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# Minimum saturated subgraphs of tripartite graphs

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May 12, 2014

A Thesis Submitted in Partial Fulfillment of the Requirements for the  
Degree of Master of Science in Applied and Computational Mathematics

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## Abstract

Let  $F$  and  $H$  be graphs. A subgraph  $G$  of  $H$  is an  $F$ -saturated subgraph of  $H$  if  $F$  is not a subgraph of  $G$  and  $F$  is a subgraph of  $G + e$  for any edge  $e \in E(H) \setminus E(G)$ . The saturation number of  $F$  in  $H$  is the minimum number of edges in a  $F$ -saturated subgraph of  $H$ . We denote the saturation number of  $F$  in  $H$  as  $\text{sat}(H, F)$ . In this thesis we review the history of saturated subgraphs, and prove new results on saturated subgraphs of tripartite graphs. Let  $K_{a,b,c}$  be a complete tripartite graph, with partite sets of size  $a$ ,  $b$ , and  $c$ . Specifically, we determine  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})$ , for  $n_1 \geq n_2 \geq n_3$ , when  $n_2$  bounded by a linear function of  $n_3$ . We also examine the special case when  $\ell = 1$  and determine  $\text{sat}(K_{n_1, n_2, n_3}, K_3)$  for  $n_1 \geq n_2 \geq n_3$ , and  $n_3$  sufficiently large. We also consider two natural variants of saturated subgraphs that arise in the tripartite setting. We examine the behavior of these extensions using illustrative examples to highlight the differences between these variations and the original problem.

# Contents

List Of Figures	iii
<b>1 Introduction</b>	<b>1</b>
1.1 Saturation Numbers and Extremal Numbers . . . . .	1
1.2 Saturated Subgraphs . . . . .	1
1.3 Natural Extensions . . . . .	3
1.4 Overview of Thesis . . . . .	5
1.5 Notation and Definitions . . . . .	6
<b>2 Prior work</b>	<b>8</b>
2.1 Extremal Numbers and Saturation Numbers . . . . .	8
2.2 Bipartite, Tripartite, and Multipartite Host Graphs . . . . .	10
<b>3 Main Results: <math>\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})</math></b>	<b>15</b>
3.1 Constructions . . . . .	15
3.2 Main Proof . . . . .	18
<b>4 Constructions for other Forbidden Graphs</b>	<b>28</b>
<b>5 Colored-Saturation and Ordered-Saturation</b>	<b>34</b>
<b>6 Future Work</b>	<b>40</b>
<b>References</b>	<b>43</b>

# List of Figures

1	Colorings of $C_4$ . . . . .	4
2	A $(C_4, c_2)$ - ordered saturated subgraph of $K_{n,n,n}$ . . . . .	5
3	A $(C_4, c_2)$ -vertex-color saturated subgraph of $K_{n,n,n}$ . . . . .	5
4	A $K_{(\ell,m)}$ -order saturated subgraph of $K_{(n_1,n_2)}$ . . . . .	12
5	A $K_{\ell,m}$ -saturated subgraph of $K_{n,n}$ . . . . .	13
6	Construction 1, a $K_{\ell,\ell,\ell}$ -saturated subgraph of $K_{n_1,n_2,n_3}$ . . . . .	15
7	Construction 2, A $K_{\ell,\ell,\ell}$ -saturated subgraph of $K_{n_1,n_2,n_3}$ . . . . .	16
8	Construction 3, a $K_{\ell,m,m}$ -saturated subgraph of $K_{n_1,n_2,n_3}$ . . . . .	29
9	Construction 4, a $K_{\ell,m,p}$ -saturated subgraph of $K_{n_1,n_2,n_3}$ . . . . .	31
10	Construction 5, a $C_4$ -saturated subgraph of $K_{n,n,n}$ . . . . .	32
11	Construction 6, a $K_{(\ell,m,p)}$ -ordered saturated subgraph of $K_{(n_1,n_2,n_3)}$ . . . . .	35
12	Construction 7, a $(C_4, c_2)$ -ordered-saturated subgraph of $(K_{n_1,n_2,n_3}, c_H)$ . . .	37
13	Construction 8, a $(C_4, c_2)$ colored-saturated subgraph of $K_{n_1,n_2,n_3}$ . . . . .	38

# 1 Introduction

## 1.1 Saturation Numbers and Extremal Numbers

Let  $F$  and  $G$  be graphs. We say that  $G$  is an  $F$ -free if  $F$  is not a subgraph of  $G$ . We will call  $F$  the *forbidden* graph. We say that  $G$  is  $F$ -saturated if  $G$  is  $F$ -free, and  $F$  is a subgraph of  $G + e$  for any edge  $e \in E(\overline{G})$ . The *extremal number*, denoted  $\text{ex}(n, F)$ , is the maximum size of an  $n$ -vertex  $F$ -saturated graph. We let  $\text{Ex}(n, F)$  denote the set of  $n$ -vertex  $F$ -saturated graphs with size  $\text{ex}(n, F)$ . The *saturation number*, denoted  $\text{sat}(n, F)$ , is the minimum size of an  $n$ -vertex  $F$ -saturated graph. We let  $\text{Sat}(n, F)$  be the set of  $n$ -vertex  $F$ -saturated graphs with size  $\text{sat}(n, F)$ .

In 1907 Mantel [15] determined the extremal number for  $K_3$ . In 1941, Turán [18] determined  $\text{ex}(n, K_\ell)$  and characterized the graphs in  $\text{Ex}(n, K_\ell)$ . In 1946, Erdős and Stone [8] determined the asymptotic values of  $\text{ex}(n, K_{\ell, \dots, \ell})$ . Later, in [7], Erdős and Simionovits proved a corollary to the Erdős-Stone result, and determined the asymptotic value of  $\text{ex}(n, F)$ .

The first result on saturation numbers was in 1964 when Erdős, Hajnal, and Moon [6] determined  $\text{sat}(n, K_\ell)$  and characterized the graphs in  $\text{Sat}(n, K_\ell)$ . Later, Kászonyi and Tuza [13] provided a general construction and upper bound for  $\text{sat}(n, F)$ . In particular they found that saturation numbers are linear in  $n$ , the order of the host graph.

## 1.2 Saturated Subgraphs

Let  $H$  and  $F$  be fixed graphs, and let  $G$  be a spanning subgraph of  $H$ . We say that  $G$  is an  $F$ -free subgraph of  $H$  if  $F$  is not a subgraph of  $G$ . In this paper we will call  $H$  the *host* graph, and we will call  $F$  the *forbidden* graph. We say that  $G$  is an  $F$ -saturated subgraph of  $H$  if  $G$  is  $F$ -free and for any edge  $e \in E(H) \setminus E(G)$ ,  $F$  is a subgraph of  $G + e$ . The *extremal number* of  $F$  in  $H$ , denoted  $\text{ex}(H, F)$ , is the maximum size of an  $F$ -saturated subgraph of  $H$ . We also let  $\text{Ex}(H, F)$  be the set of  $F$ -saturated subgraphs of  $H$  that have size  $\text{ex}(H, F)$ . The

*saturation number* of  $F$  in  $H$ , denoted  $\text{sat}(H, F)$ , is the minimum size of an  $F$ -saturated subgraph of  $H$ . We let  $\text{Sat}(H, F)$  be the set of  $F$ -saturated subgraphs of  $H$  that have size  $\text{sat}(H, F)$ .

In the late 1960s, Bollobás and Wessel [1, 2, 19, 20] independently determined the saturation number for bipartite graphs given a few extra conditions. If we color the  $n_1$  and  $\ell$  sets with one color, and the  $n_2$  and  $m$  sets with another we see that the minimum size of a  $K_{\ell, m}$ -ordered-saturated subgraphs of  $K_{n_1, n_2}$  is  $(\ell - 1)n_2 + (m - 1)n_1 - (\ell - 1)(m - 1)$  [2]. It is only recently that the unordered saturation number for complete bipartite graphs has been considered.

In 2012, Moshkovitz and Shapira [16] considered saturation in  $d$ -uniform  $d$ -partite hypergraphs. When  $d = 2$ , this reduces to saturation in bipartite graphs. A graph is *weakly-saturated* if there is some ordering of the missing edges such that with each edge added into the graph a new copy of the forbidden graph becomes a subgraph. They observed that weakly-saturated subgraphs of bipartite graphs are smaller than weakly-ordered-saturated subgraphs of bipartite graphs, and the same holds for saturated subgraphs of bipartite graphs and ordered-saturated subgraphs of bipartite graphs. They provided a construction showing that  $\text{sat}(K_{n, n}, K_{\ell, m}) \leq (\ell + m - 2)n - \left\lfloor \left( \frac{(\ell + m - 2)}{2} \right)^2 \right\rfloor$  and conjecture that, for  $n$  large enough, the bound is sharp. In early 2014 Gan, Korándi and Sudakov [11] showed that  $\text{sat}(K_{n, n}, K_{\ell, m}) \geq (\ell + m - 2)n - (\ell + m - 2)^2$ . They also examined  $K_{2, 3}$ -saturated subgraphs of  $K_{n, n}$ , the first nontrivial case, and proved that Moshkovitz and Shapira's bound is sharp.

In 2013, Ferrara, Jacobson, Pfender, and Wenger [10] examined  $K_3$ -saturated multipartite graphs. They determined  $\text{sat}(K_{n, \dots, n}, K_3)$ . Moreover, they proved that  $\text{Sat}(K_{n, \dots, n}, K_3)$  contains exactly two graphs up to isomorphism, and that  $\text{Sat}(K_{n, n, n}, K_3)$  contains one graph.

In [3], Bondy, Shen, Thomassé, and Thomassen proved several results about extremal numbers when the host graph is a complete multipartite whose partite sets are finite. In particular they examined the edge densities where a multipartite graph can no longer be



triangle-free. In the tripartite case, they proved that the edge density is the golden ration, and when there are infinitely many partite sets, the density exists and is  $\frac{1}{2}$ . In [17], Pfender determined that if there are enough partite sets then the maximum density of  $K_\ell$ -free subgraphs of multipartite graphs is  $\frac{\ell-2}{\ell-1}$ .

### 1.3 Natural Extensions

In this thesis we will focus on  $\text{sat}(H, F)$  when  $H$  is a complete tripartite graph. Since we are examining tripartite graphs, our graphs have a natural 3-coloring. This leads us to consider some variants where we consider particular proper colorings of the forbidden graph. Let  $c_H : V(H) \rightarrow [k]$  be a coloring of  $H$ , and let  $c_F : V(F) \rightarrow [k]$  be a coloring of  $F$ . We will let  $K_{(n_1, n_2)}$  be a copy of  $K_{n_1, n_2}$  colored by  $c(v_i^s) = i$ , for  $v_i^s \in V_i$ . Likewise, let  $K_{(n_1, \dots, n_k)}$  be a copy of  $K_{n_1, \dots, n_k}$  colored by  $c(v_i^s) = i$  for  $v_i^s \in V_i$ .

Let  $G$  be a spanning subgraph of  $H$ , and let  $G$  inherit the coloring of  $H$ . We now adjust the notions of  $F$ -free, and  $F$ -saturated so that they respect the colorings of  $H$  and  $F$ . We say that  $G$  is  $(F, c_F)$ -ordered-free if every copy of  $F$  contained in  $G$  does not have the coloring  $c_F$ . We say that  $G$  is  $(F, c_F)$ -ordered-saturated if  $G$  is  $(F, c_F)$ -ordered free and a copy of  $F$  with coloring  $c_F$  is a subgraph of  $G + e$  for any edge  $e \in E(H) \setminus E(G)$ . The *ordered-saturation number* of  $(F, c_F)$  in  $(H, c_H)$ , denoted  $\overrightarrow{\text{sat}}((H, c_H), (F, c_F))$ , is the minimum size of an  $(F, c_F)$ -ordered-saturated subgraph of  $(H, c_H)$ .

A weaker notion is *colored-saturation*; let  $c_F : V(F) \rightarrow [k]$  be a coloring of  $F$ , and let  $c_H : V(H) \rightarrow [k]$  be a coloring of  $H$ . We say that  $G$  is an  $(F, c_F)$ -colored-free subgraph of  $(H, c_H)$  if:

- $F$  is not a subgraph of  $G$ , or
- if  $F'$  is a subgraph of  $G$  that is isomorphic to  $F$ , then for any permutation  $\sigma : [k] \rightarrow [k]$  then  $c_H|_{V(F')} \neq \sigma(c_F)$ .

That is, the coloring of  $F'$  is not the coloring of  $F$ , even allowing for the relabeling of colors classes. We say that  $G$  is  $(F, c_F)$ -colored-saturated if  $G$  is  $(F, c_F)$ -colored-free and for any edge  $e \in E(H) - E(G)$  then there exists a permutation  $\sigma : [k] \rightarrow [k]$  such that  $G + e$  contains a copy of  $F$  with coloring  $\sigma(c_F)$ . The *colored-saturation* number of  $(F, c_F)$  in  $H$ , denoted  $\text{sat}((H, c_H), (F, c_F))$ , is the minimum size of an  $(F, c_F)$ -colored-saturated subgraph of  $(H, c_H)$ .

We will now provide a few examples to illustrate the differences between saturated subgraphs, colored-saturated subgraphs, and ordered-saturated subgraphs. We will use  $C_4$  as the forbidden graph, and  $K_{n,n,n}$  as our host. For  $i \in [3]$  let  $V_i$  be a partite set of  $K_{n,n,n}$ . Let  $c_H : V(K_{n,n,n}) \rightarrow \{1, 2, 3\}$ , so that for  $i \in \{1, 2, 3\}$ , if  $v \in V_i$ , then  $c_H(v) = i$ . There are two ways, up to relabeling, to properly color  $C_4$  using at most three colors. Call them  $c_2 : V(C_4) \rightarrow \{1, 2\}$  and  $c_3 : V(C_4) \rightarrow \{1, 2, 3\}$ . We show them in Figure 1.

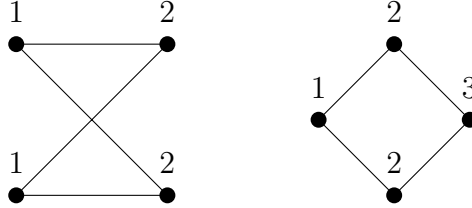


Figure 1: The colorings of  $C_4$ :  $c_2$  on the left, and  $c_3$  on the right.

Consider the  $(C_4, c_2)$ -ordered saturated subgraph of  $(K_{n,n,n}, c_H)$  shown in Figure 2. The graph shown contains many copies of  $C_4$ , and the complete joins between  $V_3$  and  $V_1$ , and between  $V_3$  and  $V_2$  contain copies of  $C_4$  that are colored with only two colors. However, the graph contains no copies of  $C_4$  with the specific coloring  $c_2$ . Figure 2 is free of that particular coloring of  $C_4$ .

