Rochester Institute of Technology

RIT Digital Institutional Repository

1994

Stability of parametrically forced linear systems

Andrew J. Leccese

Stability of Parametrically Forced Linear Systems

Andrew J. Leccese

A Thesis Submitted in Partial Fulfillment
of the Requirements for the Degree of
MASTER OF SCIENCE
in
Mechanical Engineering

Approved by: Prof. J. S. Torok (Thesis Advisor)

Prof. H. Ghoneim

Prof. C. W. Haines (Department Head)

Department of Mechanical Engineering
College of Engineering
Rochester Institute of Technology

May 1994

Permission Grant

I, Andrew J. Leccese, hereby grant permission to the Wallace Memorial Library of the Rochester Institute of Technology to reproduce this thesis in whole or in part. This permission is extended as long as no reproduction will be used for commercial use or profit.

Date: _6 - 7 - 94_

Abstract

The stability analysis of constant-coefficient linear systems is extended to systems with periodically-varying coefficients. Although this theory is mathematically well-understood, little work has been done regarding its application to physical problems. All previous results are based on asymptotic analysis. A review of the theory of parametrically-forced linear systems will be presented, followed by a detailed stability analysis of a pendulum with a harmonically moving base.

List of Symbols Used

A,B,... - capital letters generally denote matrices - square brackets are placed around the letter when needed for clarity

$a_{ij}, b_{ij}, ...$ - element in the matrix using the same letter of the alphabet

$\vec{x}^i, \vec{y}^i$ - vectors, denoted with arrow, superscript used where necessary for clarity

$x_i, y_i, ...$ - generally elements of vectors, denoted with subscript

λ - eigenvalue, also called a characteristic factor or characteristic multiplier

$\vec{v}$ - generally an eigenvector (modal vector)

r_i - characteristic exponent as defined by $\lambda_i = e^{Tb_{ii}} = e^{Tr_i}$

Table of Contents

CHAPTER 1

Introduction: Stability Theory

An acceptable and physically-realizable dynamic system must satisfy the three basic criteria of stability, accuracy, and a satisfactory transient response. Of these criteria, stability is the most important specification of a system. If a physical system is unstable, other properties such as transient response and steady-state errors are only secondary, if they are relevant at all. A specific transient response or steady-state error requirement can not be predicted for an unstable system. There are many definitions of stability, depending upon the kind of system or the point of view. In this investigation, stability was taken as a bounded response as time approaches infinity.

A linear time-invariant system is *stable* if the natural response approaches zero as time approaches infinity. This definition is also known as asymptotic stability. Since the total response is the sum of the forced and natural responses, the definition of stability implies that only the forced response remains as the natural response approaches zero. A linear system with time-varying system parameters is said to be stable if the response, for all initial conditions, remains bounded as time approaches infinity.

A linear system is said to be *unstable*, if the response grows without bound as time approaches infinity. If the system response neither decays nor grows, but remains constant or oscillates, then the linear system is referred to as *marginally stable*.

Physically, an unstable system can cause damage to the system, to adjacent property, or even to human life. Thus the criterion of stability is even more important than its quality of performance. In fact, some unstable systems are not even physically-realizable, such as trying to balance a pencil on its tip.

In order to effectively analyze the stability of dynamic systems, it becomes necessary to formulate precise definitions of the notion of stability. Since dynamic systems are mathematically modeled using differential equations, stability is characterized by the nature of the associated solutions.

Ideally, one would like to explicitly compute all solutions to a differential equation or a system of differential equations. However, there are actually very few equations (beyond linear equations with constant coefficients) that allow explicit solution in terms of analytic functions. In this investigation, we study some of the qualitative aspects of the solutions of differential equations. The objective will be to analyze the properties of the solutions without explicitly solving for them. These ideas were first advanced by the independent work of two mathematicians, A.M. Lyapunov and Henri Poincare at the turn of the century. Their ideas remain very applicable to this day.

Numerical analysis allows calculation of a specific solution to a differential equation. The computations only give results corresponding to a specific set of initial conditions. This is fine if we desire information on only one specific solution. However, in many problems, for example, in the design of complex systems, automatic controls, and so forth, we want to extract information of a qualitative nature about all the possible solutions to a set of differential equations.

