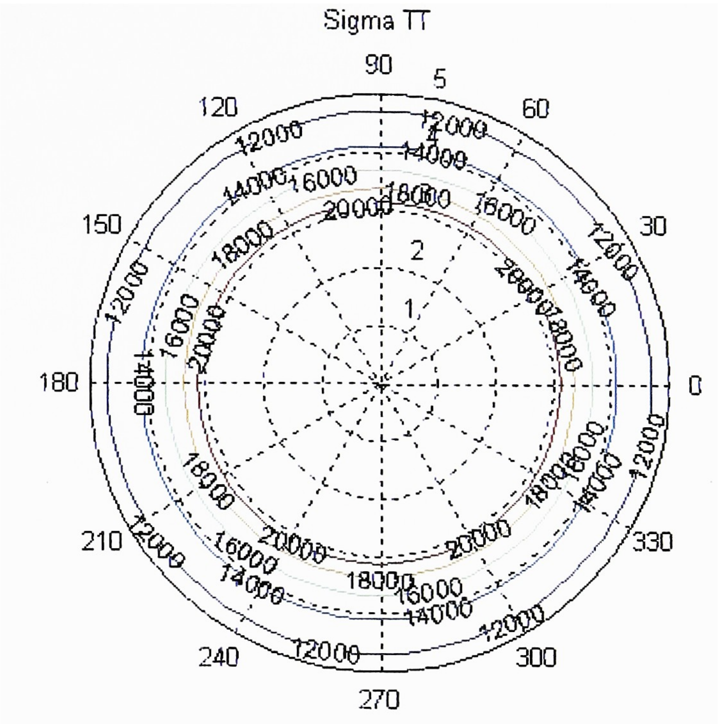


Figure 30: Stress Plot of $\sigma_{\theta\theta}$ with External Pressure Equal to Zero with $R_i = 3$, $R_o = 5$, $pi = 10000$

The contour plots of σ_{rr} and $\sigma_{\theta\theta}$ shown in figures 29 and 30 make sense, since σ_{rr}goes from -10000 at R_i to 0 at R_o per our boundary conditions and everything is in compression. $\sigma_{\theta\theta}$ also makes sense since everything is in tension as we expect an internal pressure to do as it tries to expand the cylinder. The values also go from largest at R_i to smallest at R_o as the R_o^2/r^2 in the denominator indicates.

Case2: Internal Pressure Equal to Zero. Setting p_i-0

$$\sigma_{rr} = \frac{R_o^2 p_o}{R_o^2 - R_i^2}(\frac{R_i^2}{r^2} - 1) \tag{324}$$

$$\boldsymbol{\sigma_{\theta\theta}} = -\frac{p_o R_o^2}{(R_0^2 - R_i^2)}(\frac{R_i^2}{r^2} + 1) \tag{325}$$

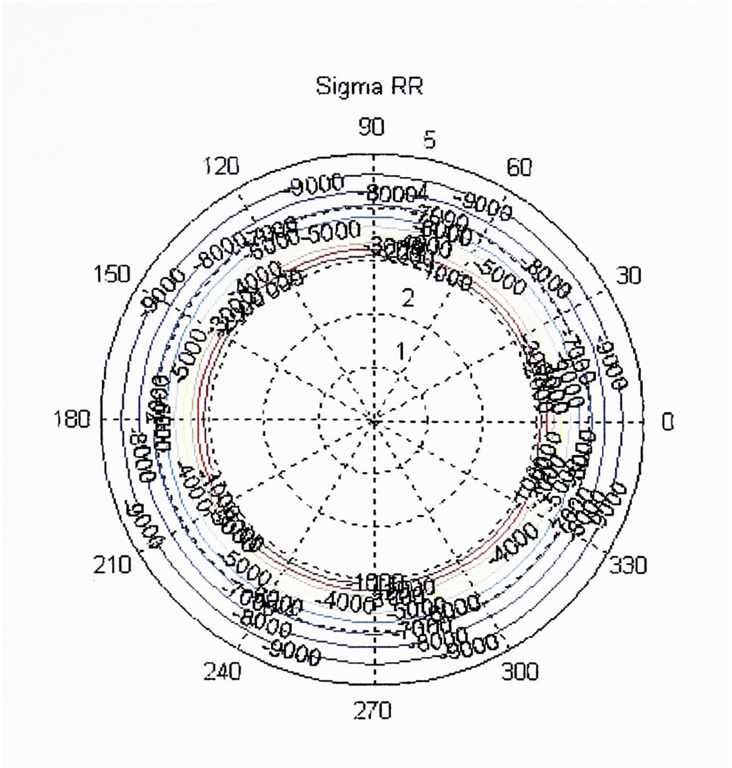


Figure 31: Stress Plot of σ_{rr} with Internal Pressure Equal to Zero with $R_i = 3$, $R_o = 5$, $po = 10000$

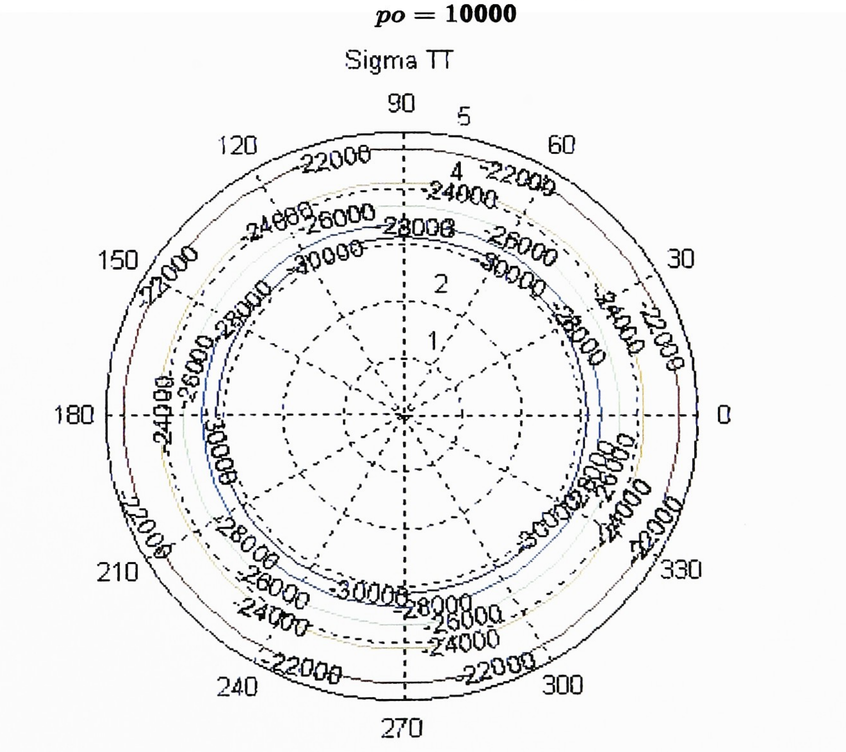


Figure 32: Stress Plot of $\sigma_{\theta\theta}$ with Internal Pressure Equal to Zero with $R_i = 3$, $R_o = 5$, $po = 10000$

Once again, our contour plot of σ_{rr}, figure 31, makes sense since it satisfies our boundary conditions, going from 10000 at R_o to zero at R_i. In the case of $\sigma_{\theta\theta}$, figure 32, we have R_i^2/r^2 this time. Since our R_i is in the numerator this time, our values go from largest at R_i to smallest at R_o and everything is in compression as we would expect an external pressure to do as it tries to smoosh the cylinder.

Case3: External Pressure On a Solid Circular Cylinder. Setting a=0

$$\sigma_{rr} = \sigma_{\theta\theta} = -p_o \tag{326}$$

Narrow, Simply Supported Beam Under Its Own Weight [5]

Looking at a simply supported beam of unit width under its own weight as shown in figure 1 and assuming that the supports run the whole depth of the beam we have the following.

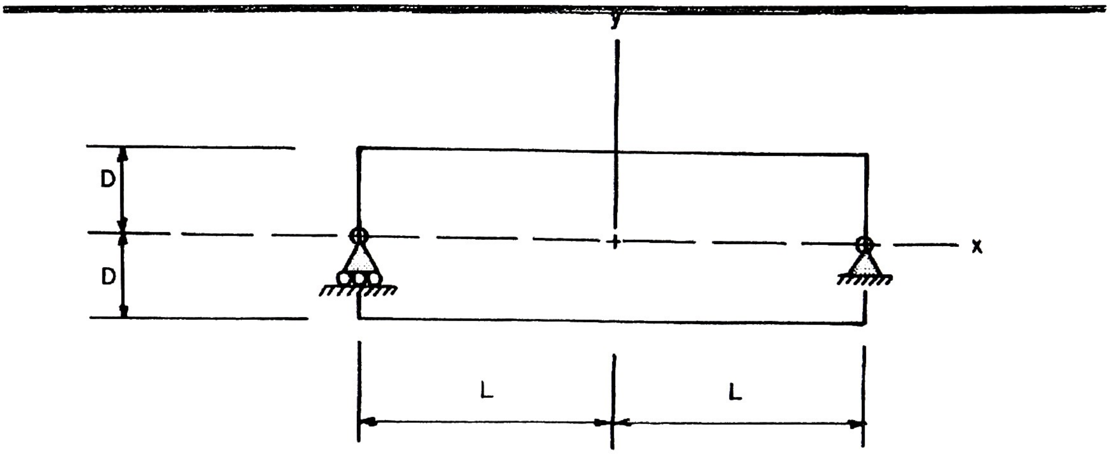


Figure 33: Narrow, Simply Supported Beam Under Its Own Weight [5]

The body forces are given by

$$F = \text{-}\rho g \tag{327}$$

in the y direction which gives us a potential function of

$$V = \rho g y \tag{328}$$

equation (67) becomes

$$\nabla^4 \phi = -(1-\nu)\nabla^2(\rho g y) \tag{329}$$

The boundary conditions of the beam are

$$\tau_{xy}(x, \pm D) = \sigma_{yy}(x, \pm D) = 0 \tag{330}$$

$$\sigma_{xx}(\pm L, y) = 0 \tag{331}$$

$$\int_{-D}^{D} \tau_{xy}(-L, y)dy = -2\rho g D L \tag{332}$$

$$\int_{-D}^{D} \tau_{xy}(L, y)dy = 2\rho g D L \tag{333}$$

The next step is to find the stress function so we can calculate the stresses from the following equations

$$\sigma_{xx} = \rho g y + \frac{\delta^2 \phi}{\delta y^2} \tag{334}$$

$$\sigma yy = \rho g y + \frac{\delta^2 \phi}{\delta x^2} \tag{335}$$

$$\tau_{xy} = -\frac{\delta^2 \phi}{\delta x \delta y} \tag{336}$$

Examining what the probable form of the Airy Stress Function might be, we realize that a reasonable starting guess is that the stress will most likely be an even function in x and an odd function in y. Recalling the polynomial functions developed earlier we might start with the following stress function

$$\phi = C_{21}x^2y + C_{23}x^2y^3 + C_{03}y^3 + C_{05}y^5 \tag{337}$$

The first and third terms satisfy equation (67) individually giving 0=0, but the third and fourth terms must satisfy equation (67) in combination with each other. Substituting equation (337) into equations (334) through (336)

$$\sigma_{xx} = \rho g y + 6C_{23}x^2y + 6C_{03}y + 20C_{05}y^3 \tag{338}$$

$$\sigma yy = \rho g y + + 2C_{21}y + 2C_{23}y^3 \tag{339}$$

$$\tau_{xy} = -2C_{21}x - 6C_{23}xy^2 \tag{340}$$

Applying the boundary condtions at +D we get

$$\tau_{xy}(x, D) = -2C_{21}x - 6C_{23}xD^2 = 0 \tag{341}$$

$$\sigma_{yy}(x, D) = \rho g D + 2C_{21}D + 2C_{23}D^3 = 0 \tag{342}$$

which gives

$$C_{21} = -\frac{3}{4}\rho g \tag{343}$$

$$C_{23} = \frac{1}{4}\frac{\rho g}{D^2} \tag{344}$$

Applying equation (67) we find

$$C_{05} = -\frac{1}{5}C_{23} \tag{345}$$

wich equals to

$$C_{05} = -\frac{1}{20}\frac{\rho g}{D^2} \tag{346}$$

Finally, we need to look at boundary condtion (331) which cannot be satisfied identically in a pointwise fashion by equation (338). Therefore we invoke St Venants principle where