We will now consider the  $(C_4, c_2)$  colored-saturated subgraph of  $K_{(n,n,n)}$  shown in Figure 3. Again the graph clearly contains copies of  $C_4$ . However, all copies of  $C_4$  present are colored with three colors, and we are only concerned with copies of  $C_4$  with two colors.

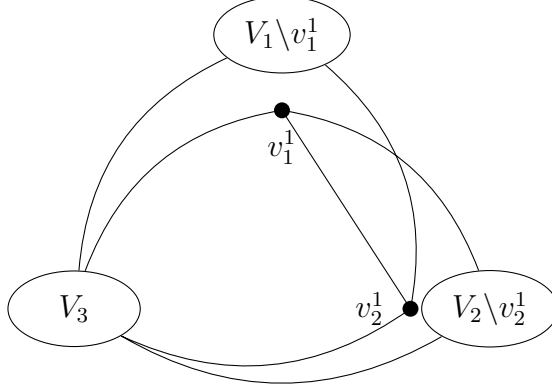


Figure 2: A  $(C_4, c_2)$ - ordered saturated subgraph of  $K_{n,n,n}$ . The lines denote complete joins between sets.

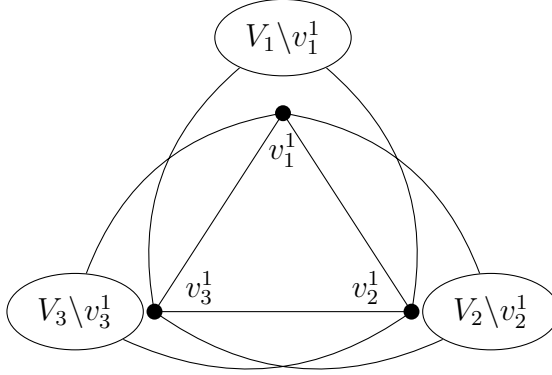


Figure 3: A graph that is  $(C_4, c_2)$ -colored-saturated subgraph of  $K_{n,n,n}$ . Lines denote complete joins between sets.

## 1.4 Overview of Thesis

In Section 2 we provide a history of prior work on saturated subgraphs. We start with a review of results of saturated subgraphs of complete graph in section 2.1. We follow this with an explanation of results concerning saturated subgraphs of multipartite graphs with two or more partite sets.

In Section 3 we present the main results of this thesis. In Section 3.1 we determine upper bounds on  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})$ . In 3.2 we prove that the bounds are sharp and determine the members of  $\text{Sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})$ . In this section we also examine the special case of  $\ell = 1$ .

In Section 4 we consider  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, m, p})$ , and provide upper bounds on what appear to be the two cases. The first, when  $\ell > m = p$ , and the second when  $\ell \geq m > p$ . We also determine  $\text{sat}(K_{n_1, n_2, n_3}, C_4)$ .

In Section 5 we discuss ordered-saturation and colored-saturation. We provide an upper bound on  $\overrightarrow{\text{sat}}(K_{(n_1, n_2, n_3)}, K_{(\ell, m, p)})$ . We also determine  $\overrightarrow{\text{sat}}(K_{(n_1, n_2, n_3)}, K_{(2, 2, 0)})$ , and the two cases of colored-saturation of  $C_4$ .

Finally, in Section 6 we discuss the next steps to take and future work beyond those steps.

## 1.5 Notation and Definitions

In this section we will provide the necessary definitions and notation for this thesis. A *graph*  $G$  consists of two sets: a *vertex set*, denoted  $V(G)$ , and an *edge set*, denoted  $E(G)$ . Each edge is an unordered pair of vertices. The two vertices in an edge are called the *endpoints* of the edge, and we say that a vertex is *incident* to an edge if it is an endpoint of that edge. An edge *joins* its two endpoints. A vertex is *adjacent* to another vertex if they are both incident to a common edge. The *neighborhood* of a vertex  $v$ , denoted  $N(v)$ , is the set of vertices that are adjacent to  $v$ . The *degree* of a vertex  $v$ , denoted  $d(v)$ , is the size of the neighborhood of  $v$ , so  $d(v) = |N(v)|$ . The minimum degree of graph  $G$ , denoted  $\delta(G)$ , is minimum degree of all vertices in  $G$ , so  $\delta(G) = \min_{v \in V(G)} d(v)$ . We also let  $[n] = \{1, \dots, n\}$ .

The *complete graph* on  $n$  vertices, denoted  $K_n$ , is a graph with every possible edge. The *cycle* on  $n$  vertices, denoted  $C_n$ , is the graph where  $V(C_n) = \{v_1, \dots, v_n\}$  and  $E(C_n) = \{v_1v_2, \dots, v_iv_{i+1}, \dots, v_{n-1}v_n, v_nv_1\}$ . The *path* on  $n$  vertices, denoted  $P_n$ , is the graph where  $V(P_n) = \{v_1, \dots, v_n\}$  and  $E(P_n) = \{v_1v_2, \dots, v_iv_{i+1}, \dots, v_{n-1}v_n\}$ . The *complement* of a graph  $G$ , denoted  $\overline{G}$ , is a graph with  $V(\overline{G}) = V(G)$  and  $E(\overline{G}) = E(K_{|V(G)|}) \setminus E(G)$ .

A *bipartite graph* is a graph where we can partition  $V(G)$  into two sets that we call  $V_1(G)$  and  $V_2(G)$ , such that every edge in  $G$  has one endpoint in  $V_1(G)$  and its other endpoint in

$V_2(G)$ . A *complete bipartite graph*, denoted  $K_{a,b}$ , is a bipartite graph such that  $|V_1(K_{a,b})| = a$ ,  $|V_2(K_{a,b})| = b$ , and  $E(K_{a,b}) = \{uv : u \in V_1(K_{a,b}), v \in V_2(K_{a,b})\}$ . A *tripartite graph* is a graph where we can partition  $V(G)$  into three sets, that we call  $V_1(G)$ ,  $V_2(G)$ , and  $V_3(G)$  such that any edge in  $G$  has its endpoints in different sets. A *complete tripartite graph*, denoted  $K_{a,b,c}$ , is a tripartite graph such that  $|V_1(K_{a,b,c})| = a$ ,  $|V_2(K_{a,b,c})| = b$ ,  $|V_3(K_{a,b,c})| = c$ , and  $E(K_{a,b,c}) = \{uv : u \in V_1(K_{a,b,c}), v \in (V_2(K_{a,b,c}) \cup V_3(K_{a,b,c}))\} \cup \{uv : u \in V_2(K_{a,b,c}), v \in V_3(K_{a,b,c})\}$ . Let  $\underbrace{K_{k,\dots,k}}_{r \text{ times}}$  be the complete multipartite graph on  $rk$  vertices with  $r$  partite sets of size  $k$ . Where no confusion will arise, we denote a partite set  $V_i(G)$  as  $V_i$ . We extend the notion of the minimum degree of a graph to the minimum degree of a partite set by letting  $\delta_i = \min_{v \in V_i} d(v)$ . We also will let  $N_i(v) = \{u : u \in N(v) \cap V_i\}$ .

An *independent set* is a set of vertices such that no pair of vertices in the set are adjacent. The *independence number* of a graph  $G$ , denoted  $\alpha(G)$ , is maximum size of an independent set in  $G$ .

We call a function  $c_G : V(G) \rightarrow [k]$  a *k-vertex coloring*, or a *k-coloring* of  $G$ . A coloring is a *proper coloring* if adjacent vertices have different colors. The *chromatic number* of a graph  $G$ , denoted  $\chi(G)$ , is the smallest number of colors needed to properly color  $G$ . All colorings in this paper are proper colorings.

Let  $G$  and  $G'$  be graphs. We say that  $G'$  is a *subgraph* of  $G$  if  $V(G') \subseteq V(G)$ , and  $E(G') \subseteq E(G)$ . If  $G'$  is a subgraph of  $G$  and  $V(G') = V(G)$  then  $G'$  is a *spanning subgraph* of  $G$ . We call  $G'$  an *induced subgraph* of  $G$ , if  $V(G') \subseteq V(G)$ , and  $E(G') = \{uv : u \in V(G'), v \in V(G'), uv \in E(G)\}$ . If  $V(G') = S$  then we denote the induced subgraph as  $G[S]$ .

## 2 Prior work

We direct the reader to [9] for an excellent survey on saturation numbers and questions concerning them.

### 2.1 Extremal Numbers and Saturation Numbers

The study of extremal numbers began with Mantel [15] who determined  $\text{ex}(n, K_3)$ . Later, Turán [18] determined both  $\text{ex}(n, K_\ell)$  and  $\text{Ex}(n, K_\ell)$ . Let  $T_\ell(n)$  be the complete  $\ell$ -partite graph with  $n$  vertices such that each partite set has size  $\lfloor \frac{n}{\ell} \rfloor$  or  $\lceil \frac{n}{\ell} \rceil$ . We call the graph  $T_\ell(n)$  a *Turán graph*.

**Theorem 1** (Turán [18], 1941).

$$\text{Ex}(n, K_\ell) = \{T_{\ell-1}(n)\}$$

and thus,

$$\text{ex}(n, K_\ell) = |E(T_{\ell-1}(n))|.$$

Due to this result, questions concerning extremal numbers are frequently called Turán's problem or Turán-type problems.

Erdős and Stone [8] generalized Turán's Theorem to complete balanced multipartite graphs.

**Theorem 2** (Erdős and Stone [8], 1946).

$$\text{ex}(n, K_{\underbrace{\ell, \dots, \ell}_{r \text{ times}}}) = \frac{r-2}{r-1} \binom{n}{2} + o(n^2).$$

Erdős and Simonovits, in [7], later extended Theorem 2.

**Corollary 3** (Erdős and Simonovits [7], 1966).

$$\text{ex}(n, F) = \frac{\chi(F) - 2}{\chi(F) - 1} \binom{n}{2} + o(n^2).$$

Corollary 3 follows from the fact that  $F$  cannot be a subgraph of  $T_{\chi(F)-1}(n)$ , which provides a lower bound. The upper bound is given by Theorem 2 since extremal numbers are monotone.

In 1964, Erdős, Hajnal, and Moon [6] considered the minimum size of a  $K_\ell$ -saturated graph. Let  $A_\ell(n)$  be a graph on  $n$  vertices such that  $A_\ell(n) = K_{\ell-2} \vee \overline{K_{n-\ell+2}}$ . Where  $G_1 \vee G_2$  is the graph such that  $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$  and  $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in (G_2)\}$

**Theorem 4** (Erdős, Hajnal, and Moon [6], 1964). *If  $n$  and  $\ell$  are integers such that  $2 \leq \ell$ , then*

$$\text{Sat}(n, K_\ell) = \{A_\ell(n)\},$$

*and thus*

$$\text{sat}(n, K_\ell) = (\ell - 2)(n - \ell + 2) + \binom{\ell - 2}{2}.$$

In [13], Kászonyi and Tuza provided the first general upper bound on  $\text{sat}(n, F)$  for any graph  $F$ . To do so, they had to consider a more general setting of saturation. Here, we present a restricted version of their theorem.

**Theorem 5** (Kászonyi and Tuza [13]). *Let  $F$  be a graph with independence number  $\alpha(F)$ . Let  $d$  be the minimum number of neighbors that any vertex has in any independence set of size  $\alpha(F)$ . The saturation number of  $F$  is bounded by*

$$\text{sat}(n, F) \leq (|V(F)| - \alpha(F) - 1)n + \frac{1}{2}(d - 1)(n - (|V(F)| - \alpha(F) - 1)).$$

Let  $\mathcal{F}$  be a family of graphs. We say that  $G$  is  $\mathcal{F}$ -free if  $G$  does not contain any member of  $\mathcal{F}$  as a subgraph. Also, we say that  $G$  is  $\mathcal{F}$ -saturated if  $G$  is  $\mathcal{F}$ -free and  $G + e$  contains some member of  $\mathcal{F}$  as a subgraph for any edge  $e \in E(\overline{G})$ . Let,  $\text{sat}(n, \mathcal{F})$  denote the minimum size of an  $n$ -vertex  $\mathcal{F}$ -saturated graph. A significantly abridged version of the Kászonyi-Tuza Theorem is, for all  $\mathcal{F}$ , that  $\text{sat}(n, \mathcal{F}) < cn$  for some  $c = c(\mathcal{F})$ .

## 2.2 Bipartite, Tripartite, and Multipartite Host Graphs

So far we have looked at the study of saturation where the host structure is  $K_n$ . However, we can also consider other host structures, and consider saturated subgraphs.

We can ask, like Zarankiewicz in [21], about the number of 1s in an  $n_1 \times n_2$  0,1-matrix that does not contain an  $\ell \times m$ -submatrix of all 1s. The *Zarankiewicz number*, denoted  $z(n_1, n_2; \ell, m)$ , is the maximum number of 1s an  $n_1 \times n_2$  0,1-matrix can contain such that there is no  $\ell \times m$  submatrix of all 1s. So far, the best result on Zarankiewicz numbers is from Kővári, Sós, and Turán [14] and Hyltén-Cavallius [12].

**Theorem 6** (Kővári, Sós, and Turán, Hyltén-Cavallius [14, 12] 1954, 1958).

$$z(n_1, n_2; \ell, m) < (\ell - 1)^{\frac{1}{m}}(n_1 - m + 1)n_2^{1 - \frac{1}{m}} + (m - 1)n_2.$$

An  $n_1 \times n_2$  0,1-matrix can be thought of as a bipartite graph with sets  $n_1$  and  $n_2$ , where two vertices are adjacent if the corresponding entry in the matrix is an 1. We can also think of the Zarankiewicz number as an extremal number, as such we can also look at the saturation number as well. Bollobás and Wessel [1, 2, 19, 20] independently determined  $\overrightarrow{\text{sat}}(K_{(n_1, n_2)}, K_{(\ell, m)})$ .

**Theorem 7** (Bollobás, Wessel [1, 2, 20, 19], 1967).

$$\overrightarrow{\text{sat}}(K_{(n_1, n_2)}, K_{(\ell, m)}) = (\ell - 1)n_2 + (m - 1)n_1 - (\ell - 1)(m - 1).$$



We will now present Bollobás's argument from [2].

*Proof.* Let  $G$  be a bipartite graph, with partite sets  $V_1$ , with size  $n_1$  and  $V_2$  with size  $n_2$ , such that the addition of an edge between its partite sets increases the number of copies of  $K_{(\ell,m)}$  contained with  $G$ , it is important to note that  $G$  need not be  $K_{(\ell,m)}$ -free, that if  $G$  may contain a set of size  $\ell$  in  $V_1$  that is completely joined to a set of size  $m$  in  $V_2$ . We first show by induction on  $k$  that  $G$  must contain at least  $(\ell - 1)n_2 + (m - 1)n_1 - (\ell - 1)(m - 1)$  edges.