Moreover, we often want to know whether a certain property of these solutions remains unchanged if the system is subjected to various types of changes (usually called perturbations). For such purposes, the computer and the calculation of a few specific solutions do not provide a satisfactory answer. These qualitative studies are also important from the practical point of view, because in most problems (including simple spring-mass or pendulum problems) the differential equations and the measurement of initial values and various other data involve approximations. Indeed, in almost every mathematical model of a physical problem a number of effects have been neglected. It is therefore important to study how sensitive the particular model is to small perturbations or changes in initial conditions and of various parameters. Another drawback in the use of numerical approximation is that often it is of interest to show that a solution of a differential equation tends to zero as $t \rightarrow \infty$. While a numerical approximation method may suggest that this is true, it cannot be used to prove it.

One qualitative phenomenon of interest of great importance is the notion of stability of a certain solution of a differential equation. This investigation is devoted primarily to the study of this property and conditions under which a solution is stable. This concept will be motivated and precisely defined in the next section.

The objective will be to concentrate on dynamic systems with periodically-varying coefficients. A full discussion of systems with variable coefficients is beyond the scope of this work. Besides, many interesting and complicated dynamic systems known as parametrically-excited systems are modeled by differential equations with periodically-varying coefficients. One example is a standard pendulum with a periodically-moving base. Such systems are mathematically well-understood, but very few engineers are exposed to such analysis.

The discussion begins with a review of linear systems theory. For completeness, the stability of constant coefficient systems will also be reviewed. The stability of systems with periodic coefficients will be treated as two separate cases. The one-dimensional theory will be discussed in detail to provide insight to the higher-dimensional theory. For clarity of exposition, the two-dimensional case will be representative of the theory in higher dimensions. Finally, a detailed analysis of a parametrically forced pendulum will be documented.

CHAPTER 2

2.1 Fundamental Solutions of Linear Systems

As a background for the discussion to follow, we will start with the system

$$\dot{\vec{x}} = A\ \vec{x}$$

where $\vec{x}$ is an n-dimensional vector with variable components which are real numbers, which can be expressed explicitly as:

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} \quad \text{and} \quad \dot{\vec{x}} = \begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t)` \end{Bmatrix}$$

Also, A is an n by n matrix in which each component a_{ij} ($1 \leq i,j \leq n$) may be either constant or a continuous function of time, that is,

$$a_{ij} = a_{ij}(t) \quad \text{with } (1 \leq i,j \leq n)$$

Since this is a linear homogeneous system, two solutions $\vec{x}^1$ and $\vec{x}^2$ which satisfy the relations $\dot{\vec{x}}^1 = A\vec{x}^1$ and $\dot{\vec{x}}^2 = A\vec{x}^2$ can be combined to form the solution $\alpha\vec{x}^1 + \beta\vec{x}^2$ which will also satisfy the relation $\dfrac{d}{dt}(\alpha\vec{x}^1 + \beta\vec{x}^2) = A(\alpha\vec{x}^1 + \beta\vec{x}^2)$.

2.2 Fundamental Solutions

A collection of vectors $\vec{v}^1, \vec{v}^2, \ldots, \vec{v}^n$ are called linearly independent if

$$c_1\vec{v}^1 + c_2\vec{v}^2 + \ldots + c_n\vec{v}^n = \vec{0}$$

only when

$$c_1 = c_2 = \ldots = c_n = 0$$

In other words, any one vector can not be written as a linear combination of the others.

Since we have a n-dimensional system, there will be n linearly independent solutions. Now let $\vec{x}^1, \vec{x}^2, \ldots, \vec{x}^n$ be the n linearly independent solutions of the system $\dot{\vec{x}} = A\vec{x}$. Therefore $\dot{\vec{x}}^i = A\vec{x}^i$ for each i and $\vec{x}^1(t), \vec{x}^2(t), \ldots, \vec{x}^n(t)$ are linearly independent for all time. Then $\{\vec{x}^1, \vec{x}^2, \ldots, \vec{x}^n\}$ is a fundamental set of solutions to $\dot{\vec{x}} = A\vec{x}$. When arranged as columns,

$$X(t) = \left[\vec{x}^1(t) \ \vdots \ \vec{x}^2(t) \ \vdots \ \cdots \ \vdots \ \vec{x}^n(t)\right]$$

is called a fundamental matrix of $\dot{\vec{x}} = A\vec{x}$ (or the fundamental solution if the system is one dimensional). Since $\vec{x}^1, \vec{x}^2, \ldots, \vec{x}^n$ are linearly independent, $\det(X(t)) \neq 0$ for all t. Therefore $X(t)^{-1}$ will exist for all t.