$$\int_{-D}^{D} \sigma_{xx}(L, y)dy = 0 \quad (347)$$

$$\int_{-D}^{D} \sigma_{xx}(L, y)ydy = 0 \quad (348)$$

Evaluating equation (347) using equation (338) we find that it is satisfied since our chosen function for σ_{xx} is odd in y and our limits are -D to D giving us 0=0. Applying equation (348) we get

$$\frac{2}{3}\rho g D^3 + \rho g L^2 D + 4C_{03}D^3 - \frac{2}{5}\rho g D^3 = 0 \quad (349)$$

from which

$$C_{03} = -\rho g(\frac{1}{15} + \frac{1}{4}\frac{L^2}{D^2}) \quad (350)$$

Evaluating our stresses in terms of our constants, we get

$$\sigma_{xx} = 3\rho g[\frac{1}{5} + \frac{1}{2D^2}(x^2 - L^2)]y - \rho g\frac{y^2}{D^2} \quad (351)$$

$$= \frac{\rho g D}{I}[\frac{2}{5}D^2 y - \frac{2}{3}y^3 + (x2 - L^2)\text{y}] \quad (352)$$

$$\sigma_{yy} = -\frac{1}{2}\rho g[1 - \frac{y^2}{D^2}]y \quad (353)$$

$$= \frac{\rho g D}{I}[-\frac{D2y}{3} + \frac{y^3}{3}] \quad (354)$$

$$\tau_{xy} = \frac{3}{2}\rho g[1 - \frac{y^2}{D^2}]x \quad (355)$$

$$= \frac{\rho g D}{I}[D^2 x - y^2 x] \quad (356)$$

where I is the moment of inertia/unit width about the centroidal axis=$\frac{2}{3}D^3$.

The plot of σ_{xx} shown in figure 34 makes sense since everything above the line y=0 is in compression and everything below the line y=0 is in tension. There is also no stress at the the x ends of the beams which satisfies our boundary conditions.

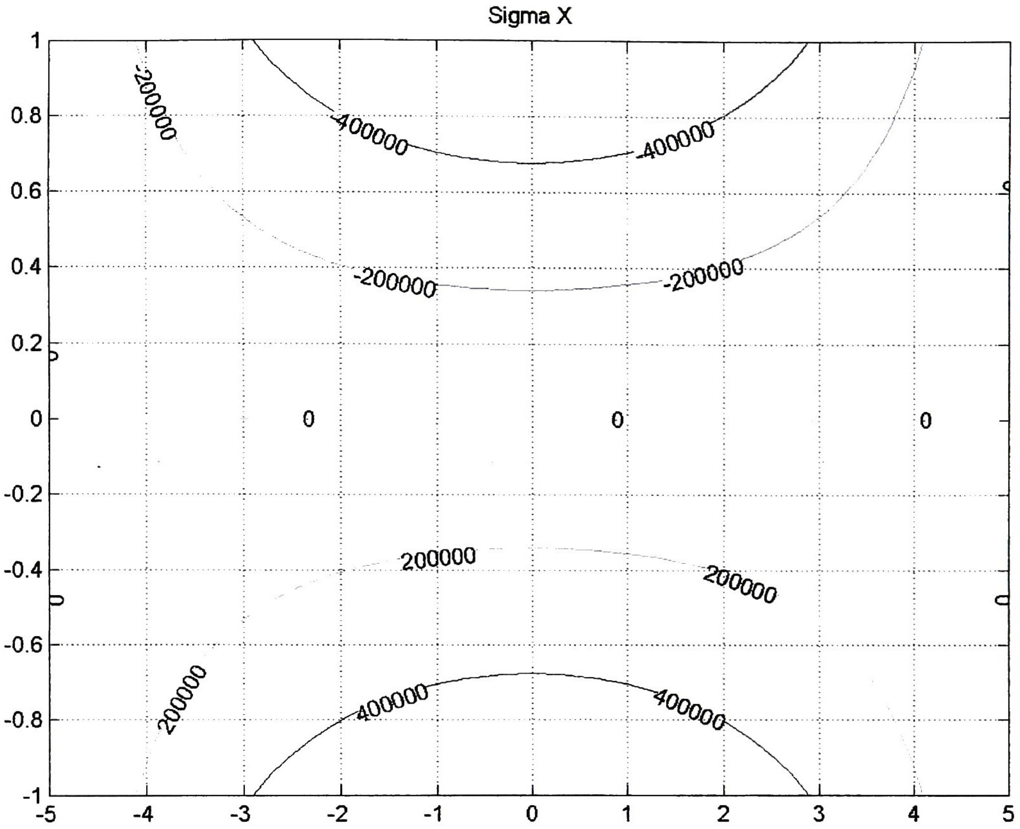


Figure 34: Stress Plot of σ_{xx} for Simply Supported Beam Under Its Own Weight

Figure 35 shows the contour plot of σ_{yy} which makes sense due to the fact that the support pins are on the line y=0 and everything below the pins is hanging and hence getting pulled in tension by gravity, while everything above pins is also getting pulled by gravity but is in compression since it is not hanging down but "sitting on top" of everything else trying to compress what's below it. The stress at y=+/-1 is also equal to zero which satisfies our boundary conditions.

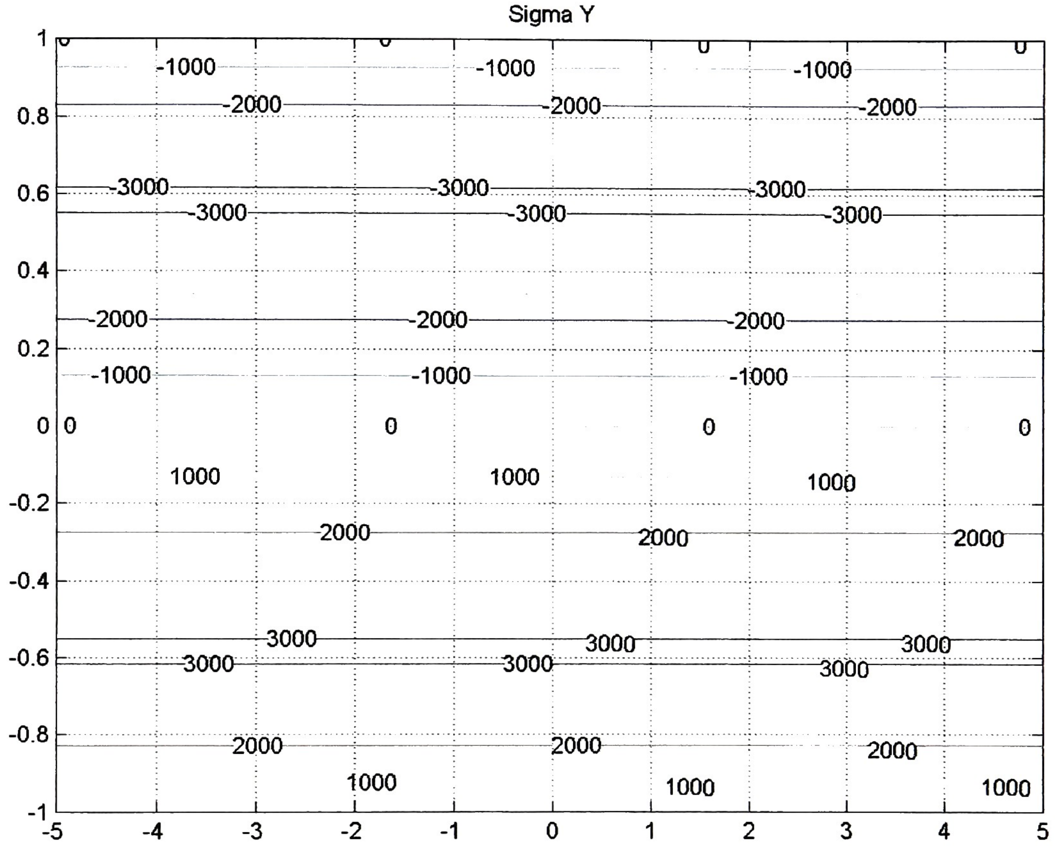


Figure 35: Stress Plot of σ_{yy} for Simply Supported Beam Under Its Own Weight

LOADING OF DOOR SUPPORT BY GAS SHOCK

Background

Universal Instruments designs machines for automated circuit board assembly lines, designing high speed pick and place machines with average throughput ranges betwen 30,000 and 80,000 components per hour ranging in size from .001x .002 inches up to 30 x 30 mm. Components are placed using a moving gantry system with an attached head which picks, orients, and then places the component within +/- 45 μm.

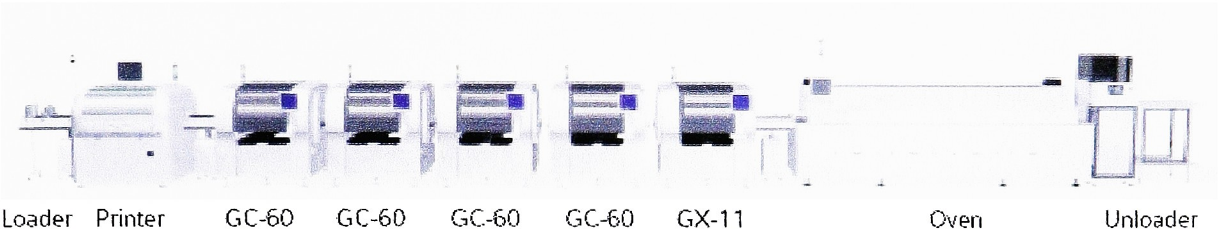


Figure 36: A Sample Line Using Five of Universals Machines

Design Study of Access Doors

In our particular design study we will be fosusing on the design of the access doors, or more specifically, the supporting structure for the access doors and how we could possibly use Airy Stress Functions as part of our analysis by comparing results obtained through the use of Airy Stress Functions to our Finite Element Analysis results.

Determination of Boundary Conditions

The access doors consist of two covers hinged together with a gas shock attached to the upper door providing both mechanical assistance during opening and support to keep the covers in the open position as shown in figures 37 and 38.

Figure 37: General view of cover package with one door open and one closed

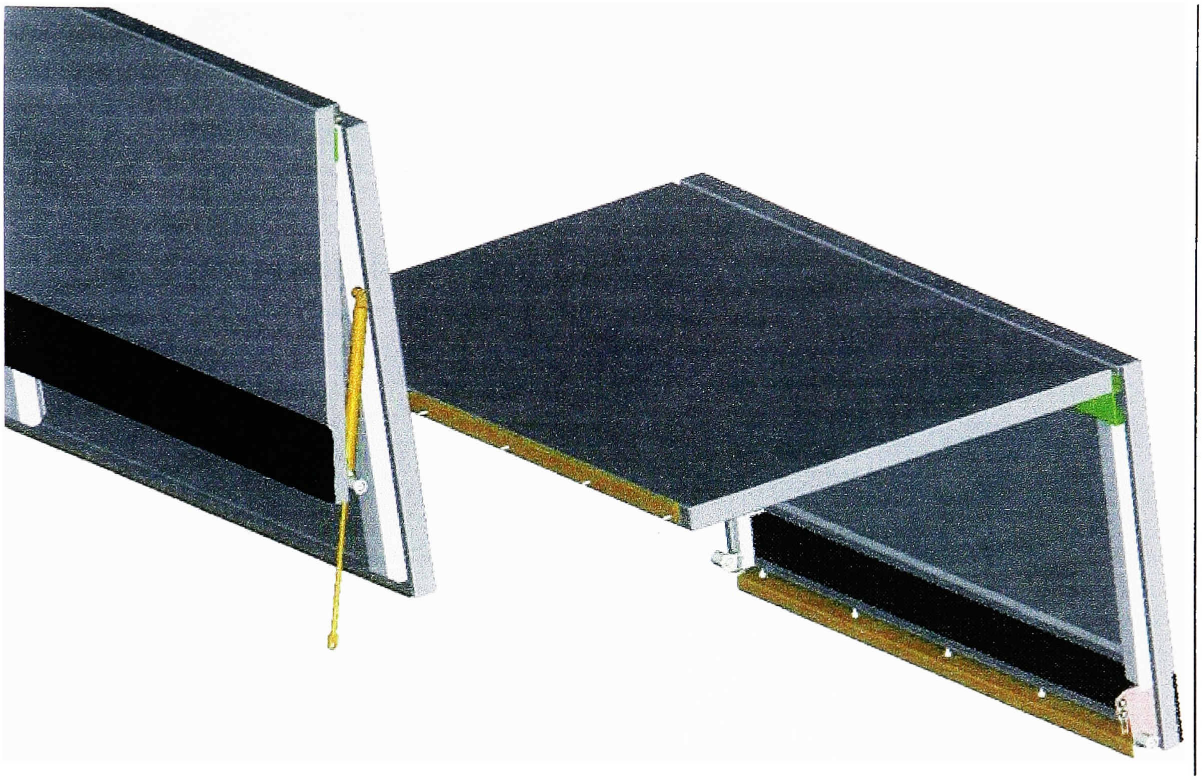


Figure 38: Doors alone, one side open and one side closed

The first step was to design the structure of the two doors to meet all design requirements -- lightweight, no visible hardware or hinges, bronzed ESD coated acrylic window across front of lower door, easily manufacturable, provides guarding to our component feeders, provides proper safety interlock switch engagement, cost effective. In tandem with designing doors that met all those requirements an analysis of the dyanimics of the doors and gas shocks was done in order to determine the parameters for our gas shock. All these things together gave us enough information to analyze our supporting structure.