The base case when  $\ell$  and  $m$  are both 1 is trivial. We also note that we can exploit symmetry, so we only need to show that if the property holds for  $\ell$  or less, and  $m$  or less, then it holds for  $\ell + 1$  and  $m$ .

Let  $\overline{G}$  be the bipartite complement of  $G$ , that is a graph such that  $E(\overline{G}) = \{uv : u \in V_1, v \in V_2, uv \notin E(G)\}$ . We will show that there are at most  $(n_1 - \ell)(n_2 - m + 1)$  edges in  $\overline{G}$ . For each edge in  $\overline{G}$  will assign a total weight of  $n_1$  to  $V(G)$ . For each edge  $e \in E(\overline{G})$ , there is a copy of  $K_{\ell+1,m}$  in  $G$  with the set of size  $\ell + 1$  in  $V_1$  and the set of size  $m$  in  $V_2$  that is completed by its addition. We assign weight  $n_1 - \ell$  to the vertex that is adjacent to  $e$  in  $V_1$ , and we assign weight 1 to each of the other  $\ell$  vertices in  $V_1$  that form the copy of  $K_{(\ell+1,m)}$ .

Let  $v$  be an arbitrary vertex in  $V_1$ , and let  $v$  have  $r$  neighbors in  $\overline{G}$ . Any edge in  $\overline{G}$  that is not incident to  $v$  that adds weight to  $v$  cannot be incident of the  $r$  neighbors of  $v$ . Now consider  $\hat{G}$ , a subgraph of  $\overline{G}$ , such that  $v$  and its  $r$  neighbors have been removed. The edges in  $\hat{G}$  are the only edges in  $\overline{G}$  that can assign weight 1 to  $v$ . By the induction hypothesis there are at most  $((n_1 - 1) - \ell + 1)((n_2 - r) - m + 1)$  edges in  $\hat{G}$ . Thus, the total weight on  $v$  is at most  $(n_1 - \ell)(n_2 - m + 1)$ . We also know that the total weight in  $\overline{G}$  is  $n_1|E(\overline{G})|$ . Therefore,

$$n_1|E(\overline{G})| \leq n_1(n_1 - \ell)(n_2 - m + 1).$$

This means that  $\overrightarrow{\text{sat}}(K_{(n_1,n_2)}, K_{(\ell,m)}) \geq (\ell - 1)n_2 + (m - 1)n_1 - (\ell - 1)(m - 1)$ .

Let  $G_b$  be a bipartite graph with partite sets  $V_1$  and  $V_2$ , such that  $|V_1| = n_1$  and  $|V_2| = n_2$ . Let  $S_1$  be a set of  $\ell - 1$  vertices that is a subset of  $V_1$ , and let  $S_2$  be a set of  $m - 1$  vertices that is a subset of  $V_2$ . Join  $S_1$  to  $V_2$  and join  $S_2$  to  $V_1$ . We note that  $G_b$  is clearly saturated a  $K_{(\ell,m)}$  saturated subgraph of  $K_{(n_1,n_2)}$ .  $\square$

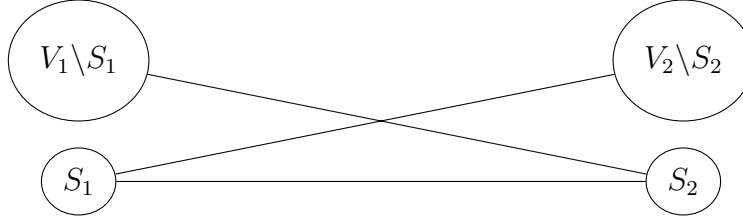


Figure 4: A  $K_{(\ell,m)}$ -order saturated subgraph of  $K_{(n_1,n_2)}$ . Lines denote complete joins.

In [16], Moshkovitz and Shapira, after considering the more general problem of weakly saturated hypergraphs, remarked that there are constructions that show that  $\text{sat}(K_{n,n}, K_{\ell,m})$  is smaller than  $\overrightarrow{\text{sat}}(K_{(n,n)}, K_{(\ell,m)})$  when  $\ell \neq m$ . After giving an example of a  $K_{\ell,m}$ -saturated subgraph of  $K_{n,n}$  they made the following conjecture:

**Conjecture 8** (Moshkovitz and Shapira [16], 2012+). *For  $\ell > m$  and  $n$  large enough*

$$\text{sat}(K_{n,n}, K_{\ell,m}) = (\ell + m - 2)n - \left\lfloor \left( \frac{\ell + m - 2}{2} \right)^2 \right\rfloor.$$

Figure 5 shows the construction Moshkovitz and Shapira provided. In this construction,  $S_1$  and  $S_2$  have size  $m - 1$ , and the sets  $T_1$  and  $T_2$  have size  $\lfloor \frac{\ell+m-2}{2} \rfloor$ .

Recently, Gan, Korándi, and Sudakov [11] considered Conjecture 8, and provided a lower bound.

**Theorem 9** (Gan, Korándi, and Sudakov [11], 2014+). *For  $m \leq \ell \leq n$ ,*

$$\text{sat}(K_{n,n}, K_{\ell,m}) \geq (\ell + m - 2)n - (\ell + m - 2)^2.$$

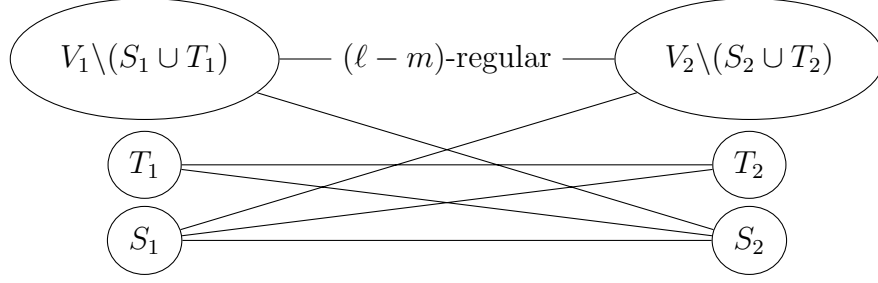


Figure 5: A  $K_{\ell,m}$ -saturated subgraph of  $K_{n,n}$ . Lines denote complete joins.

Unlike the proof of Theorem 7, Gan, Korándi and Sudakov [11] did not use an argument based on edge weights to prove Theorem 9. Instead, they partitioned the bipartite graph by vertex degrees. Using additional properties of  $K_{\ell,m}$ -saturated subgraphs of  $K_{n,n}$  they were able to directly count the edges in the graph.

In [10], Ferrara, Jacobson, Pfender, and Wenger determined  $\text{sat}(K_{\underbrace{n, \dots, n}_{r \text{ times}}, K_3)$ . Of particular interest to this thesis they determined  $\text{sat}(K_{n,n,n}, K_3)$ . In that paper they also provided the set of graphs that have size  $\text{sat}(K_{\underbrace{n, \dots, n}_{r \text{ times}}, K_3)$ .

**Theorem 10** (Ferrara, Jacobson, Pfender, and Wenger [10], 2013+). *If  $r \geq 3$  and  $n \geq 100$ , then*

$$\text{sat}(K_{\underbrace{n, \dots, n}_{r \text{ times}}, K_3) = \min\{2rn + n^2 - 4r - 1, 3rn - 3n - 6\}.$$

**Theorem 11** (Ferrara, Jacobson, Pfender, and Wenger [10], 2013+).

$$\text{sat}(K_{n,n,n}, K_3) = 6n - 6.$$

The proof of Theorem 11 begins by establishing the nature of neighborhoods of vertices of small degree in a  $K_3$ -saturated subgraph of  $K_{n,n,n}$ . This enabled them to count edges between the sets opposite a vertex of low degree. The main results from this paper are built around this technique.

In [3], Bondy, Shen, Thomassé, and Thomassen proved several results about extremal

numbers when the host graph is a complete multipartite whose partite sets are finite. In particular they examined the edge densities where a graph can no longer be triangle-free. If only the edge count was considered then the results would trivially reduce to Mantel's result from [15]. By considering the maximum minimum edge density between partite sets the fact that the host graph is a multipartite graph is forced to be an important factor.

**Theorem 12** (Bondy, Shen, Thomassé, and Thomassen [3], 2006). *The maximum minimum edge density between partite sets in a  $K_3$ -saturated subgraph of  $K_{n_1, n_2, n_3}$  is the golden ratio.*

**Theorem 13** (Bondy, Shen, Thomassé, and Thomassen [3], 2006). *If  $n_i$  is finite for all  $i \in \mathbb{Z}^+$  then the maximum minimum edge density between partite sets in a  $K_3$ -saturated subgraph of  $K_{n_1, n_2, \dots}$  is  $\frac{1}{2}$ .*

In [17], Pfender determined that if there are enough partite sets then the maximum density of  $K_\ell$ -free subgraphs of multipartite graphs is  $\frac{\ell-2}{\ell-1}$ .

In [5], Conlon examined  $\text{ex}(Q_n, F)$  for a variety of graphs  $F$ , and provided a unified approach to some forbidden graphs that previously required separate techniques. In [4], Choi and Guan examined  $\text{sat}(Q_n, C_4)$ , and provided an improved upper bound of  $(\frac{1}{4} + \epsilon)n2^{n-1}$ .

### 3 Main Results: $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})$

We begin this section by providing some constructions that provide an upper bound for  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})$ . We then determine  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})$  and  $\text{Sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})$ .

#### 3.1 Constructions

Let the three partite sets of  $K_{n_1, n_2, n_3}$  be  $V_1$ ,  $V_2$ , and  $V_3$ ,  $V_i = \{v_i^1, v_i^2, \dots, v_i^{n_i}\}$ . In this section all arithmetic in subscripts is performed modulo 3.

**Construction 1.** For each  $i \in [3]$ , let  $S_i = \{v_i^1, \dots, v_i^\ell\}$ . For all  $i$ , join  $S_i$  to  $V_{i+1}$ , and  $V_{i+2}$ , and then remove the edges  $v_1^1 v_2^1$ ,  $v_1^1 v_3^1$ , and  $v_2^1 v_3^1$ . See Figure 6. We call this graph  $G_\ell^0(n_1, n_2, n_3)$ . Thus,

$$E(G_\ell^0(n_1, n_2, n_3)) = \{v_i^r v_j^s : i \in [3], j \in [3], i \neq j, r \leq \ell \text{ or } s \leq \ell\} \setminus \{v_1^1 v_2^1, v_1^1 v_3^1, v_2^1 v_3^1\}.$$

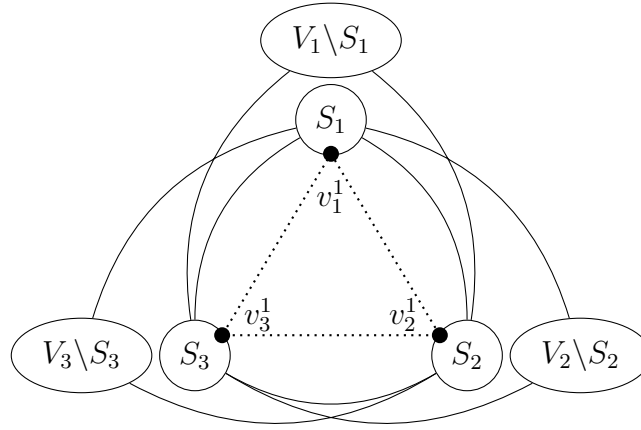


Figure 6: Construction 1: A  $K_{\ell, \ell, \ell}$ -saturated subgraph of  $K_{n_1, n_2, n_3}$ . Solid lines denote complete joins between sets, and dashed lines denote edges that have been removed.

**Construction 2.** We now will construct a family of three graphs, let  $i \in [3]$ , and let  $G_\ell^i(n_1, n_2, n_3)$  be a graph such that  $n_i > \ell + 1$ . For each  $j \in [3]$ , let  $S_j = \{v_j^1, \dots, v_j^\ell\}$ . Join

$S_j$  to  $V_{j+1}$ , and  $V_{j+2}$ , and then remove the edges  $v_i^1 v_{i+1}^1$ ,  $v_i^2 v_{i+2}^1$ , and  $v_{i+1}^1 v_{i+2}^1$ . See Figure 7. Thus,

$$E(G_\ell^i(n_1, n_2, n_3)) = \{v_j^r v_k^s : j \in [3], k \in [3], j \neq k, r \leq \ell \text{ or } s \leq \ell\} \setminus \{v_i^1 v_{i+1}^1, v_i^2 v_{i+2}^1, v_{i+1}^1 v_{i+2}^1\}.$$

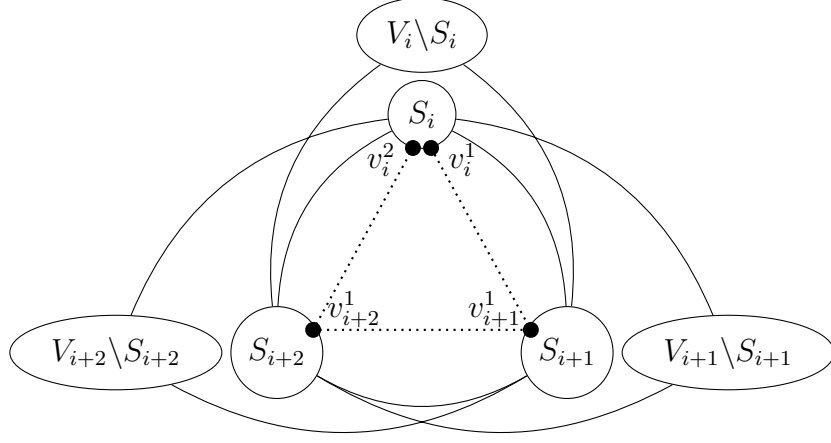


Figure 7: Construction 2: A  $K_{\ell, \ell, \ell}$ -saturated subgraph of  $K_{n_1, n_2, n_3}$ . Solid lines denote complete joins between sets, and dashed lines denote edges that have been removed.

**Theorem 14.** For  $n_1 \geq n_2 \geq n_3 > \ell + 1$ , the graphs in Construction 1 and Construction 2 are  $K_{\ell, \ell, \ell}$ -saturated subgraphs of  $K_{n_1, n_2, n_3}$ . Thus,

$$\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell}) \leq 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3.$$

*Proof.* We note that for  $i \in \{0, 1, 2, 3\}$

$$|E(G_\ell^i(n_1, n_2, n_3))| = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3.$$

We will now show that the four graphs are  $K_{\ell, \ell, \ell}$ -saturated. There are two cases to consider, which correspond to Construction 1 and Construction 2.