Suppose that:

$$X(0) = I_{n \times n} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

which means that:

$$\vec{x}^1(0) = \hat{e}^1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{Bmatrix}, \quad \vec{x}^2(0) = \hat{e}^2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{Bmatrix}, \ldots\ldots, \quad \vec{x}^n(0) = \hat{e}^n = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{Bmatrix}$$

When this is true then X(t) is not only a fundamental matrix, it is also referred to as the principal fundamental matrix.

In general, if X(t) is a fundamental matrix, then the matrix defined by $X(t) \cdot X(0)^{-1}$ will be a principal fundamental matrix. Additionally, any matrix defined by $X(t) \cdot C$ will also be a fundamental matrix provided that C is a non-singular matrix $(\det(C) \neq 0)$.

Also it should be noted that the fundamental matrix X(t) is commonly written as $\Phi(t)$ with columns denoted as $\vec{\phi}^i(t)$ instead of $\vec{x}^i(t)$. For completeness Φ can be written as

$$\Phi(t) = \begin{bmatrix} \vec{\phi}^1 & \vdots & \vec{\phi}^2 & \vdots & \cdots & \vdots & \vec{\phi}^n \end{bmatrix}$$

For any system with constant coefficients of the form:

$$\dot{\vec{x}} = Ax$$

the principal fundamental matrix is defined as:

$$X(t) = e^{At}$$

The meaning of e raised to the power of a matrix is explained in the Appendix.

For time varying coefficients, the fundamental matrix can be determined by the following procedure. First a general solution with n constants must be found:

$$\vec{x}(t) = \begin{Bmatrix} \alpha_1 f_1(t) \\ \alpha_2 f_2(t) \\ \vdots \\ \alpha_n f_n(t) \end{Bmatrix}$$

then choose the constants α_k such that $\vec{x}^i(0) = \hat{e}^i$ or in other words:

$$X(0) = I_{nxn} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

Now X(t), the principal fundamental matrix, is known so any solution (depending on arbitrary initial conditions $\vec{x}(0)$) is given by:

$$\vec{x}(t) = X(t) \cdot \vec{x}(0)$$

2.3 Examples of First Order Linear Systems

1) A one dimensional system with a constant, instead of a time dependent, coefficient would be $\dot{x}=ax$.

The fundamental matrix (or the fundamental solution in this case) would be:

$$X(t)=e^{\int a \cdot dt}c=e^{a \cdot t}c.$$

Now pick c such that $X(0)=1$. Substituting $t=0$ results in $c=1$ to satisfy this condition. Therefore $x(t)=X(t)x(0)=e^{at}x(0)$. This results in the fundamental solution $X(t)=e^{at}$.

2) Now look at the system $\dot{\vec{x}} = A\vec{x}$ where A is an n by n constant matrix. If X(t) is the principal fundamental matrix, then:

$$X(0)=e^{[0]_{n \times n}} = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}_{n \times n}$$

it is also true that $X(t)=e^{At}=\exp[At]$.

3) Now consider the following system:

$$\dot{\vec{x}} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = A\vec{x}$$

The two equations of interest are:

$$\dot{x}_1=x_1+x_2$$
$$\dot{x}_2=4x_1+x_2$$

The general solutions are:

$$x_1(t)=c_1e^{3t}+c_2e^{-t}$$
$$x_2(t)=2c_1e^{3t}-2c_2e^{-t}$$

or for each of the n linearly independent solutions (2 in this case):

$$\vec{x}^i = \left\{ \begin{matrix} x_1 \\ x_2 \end{matrix} \right\} = \left\{ \begin{matrix} c_1 e^{3t} + c_2 e^{-t} \\ 2c_1 e^{3t} - 2c_2 e^{-t} \end{matrix} \right\}$$