__Dynamic Analysis__

In doing the dynamic analysis we set our origin at the axis of the center hinge and tracked the movement of the following points -- axis of pivoting hinge between two doors, center of gravity of upper door, center of gravity of lower door, fixed gas shock mounting point in our support structure, moving mounting point of gas shock in our upper door, roller attached to the lower door.

Some assumptions were made to make the analysis easier, yet still applicable. The first assumption we made was to break the motion of the doors into two sections determined by the roller attached to the lower door. In the first section the roller rode vertically along a lip in the upper corner cover as shown in green part of figure 39.

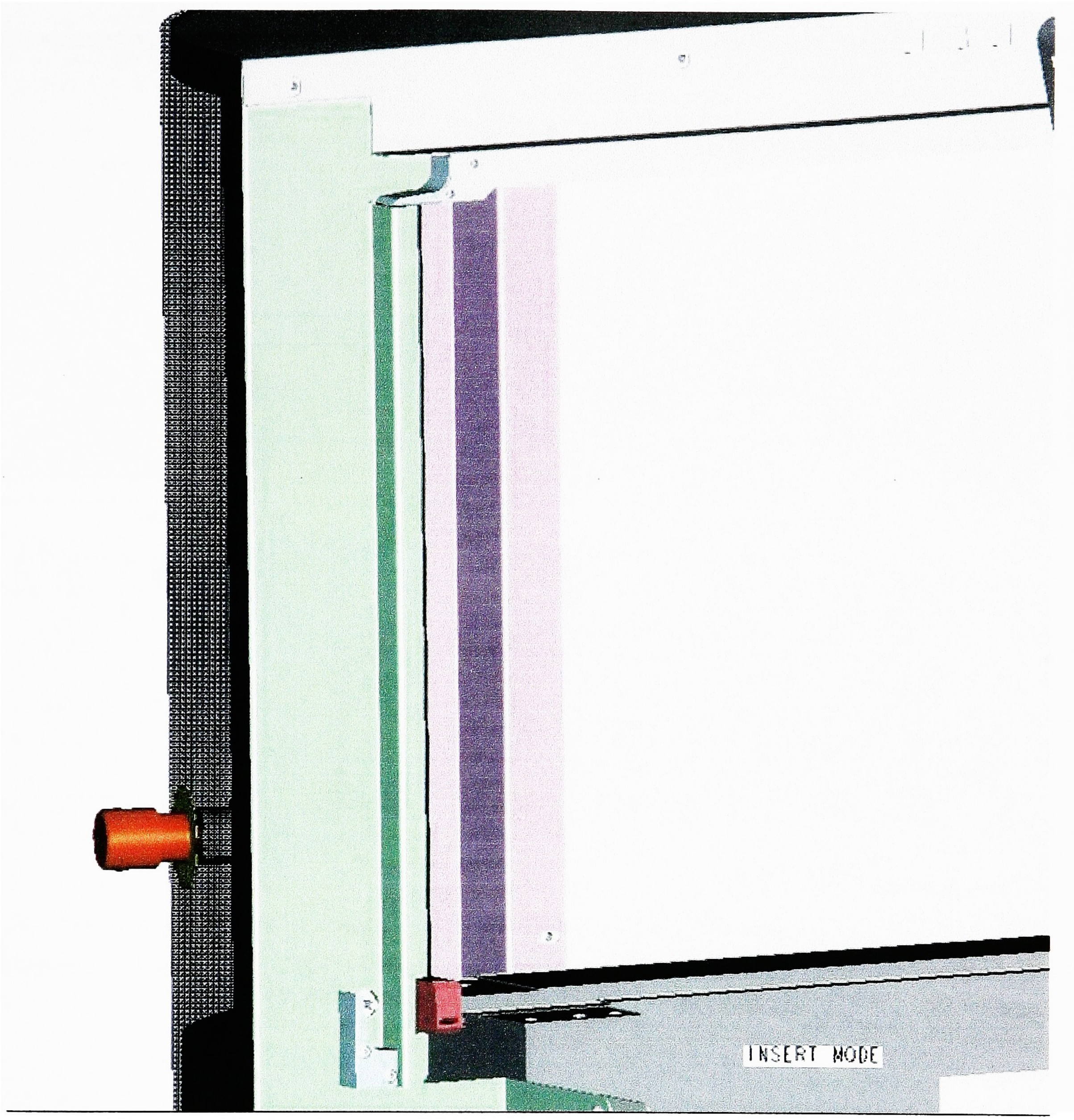


Figure 39: Picture of guiding lip for roller attached to lower door

In the second section we made the assumption that once the roller rose above the upper limit of the lip that it remained at this hieght and only traveled horizontally across the lip of the light grey part in figure 39. In reality this does not happen as upon breaking the upper limit of the lip the lower door begins to swing freely eventually coming to rest with it's Center of Gravity directly below the moving hinge point between the two doors.

The second assumption made was in relation to the gas shocks -- further discussed in the next section -- where we the assumption that as the gas shock moved through its stroke from fully compressed to fully extended the force varied linearly. The technically correct analysis would have taken into the account that we are actually compressing a gas and used some of the gas law equations. To further solidify our assumption as valid we spoke with the manufacturer who also felt that this was a reasonable assumption given our application.

Using the angle made by the upper door and a horizontal plane running through our origin as our driving variable, we used excel to carry out all the calculations and make a plot of the motion of all the points we were tracking as shown in figures 40 and 41.