**Case 1:** Consider  $G_\ell^0(n_1, n_2, n_3)$ . For each  $i \in [3]$ , let  $T_i$  be an  $\ell$ -vertex subset of  $V_i$ . If

$T_i = S_i$  and  $T_j = S_j$  for  $i \in [3]$  and  $j \in [3]$  such that  $i \neq j$ , then  $v_i^1 \in T_i$  and  $v_j^1 \in T_j$  and  $T_i$  and  $T_j$  are not completely joined. If  $T_i \neq S_i$  and  $T_j \neq S_j$  for  $i \in [3]$  and  $j \in [3]$  such that  $i \neq j$ , then there exist  $v_i^r \in T_i$  and  $v_j^s \in T_j$  such that  $\ell < r \leq n_i$  and  $\ell < s \leq n_j$ , and  $T_i$  and  $T_j$  are not completely joined. By the pigeonhole principal either: there exists some  $i \in [3]$ ,  $j \in [3]$ ,  $i \neq j$  such that  $T_i = S_i$  and  $T_j = S_j$ ; or there exists some  $i \in [3]$ ,  $j \in [3]$ ,  $i \neq j$  such that  $T_i \neq S_i$  and  $T_j \neq S_j$ . Therefore,  $G_1[T_1 \cup T_2 \cup T_3]$  is not isomorphic to  $K_{\ell,\ell,\ell}$ , and  $G_1$  is  $K_{\ell,\ell,\ell}$ -free.

There are two types of non-edges in  $G_1$ : the first is  $v_i^1 v_{i+1}^1$  for  $i \in [3]$ ; the second is  $v_i^r v_{i+1}^s$  for  $\ell < r \leq n_i$  and  $\ell < s \leq n_{i+1}$  for  $i \in [3]$ . Adding  $v_i^1 v_{i+1}^1$  yields a copy of  $K_{\ell,\ell,\ell}$  on  $\{v_i^1, \dots, v_i^\ell\} \cup \{v_{i+1}^1, \dots, v_{i+1}^\ell\} \cup \{v_{i+2}^2, \dots, v_{i+2}^{\ell+1}\}$ . Adding  $v_i^r v_{i+1}^s$  yields a copy of  $K_{\ell,\ell,\ell}$  on  $(\{v_i^2, \dots, v_i^\ell\} \cup \{v_i^r\}) \cup (\{v_{i+1}^2, \dots, v_{i+1}^\ell\} \cup \{v_{i+1}^s\}) \cup \{v_{i+2}^1, \dots, v_{i+2}^\ell\}$ . Therefore,  $G_\ell^0(n_1, n_2, n_3)$  is a  $K_{\ell,\ell,\ell}$ -saturated subgraph of  $K_{n_1, n_2, n_3}$ .

**Case 2:** Let  $i \in [3]$  and consider  $G_\ell^i(n_1, n_2, n_3)$ . For each  $j \in [3]$ , let  $T_j$  be an  $\ell$ -vertex subset of  $V_j$ . If  $T_j = S_j$  and  $T_k = S_k$  for  $j \in [3]$ , and  $k \in [3]$  such that  $j \neq k$ , then  $v_j^1 \in T_j$  and  $v_k^2 \in T_j$ , and  $v_k^1 \in T_k$  and  $v_k^2 \in T_k$ . Thus,  $T_j$  and  $T_k$  are not fully joined. If  $T_j \neq S_j$  and  $T_k \neq S_k$ , then there exists  $v_j^r \in T_j$  and  $v_k^s \in T_k$  such that  $\ell < r \leq n_j$  and  $\ell < s \leq n_k$ , and these vertices are not adjacent. By the pigeonhole principal either: there exists some  $j \in [3]$ ,  $k \in [3]$ ,  $j \neq k$  such that  $T_j = S_j$  and  $T_k = S_k$ ; or there exists some  $j \in [3]$ ,  $k \in [3]$ ,  $j \neq k$  such that  $T_j \neq S_j$  and  $T_k \neq S_k$ . Therefore,  $G_\ell^i(n_1, n_2, n_3)[T_1 \cup T_2 \cup T_3]$  is not isomorphic to  $K_{\ell,\ell,\ell}$  and  $G_\ell^i(n_1, n_2, n_3)$  is  $K_{\ell,\ell,\ell}$ -free for each  $i \in [3]$ .

There are five types of non-edges in  $G_\ell^i(n_1, n_2, n_3)$ :  $v_i^r v_{i+1}^s$  for  $\ell < r \leq n_i$  and  $\ell < s \leq n_{i+1}$ ,  $v_i^r v_{i+2}^s$  for  $\ell < r \leq n_i$  and  $\ell < s \leq n_{i+2}$ ,  $v_{i+1}^r v_{i+2}^s$  for  $\ell < r \leq n_{i+1}$  and  $\ell < s \leq n_{i+2}$ ,  $v_i^j v_{i+j}^1$  for  $j \in \{1, 2\}$ , and  $v_{i+1}^1 v_{i+2}^1$ . Adding  $v_i^r v_{i+1}^s$  yields a copy of  $K_{\ell,\ell,\ell}$  on  $\{v_{i+2}^1, \dots, v_{i+2}^\ell\} \cup \{v_{i+1}^2, \dots, v_{i+1}^\ell\} \cup \{v_{i+1}^s\} \cup \{v_i^1\} \cup \{v_i^3, \dots, v_i^\ell\} \cup \{v_i^r\}$ . Adding  $v_i^r v_{i+2}^s$  yields a copy of  $K_{\ell,\ell,\ell}$  on  $\{v_{i+2}^2, \dots, v_{i+2}^\ell\} \cup \{v_{i+2}^s\} \cup \{v_{i+1}^1, \dots, v_{i+1}^\ell\} \cup \{v_i^2, \dots, v_i^\ell\} \cup \{v_i^r\}$ . Adding  $v_{i+1}^r v_{i+2}^s$  yields a copy of  $K_{\ell,\ell,\ell}$  on  $\{v_{i+2}^2, \dots, v_{i+2}^\ell\} \cup \{v_{i+2}^t\} \cup \{v_{i+1}^2, \dots, v_{i+1}^\ell\} \cup \{v_{i+1}^s\} \cup \{v_i^1, \dots, v_i^\ell\}$ . Adding  $v_i^j v_{i+j}^1$

yields a copy of  $K_{\ell,\ell,\ell}$  on  $\{v_i^1, \dots, v_i^\ell\} \cup \{v_{i+j}^1, \dots, v_{i+j}^\ell\} \cup \{v_{i-j}^2, \dots, v_{i-j}^{\ell+1}\}$ . Adding  $v_{i+1}^1 v_{i+2}^1$  yields a copy of  $K_{\ell,\ell,\ell}$  on  $\{v_{i+1}^1, \dots, v_{i+1}^\ell\} \cup \{v_{i+2}^1, \dots, v_{i+2}^\ell\} \cup \{v_i^3, \dots, v_i^{\ell+2}\}$ .  $\square$

### 3.2 Main Proof

Let  $G$  be a  $K_{\ell,\ell,\ell}$ -saturated subgraph of  $K_{n_1,n_2,n_3}$ . Let  $\delta_i = \min_{v \in V_i} d(v)$ .

**Observation 1.** If  $v$  and  $u$  are not adjacent and are in different partite sets, then they must have at least  $\ell$  common neighbors.

Hence, a vertex  $v$  must be adjacent at least  $\ell$  vertices in the other two sets, or to all of one set and at least  $\ell - 1$  vertices in the other. Thus, the  $\delta(g) \geq 2\ell$ .

**Lemma 1.** *If  $G$  is a  $K_{\ell,\ell,\ell}$ -saturated subgraph of  $K_{n_1,n_2,n_3}$  and there is a vertex  $v \in V_i$  with non-neighbors in both  $V_{i+1}$  and  $V_{i+2}$ , then there are at least  $\ell(n_{i+1} + n_{i+2}) - \ell\delta_i + \ell^2 - 1$  edges joining  $V_{i+1}$  and  $V_{i+2}$ . Furthermore, there are at least  $\ell^2 - 1$  edges induced by the neighborhood of a vertex of minimum degree in  $V_i$ .*

*Proof.* Let  $v_i$  be a vertex in  $V_i$  such that  $d(v_i) = \delta_i$ . Let  $v_i$  have  $a$  neighbors in  $V_{i+1}$  and  $b$  neighbors in  $V_{i+2}$ . Hence, there are at least  $\ell(n_{i+1} - a) + \ell(n_{i+2} - b) = \ell(n_1 + n_2) - \ell\delta_i$  edges joining  $V_{i+1} \setminus N(v_i)$  and  $V_{i+2} \setminus N(v_i)$ . We also know that there are at least  $\ell^2 - 1$  edges joining the neighbors of  $v_i$ . Thus, there are at least  $\ell(n_1 + n_2) - \ell\delta_i + \ell^2 - 1$  edges joining  $V_{i+1}$  and  $V_{i+2}$ .  $\square$

**Theorem 15.** *If  $n_1 \geq n_2 \geq n_3 \geq 6\ell^3 + 13\ell^2 + 7\ell - 1$  and  $\ell(n_2 + n_3) - 2\ell^2 + \ell + 1 > n_1$ , then*

$$\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell}) = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3,$$

and

$$\text{Sat}(K_{n_1,n_2,n_3}, K_{\ell,\ell,\ell}) = \{G_\ell^0(n_1, n_2, n_3), G_\ell^1(n_1, n_2, n_3), G_\ell^2(n_1, n_2, n_3), G_\ell^3(n_1, n_2, n_3)\}.$$



*Proof.* Let  $G$  be a  $k_{\ell,\ell,\ell}$ -saturated subgraph of  $K_{n_1,n_2,n_3}$ . We first need to show that vertices of degree  $\delta_i$  have non-neighbors in both other sets. We note that  $n_3$  is large enough so that a complete join between any pair of sets results in more than  $2\ell(n_1 + n_2 + n_3)$  edges. For the remainder of this proof we will assume that vertices of degree  $\delta_i$  have non-neighbors in both other sets.

There are at least  $\delta_1 n_1$  edges incident to  $V_1$ . By Lemma 1, there are at least  $\ell(n_2 + n_3) - \ell\delta_1 + \ell^2 - 1$  edges joining  $V_2$  and  $V_3$ . If  $\delta_1 \geq 4\ell$ , then

$$\begin{aligned} |E(G)| &\geq \delta_1 n_1 + \ell(n_2 + n_3) - \ell\delta_1 + \ell^2 - 1 \\ &> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3. \end{aligned}$$

Thus, we can assume that  $\delta_1 < 4\ell$ .

Note that  $2\ell n_2 \geq \ell(n_2 + n_3) > n_1 + 2\ell^2 - \ell - 1$ . Also, by Lemma 1 there are at least  $\ell(n_1 + n_3) - \ell\delta_2 + \ell^2 - 1$  edges joining  $V_1$  and  $V_3$ . Thus, if  $\delta_2 \geq 2\ell^2 + 3\ell + 1$ , then

$$\begin{aligned} |E(G)| &\geq \delta_2 n_2 + \ell(n_1 + n_3) - \ell\delta_2 + \ell^2 - 1 \\ &> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3. \end{aligned}$$

For the remainder of the proof we will assume that  $\delta_2 \leq 2\ell^2 + 3\ell$ .

Suppose that  $v_2 \in V_2$  has  $d(v_2) = \delta_2$ . Since,  $v_2$  has at least  $\ell$  neighbors in  $V_3$ , it follows that  $v_2$  has at no more than  $2\ell^2 + 2\ell$  neighbors in  $V_1$ . If  $v_2$  is adjacent to all the vertices in  $V_1$  with degree less than  $4\ell + 1$ , then there are at least  $(4\ell + 1)(n_1 - 2\ell^2 - 2\ell) + 2\ell(2\ell^2 + 2\ell)$  edges incident to  $V_1$  since  $4\ell \geq \delta_1 \geq 2\ell$ . When we add the edges joining  $V_2$  and  $V_3$  from

Lemma 1 we have

$$\begin{aligned}
|E(G)| &\geq (4\ell + 1)(n_1 - (2\ell^2 + 2\ell)) + 2\ell(2\ell^2 + 2\ell) + \ell(n_2 + n_3) - 3\ell^2 - 1 \\
&\geq 4\ell n_1 + n_1 + \ell(n_2 + n_3) - 4\ell^3 - 11\ell^2 - 2\ell - 1 \\
&> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3.
\end{aligned}$$

For the remainder of the proof we will assume that there is a vertex in  $V_1$  with degree less than or equal to  $4\ell$  that is not adjacent to  $v_2$ .

Let  $v_2 \in V_2$ ,  $v_3 \in V_3$  and  $u_1 \in V_1$  such that  $d(v_2) = \delta_2$ ,  $d(v_3) = \delta_3$ , and  $d(u_1) \leq 4\ell$  where  $u_1$  and  $v_2$  are not adjacent. Since  $u_1$  and  $v_2$  are not adjacent, they must have at least  $\ell$  common neighbors. Hence,  $|N_3(u_1) \cup N_3(v_2)| \leq 2\ell^2 + 4\ell$ , and we also know that  $N_3(u_1) \cup N_3(v_2)$  is incident to at least  $\ell(n_1 - 2\ell^2 - 2\ell) + \ell(n_2 - 3\ell)$  edges. The rest of  $V_3$  must be incident to at least  $\delta_3(n_3 - 2\ell^2 - 4\ell)$  edges. This, along with the edges joining  $V_1$  and  $V_2$  from Lemma 1, means that if  $\delta_3 > 2\ell$ , then

$$\begin{aligned}
|E(G)| &\geq \delta_3(n_3 - 2\ell^2 - 4\ell) + \ell(n_1 - 2\ell^2 - 2\ell) + \ell(n_2 - 3\ell) + \ell(n_1 + n_2) - \ell\delta_3 + \ell^2 - 1 \\
&\geq 2\ell n_1 + 2\ell n_2 + \delta_3 n_3 - \delta_3(2\ell^2 + 5\ell) - 2\ell^3 - 4\ell^2 - 1 \\
&> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3.
\end{aligned}$$

For the remainder of the proof we will assume that  $\delta_3 = 2\ell$ .

Let  $d(v_2) = \delta_2$  and  $d(v_3) = \delta_3$ . Note that  $|N_1(v_2) \cup N_1(v_3)|$  is at most  $2\ell^2 + 3\ell$  and that  $N_1(v_2) \cup N_1(v_3)$  is incident to at least  $\ell(n_3 - 2\ell^2 - 2\ell) + \ell(n_2 - \ell)$  edges. The rest of  $V_1$  is incident to at least  $\delta_1(n_1 - 2\ell^2 - 3\ell)$  edges. This, along with the edges joining  $V_2$  and  $V_3$

from Lemma 1, means that, if  $\delta_1 > 2\ell$  then,

$$\begin{aligned}
|E(G)| &\geq \delta_1(n_1 - 2\ell^2 - 3\ell) + \ell(n_3 - 2\ell^2 - 2\ell) + \ell(n_2 - \ell) + \ell(n_2 + n_3) - \ell\delta_1 + \ell^2 - 1 \\
&\geq \delta_1 n_1 + 2\ell n_2 + 2\ell n_3 - \delta_1(2\ell^2 + 4\ell) - 2\ell^3 - 2\ell^2 - 1 \\
&> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3.
\end{aligned}$$

For the remainder of this section of the proof we will assume that  $\delta_1 = 2\ell$ .