For such a system we want

$$\vec{x}^i(0) = \hat{e}^i$$

which results in the relations:

$$\vec{x}^1(0) = \left\{ \begin{matrix} c_1 + c_2 \\ 2c_1 - 2c_2 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\}$$

and

$$\vec{x}^2(0) = \left\{ \begin{matrix} c_1 + c_2 \\ 2c_1 - 2c_2 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\}$$

for $\vec{x}^1$ the results are:

$$\begin{matrix} c_1 = 0.5 \\ c_2 = 0.5 \end{matrix} \Rightarrow \left\{ \begin{matrix} 0.5 \\ 0.5 \end{matrix} \right\}$$

and:

$$\vec{x}^1 = \left\{ \begin{matrix} 0.5e^{3t} + 0.5e^{-t} \\ e^{3t} - e^{-t} \end{matrix} \right\}$$

For $\vec{x}^2$ the results are:

$$\begin{matrix} c_1 = 0.25 \\ c_2 = -0.25 \end{matrix} \Rightarrow \left\{ \begin{matrix} 0.25 \\ -0.25 \end{matrix} \right\}$$

and:

$$\vec{x}^2 = \left\{ \begin{matrix} 0.25e^{3t} - 0.25e^{-t} \\ +0.5e^{3t} + 0.5e^{-t} \end{matrix} \right\}$$

Therefore the fundamental matrix is:

$$X(t) = \begin{bmatrix} 0.5e^{3t} + 0.5e^{-t} & 0.25e^{3t} - 0.25e^{-t} \\ e^{3t} - e^{-t} & 0.5e^{3t} + 0.5e^{-t} \end{bmatrix}$$

4) Now consider a 1 by 1 system with a variable coefficient, $\dot{x} = a(t)x$. In a one dimensional system, the general solution is always:

$$X(t) = e^{\int a(t)dt} c$$

where c is a constant

Since the system is one dimensional, n=1 and the fundamental matrix is simply the scalar:

$$X(t) = e^{\int a(t)dt} c$$

If c is chosen such that $X(0)=1$, then $\bar{x} = X(t)x(0)$ is the solution which will satisfy the given differential equation and initial conditions.

5) Let, for example, the system be $\dot{x} = \cos(t)x$. Again, as shown in the last example, for a one dimensional system the fundamental solution is:

$$X(t) = e^{\int \cos(t)} c = e^{\sin(t)} c$$

Again we want $X(0)=1$ so setting $e^0 c = 1$ results in $c=1$. This results in $X(t) = e^{\sin(t)}$, which is the principal fundamental solution. Therefore $x(t) = X(t)x(0) = e^{\sin(t)}x(0)$.

6) Next, consider the following system:

$$\dot{\bar{x}} = \begin{bmatrix} \sin(t) & 0 \\ 0 & \cos(t) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = A(t)\bar{x}$$

The two equations of interest are:

$$\dot{x}_1 = \sin(t)x_1$$
$$\dot{x}_2 = \cos(t)x_2$$

Since each equation is one dimensional the general solutions will be:

$$x_1(t) = e^{-\cos(t)} c_1$$
$$x_2(t) = e^{\sin(t)} c_2$$

or for each of the n linearly independent solutions (2 in this case):

$$\bar{x}^i = \left\{ \begin{matrix} x_1 \\ x_2 \end{matrix} \right\} = \left\{ \begin{matrix} e^{-\cos(t)} c_1 \\ e^{\sin(t)} c_2 \end{matrix} \right\}$$

For such a system we want $\bar{x}^i(0) = \hat{e}^i$ which results in the relations:

$$\bar{x}^1(0) = \left\{ \begin{matrix} e^{-1} c_1 \\ c_2 \end{matrix} \right\} = \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\}$$

and

$$\bar{x}^2(0) = \left\{ \begin{matrix} e^{-1} c_1 \\ c_2 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\}$$

For $\bar{x}^1$ the result is:

$$\begin{matrix} c_1 = e \\ c_2 = 0 \end{matrix} \Rightarrow \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\}$$

and:

$$\bar{x}^1 = \left\{ \begin{matrix} e^{(-\cos(t)+1)} \\ 0 \end{matrix} \right\}$$