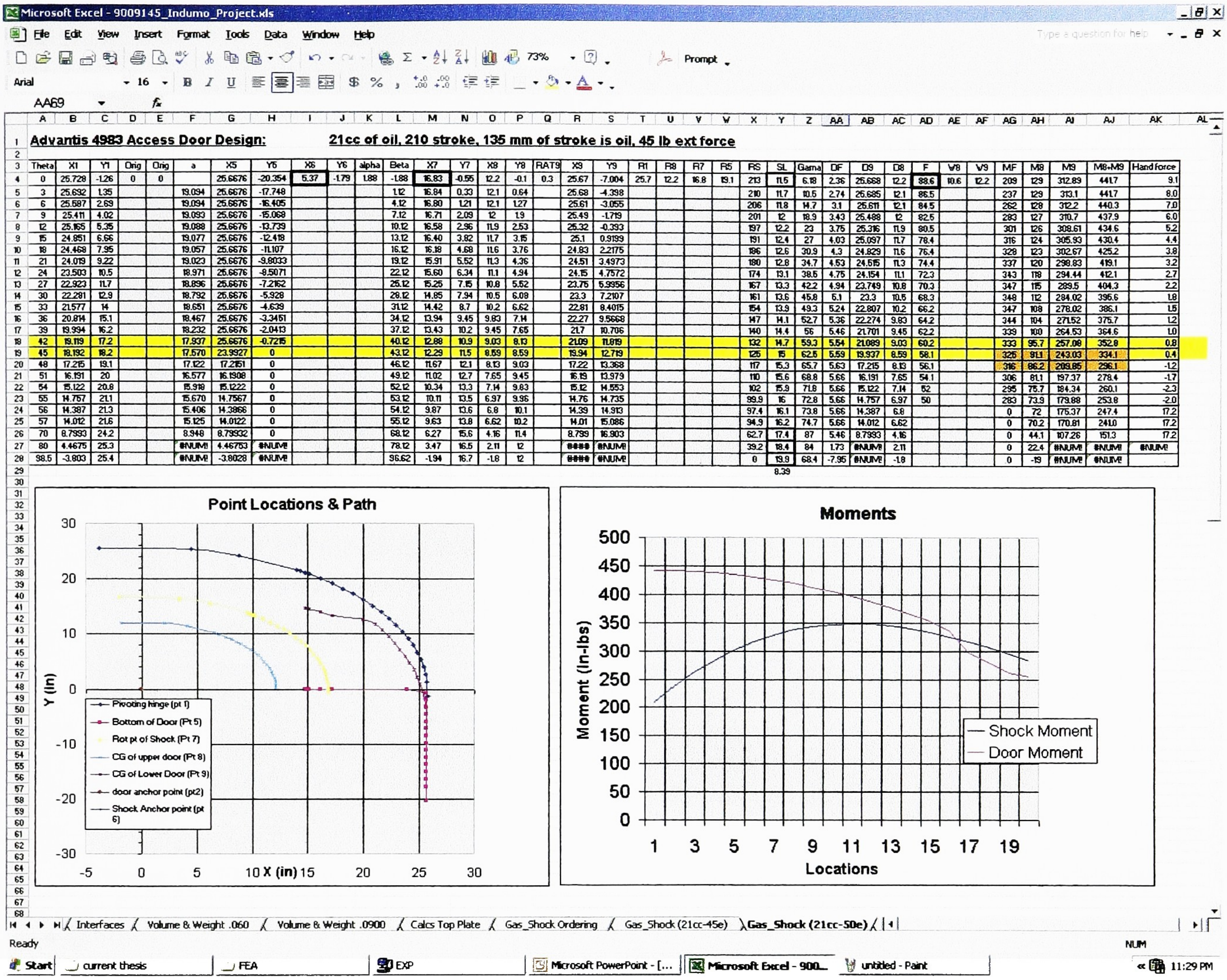


Figure 40: Sample spreadsheet of all calculations performed

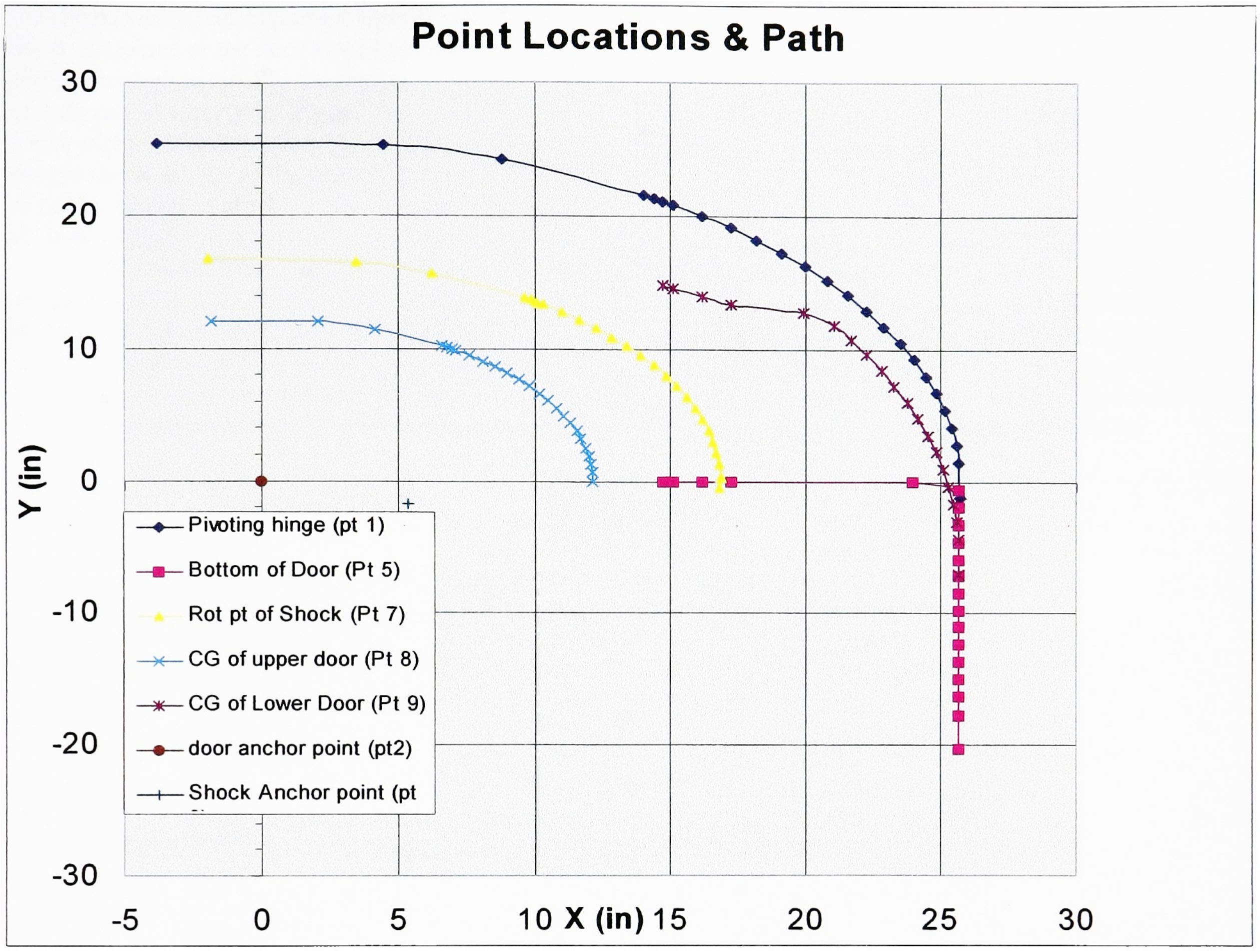


Figure 41: Plot of the motion of all tracked points to help verify and visualize calculations

<u>Gas Shock</u>

Once the dynamic analysis was completed, it was now time to move onto the forces. In our analysis, the moments generated by the Center of Gravities of the two doors about the origin added together had to be counteracted by the moment generated by the gas shock about the origin. Since the dynamic analysis was now completed, we could determine the moments generated by the Center of Gravities and since we tracked the two mounting points of the gas shock, we could also determine how the moment arm of our gas shock varied throughout the range of motion allowing us to determine the force required by the gas shock to meet our design intentions.

When ordering our gas shocks, certain design parameters were fixed, while others were variable and driven by some of our fixed parameters. The end mounting conditions and the max and min length of the shock which in turn gives us our requred stroke length were the fixed parameters. The shocks we were ordering were a mixture of gas and oil. As the shock moved from compressed to extended it would first move rapidly through the gas portion and then, upon entering the oil portion the velocity would drop dramatically. The amount of oil in the gas shock was important as it determined at what point in the stroke the oil would engage and the percent increase in force as the shock went from the fully extended to fully compressed.

In ordering our gas shock, we could specify the fully extended force, which was initially determined by how much force would give a strong enough moment to keep the doors in the up position. In order to determine the desired amount of oil we added another design parameter in which we wanted to oil to engage just before the roller broke the upper limit of the lip and the lower door started to swing freely. This was determined by calculating the lenght of the gas shock just before the upper limit was broke and then subtracting that length from the fully extended length in order to determine what portion of the stroke needed to be traveling through oil.

Giving the manufacturer the desired extended force and desired amount of stroke in the oil allowed him to give us the fully compressed force. The fully compressed force allowed us to determin the moment generated by the gas shock, which was important since it had to be less than the moment generated by the Center of Gravities of the doors combined or the door would not want to remain in the down position. Meeting this requirement however was still not enough to solidify the design. We also needed to know how much force was needed to get the door moving initially and at what point the gas shock would take over. In order to do that we calculated an imaginary hand force which acted vertically upward at the tracked roller point. Knowing the moments generated by both the doors and the gas shock as well as the moment arm of this imaginary hand force allowed us to calculated how much force would be needed at at what point this hand force was no longer needed. Our goal was to need no more than 10 lbs of force to lift the doors from the down position and to have the gas shock take over completed before it hit the oil.

In order to help visualize the hand force requred, we plotted the moments generated by the doors against the moment generated by the gas shock as shown in figure 42. Location 1 is with the doors in the down position and location 19 is with the doors in the up position. The point of intersection of these two lines is where the gas shock takes over and starts to move the doors on it's own. The difference between moments at position 1 gives a visual feel of how much of an advantage the doors have over the gas shock in the down position. Luckly, the moment arm of our imaginary hand force about the origin is quite large, which helps keep the required hand force down.

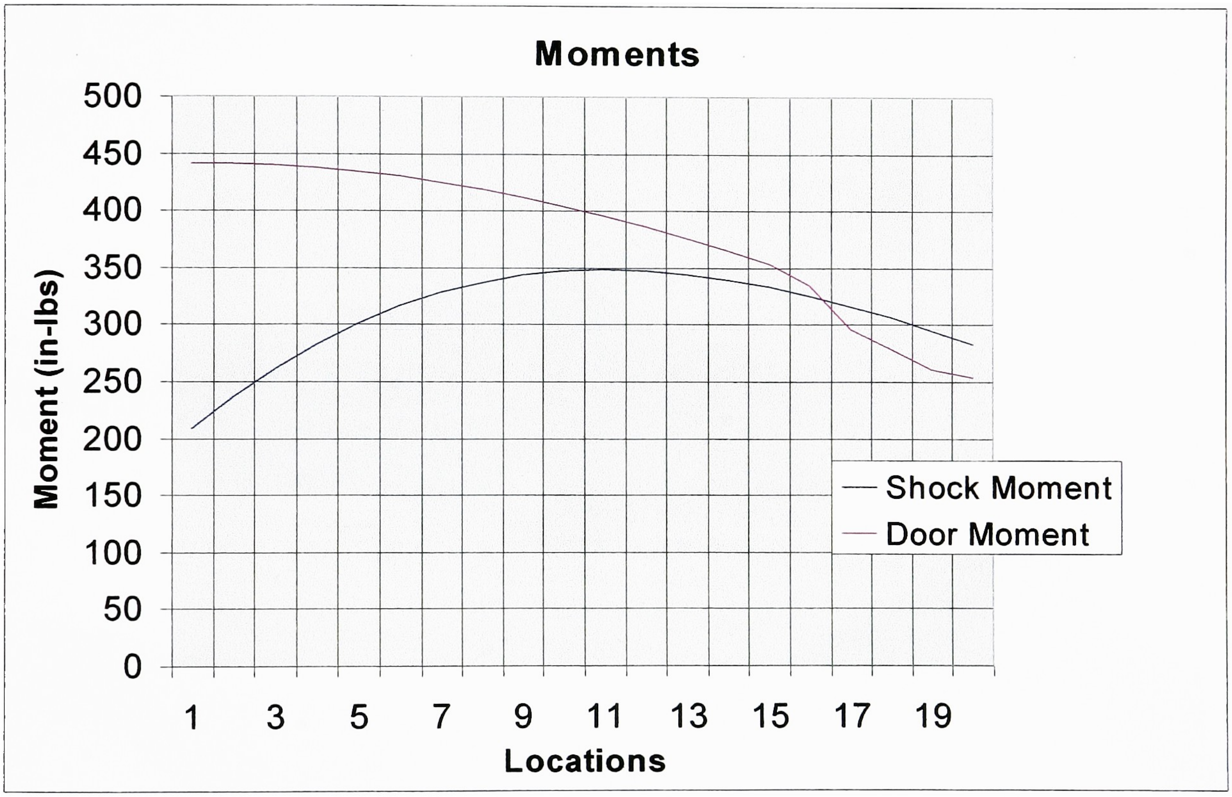


Figure 42: Plot of the moments created by the CG of the two doors added together vs the opposing moment generated by the gas shock

In figure 40, the yellow row represents the point where we want the oil to engage, and the orange blocks represent the calculated point where the gas shock starts to take over and no more hand force is required. Varying the specified extended force was a delicate balance between providing enough force to keep the doors in the up position while at the same time ensuring that the doors would remain in the down position with less than 10 pounds needed to get them moving.

Once all things were considered, we had selected a range of gas shocks with varying degrees of force and varying amounts of oil. The next step was to examine the deflection caused by the force of the gas shock in the vertical direction, we examined the point where the gas shock will be applying it's maximum force straight up and down as this is the worst case scenerio for vertical deflection. After analyzing the vertical component of the force applied by the gas shock as it moved through it's various angles, we were able to determine the max amount of force applied in the vertical direction which we would then use for our FEA. The load used for the FEA was 65 lbs.

FEA and Airy Stress Function Comparison

In doing the comparison between the FEA results and Airy Stress function results, I wanted to see how close they meshed up. Since we have an Airy Stress Function for a cantilever beam loaded at the End I went back and ran the FEA with the 65 lb load at the end of the beam. The final beam looked like figure 43, but in order to get a better model for FEA and Airy Stress Function we will be leaving off the rounded lip and only analyzing the continious cross section portion of the underlying beam structure as shown in figures 44 and 45 when doing the Airy Stress Function analysis. The FEA analysis will be run including the capped end of the beam as shown in figure 46 since when the cover is assembled the capped end is flush against the corner structure providing a nice surface that we can choose to hold fixed.