Let  $d(v_1) = 2\ell$  and  $d(v_3) = 2\ell$ . Hence,  $|N_2(v_1) \cap N_2(v_2)| \leq 2\ell$ , and  $N_2(v_1) \cap N_2(v_2)$  is incident to at least  $\ell(n_1 - \ell) + \ell(n_3 - \ell)$  edges. The rest of  $V_2$  is incident to at least  $\delta_2(n_2 - 2\ell)$  edges. This, along with the edges that join  $V_1$  and  $V_3$  from Lemma 1,  $\delta_2 > 2\ell$ , then

$$\begin{aligned}
|E(G)| &\geq \delta_2(n_2 - 2\ell) + \ell(n_1 - \ell) + \ell(n_3 - \ell) + \ell(n_1 + n_3) - \ell\delta_2 + \ell^2 - 1 \\
&\geq 2\ell n_1 + \delta_2(n_2 - 2\ell) + 2\ell n_3 - \delta_2\ell - \ell^2 - 1 \\
&> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3.
\end{aligned}$$

Thus, for the remainder of the proof we will assume that  $\delta_1 = \delta_2 = \delta_3 = 2\ell$ .

By Lemma 1, there are at least  $\ell(n_{i+1} + n_{i+2}) - \ell^2 - 1$  edges joining  $V_{i+1}$  and  $V_{i+2}$  for  $i \in [3]$ , so  $|E(G)| \geq 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3$ . By Theorem 14 we know that  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell}) \leq 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3$ . Thus, if  $G \in \text{Sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell})$ , then  $|E(G)| = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3$ . We now determine the structure of  $G$ . Let  $v_i \in V_i$  be a vertex of degree  $2\ell$ . Thus,  $v_i$  has  $\ell$  neighbors in both  $V_{i+1}$  and  $V_{i+2}$ , and  $G$  contains all edges joining  $N_{i+1}(v_i)$  to  $V_{i+2} \setminus N_{i+2}(v_i)$  and all edges joining  $N_{i+2}(v_i)$  to  $V_{i+1} \setminus N_{i+1}(v_i)$ . Therefore, the vertices of degree  $2\ell$  in  $G$  form an independent set. Let  $S = N(v_1) \cup N(v_2) \cup N(v_3)$  and let  $S_i = S \cap V_i$ . Since  $S_i$  contains the  $\ell$  common neighbors of  $v_{i+1}$  and  $v_{i+2}$  we know that  $|S_i| = \ell$ . By Lemma 1, there are at least  $\ell^2 - 1$  edges joining  $S_i$  and  $S_{i+1}$ . Furthermore, since  $|E(G)| = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3$  there are exactly

$\ell^2 - 1$  edges joining  $S_i$  and  $S_{i+1}$ . Let  $u_i^1 \in S_i$ ,  $u_{i+1}^1 \in S_{i+1}$ ,  $u_i^2 \in S_i$ , and  $u_{i+1}^2 \in S_{i+1}$ . Suppose  $G \notin \{G_\ell^0(n_1, n_2, n_3), G_\ell^1(n_1, n_2, n_3), G_\ell^2(n_1, n_2, n_3), G_\ell^3(n_1, n_2, n_3)\}$ . Thus, the three nonedges in  $G[S]$  do not form  $K_3$  or  $P_4$ . Without loss of generality, assume  $u_i^1 u_{i+1}^1$  is a nonedge and the other two nonedges in  $G[S]$  are incident to  $u_i^2$  and  $u_{i+1}^2$ , respectively. Let  $H$  be a subgraph of  $G + v_i v_{i+1}$  that is isomorphic to  $K_{\ell, \ell, \ell}$ . Thus,  $H$  must contain  $v_i$ ,  $v_{i+1}$  and  $S_{i+2}$ . Therefore,  $H$  cannot contain  $u_i^2$  or  $u_{i+1}^2$ . However,  $N_{i+1}(v_i) = S_{i+1} \cup \{v_{i+1}\}$ , and  $N_i(v_{i+1}) = S_i \cup \{v_i\}$  we know that  $H$  must contain  $S_i \setminus \{u_i^2\}$ , and  $S_{i+1} \setminus \{u_{i+1}^2\}$ . However, that means that  $H$  contains the nonedge  $u_i^1 u_{i+1}^1$ , and cannot be isomorphic to  $K_{\ell, \ell, \ell}$ . Therefore,  $G \in \{G_\ell^0(n_1, n_2, n_3), G_\ell^1(n_1, n_2, n_3), G_\ell^2(n_1, n_2, n_3), G_\ell^3(n_1, n_2, n_3)\}$ .  $\square$

We will now remove the bound on  $n_1$  imposed by Theorem 15. To remove the bound on  $n_1$ , we will impose a bound on  $n_2$ .

**Theorem 16.** *For all positive integers  $c$  and  $\ell$  there exists an  $N = N(c, \ell)$  such that if  $n_1 \geq \ell(n_2 + n_3) - 2\ell^2 + \ell + 1$ ,  $n_2 < cn_3$ , and  $n_3 > N$ , then*

$$\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell}) = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3,$$

and

$$\text{Sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell}) = \{G_\ell^0(n_1, n_2, n_3), G_\ell^1(n_1, n_2, n_3), G_\ell^2(n_1, n_2, n_3), G_\ell^3(n_1, n_2, n_3)\}.$$

In particular  $N(c, \ell) = \ell \binom{2\ell}{\ell} \binom{4c\ell + 2\ell}{\ell} + 16c\ell^2 + 5\ell^2 - 2$  suffices.

*Proof.* Let  $G$  be a  $K_{\ell, \ell, \ell}$ -saturated subgraph of  $K_{n_1, n_2, n_3}$ . If  $\delta_1 > 4\ell$ , then

$$\begin{aligned} E(G) &\geq \delta_1 n_1 \geq (4\ell + 1)n_1 \\ &\geq 2\ell(n_1 + n_2 + n_3) + n_1 - 4\ell^2 + 2\ell + 2 \\ &> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3. \end{aligned}$$

For the remainder of this proof we will assume that  $\delta_1 \leq 4\ell$ . Thus, a vertex of minimum degree from  $V_1$  has non-neighbors in both  $V_2$  and  $V_3$ .

By Lemma 1, there are at least  $\ell(n_2 + n_3) - \delta_1 + \ell^2 - 1$  edges joining  $V_2$  and  $V_3$ . If  $\delta_1 > 2\ell$ , then

$$\begin{aligned} |E(G)| &\geq \delta_1 n_1 + \ell(n_2 + n_3) - \ell\delta_1 + \ell^2 - 1 \\ &> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3. \end{aligned}$$

For the remainder of this proof we will assume that  $\delta_1 = 2\ell$ .

A vertex  $v \in V_1$  is *removable* if  $d(v) = 2\ell$  and  $G - v$  is a  $K_{\ell,\ell,\ell}$ -saturated subgraph of  $K_{n_1-1, n_2, n_3}$ . If there are more than  $\ell$  vertices in  $V_1$  with degree  $2\ell$  that have the same neighbors, then there are removable vertices. Let  $v$  be a vertex with  $d(v) = 2\ell$  that shares a common neighborhood with at least  $\ell$  other vertices of degree  $2\ell$ . Any edge between  $V_2$  and  $V_3$  that would form a  $K_{\ell,\ell,\ell}$  using  $v$  could be formed using the  $\ell$  other vertices that have the same neighborhood as  $v$ . Let us suppose that there are no removable vertices. Thus, there are no sets of size  $\ell + 1$  vertices in  $V_1$ , all having degree  $2\ell$ , with a common neighborhood.

First assume that,  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_2(v) \right| \geq 2\ell$ , and  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_3(v) \right| \geq 2\ell$ . It follows from Observation 1 that there are at least  $2\ell(n_2 - 2\ell)$  edges incident to  $\bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_3(v)$ , likewise there are at least  $2\ell(n_3 - 2\ell)$  edges incident to  $\bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_2(v)$ . Hence, there are at least  $2\ell(n_2 + n_3) - 8\ell^2$  edges joining  $V_2$  and  $V_3$ . If  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_2(v) \right| = 2\ell$ , and  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_3(v) \right| = 2\ell$ , then by the pigeonhole principal there are at most  $\ell \binom{2\ell}{\ell}^2$  vertices of degree  $2\ell$  in  $V_1$ . Thus,

$$\begin{aligned} |E(G)| &\geq 2\ell n_1 + n_1 - \ell \binom{2\ell}{\ell}^2 + 2\ell(n_2 + n_3) - 8\ell^2 \\ &> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3. \end{aligned}$$

Furthermore, if  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_2(v) \right| \geq 2\ell$  or  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_3(v) \right| \geq 2\ell$ , then we pick up an additional

$n_2$  or  $n_3$ . Letting  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_2(v) \right| \geq 2\ell$  results in fewer additional edges at an additional  $n_3$  edges, so

$$\begin{aligned} |E(G)| &\geq 2\ell n_1 + 2\ell(n_2 + n_3) - 8\ell^2 - 2 + n_3 - 2\ell \\ &> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3. \end{aligned}$$

Thus,  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_3(v) \right| < 2\ell$ , or  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_3(v) \right| < 2\ell$ . We note  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_3(v) \right| < 2\ell$  results in fewer edges, thus for the remainder of the proof we will assume that  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_3(v) \right| < 2\ell$ .

Let  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_3(v) \right| = x$ , and let  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_2(v) \right| = n_2 - y$ . We note that  $\ell \leq x < 2\ell$ . Thus,

$$\begin{aligned} |E(G)| &\geq 2\ell n_1 + (n_2 - y)(n_3 - x) + xy \\ &\geq 2\ell(n_1 + n_2 + n_3) + n_2 n_3 - 2\ell n_2 - 2\ell n_3 - x n_2 - y n_3 + 2xy \\ &> 2\ell(n_1 + n_2 + n_3) + n_2(n_3 - 2\ell) - 2\ell n_3 - y n_3 \end{aligned}$$

Thus, if  $y \leq n_2(1 - \frac{4\ell}{n_3}) - 2\ell$ , then  $|E(G)| > 2\ell(n_1 + n_2 + n_3)$ .

Let  $\left| \bigcup_{\substack{v \in V_1, \\ d(v)=2\ell}} N_2(v) \right| = n_2 - y = y'$ . Since  $n_2 \leq cn_3$ , we have that  $\ell \leq y' < 4\ell c + 2\ell$ . If both  $x = \ell$  and  $y' = \ell$ , then there are at most  $\ell$  vertices of degree  $2\ell$  in  $V_1$ . Thus, using Lemma 1, we know that

$$\begin{aligned} |E(G)| &\geq (2\ell + 1)(n_1 - \ell) + 2\ell^2 + \ell(n_2 + n_3) - \ell^2 - 1 \\ &\geq 2\ell(n_1 + n_2 + n_3) - 3\ell^2. \end{aligned}$$

Thus, we can assume that if  $x = y = \ell$ , then there are removable vertices in  $G$ . Hence, for the remainder of this proof we will assume that  $x > \ell$  or  $y' > \ell$ . Since there are at most  $\ell$  vertices of degree  $2\ell$  from  $V_1$  that can share a common neighborhood, we can place an upper

bound on the vertices of degree  $2\ell$  in  $V_1$ . We note that there are  $\binom{x}{\ell}\binom{y'}{\ell}$  ways to choose a neighborhood for a vertex of degree  $2\ell$ , since at most  $\ell$  vertices share a neighborhood, there are at most  $\ell\binom{x}{\ell}\binom{y'}{\ell}$  vertices of degree  $2\ell$  in  $V_1$ . Thus,

$$\begin{aligned}
|E(G)| &\geq 2\ell n_1 + n_1 - \ell\binom{x}{\ell}\binom{y'}{\ell} + x(n_2 - y') + y'(n_3 - x) \\
&\geq 2\ell n_1 + \ell n_2 + \ell n_3 + n_1 + n_3 - 2xy' - \ell\binom{x}{\ell}\binom{y'}{\ell} \\
&\geq 2\ell(n_1 + n_2 + n_3) + n_3 - 4\ell(4\ell c + 2\ell) - \ell\binom{2\ell}{\ell}\binom{4\ell c + 2\ell}{\ell} \\
&> 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3.
\end{aligned}$$

Therefore, if  $n_2 \leq cn_3$ , and  $n_3 \geq \max\{\ell\binom{2\ell}{\ell}\binom{4\ell c + 2\ell}{\ell} + 16\ell^2 c + 5\ell^2 - 2, 6\ell^3 + 13\ell^2 + 7\ell - 1\}$ , then  $G$  must have a removable vertex.

We now prove the result using induction on  $n_1$ . We note that Theorem 15 serves as our base case. Assume that  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell}) = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3$ , and let  $\text{Sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell}) = \{G_\ell^0(n_1, n_2, n_3), G_\ell^1(n_1, n_2, n_3), G_\ell^2(n_1, n_2, n_3), G_\ell^3(n_1, n_2, n_3)\}$ . Let  $G$  be a  $K_{\ell, \ell, \ell}$ -saturated subgraph of  $K_{n_1+1, n_2, n_3}$ . Let  $v_1$  be a removable vertex in  $G$ . Let  $N_2(v_1) = \{u_2^1, \dots, u_2^\ell\}$ , and  $N_3(v_1) = \{u_3^1, \dots, u_3^\ell\}$ . Since  $G$  is  $K_{\ell, \ell, \ell}$ -saturated,  $\{u_2^1, \dots, u_2^\ell\}$  must be joined to  $V_3 \setminus u_3^1$ , and  $\{u_3^1, \dots, u_3^\ell\}$  is joined to  $V_2 \setminus u_2^1$ . Let  $G' = G - v_1$ . By the induction hypothesis, there are  $2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3$  edges in  $G'$ , and  $G'$  must be isomorphic to some member of  $\{G_\ell^0(n_1, n_2, n_3), G_\ell^1(n_1, n_2, n_3), G_\ell^2(n_1, n_2, n_3), G_\ell^3(n_1, n_2, n_3)\}$ . Thus,  $\{u_2^1, \dots, u_2^\ell\}$  and  $\{u_3^1, \dots, u_3^\ell\}$  are  $S_2$  and  $S_3$  respectively. Therefore,  $|E(G)| = 2\ell((n_1+1) + n_2 + n_3) - 3\ell^2 - 3$ , and  $G \in \{G_\ell^0(n_1, n_2, n_3), G_\ell^1(n_1, n_2, n_3), G_\ell^2(n_1, n_2, n_3), G_\ell^3(n_1, n_2, n_3)\}$ .  $\square$

If  $\ell = 1$ , then we have much more control over the behavior of  $K_3$ -saturated subgraphs of  $K_{n_1, n_2, n_3}$ . This allows us to remove the bound on  $n_2$  and simply leave a lower bound on  $n_3$ .