For $\bar{x}_2$ the result is:

$$\begin{matrix} c_1 = 0 \\ c_2 = 1 \end{matrix} \Rightarrow \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\}$$

and:

$$\bar{x}^2 = \left\{ \begin{matrix} 0 \\ e^{\sin(t)} \end{matrix} \right\}$$

Therefore the fundamental matrix is:

$$X(t) = \begin{bmatrix} e^{-\cos(t)+1} & 0 \\ 0 & e^{\sin(t)} \end{bmatrix}$$

7) Finally, consider the following system:

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} \cos(t) & 0 \\ \sin(t) & -1+\cos(t) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

where the equations of interest are:

$$\dot{x}_1 = \cos(t)x_1$$
$$\dot{x}_2 = \sin(t)x_1 + (-1+\cos(t))x_2$$

The first equation is one dimensional resulting in the general solution:

$$x_1(t) = e^{\sin(t)}c_1$$

substituting this back into the second equation results in:

$$\dot{x}_2 = c_1 e^{\sin(t)} + (-1+\cos(t))x_2$$

which can be solved yielding:

$$x_2 = e^{-t+\sin(t)}c_2 + c_1 e^{-t+\sin(t)} \int_0^t e^{u-\sin(u)} e^{\sin(u)} \, du$$

a more detailed procedure to get the above result is shown in Appendix A. The result is:

$$x_2 = e^{-t+\sin(t)}c_2 + c_1 e^{-t+\sin(t)} \int_0^t e^u \, du$$
$$x_2 = c_1 e^{\sin(t)} + c_3 e^{-t+\sin(t)}$$

which is the general solution of the second equation. Therefore for each of the n linearly independent solutions (again, 2 in this case) we have:

$$\bar{x}^i = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 e^{\sin(t)} \\ e^{\sin(t)}(c_1 + c_3 e^{-t}) \end{Bmatrix}$$

For such a system we want $\bar{x}^i(0) = \hat{e}_i$, which results in the relations:

$$\bar{x}^1(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_1 + c_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

and

$$\vec{x}^2(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} c_1 \\ c_1 + c_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

For $\vec{x}^1$ the result is:

$$\begin{matrix} c_1 = 1 \\ c_3 = -1 \end{matrix} \Rightarrow \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

and:

$$\vec{x}^1 = \begin{Bmatrix} e^{\sin(t)} \\ e^{\sin(t)}(1 - e^{-t}) \end{Bmatrix}$$

For $\vec{x}^2$ the result is:

$$\begin{matrix} c_1 = 0 \\ c_1 + c_3 = 1 \end{matrix} \Rightarrow \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

and:

$$\vec{x}^2 = \begin{Bmatrix} 0 \\ e^{-t + \sin(t)} \end{Bmatrix}$$

Therefore the principal fundamental matrix is:

$$X(t) = \begin{bmatrix} e^{\sin(t)} & 0 \\ e^{\sin(t)}(1 - e^{-t}) & e^{-t + \sin(t)} \end{bmatrix}$$

CHAPTER 3

3.1 Constant Coefficient Systems

Consider a system of differential equations of the form:

$$\dot{\vec{x}} = A\vec{x}$$

A is a matrix of constant coefficients but $\dot{\vec{x}}$ and $\vec{x}$ are both functions of time, hence the dot on $\dot{\vec{x}}$ denoting a derivative with respect to time. Therefore the system could be written as:

$$\dot{\vec{x}}(t) = A\vec{x}(t)$$

The time dependent nature should be understood and will frequently not be expressed explicitly.

The eigenvectors, or modal vectors and the solution of the given system of differential equations are both dependent on each other. Therefore, with this in mind we can look at the form of the coefficient matrix. For the previously given system:

$$\dot{\vec{x}} = A\vec{x}$$

which, for a two-dimensional system, could also be written is the following form:

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$$
$$\dot{x}_2 = a_{21}x_2 + a_{22}x_2$$

In order to determine the stability of systems such as these one needs to investigate the eigenvalues and eigenvectors of the system. Eigenvalues are the values, denoted as λ or λ_i, which satisfy the relation:

$$A\vec{x} = \lambda\vec{x}$$

for some non-zero vector $\vec{x}$.

A necessary and sufficient condition for the existence of such a solution is the relation:

$$\det(A-\lambda I) = |A-\lambda I| = 0$$

As long as this relation is true, the system will have a non-trivial solution.