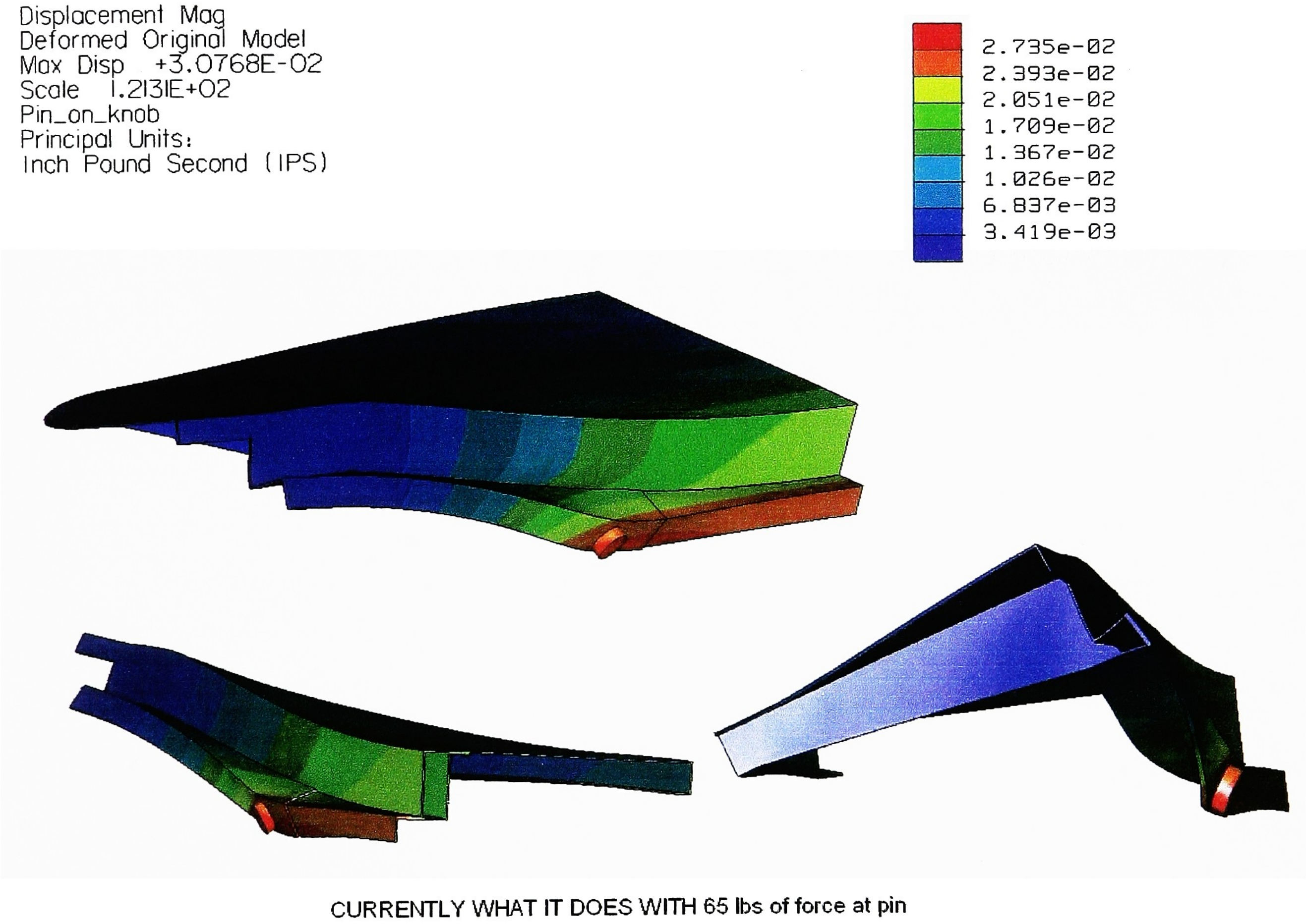


Figure 43: FEA displacement analysis done on final beam structure

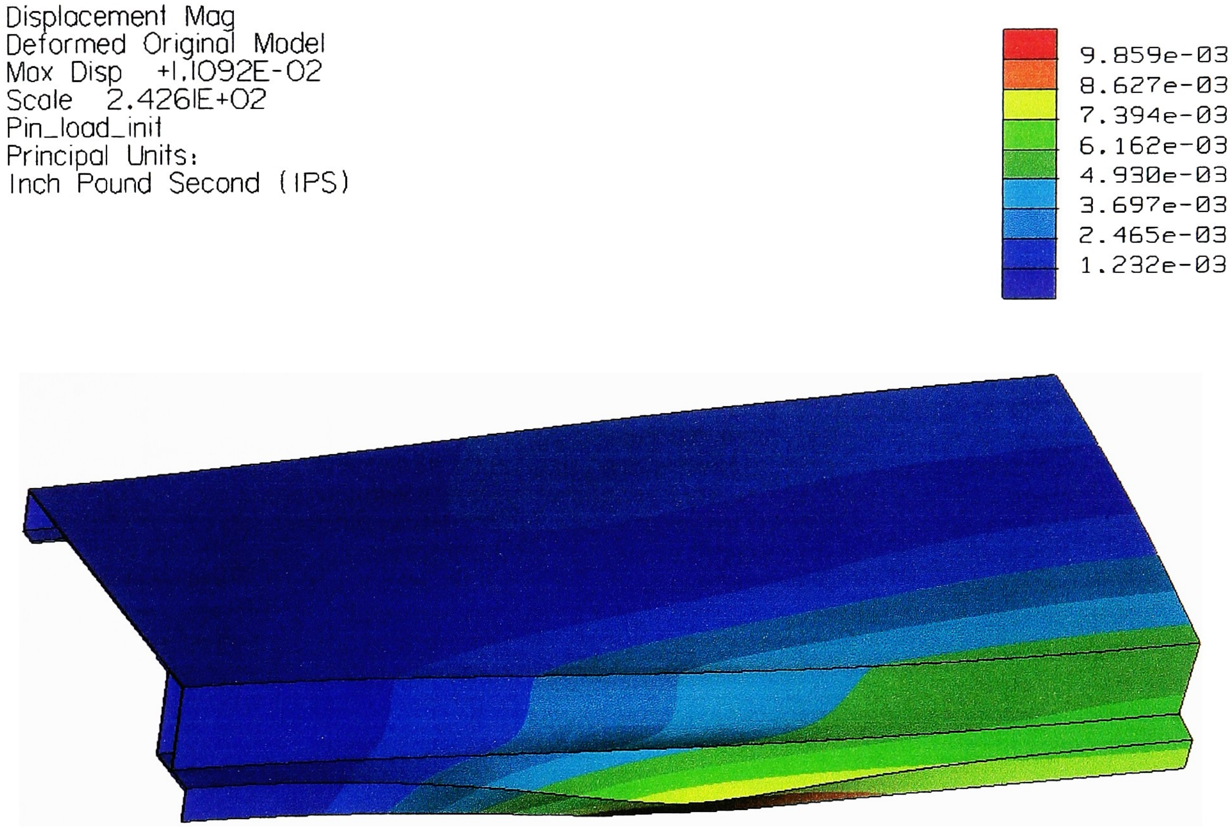


Figure 44: FEA displacement analysis of continious cross section portion of structure

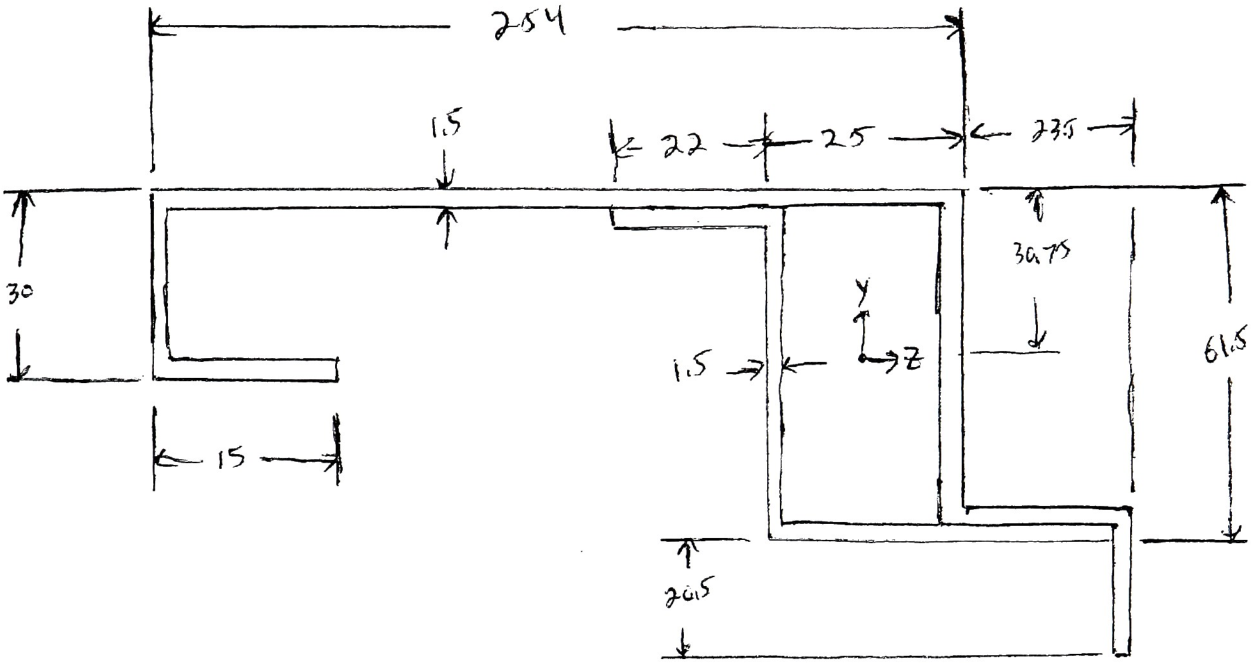


Figure 45: Cross Section (mm)

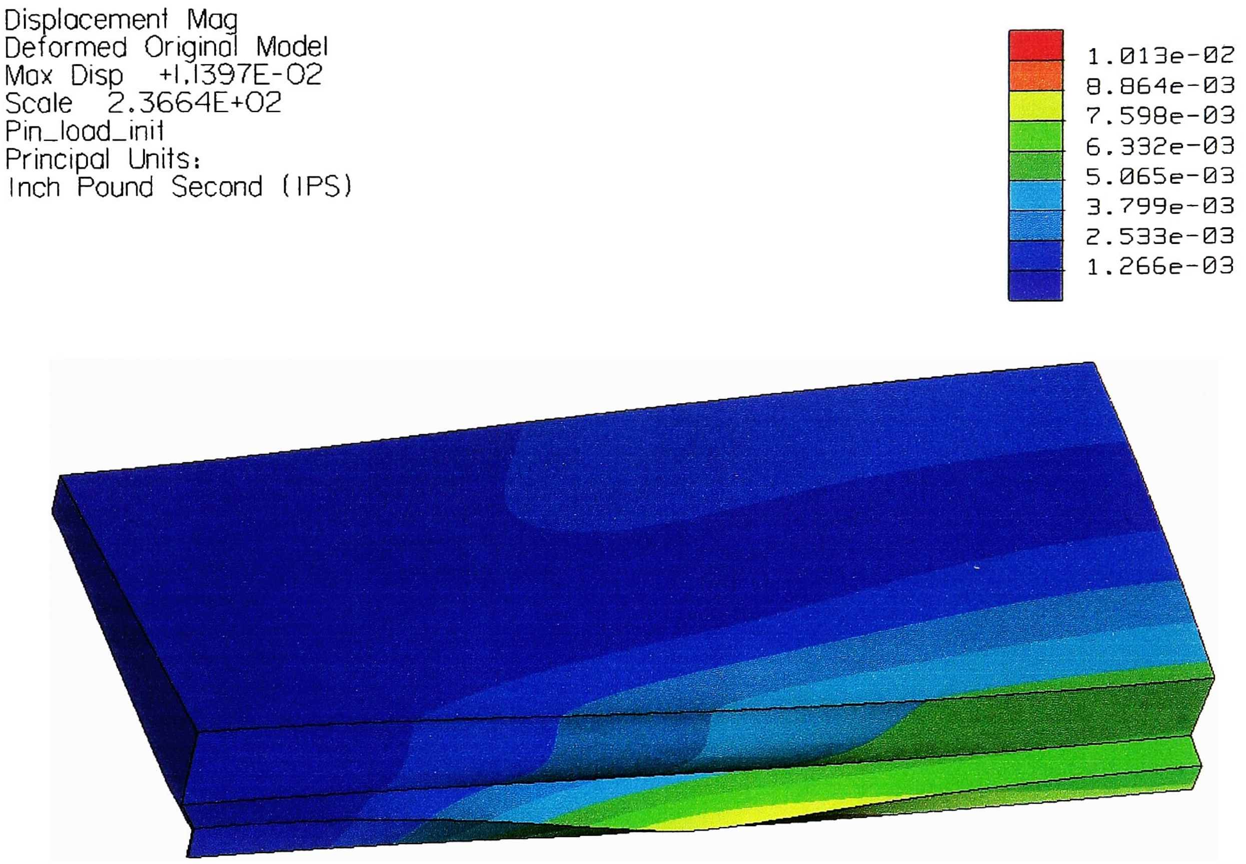


Figure 46: FEA displacement analysis of continious cross section including capped end

Airy Stress Function Analysis

Looking at our Airy Stress function for a Cantilever Beam Loaded at End

$$\phi = bxy + dxy^3 \tag{357}$$

and the corresponding stress equations where b and d are constants.

$$\frac{\partial^2 \phi}{\partial^2 y} = \sigma_{xx} = 6dxy \tag{358}$$

$$\frac{\partial^2 \phi}{\partial^2 x} = \sigma_{yy} = 0 \tag{359}$$

$$\frac{\partial^2 \phi}{\partial y \delta x} = \tau_{xy} = -b - 3dy^2 \tag{360}$$