**Theorem 17.** *If  $n_1 \geq n_2 \geq n_3 \geq 25$ , then*

$$\text{sat}(K_{n_1, n_2, n_3}, K_3) = 2(n_1 + n_2 + n_3) - 6,$$

and

$$\text{Sat}(K_{n_1, n_2, n_3}, K_3) = \{G_1^0(n_1, n_2, n_3)\}.$$

*Proof.* Let  $G$  be a  $K_3$  saturated subgraph of  $K_{n_1, n_2, n_3}$ . We first note that  $\delta_i \geq 2$  for all  $i \in [3]$ , and Lemma 1 still holds. We use Theorem 15 when  $n_1 \leq n_2 + n_3$ , and notice that  $G_1$  is the only construction in the case when  $\ell = 1$ . If  $n_1 > n_2 + n_3$  then  $\delta_1 = 2$ . We now focus our efforts on showing that if  $n_1 > n_2 + n_3$ , then  $V_1$  contains a removable vertex.

If  $u$  and  $v$  have the same neighbors then  $G$  contains a removable vertex, as any edge that would form a  $K_3$  using  $u$  could use  $v$  instead. For the remainder of this proof we will assume that vertices of degree 2 have disjoint neighborhoods.

Again, let  $u$  and  $v$  be vertices in  $V_1$  such that  $d(u) = d(v) = 2$ . Let  $w$  be a common neighbor of  $u$  and  $v$ . Let  $x \in N(u) \setminus N(v)$ , and let  $y \in N(v) \setminus N(u)$ . If  $G$  contains the edge  $wx$ , or the edge  $wy$ , then  $G$  contains a copy of  $K_3$ . If  $G$  does not contain these edges then consider the addition the edge  $vx$ . If there is a copy of  $K_3$  in  $G + vx$  it must be on  $\{v, x, w\}$ , but there is no edge joining  $w$  and  $x$ . Thus, if any pair of vertices of degree 2 in  $V_1$  have disjoint neighborhoods, then  $G$  is not a  $K_3$ -saturated subgraph of  $K_{n_1, n_2, n_3}$ . For the remainder of this proof we will assume that vertices of degree 2 do not have any common neighbors.

Let  $u$  and  $v$  be vertices in  $V_1$  such that  $d(u) = d(v) = 2$  and  $u$  and  $v$  do not have any common neighbors. If there is a third vertex  $w \in V_1$  such that  $d(w) = 2$ , then by Observation 1, we know that there are at least  $n_2 + n_3 - 6$  edges incident to  $N(u)$ , likewise



for  $N(v)$  and  $N(w)$ . Thus,

$$\begin{aligned} |E(G)| &\geq 2n_1 + 3(n_2 + n_3 - 6) \\ &> 2(n_1 + n_2 + n_3). \end{aligned}$$

For the remainder of this proof we will assume that there are no more than two vertices of degree 2 in  $V_1$ .

Let  $u$  and  $v$  be the two vertices in  $V_1$  such that  $d(u) = d(v) = 2$  and  $u$  and  $v$  do not have any common neighbors. By Observation 1, we know that there are at least  $n_2 + n_3 - 4$  edges incident to  $N(u)$  and also to  $N(v)$ . Thus,

$$\begin{aligned} |E(G)| &\geq 3n_1 - 2 + 2(n_2 + n_3 - 4) \\ &> 2(n_1 + n_2 + n_3). \end{aligned}$$

Therefore,  $G$  must have a removable vertex.

We will finish our proof using induction on  $n_1$ . We note that Theorem 15 serves as our base case. Assume that  $\text{sat}(K_{n_1, n_2, n_3}, K_3) = 2\ell(n_1 + n_2 + n_3) - 6$ , and assume that  $\text{Sat}(K_{n_1, n_2, n_3}, K_3) = \{G_1^0(n_1, n_2, n_3)\}$ . Let  $G$  be a  $K_3$ -saturated subgraph of  $K_{n_1+1, n_2, n_3}$ , and let  $v_1$  be a removable vertex in  $G$ . Let  $N_2(v_1) = \{u_2^1\}$ , and  $N_3(v_1) = \{u_3^1\}$ . Since  $G$  is  $K_3$ -saturated,  $\{u_2^1\}$  must be joined to  $V_3 \setminus \{u_3^1\}$ , and  $\{u_3^1\}$  is joined to  $V_2 \setminus \{u_2^1\}$ . Let  $G' = G - v_1$ . By induction, there are  $2\ell(n_1 + n_2 + n_3) - 6$  edges in the  $G'$  and it must be isomorphic to  $G_1^0(n_1, n_2, n_3)$ . This means that  $\{u_2^1\}$ , and  $\{u_3^1\}$ , are  $S_2$ , and  $S_3$  in  $G_1^0(n_1, n_2, n_3)$ . Therefore,  $G$  is isomorphic to  $G_1^0(n_1 + 1, n_2, n_3)$ , and  $|E(G)| = 2\ell((n_1 + 1) + n_2 + n_3) - 3\ell^2 - 3$ .  $\square$

## 4 Constructions for other Forbidden Graphs

In this section we provide upper bounds for  $K_{\ell,m,p}$ -saturated subgraphs of  $K_{n_1,n_2,n_3}$ . We also consider  $C_4$ -saturated subgraphs of  $K_{n_1,n_2,n_3}$ , and find  $\text{sat}(K_{n_1,n_2,n_3}, C_4)$ . We first consider a construction for a  $K_{\ell,m,m}$ -saturated subgraph of  $K_{n_1,n_2,n_3}$ .

**Construction 3.** For each  $i \in [3]$ , let  $S_i = \{v_i^{n_i-m+1}, \dots, v_i^{n_i}\}$ . Join  $S_i$  to  $V_{i+1}$ , and  $V_{i+2}$ , and then remove the edges  $v_1^{n_1}v_2^{n_2}$ ,  $v_1^{n_1}v_3^{n_3}$ , and  $v_2^{n_2}v_3^{n_3}$ . Between  $V_1 \setminus S_1$  and  $V_2 \setminus S_2$  for  $j \in \{1, \dots, n_2 - m\}$ , join  $v_1^j$  to  $\{v_2^r : r \in \{j, \dots, j + \ell - m - 1 \bmod (n_2 - m)\}\}$ . Between  $V_1 \setminus S_1$  and  $V_3 \setminus S_3$  for  $j \in \{1, \dots, n_3 - m\}$  join  $v_1^j$  to  $\{v_3^r : r \in \{j, \dots, j + \ell - m - 1 \bmod (n_3 - m)\}\}$ . Between  $V_2 \setminus S_2$  and  $V_3 \setminus S_3$  for  $j \in \{1, \dots, n_3 - m\}$  join  $v_2^j$  to  $\{v_3^r : r \in \{j + 1, \dots, j + \ell - m \bmod (n_3 - m)\}\}$ . We call this graph  $G_3$ . Thus,

$$\begin{aligned} E(G_3) = & \left( \{v_i^r v_j^s : i \in [3], j \in [3], i \neq j, n_i - m + 1 \leq r \leq n_i \text{ or } n_j - m + 1 \leq s \leq n_j\} \right. \\ & \cup \{v_1^a v_j^b : j \in \{2, 3\}, a \in [n_j - m], b \in \{a, \dots, a + \ell - m - 1 \bmod (n_j - m)\}\} \\ & \cup \{v_2^a v_3^b : a \in [n_3 - m], b \in \{a + 1, \dots, a + \ell - m \bmod (n_3 - m)\}\} \\ & \left. \setminus \{v_1^{n_1} v_2^{n_2}, v_1^{n_1} v_3^{n_3}, v_2^{n_2} v_3^{n_3}\} \right). \end{aligned}$$

**Theorem 18.** *If  $\ell > m$  and  $n_3 \geq 2(\ell - m)$ , then  $G_3$  is a  $K_{\ell,m,m}$ -saturated subgraph of  $K_{n_1,n_2,n_3}$ , and thus*

$$\text{sat}(K_{n_1,n_2,n_3}, K_{\ell,m,m}) \leq (\ell - m)(n_2 + 2n_3) + 2m(n_1 + n_2 + n_3) - 3m(\ell - m) - 3m^2 - 3.$$

*Proof.* We start by proving that the subgraph consisting of  $(V_1 \setminus S_1) \cup (V_2 \setminus S_2) \cup (V_3 \setminus S_3)$  is  $K_{\ell-m,1,1}$ -free.

Let  $T_1$  be a  $(\ell - m)$ -vertex set in  $V_1 \setminus S_1$ . Let  $T_2$  be a 1-vertex set in  $V_2 \setminus S_2$ , and let  $T_3$  be a 1-vertex set in  $V_3 \setminus S_3$ . If  $T_1 \neq \{v_1^j \dots v_1^{j-\ell-m-1 \bmod (n_1-m)}\}$  for  $j \in [n_1 - m]$ , then

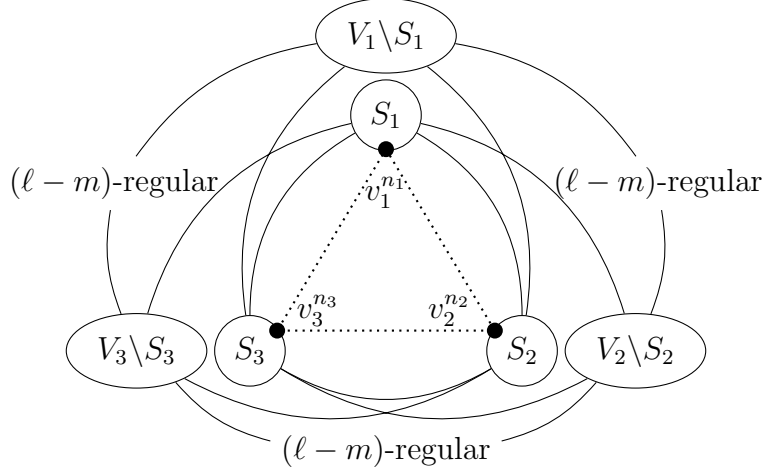


Figure 8: Construction 3: A  $K_{\ell,m,m}$ -saturated subgraph of  $K_{n_1,n_2,n_3}$ . Solid lines denote a complete join between two sets, and a dotted line denotes a removed from the join. The lines marked as  $(\ell - m)$ -regular use the particular edges described in Construction 3.

$\bigcap_{v \in T_1} N(v) = \emptyset$ . Thus,  $T_1$  must contain vertices with consecutive labels. Let  $T_1 = \{v_1^j, \dots, v_1^{j-\ell-m-1 \bmod (n_1-m)}\}$  for  $j \in [n_1 - m]$ . If  $T_2 = \{v_2^{j-\ell-m-1 \bmod (n_2-m)}\}$ , and  $T_3 = \{v_3^{j-\ell-m-1 \bmod (n_3-m)}\}$  then  $T_2$  and  $T_3$  are not joined. If  $T_2 \neq \{v_2^{j-\ell-m-1 \bmod (n_2-m)}\}$  then  $T_1$  is not completely joined to  $T_2$ . Likewise, if  $T_3 \neq \{v_3^{j-\ell-m-1 \bmod (n_3-m)}\}$  then  $T_1$  is not completely joined to  $T_3$ . Thus, if  $|T_2| = 1$  and  $|T_3| = 1$ , then  $G_3[T_1 \cup T_2 \cup T_3]$  is not isomorphic to  $K_{\ell-m,1,1}$ .

Let  $T_1$  be a 1-vertex set in  $V_1 \setminus S_1$ . Let  $T_i$  be a 1-vertex set in  $V_i \setminus S_i$  for  $i \in \{2, 3\}$ . Let  $T_j$  be a  $\ell - m$  vertex set in  $V_j \setminus S_j$  for  $j \in \{2, 3\}$  and  $i \neq j$ . Let  $T_1 = \{v_1^r\}$  for  $r \in [n_1 - m]$ . Let  $T_i = \{v_i^s\}$  for  $s \in [n_i - m]$ . If  $s \in \{r, \dots, \ell - m - 1 \bmod (n_j - m)\}$ , then  $N(v_1^r) \cap N(v_i^s) = \{s + 1, \dots, \ell - m - 1 \bmod (n_j - m)\}$ , so for any choice of  $T_j$  we know  $T_1$  and  $T_i$  will not both have  $\ell - m$  common neighbors in  $T_j$ . If  $s \notin \{r, \dots, \ell - m - 1 \bmod (n_j - m)\}$ , then  $T_1$  and  $T_j$  are not joined. Thus if  $|T_1| = 1$  then  $G_3[T_1 \cup T_2 \cup T_3]$  is not isomorphic to  $K_{\ell-m, 1, 1}$ . Therefore,  $G_3 \setminus (S_1 \cup S_2 \cup S_3)$  is  $K_{\ell-m, 1, 1}$  free.

Let  $T_i$  be an  $\ell$ -vertex set for  $i \in [3]$ . Also let  $T_{i+1}$  be an  $m$ -vertex set in  $V_{i+1}$ , and let  $T_{i+2}$  be an  $m$ -vertex set in  $V_{i+2}$ . If  $T_1 \cup T_2 \cup T_3$  contains  $v_1^{n_1}$ ,  $v_2^{n_2}$ , and  $v_3^{n_3}$ , then  $T_1$  is not

completely joined to  $T_2 \cup T_3$  and  $G_3[T_1 \cup T_2 \cup T_3]$  is not isomorphic to  $K_{\ell, m, m}$ . If  $T_1 \cup T_2 \cup T_3$  contains  $v_j^{n_j}$  and  $v_k^{n_k}$  for  $j \in [3]$  and  $k \in [3]$  and  $j \neq k$ , then  $T_j$  and  $T_k$  are not completely joined and  $G_3[T_1 \cup T_2 \cup T_3]$  is not isomorphic to  $K_{\ell, m, m}$ . If  $T_1 \cup T_2 \cup T_3$  contains  $v_j^{n_j}$  for  $j \in [3]$  and  $i \neq j$  then  $T_i$  must contain  $\ell - m + 1$  vertices from  $V_i \setminus S_i$  these vertices have at most  $m - 1$  common neighbors in the remaining set, thus  $G_3[T_1 \cup T_2 \cup T_3]$  is not isomorphic to  $K_{\ell, m, m}$ . We can now assume that  $v_i^{n_i} \in T_i$ ,  $v_{i+1}^{n_{i+1}} \notin T_{i+1}$ , and  $v_{i+2}^{n_{i+2}} \notin T_{i+2}$ . If  $T_1 \cup T_2 \cup T_3$  contains  $v_i^{n_i}$  then  $T_i$  must contain  $\ell - m$  vertices from  $V_i \setminus S_i$ . If  $G_3[T_1 \cup T_2 \cup T_3]$  is isomorphic to  $K_{\ell, m, m}$  then  $G_3 \setminus (S_1 \cup S_2 \cup S_3)$  must contain a  $K_{\ell-m, 1, 1}$ , but we know that  $G_3 \setminus (S_1 \cup S_2 \cup S_3)$  is  $K_{\ell-m, 1, 1}$  free. Therefore,  $G_3$  is a  $K_{\ell, m, m}$ -free subgraph of  $K_{n_1, n_2, n_3}$ .