To be sure that this is clear, here is an example of finding the eigenvalues for an arbitrary 2 by 2 system:

Let:

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

$$[A-\lambda I] = \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix}$$

$$|A - \lambda I| = (4-\lambda)(-3-\lambda)-(-5)(2) = \lambda^2 - \lambda - 2 = 0$$

The roots of this equation are:

$$\lambda_1 = 2$$
$$\lambda_2 = -1$$

Eigenvectors are then found by substituting one eigenvalue at a time back into the system $[A-\lambda I]\vec{x} = \vec{0}$ and then solving for a vector in terms of an unknown parameter. Here is an example continuing with this same system:

$$[A - \lambda I]\vec{x} = \begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For $\lambda_1 = 2$:

$$(4-2)x_1 - 5x_2 = 2x_1 - 5x_2 = 0$$
$$2x_1 + (-3-2)x_2 = 0$$

This results in two identical equations:

$$2x_1 - 5x_2 = 0$$

Finding a relation between x_1 and x_2 it is found that $x_2 = 0.4x_1$ which results in the eigenvector:

$$\vec{v}_1 = \begin{Bmatrix} 1.0 \\ 0.4 \end{Bmatrix}$$

Any multiple of this eigenvector is also a valid eigenvector. This is a relatively standard form since it is maximum normalized, the largest term is forced to be positive 1.

The same procedure can be performed using the second eigenvalue $\lambda_2 = -1$. This results in the eigenvector:

$$\vec{v}_2 = \left\{ \begin{matrix} 1.0 \\ 1.0 \end{matrix} \right\}$$

This is a general procedure for finding the eigenvalues and eigenvectors for a 2 by 2 system of equations. However, we want to transform the equations somewhat. Since the eigenvectors corresponding to different eigenvalues are all linearly independent, we can investigate the dynamics of the system:

$$\dot{\vec{x}} = A\vec{x}$$

by examining the dynamics of the solutions corresponding to different eigenvalues and eigenvectors.

We want to change from the coordinate system x to the coordinate system y through the linear transformation:

$$\vec{x} = P\vec{y}$$

This matrix P is composed of the eigenvectors arranged as columns. Substituting the above equation into our original set of equations results in the following equation:

$$P\dot{\vec{y}} = AP\vec{y}$$

Since P is the modal matrix, is will be non-singular, therefore P^{-1} will exist. Now pre-multiply both sides by P^{-1} which results in:

$$\dot{\vec{y}} = P^{-1}AP\vec{y}$$

By renaming the matrix $P^{-1}AP$ as a new matrix B, the transformed system can be represented as the following:

$$\dot{\vec{y}} = B\vec{y}$$
$$\text{or}$$
$$\dot{\vec{y}}(t) = B\vec{y}(t)$$

Again the time dependent nature should be understood and will frequently not be expressed explicitly. This will become our new system which will reveal the dynamics of each eigenspace. The matrix B will be as simple as possible (in canonical form) and will fit into one of six possible cases. The system will have initial conditions defined as:

$$\vec{y}_0 = P^{-1}\vec{x}_0$$

Proceeding, we look for solutions of the form:

$$\vec{y} = \vec{y}(t) = \vec{c}e^{\lambda t} = \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} e^{\lambda t}$$

Where the values denoted as c_i are constants

Knowing the expected form of the solutions, we will take a closer look at all of the six possible forms which these solutions can take for such a system.

3.2 Examples of Possible Cases

As mentioned previously, using such a 2 by 2 system, the matrix B will fit into one of six cases as follows.

Case i)

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_2 < \lambda_1 < 0$
or $0 < \lambda_2 < \lambda_1$

Here λ_1 and λ_2 are the two eigenvalues. The eigenvectors are:

$$\phi_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ and } \phi_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

A phase plot consisting of y_1 versus y_2 can be constructed to help demonstrate whether the system is stable or not. As a side note, in cases where the 2 by 2 system is a set of coupled first order equations which was derived from a single second order equation, y_2 will actually by the first derivative of the variable y_1.