We must first determine our contstants so we begin by setting our coordinate system at the center of the box section of our beam as shown in figure 45. If we attempt to follow a similar analysis as was done to develop the origonal Airy Stress function and try to look at the fact that the top and bottom of our beam are stress free

$$\tau_{xy_{top}} = -b - 3dy^2 = 0 \qquad 361$$

$$\sigma_{yy_{top}} = 0 \qquad 362$$

$$\tau_{xy_{bottom}} = b + 3dy^2 = 0 \qquad 363$$

$$\sigma_{yy_{bottom}} = 0 \qquad 364$$

we notice that our beam must be symmetric about the z axis. We do not have a cross section that is symmetric about the z axis, but we will set our coordinate system at the center of the beam, giving us the following

$$\tau_{xy_{top}} = -b - 3d(.041)^2 = 0 \qquad 365$$

$$\sigma_{yy_{top}} = 0 \qquad 366$$

$$\tau_{xy_{bottom}} = b + 3d(-.041)^2 = 0 \qquad 367$$

$$\sigma_{yy_{bottom}} = 0 \qquad 368$$

The above equations however only give us one of the necessary equations to solve for our constants

$$b = -.005043D \qquad 369$$

Our second equation comes from the evaluating our stress vector at at the free end of the beam x=0

$$\vec{\sigma}_{free\,end} = \sigma_{xx}\hat{e}_x - \tau_{xy}\hat{e}_y \qquad 370$$

$$\vec{\sigma}_{free\,end} = 6dxy\,\hat{e}_x + b + 3dy^2\hat{e}_y \qquad 371$$

$$\vec{\sigma}_{free\,end} = (b + 3dy^2)\hat{e}_y \qquad 372$$

Using our force, 65lbs or 291.2 N, that we determined from our boundary conditions and our stress vector to evalute the following

$$\int \sigma\, dA = F \qquad 373$$

which can be rewritten as the following

$$t\int(b+3dy^2)dy = -F \qquad 374$$

Since our area thickness is not the same from the bottom to the top of the beam we must do multiple integrals so we can take into account how the area is distributed about the Z axis

$$.0015\int_{-.041}^{-.0205}(b+3dy^2)dy = -291.2 \qquad 375$$

$$.0485\int_{-.0205}^{-.019}(b+3dy^2)dy = -291.2 \qquad 376$$

$$.0265\int_{-.019}^{-.0175}(b+3dy^2)dy = -291.2 \qquad 377$$

$$.003\int_{-.0175}^{.011}(b+3dy^2)dy = -291.2 \qquad 378$$

$$.018\int_{.011}^{.0125}(b+3dy^2)dy = -291.2 \qquad 379$$

$$.0045\int_{.0125}^{.038}(b+3dy^2)dy = -291.2 \qquad 380$$

$$.0265\int_{.038}^{.0395}(b+3dy^2)dy = -291.2 \qquad 381$$

$$.254\int_{.0395}^{.041}(b+3dy^2)dy = -291.2 \qquad 382$$

After running the integrals and adding them together we get

$$-291.2 = 7.91e^{-4}b + 2.51e^{6}d \qquad 383$$

Taking equations 369 and 383 and carrying out the algebra to evaluate our constants

$$B = -995966.37 \qquad 384$$

$$D = 197494819 \qquad 385$$

Plugging our evaluated constants into our stress equations

$$\sigma_{xx} = 1.185e^{9}xy \qquad 386$$

$$\sigma_{yy} = 0 \qquad 387$$

$$\tau_{xy} = 995966.37 - 592484456y^2 \qquad 388$$

Stress Function and FEA Comparison

Before making stress plots of our Airy Stress Function results, it is wise to run our FEA first so that we can then run the exact same level curves on our Airy Stress Function results. In running the FEA it is a good idea to get an Isometric view of the structure as shown in figure 47 so we can see what is happening on the whole beam, however we also need FEA run looking at the xy plane only -- showin in figures 48 and 51 -- as that is the plane of our stess plots from our Airy Stress Functions

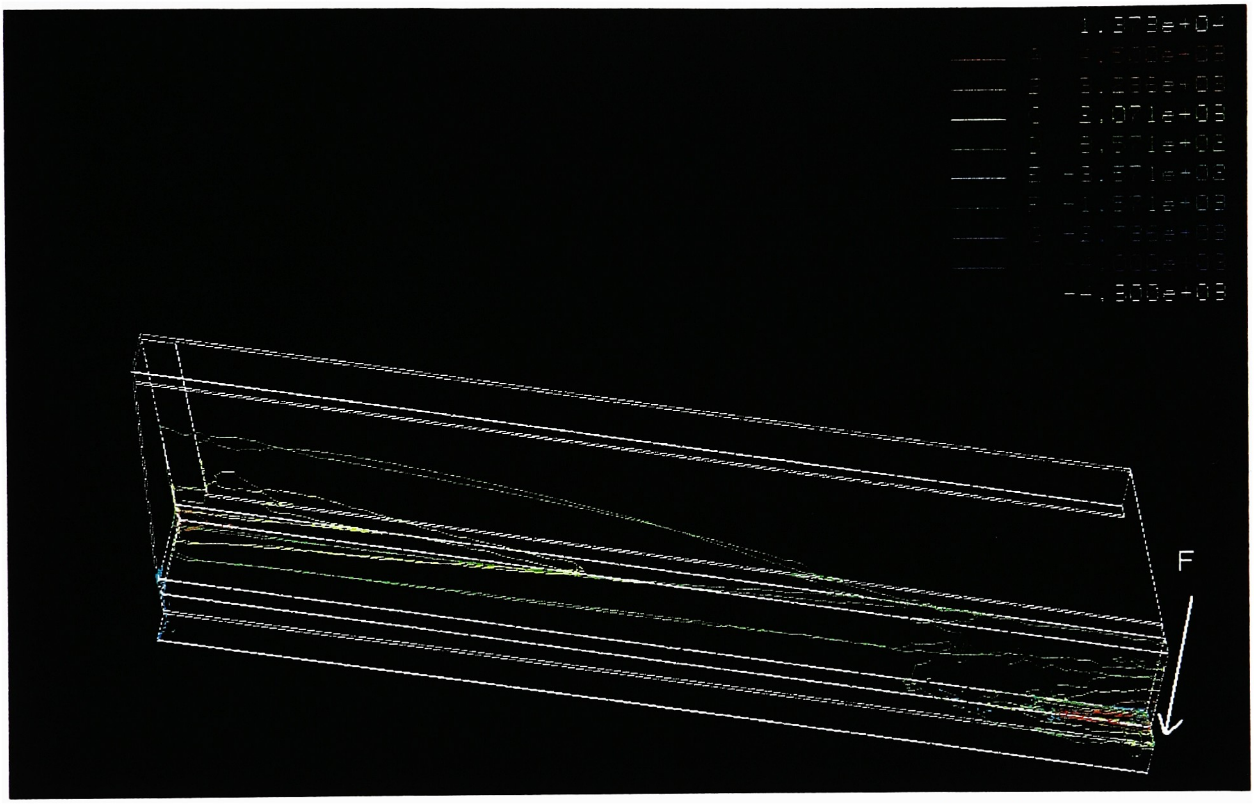


Figure 47: Stress Plot of Max Principal Stress, isometric view, IPS units

Our isometric view shows us that not only is there some beam bending going on, but there is also some torsional effects going on as can be seen in the green and yellow countour lines taking on a triangular shape sloping toward the force.. This torsion comes from the fact that the Force stems partly from the structure of the beam and partly from the fact that the force is applied only to the small L shaped flange.