There are three types of non-edges in  $G_3$ : the first is  $v_i^{n_i} v_{i+1}^{n_{i+1}}$  for  $i \in [3]$ ; the second is incident to  $v_1^a$ , for  $a \in [n_j - m]$ ; the third joins  $v_2^a$  for  $a \in [n_2 - m]$  to a vertex in  $V_3$ . Adding  $v_i^{n_i} v_{i+1}^{n_{i+1}}$  yields  $K_{\ell, m, m}$  on  $\{v_{i+2}^1, \dots, v_{i+2}^\ell\} \cup S_i \cup S_{i+1}$ . Adding  $v_1^a v_j^b$  yields  $K_{\ell, m, m}$  on  $(S_1 \setminus \{v_1^{n_1}\}) \cup \{v_1^a\} \cup \{v_1^s : s \in \{b, \dots, b - \ell + m + 1 \bmod (n_j - m)\}\} \cup (S_j \setminus \{v_j^{n_j}\}) \cup \{v_j^b\} \cup \{v_i^r : i \in \{2, 3\}, i \neq j, r \in \{n_i - m + 1, \dots, n_i\}\}$ . Adding  $v_2^a v_3^b$  yields  $K_{\ell, m, m}$  on  $S_1 \cup (S_2 \setminus \{v_2^{n_2}\}) \cup \{v_2^a\} \cup \{v_2^s : s \in \{b - 1, \dots, b - \ell + m \bmod (n_j - m)\} \cup \{v_3^b\}$ . Therefore,  $G_3$  is a  $K_{\ell, m, m}$ -saturated subgraph of  $K_{n_1, n_2, n_3}$ . We note,

$$|E(G_3)| = (\ell - m)(n_2 + 2n_3) + 2m(n_1 + n_2 + n_3) - 3m(\ell - m) - 3m^2 - 3. \quad \square$$

We now construct a  $K_{\ell, m, p}$ -saturated subgraph of  $K_{n, n, n}$ .

**Construction 4.** For each  $j \in [3]$  let  $S_j$  be an  $(m - 1)$ -vertex subset of  $V_j$ . For each  $j \in [3]$  Join  $V_i$  to  $S_{i+1}$  and  $S_{i+2}$ . Also, let  $t = \lfloor \frac{\ell - m}{2} \rfloor$ , and for each  $j \in [3]$  let  $T_i$  be a  $t$ -vertex subset of  $V_j \setminus S_j$ . Let  $T_1 \cup T_2 \cup T_3$  span a  $K_{t, t, t}$ . For each  $i \in [3]$  and  $j \in [3]$  such that  $i \neq j$ , let  $V_i \setminus (S_i \cup T_i)$  span a  $(\ell - m)$ -regular bipartite graph. Call this graph  $G_4$

We note that there are many realizations of  $(\ell - m)$ -regular bipartite graphs, and unlike Construction 3 there are no further restrictions on the sparse part of this construction.

**Theorem 19.** *If  $\ell \geq m > p$ , then the graph from Construction 4 is a  $K_{\ell,m,p}$ -saturated subgraph of  $K_{n,n,n}$ . Thus*

$$\text{sat}(K_{n,n,n}, K_{\ell,m,p}) \leq 3(\ell + m - 2)n - 3(m - 1)(\ell - 1) + 3 \left\lfloor \frac{\ell - m}{2} \right\rfloor^2 - 3(\ell - m) \left\lfloor \frac{\ell - m}{2} \right\rfloor.$$

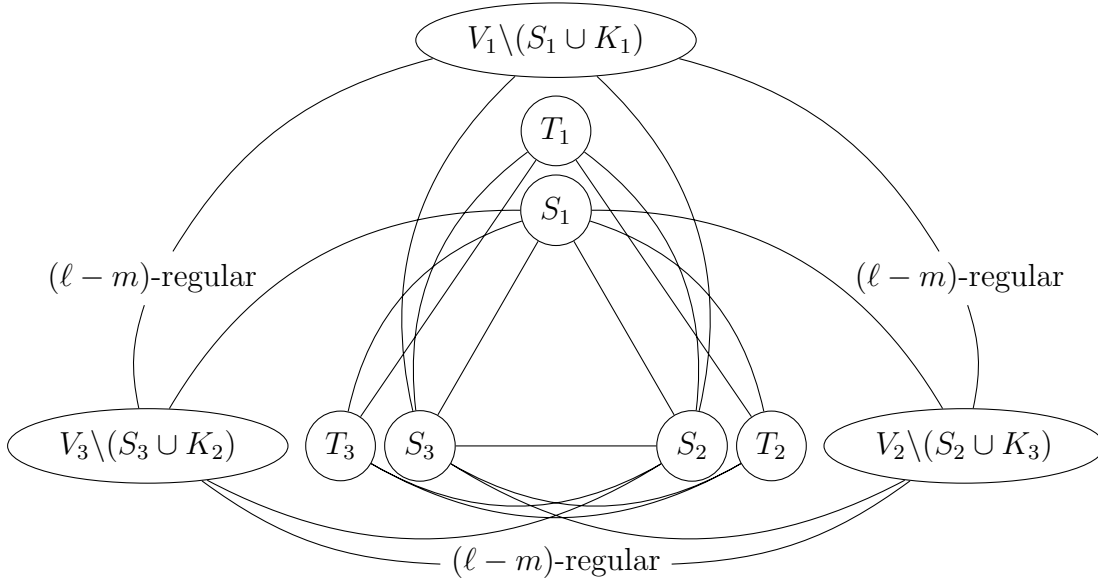


Figure 9: Construction 4: A  $K_{\ell,m,p}$ -saturated subgraph of  $K_{n_1,n_2,n_3}$ . Solid lines between sets denote complete joins.

*Proof.* For all  $i \in [3]$ , there are only  $m - 1$  members of  $V_i$  with at least  $\ell$  common neighbors in the other two sets, so we cannot form a set of size  $m$  with enough common neighbors. Thus,  $G_4$  is  $K_{\ell,m,p}$ -free.

There are two types of non-edges in  $G_4$ : the first is, for some  $i \in [3]$  and some  $j \in [3]$ , an edge joining some vertex  $v \in T_i$  to a vertex  $u \in V_j \setminus (S_j \cup T_j)$  for  $i \neq j$ ; the second is an edge joining some vertex  $w \in V_i \setminus (S_i \cup T_i)$  to a vertex  $u \in V_j \setminus (S_j \cup T_j)$  for some  $i \in [3]$  and some  $j \in [3]$ . Let  $k \in [3]$  such that  $k \neq i$  and  $k \neq j$ . Adding the first type of edge yields a copy of  $K_{\ell,m,m-1}$  on  $S_j \cup \{u\} \cup S_i \cup \{v\} \cup N_i(u) \cup S_k$ . Adding the second type of edge yields a copy of  $K_{\ell,m,m-1}$  on  $S_j \cup \{u\} \cup S_i \cup \{w\} \cup N_i(u) \cup S_k$ . Since  $p < m$  we know that  $K_{\ell,m,p}$  is

a subgraph of  $K_{\ell,m,m-1}$ . Thus,  $G_4$  is a  $K_{\ell,m,p}$ -saturated subgraph of  $K_{n_1,n_2,n_3}$ . We note that

$$|E(G_4)| = 3(\ell + m - 2)n - 3(m - 1)(\ell - 1) + 3 \left\lfloor \frac{\ell - m}{2} \right\rfloor^2 - 3(\ell - m) \left\lfloor \frac{\ell - m}{2} \right\rfloor. \quad \square$$

We will now consider a  $C_4$ -saturated subgraph of  $K_{n,n,n}$ . In Section 5 we will consider some variants of saturated subgraphs and use  $C_4$  as an example, so we determine  $\text{sat}(K_{n,n,n}, C_4)$  to compare with those results.

**Construction 5.** For each  $i \in [3]$ , join  $V_i$  to  $v_{i+1}^1$ . Let us call this graph  $G_5$ .

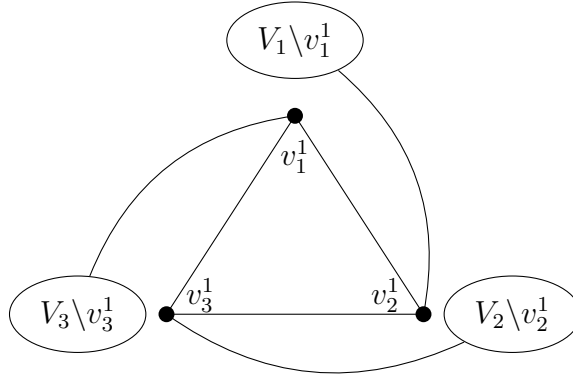


Figure 10: Construction 5: A  $C_4$ -saturated subgraph of  $K_{n,n,n}$ . Lines denote complete joins.

**Theorem 20.** For all  $n$ ,  $\text{sat}(K_{n,n,n}, C_4) = 3n$ .

*Proof.* We note that the only cycle in  $G_s$  is a copy of  $C_3$ , so  $G_s$  is  $C_4$ -free.

There is one types of edge that we can add to  $G_s$ :  $v_i^s v_{i+1}^t$  for  $i \in [3]$ ,  $1 \leq s \leq n$ , and  $1 < t \leq n$ . Adding  $v_i^s v_{i+1}^t$  results in a copy of  $C_4$  on  $\{v_i^s, v_{i+1}^t, v_{i+2}^1, v_{i+1}^1\}$ .

Let  $G$  be a  $C_4$ -saturated subgraph of  $K_{n,n,n}$ . Clearly  $G$  is connected, as every pair of vertices is either adjacent or joined by a path of length 3. Hence,  $G$  must have a spanning tree as a subgraph, so  $|E(G)| \geq 3n - 1$ . If  $|E(G)| = 3n - 1$ , then  $G$  is a tree. Between any two vertices in a tree there is a unique path. Let  $u$  and  $v$  be vertices connected by a path of length 3, let  $w$  be one of the two internal vertices on this path. Since the path from  $u$  to  $v$

is unique, we know that  $w$  cannot be adjacent to both  $u$  and  $v$ , nor can it be connected by a path of length 3 to either  $u$  or  $v$ . Thus,  $G$  cannot be a tree and so  $|E(G)| \geq 3n$ . Therefore,  $\text{sat}(K_{n,n,n}, C_4) = 3n$ .  $\square$

## 5 Colored-Saturation and Ordered-Saturation

In this section, we look at the natural variants of saturation that arise in multipartite graphs and look at ordered-saturation and colored-saturation. If  $F$  is a balanced complete multipartite graph, then ordered-saturation, colored-saturation, and saturation are all the same. Also, if  $F$  is a complete multipartite graph, then colored-saturation and saturation are the same. We now will restate the definitions of ordered-saturation and colored-saturation.

Let  $G$  be a spanning subgraph of  $H$ , and let  $G$  inherit the coloring of  $H$ . We now adjust the notions of  $F$ -free, and  $F$ -saturated so that they respect the colorings of  $H$  and  $F$ . We say that  $G$  is  $(F, c_F)$ -ordered-free if every copy of  $F$  contained in  $G$  does not have coloring  $c_F$ . We say that  $G$  is  $(F, c_F)$ -ordered-saturated if  $G$  is  $(F, c_F)$ -ordered free and a copy of  $F$  with coloring  $c_F$  is a subgraph of  $G + e$  for any edge  $e \in E(H) \setminus E(G)$ . The *ordered-saturation number* of  $(F, c_F)$  in  $(H, c_H)$ , denoted  $\overrightarrow{\text{sat}}((H, c_H), (F, c_F))$ , is the minimum size of an  $(F, c_F)$ -ordered-saturated subgraph of  $(H, c_H)$ .

A weaker notion is *colored-saturation*; let  $c_F : V(F) \rightarrow [k]$  be a coloring of  $F$ , and let  $c_H : V(H) \rightarrow [k]$  be a coloring of  $H$ . We say that  $G$  is  $(F, c_F)$ -colored-free subgraph of  $(H, c_H)$  if:

- $F$  is not a subgraph of  $G$ , or
- If  $F'$  is a subgraph of  $G$  that is isomorphic to  $F$ , then for any permutation of  $\sigma : [k] \rightarrow [k]$  then  $c_H|_{V(F)} \neq \sigma(c_F)$ .

That is, the coloring of  $F'$  is not the coloring of  $F$ , possibly with relabeling of colors classes. We say that  $G$  is  $(F, c_F)$ -colored-saturated if  $G$  is  $(F, c_F)$ -colored-free and for any edge  $e \in E(H) - E(G)$  then there exists a permutation  $\sigma : [k] \rightarrow [k]$  such that  $G + e$  contains a copy of  $F$  with coloring  $\sigma(c_F)$ . The colored-saturation number of  $(F, c_F)$  in  $H$ , denoted  $\text{sat}((H, c_H), (F, c_F))$ , is the minimum size of an  $(F, c_F)$ -colored-saturated subgraph of  $(H, c_H)$ .



Remember that  $K_{(n_1, \dots, n_k)}$  is a copy of  $K_{n_1, \dots, n_k}$  colored by  $c(v_i^s) = i$  for  $v_i^s \in V_i$ . We now construct a  $K_{(\ell, m, p)}$ -ordered saturated subgraph of  $K_{(n_1, n_2, n_3)}$ .