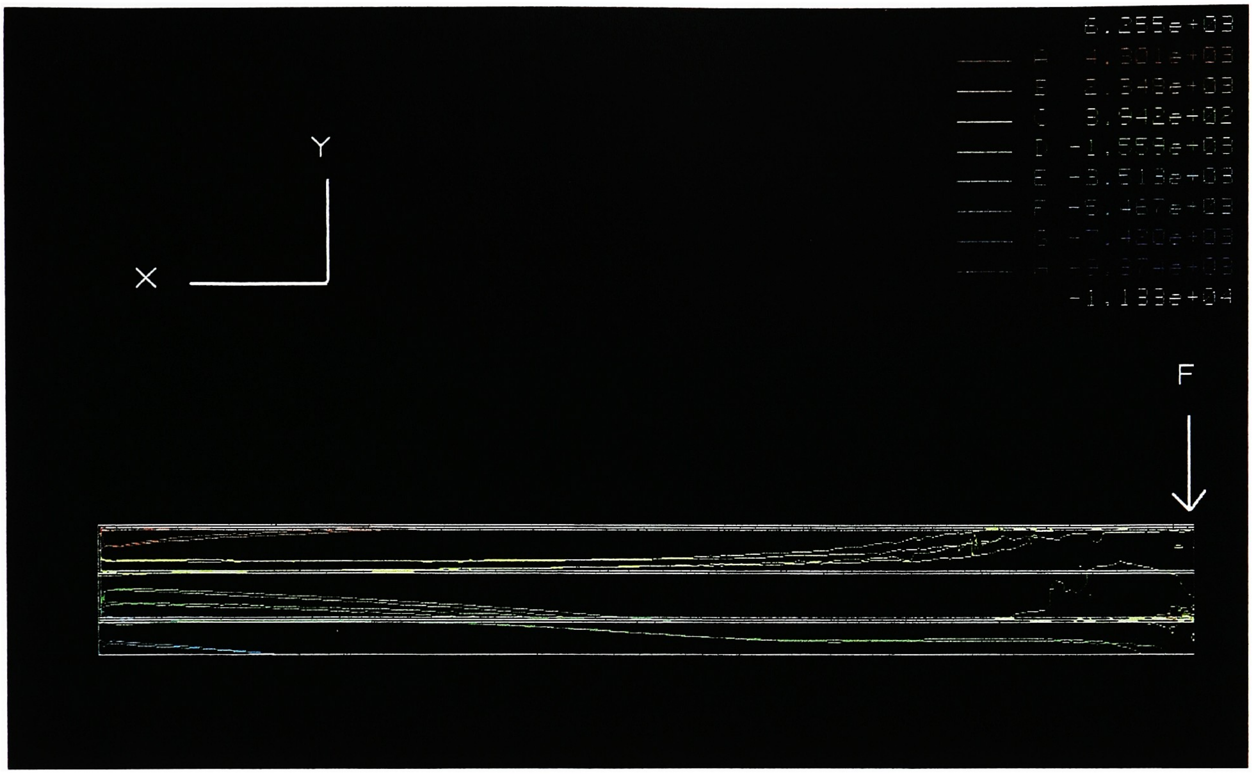


Figure 48: σ_{xx} shown in the xy plane, FEA, IPS units

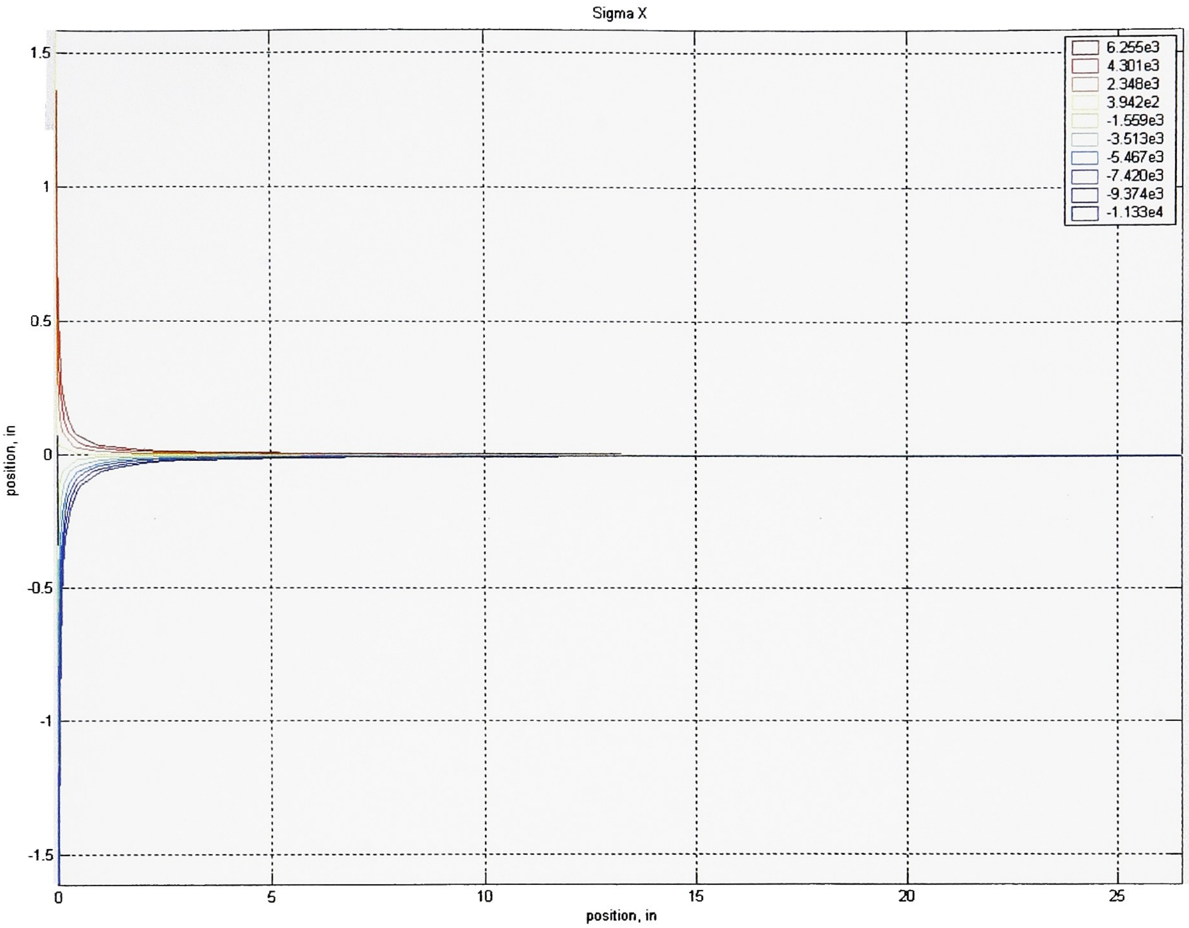


Figure 49: σ_{xx}Stress Plot, Airy Stress Function, Full Beam

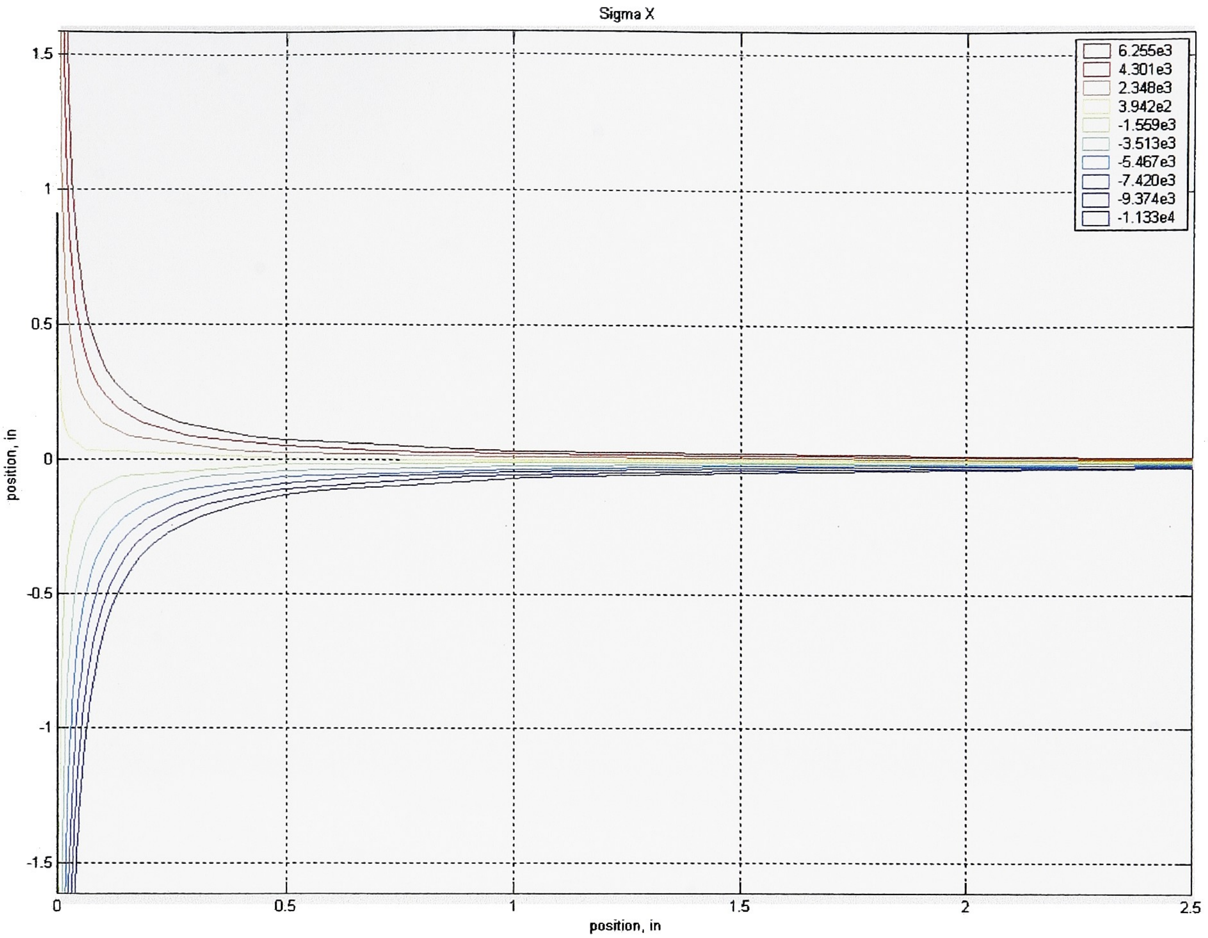


Figure 50: σ_{xx} Stress Plot, Airy Stress Function, First 2.5 inches of Beam

Relatively speaking, the Airy Stress Plots show similar features to the FEA stress plots for σ_{xx}. The contour lines take the shape similar to the functions y=$\frac{1}{x}$ and y=-$\frac{1}{x}$ in both the Airy plots and the FEA plots, although in the Airy plot, all the stress lines begin relatively close to the origin and in the FEA plots some of the contour lines aren't even visible till roughly 18 inches out. This might be caused by the capped end in the FEA model.

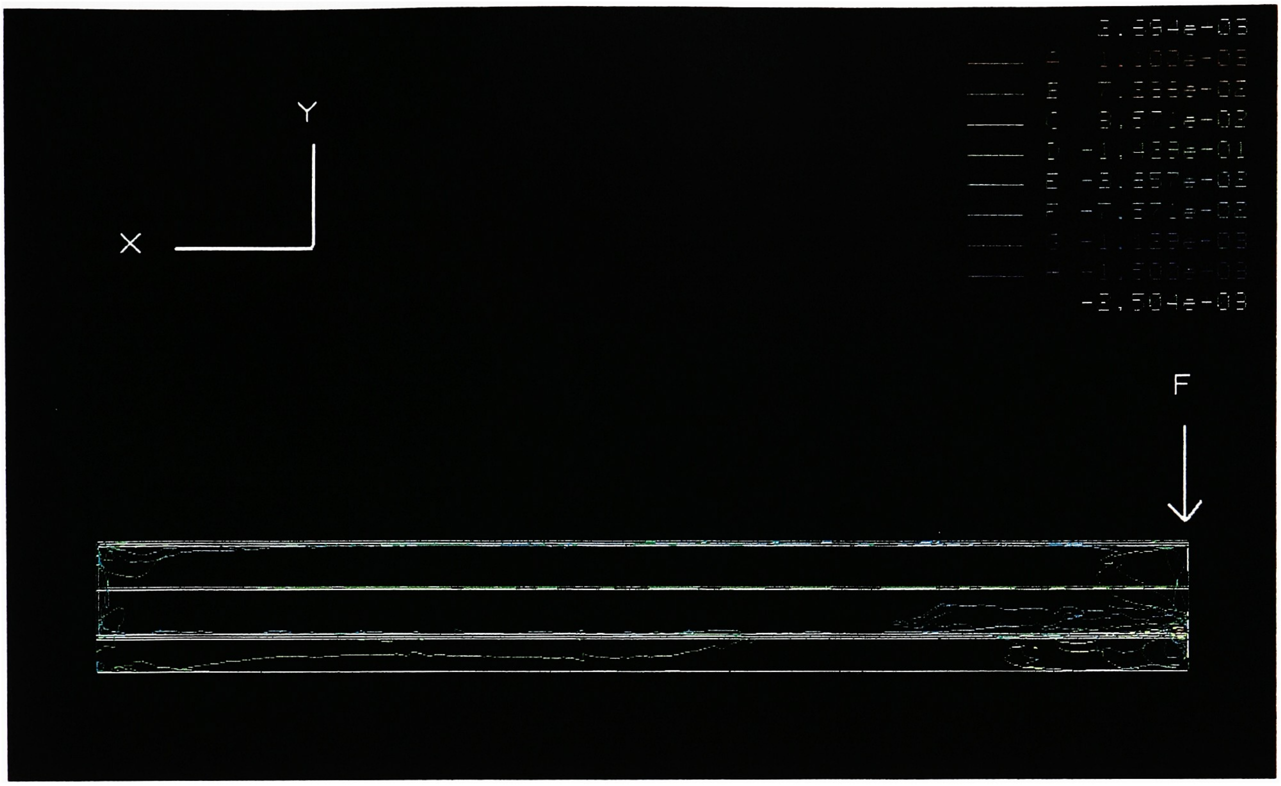


Figure 51: τ_{xy} shown in the xy plane

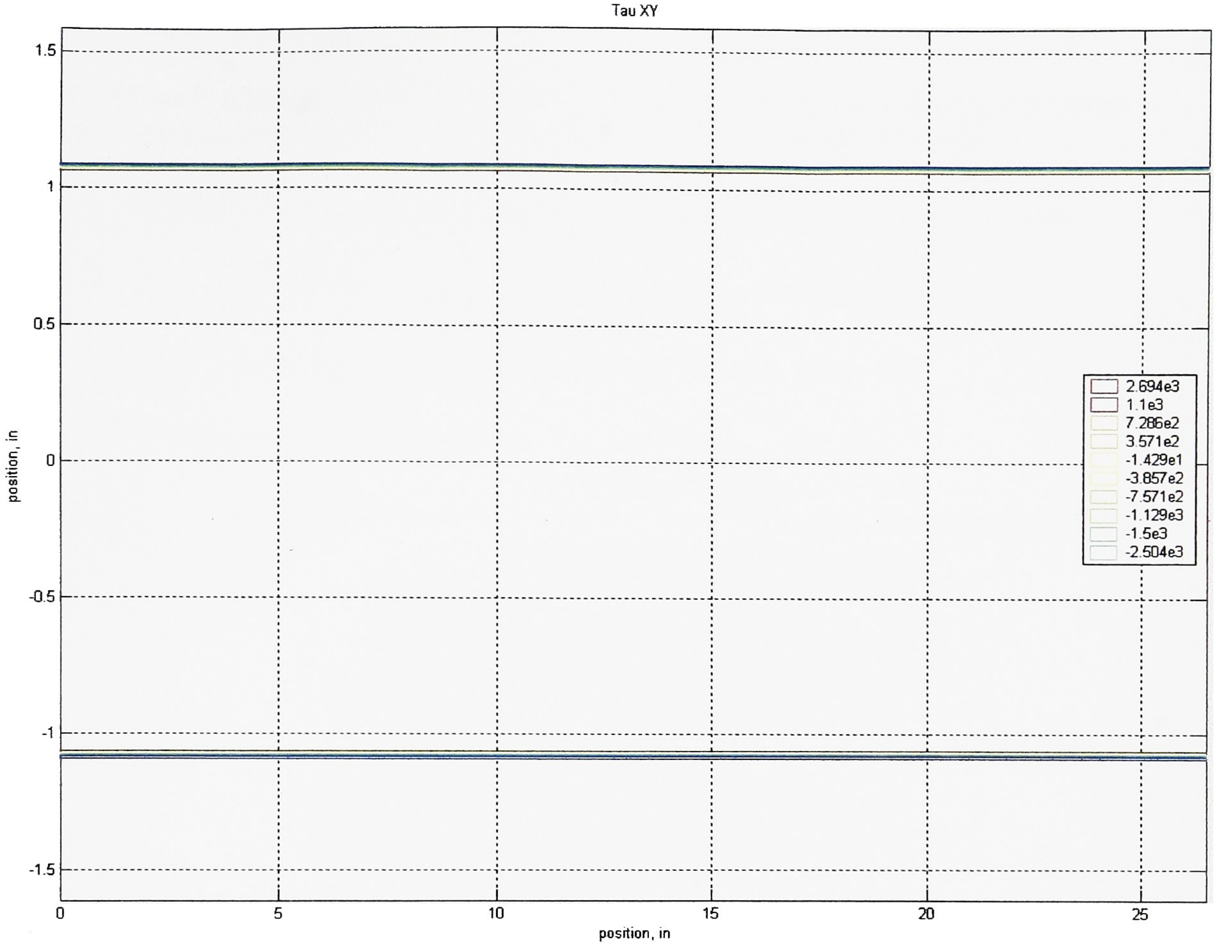


Figure 52: τ_{xy}, Airy Stress Function,

Once again, we have semi-similar plots. In the FEA our contour lines remain relatively level, which is exactly what we have in our Airy Stress Plot. The major difference seen though, is that the contour lines in the FEA plot are shifted in posotive y direction compared to the Airy Stress contour lines. I believe that this happens because of the stress function chosen, which makes everything symmetric about the x axis and the fact that the rectangular box section is not centered on the x axis but shifted vertically in the y direction.

Overall the stress plots are similar, however I do not think that the Airy Stress Function chosen can adequately show enough about the beam to feel confident in a design, but perhaps might be enough if you want to get general feeling for what is going on.

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[6] Timoshenko, S. Theory of Elasticity. New York and London: McGraw Hill, 1934.

[7] Guran, Ardeshir and Teodor M. Atanackovic. Theory of Elasticity for Scientists and Engineers. Boston: Birkhauser, 2000.

APPENDICES

Matlab Code

Cantilever Beam Loaded at End

```
clear
clc
%need to make sure your mesh is square as you have y^3
t=1; %set the thickness to this
h=2; %set the height to this\
L=2; %set the length equal to this
F=10; % this is the load applied in correct units
[x,y]=meshgrid(0:.1:L, -h/2:.1:h/2);
b=-(3*F)/(2*t*h); %this evaluates the constant b in terms of known values
d=(2*F)/(t*h^3); %this evaluates d in terms of known values
phi=b*x.*y+d*x.*y.^3;
[m,n]=contour(x,y,phi,20);
clabel(m,n);
title('phi')
grid on
figure(2); %this creates a second figure
Sigma_x=(12*F)/(t*h^3)*x.*y; %this is the equation for Sigma x
[o,p]=contour(x,y,Sigma_x,10);
clabel(o,p);
title('Sigma X')
grid on
figure(3);
Tau_xy=(F/2)*(3/(t*h)-(12/(t*h^3)*y.^2));
[q,r]=contour(x,y,Tau_xy,5);
clabel(q,r);
title('Tau XY')
grid on
```

Curved Beam Under the Action of Couples

```
%Curved Beam loaded by couples, stress along x-axis
clear
clc
r1=2
r2=4
t=1
M=10
[th,r]=meshgrid((0:5:360)*pi/180,r1:.05:r2);
[X Y]=pol2cart(th,r);
K=[(r2^2-r1^2)^2-4*r1^2*r2^2*(log((r2/r1))^2)]*t;
c1=4*M/K*r1^2*r2^2*log(r2/r1);
c2=-M/K*[r2^2-r1^2+2*(r2^2*log(r2)-r1^2*log(r1))];
c3=2*M/K*(r2^2-r1^2);
Sigma_RR=c1./(r.^2)+2*c2+2*c3*log(r)+c3;
%For some reason if the difference between r1 and r2 is 1 it plots nothing.
Sigma_TT=-c1./(r.^2)+2*c2+2*c3*log(r)+3*c3;
h=polar([0 2*pi],[r1 r2]);
delete(h)
hold on
[c,h]=contour(X,Y,Sigma_RR);
clabel(c,h);
title('Sigma RR')
figure(2)
j=polar([0 2*pi],[r1 r2]);
delete(j)
hold on
[f,g]=contour(X,Y,Sigma_TT);
clabel(f,g)
title('Sigma TT')
```

```
clear
clc
r1=.05;
r2=.2;
[th,r]=meshgrid((0:5:360)*pi/180,r1:.05:r2);
[X Y]=pol2cart(th,r);
M=5;
t=1;
c1=M/(2*pi*t);
Sigma_RT=c1./r.^2;
h=polar([0 2*pi],[r1 r2]);
delete(h)
hold on
[c,h]=contour(X,Y,Sigma_RT);
clabel(c,h);
title('Sigma RT')
```

Third Degree Polynomial

```
clear
clc
%need to make sure your mesh is square
[x,y]=meshgrid(-10:.2:10);% This does square at intervals of .1 from -2 to 2
a=1;
b=1;
c=1;
d=1;
subplot(2,2,1); %plots plot in top left
Sigma_x=2*c*x+6*d*y; %this is the equation for Sigma x
[o,p]=contour(x,y,Sigma_x);
clabel(o,p)
title('Sigma X')
grid on
subplot(2,2,2) %plots in top right
Sigma_y=6*a*x+2*b*y;
[q,r]=contour(x,y,Sigma_y);
clabel(q,r)
title('Sigma Y')
grid on
subplot(2,2,3) %plots in bottom left
Tau_xy=-2*b*x-2*c*y;
[s,t]=contour(x,y,Tau_xy);
clabel(s,t)
title('Tau XY')
grid on
```

Fourth Degree Polynomial

```
clear
clc
%need to make sure your mesh is square
[x,y]=meshgrid(-20:.2:20);% This does square at intervals of.1 from -2 to 2
a=-1;
b=1;
c=-1;
d=-1;
e=-(a+c/3);
% subplot(2,2,1); %plots first one in top left
% phi=a*x.^4+b*x.^3.*y+c*x.^2.*y.^2+d*x.*y.^3+e*y.^4;
% [contour(x,y,phi);
% title('phi')
subplot(2,2,1); %plots second plot in top right
Sigma_x=2*c*x.^2+6*d*x.*y-12*a*y.^2-4*c*y.^2; %this is the equation for Sigma x
[m,n]=contour(x,y,Sigma_x);
clabel(m,n)
title('Sigma X')
grid on
subplot(2,2,2) %plots third plot in bottom left
Sigma_y=12*a*x.^2+6*b*x.*y+2*c*y.^2;
[o,p]=contour(x,y,Sigma_y);
clabel(o,p)
title('Sigma Y')
grid on
subplot(2,2,3) %plots fourth plot in bottom right
Tau_xy=-3*b*x.^2-4*c*x.*y-3*d*y.^2;
[q,r]=contour(x,y,Tau_xy);
clabel(q,r)
title('Tau XY')
grid on
```

Fifth Degree Polynomial

```
clear
clc
%need to make sure your mesh is square
[x,y]=meshgrid(-25:.2:25);% This does square at intervals of .1 from -2 to 2
a=1;
b=1;
c=-5;
d=1;
% subplot(2,2,1); %plots first one in top left
% phi=a*x.^5+b*x.^4.*y+c*x.^3.*y.^2+d*x.^2.*y.^3+e*x.*y.^4+f*y.^5;
% contour(x,y,phi);
% title('phi')
subplot(2,2,1); %plots second plot in top right
Sigma_x=2*c*x.^3+6*d*x.^2.*y-12*(5*a+c)*x.*y.^2-4*(b+d)*y.^3; %this is the equation for Sigma x
[m,n]=contour(x,y,Sigma_x);
clabel(m,n)
title('Sigma X')
subplot(2,2,2) %plots third plot in bottom left
Sigma_y=20*a*x.^3+12*b*x.^2.*y+6*x.*y.^2+2*d*y.^3;
[o,p]=contour(x,y,Sigma_y);
clabel(o,p)
title('Sigma Y')
subplot(2,2,3) %plots fourth plot in bottom right
Tau_xy=-4*b*x.^3-6*c*x.^2.*y-6*d*x.*y.^2+4*(5*a+c)*y.^3;
[q,r]=contour(x,y,Tau_xy);
clabel(q,r)
title('Tau XY')
```

Pressure Vessels

```
%Pressure Vessels
clear
clc
ri=3
ro=5
[th,r]=meshgrid((0:5:360)*pi/180,ri:.05:ro);
[X Y]=pol2cart(th,r);
%For External Pressure equal to zero
figure(1)
title('Po=0')
pi=10000
po=0
Sigma_RR_po=(ri^2*pi)/(ro^2-ri^2)*(1-(ro^2./r.^2));
Sigma_TT_po=(pi*ri^2)/(ro^2-ri^2)*(1+(ro^2./r.^2));
h=polar([0 2*pi],[ri ro]);
delete(h)
hold on
[c,h]=contour(X,Y,Sigma_RR_po);
clabel(c,h);
title('Sigma RR')
figure(2)
j=polar([0 2*pi],[ri ro]);
delete(j)
hold on
[f,g]=contour(X,Y,Sigma_TT_po);
clabel(f,g)
title('Sigma TT')

%This portion is for pi=0

figure(3)
pi1=0
po1=10000
Sigma_RR_pi=(ro^2*po1)/(ro^2-ri^2)*((ri^2./r.^2)-1);
Sigma_TT_pi=-(po1*ro^2)/(ro^2-ri^2)*(1+(ri^2./r.^2));
h=polar([0 2*pi],[ri ro]);
delete(h)
hold on
[c,h]=contour(X,Y,Sigma_RR_pi);
clabel(c,h);
title('Sigma RR')
figure(4)
```

```
j=polar([0 2*pi],[ri ro]);
delete(j)
hold on
[f,g]=contour(X,Y,Sigma_TT_pi);
clabel(f,g)
title('Sigma TT')
```

Simply Supported Beam Under Its Own Weight

```
clear
clc
D=1; %thickness
L=5; %set the length equal to this
[x,y]=meshgrid(-L:.1:L, -D:.1:D);
rho=490.752; %Using steel, units are lb/ft^3
g=32.2; %ft/s^2
I=(2/3)*D^3;
Sigma_x=(rho*g*D/I)*((2/5)*D^2*y-(2/3)*y.^3+(x.^2-L^2).*y); %this is the equation for Sigma x
[m,n]=contour(x,y,Sigma_x);
clabel(m,n)
grid on
title('Sigma X')
figure(2);
Sigma_y=(rho*g*D/I)*(-D^2*y/3+(y.^3)/3);
[o,p]=contour(x,y,Sigma_y);
clabel(o,p)
grid on
title('Sigma Y')
figure(3)
Tau_xy=(rho*g*D/I)*(D^2*x-y.^2.*x);
[r,s]=contour(x,y,Tau_xy);
clabel(r,s)
grid on
title('Tau XY')
```

Simply Supported Uniformly Loaded Beam

```
clear
clc
%need to make sure your mesh is square
q=10; %set the distributed load to this
t=1; %set the thickness to this
h=1; %set the height to this\
L=4; %set the length equal to this
[x,y]=meshgrid(-L/2:.1:L/2, -h/2:.1:h/2);
I=h^3/12;
a=-q/4;
b=-(3*q)/(4*h);
c=(q/240*I)*(2*h^2-5*L^2);
d=0;
e=0;
f=q/(h^3);
g=-(e+f)/5;
% phi=a*x.^2+b*x.^2.*y+c*y.^3+d*x.^4+e*x.^4.*y+f*x.*y.^3+g*y.^5;
% contour(x,y,phi,10); %this contours Phi
% title('phi')
Sigma_x=(q/(8*I))*(4*x.^2-L^2).*y+(q/(60*I))*(3*h^2*y-20*y.^3); %this is the equation for Sigma x
[m,n]=contour(x,y,Sigma_x);
clabel(m,n)
grid on
title('Sigma X')
figure(2);
Sigma_y=(q/(24*I))*(-h^3+4*y.^3-3*h^2*y);
[o,p]=contour(x,y,Sigma_y);
clabel(o,p)
grid on
title('Sigma Y')
figure(3)
Tau_xy=(q*x/(8*I)).*(h^2-4.*y.^2);
[r,s]=contour(x,y,Tau_xy);
clabel(r,s)
grid on
title('Tau XY')
```