**Construction 6.** Let  $S_1 = \{v_1^1, \dots, v_1^\ell\}$ . Let  $S_2 = \{v_2^1, \dots, v_2^m\}$ , and let  $S_3 = \{v_3^1, \dots, v_3^p\}$ . Join  $S_i$  to  $V_{i+1}$  and to  $V_{i+2}$ , and then remove the edges  $v_1^1 v_2^1$ ,  $v_1^1 v_3^1$ , and  $v_2^1 v_3^1$ . Let us call this graph  $G_o$ . Thus,

$$E(G_o) = (\{v_1^r v_2^s : r \leq \ell \text{ or } s \leq m\} \cup \{v_1^r v_3^s : r \leq \ell \text{ or } s \leq p\} \cup \{v_2^r v_3^s : r \leq m \text{ or } s \leq p\}) \setminus \{v_1^1 v_2^1, v_1^1 v_3^1, v_2^1 v_3^1\}.$$

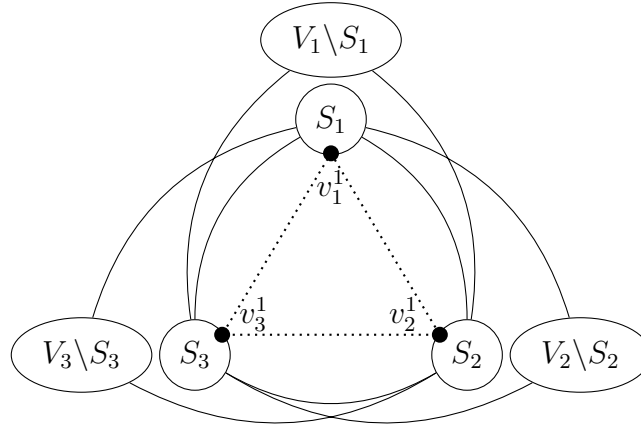


Figure 11: Construction 6: A  $K_{(\ell, m, p)}$ -ordered saturated subgraph of  $K_{(n_1, n_2, n_3)}$ . Solid lines denote a complete join between two sets, and dotted lines denote edges that have been removed.

**Theorem 21.**  $G_o$  is a  $K_{(\ell, m, p)}$ -ordered saturated subgraphs of  $K_{(n_1, n_2, n_3)}$ , and thus

$$\begin{aligned} \overrightarrow{\text{sat}}(K_{(n_1, n_2, n_3)}, K_{(\ell, m, p)}) &\leq \ell(n_2 + n_3 - (m + p)) + p(n_1 + n_2 - (\ell + m)) \\ &\quad + m(n_1 + n_3 - (\ell + p)) + mp + \ell p + \ell m - 3. \end{aligned}$$

*Proof.* Let  $T_1$  be a set of size  $\ell$  in  $V_1$ . Thus,  $T_1$  contains either  $v_1^1$  or  $v_1^r$  for some  $r \in \{\ell + 1, \dots, n_1\}$ . Likewise, let  $T_2$  be a set of size  $m$  in  $V_2$ , which must contain either  $v_2^1$  or  $v_2^r$

for some  $r \in \{m+1, \dots, n_2\}$ . Finally, let  $T_3$  be a set of size  $p$  in  $V_3$  which must contain either  $v_3^1$  or  $v_3^r$  for some  $r \in \{p+1, \dots, n_3\}$ . The pigeonhole principal implies that  $T_1 \cup T_2 \cup T_3$  must contain a pair of nonadjacent vertices. Thus,  $G_o$  does not contain a copy of  $K_{\ell, m, p}$  where the set of size  $\ell$  is in  $V_1$ .

There are six types of non edges in  $G_o$ : first,  $v_1^1 v_2^1$ ; second,  $v_1^1 v_3^1$ ; third,  $v_2^1 v_3^1$ ; fourth,  $v_1^r v_2^s$  for  $r \in \{\ell+1, \dots, n_1\}$  and  $s \in \{m+1, \dots, n_2\}$ ; fifth,  $v_1^r v_3^t$  for  $r \in \{\ell+1, \dots, n_1\}$  and  $t \in \{p+1, \dots, n_3\}$ ; finally,  $v_2^s v_3^t$  for  $s \in \{m+1, \dots, n_2\}$  and  $t \in \{p+1, \dots, n_3\}$ .

Adding  $v_1^1 v_2^1$  yields the desired coloring of  $K_{\ell, m, p}$  on  $S_1 \cup S_2 \cup (S_3 \setminus \{v_3^1\}) \cup \{v_3^r\}$ . Adding  $v_1^1 v_3^1$  yields the desired coloring of  $K_{\ell, m, p}$  on  $S_1 \cup S_3 \cup (S_2 \setminus \{v_2^1\}) \cup \{v_2^r\}$ . Adding  $v_2^1 v_3^1$  yields the desired coloring of  $K_{\ell, m, p}$  on  $S_2 \cup S_3 \cup (S_1 \setminus \{v_1^1\}) \cup \{v_1^r\}$ . Adding  $v_1^r v_2^s$  yields the desired coloring of  $K_{\ell, m, p}$  on  $(S_1 \setminus \{v_1^1\}) \cup \{v_1^r\} \cup (S_2 \setminus \{v_2^1\}) \cup \{v_2^s\} \cup S_3$ . Adding  $v_1^r v_3^t$  yields the desired coloring of  $K_{\ell, m, p}$  on  $(S_1 \setminus \{v_1^1\}) \cup \{v_1^r\} \cup (S_3 \setminus \{v_3^1\}) \cup \{v_3^t\} \cup S_2$ . Adding  $v_2^s v_3^t$  yields the desired coloring of  $K_{\ell, m, p}$  on  $(S_2 \setminus \{v_2^1\}) \cup \{v_2^s\} \cup (S_3 \setminus \{v_3^1\}) \cup \{v_3^t\} \cup S_1$ .

Therefore,  $G_o$  is  $K_{(\ell, m, p)}$ -saturated, and

$$\begin{aligned} \text{sat}(K_{(n_1, n_2, n_3)}, K_{(\ell, m, p)}) &\leq |E(G_o)| \\ &= \ell(n_2 + n_3 - (m + p)) + p(n_1 + n_2 - (\ell + m)) \\ &\quad + m(n_1 + n_3 - (\ell + p)) + mp + \ell p + \ell m - 3. \end{aligned} \quad \square$$

At this point it is worth noting that, unlike in the bipartite setting, it appears that  $\text{sat}(K_{(n_1, n_2, n_3)}, K_{(\ell, m, p)})$  can be less than  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, m, p})$ .

We observe that ordered-saturation numbers can also be on the order of  $n^2$ . Consider  $C_4$ , in particular we will look at a 2-colored copy of  $C_4$ . A 2-colored  $C_4$ -ordered saturated subgraph of  $K_{n, n, n}$  will have a size on the order of  $n^2$ . However, a 2-colored  $C_4$  colored-saturated subgraph of  $K_{n, n, n}$  will be on the order of  $n$ .

Without loss of generality let the 2-colors used to color the  $C_4$  be the same as colors  $V_1$

and  $V_2$ .

**Construction 7.** Join  $V_1$  to  $V_3$  and join  $V_2$  to  $V_3$ . Also join  $v_1^1$  to  $V_2$  and join  $v_2^1$  to  $V_1$ . Call this graph  $G_C$ . Thus,

$$E(G_C) = \{v_3^r v_i^s : i \in \{1, 2\}, r \leq n, s \leq n\} \cup \{v_1^r v_2^s : r = 1 \text{ or } s = 1\}$$

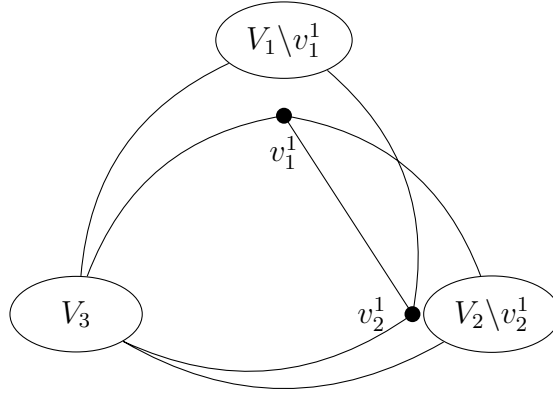


Figure 12: Construction 7: A  $(C_4, c_2)$ -ordered-saturated subgraph of  $(K_{n_1, n_2, n_3}, c_H)$ . Lines denote complete joins

**Construction 8.** Join  $v_1^1$  to  $V_2$  and  $V_3$ , join  $v_2^1$  to  $V_1$  and  $V_3$ , join  $v_3^1$  to  $V_1$  and  $V_2$ . Call this graph  $G_\chi$ . Thus,

$$E(G_\chi) = \{v_i^1 v_j^s : i \in [3], j \in [3], i \neq j\}.$$

**Theorem 22.**

$$\overrightarrow{\text{sat}}(K_{(n, n, n)}, K_{(2, 2, 0)}) = 2n^2 + 2n - 1.$$

*Proof.* We note that 2-colored  $C_4$  is the same as  $K_{2, 2, 0}$ . Since we are looking for a  $K_{2, 2}$  between  $V_1$  and  $V_2$ , Theorem 7 implies that there are at least  $2n - 1$  edges joining  $V_1$  and  $V_2$ . We also know that  $V_3$  must be completely joined to both  $V_1$  and  $V_2$ , as any edge

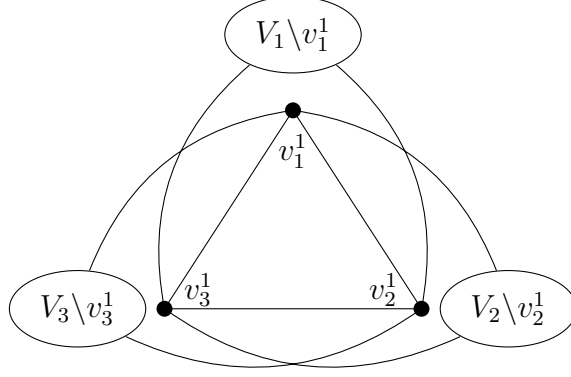


Figure 13: Construction 8: A  $(C_4, c_2)$ -colored-saturated subgraph of  $K_{n_1, n_2, n_3}$ . Lines denote complete joins

that is incident to  $V_3$  cannot be part of a graph that is contained within  $V_1 \cup V_2$ . Thus,

$$\overrightarrow{\text{sat}}(K_{(n, n, n)}, K_{(2, 2, 0)}) \leq 2n^2 + 2n - 1.$$

We will now show that  $G_C$  is a  $K_{(2, 2, 0)}$ -ordered saturated subgraph of  $K_{n, n, n}$ . To form a  $K_{(2, 2, 0)}$  we must use a  $v_1^r$  for  $1 < r \leq n$  and  $v_2^s$  for  $1 < s \leq n$ . However,  $v_1^r$  is not adjacent to  $v_2^s$ .

We note that there is one type of edge that can be added to  $G_C$ ,  $v_1^r v_2^s$  for  $1 < r \leq n$  and  $1 < s \leq n$ . Adding this edge yields a  $K_{(2, 2, 0)}$  on  $\{v_1^1, v_1^r, v_2^1, v_2^s\}$ . Thus,  $G_C$  is a  $K_{(2, 2, 0)}$ -ordered saturated subgraph of  $K_{n, n, n}$ . Therefore,

$$\overrightarrow{\text{sat}}(K_{(n, n, n)}, K_{(2, 2, 0)}) = 2n^2 + 2n - 1. \quad \square$$

**Theorem 23.**

$$\text{sat}(K_{(n, n, n)}, K_{(2, 2, 0)}) = 6n - 3.$$

*Proof.* Using Theorem 7 three times, once per pair of partite sets we see that

$$\text{sat}(K_{(n, n, n)}, K_{(2, 2, 0)}) \leq 6n - 3.$$

We now will show that  $G_\chi$  is a  $K_{(2,2,0)}$ -colored-saturated subgraph of  $K_{(n,n,n)}$ . To form a copy of  $K_{2,2,0}$  we must use  $v_i^r$  and  $v_{i+1}^s$  for  $1 < r \leq n$  and  $1 < s \leq n$ . However,  $v_i^r$  and  $v_{i+1}^s$  are not adjacent. Thus,  $G_\chi$  is  $K_{(2,2,0)}$ -colored-free. We note that there is one type of edge that can be added to  $G_\chi$ ,  $v_i^r v_{i+1}^s$  for  $1 < r \leq n$  and  $1 < s \leq n$ . Adding it yields a  $K_{2,2,0}$  on  $\{v_i^1, v_i^r, v_{i+1}^1, v_{i+1}^s\}$ . Thus  $G_\chi$  is a  $K_{(2,2,0)}$ -color-saturated subgraph of  $K_{(n,n,n)}$ . Therefore,

$$\text{sat}(K_{(n,n,n)}, K_{(2,2,0)}) = 6n - 3.$$

□

## 6 Future Work

A clear place to continue work would be in extending Theorem 16 so that  $n_2$  is no longer bounded by a linear factor of  $n_3$ .

**Conjecture 24.** *For  $n_1 \geq n_2 \geq n_3$ , and  $n_3$  sufficiently large,*

$$\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, \ell, \ell}) = 2\ell(n_1 + n_2 + n_3) - 3\ell^2 - 3.$$

In the proof of Theorem 16 we ignored edges joining neighbors of vertices of degree  $2\ell$ . The proof of this conjecture likely depends on determining the nature of the edges joining these neighborhoods, and showing that those edges prevent the graph from being  $K_{\ell, \ell, \ell}$ -saturated.

We can also attempt to extend our results by unbalancing the target graph. As we showed in Section 4, it seems that there are two cases to consider.

**Question 1.** What are  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, m, m})$  for  $\ell > m$ , and  $\text{sat}(K_{n_1, n_2, n_3}, K_{\ell, m, p})$  for  $\ell \geq m > p$ ?

Along a similar line, would be the question but in the ordered setting.

**Question 2.** What is  $\overrightarrow{\text{sat}}(K_{(n_1, n_2, n_3)}, K_{(k, \ell, m)})$ ?

While answering these questions we would see if a gap between saturations numbers and ordered-saturation exists in the tripartite case as there is in the bipartite case.

Another question in the tripartite case is what are the requirements on  $(F, c_F)$  for  $\text{sat}((K_{n_1, n_2, n_3}, c_h), (F, c_F))$  to be linear in  $n$ . The answer to such a question would provide an upper bound on  $\text{sat}((K_{n_1, n_2, n_3}, c_H), (F, c_F))$ , much like Theorem 5.

**Conjecture 25.** *If  $(F, c_F)$  has edges joining each pair of color classes, then is there a constant  $C$  such that*

$$\text{sat}((K_{n_1, n_2, n_3}, c_H), (F, c_F)) \leq Cn.$$

Along these lines it would be interesting to look at for a general bound for colored-saturation and unordered-saturation.

**Question 3.** Are there functions  $C_1 = C_1(F, c_f, n_1, n_2, n_3)$  and  $C_2 = C_2(F, n_1, n_2, n_3)$  such that  $\text{sat}(K_{n_1, n_2, n_3}, (F, c_f)) \leq C_1 n_1$  and  $\text{sat}(K_{n_1, n_2, n_3}, F) \leq C_2 n_1$ ?

A final question to ponder would be if the results from this paper be extended to a multipartite host:

**Question 4.** What is  $\text{sat}(K_{\underbrace{n, \dots, n}_{r \text{ times}}, K_{\ell, \ell, \ell}})$ ?

While this paper generalizes the work from Ferrara, Jacobson, Pfender, and Wenger in the tripartite case, it does not extend their work on multi-partite graphs. The first step in this direction should be determining  $\text{sat}(K_{n, n, n, n}, K_{2, 2, 2})$ .

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