For the case where $\lambda_2 < \lambda_1 < 0$ the phase plot will appear like figure 3.1 below:
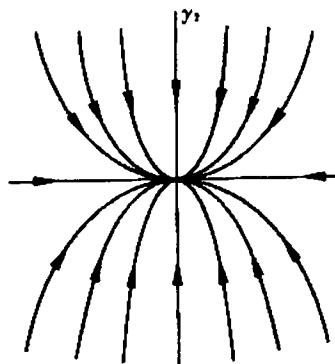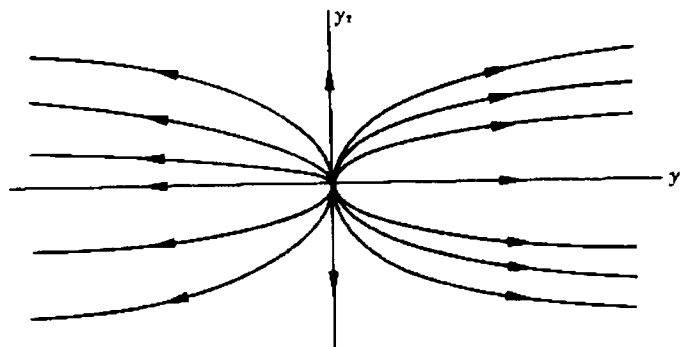

(Fig. 3.1)

For such a system the solutions are of the form:

$$\vec{y}(t) = \left\{ \begin{array}{c} y_1(t) \\ y_2(t) \end{array} \right\} = \left\{ \begin{array}{c} y_1(0)e^{\lambda_1 t} \\ y_2(0)e^{\lambda_2 t} \end{array} \right\}$$

Since λ_1 and λ_2 are both negative in this instance the solutions will decay exponentially and approach zero as $t \rightarrow \infty$. Therefore this solution is inherently stable.

When $0 < \lambda_2 < \lambda_1$ the phase plot will appear like figure 3.2 shown here:

(Fig. 3.2)

Again the solution has the same form:

$$\vec{y}(t) = \left\{ \begin{array}{c} y_1(t) \\ y_2(t) \end{array} \right\} = \left\{ \begin{array}{c} y_1(0)e^{\lambda_1 t} \\ y_2(0)e^{\lambda_2 t} \end{array} \right\}$$

However since λ_1 and λ_2 are both positive in this instance the solutions will grow exponentially and will become unstable.

Case ii)

$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

where $\lambda > 0$
or $\lambda < 0$

Here λ is the repeating eigenvalue. As before the eigenvectors are:

$$\phi_1 = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} \text{ and } \phi_2 = \left\{ \begin{array}{c} 0 \\ 1 \end{array} \right\}$$

When $\lambda > 0$ the phase plot looks like Figure 3.3:
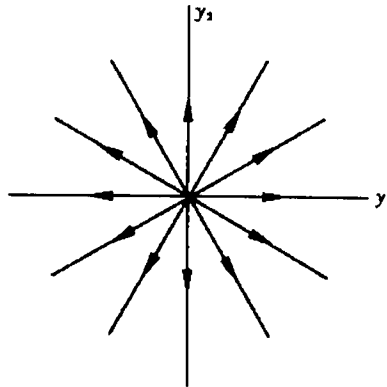
(Fig 3.3)

The solution would have the form:

$$\bar{y}(t) = \left\{ \begin{matrix} y_1(t) \\ y_2(t) \end{matrix} \right\} = \left\{ \begin{matrix} y_1(0)e^{\lambda t} \\ y_2(0)e^{\lambda t} \end{matrix} \right\}$$

Since λ is positive the solutions will increase exponentially, therefore the system is unstable. Since both y_1 and y_2 increase at the same exponential rate, the resulting phase plot shows a linear relationship between these two variables.

When $\lambda < 0$ the phase plot looks like figure 3.4:
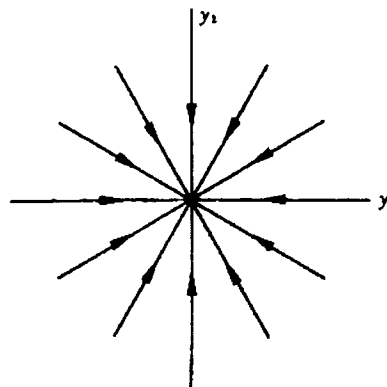
(Fig. 3.4)

Again the solution would have the form:

$$\bar{y}(t) = \left\{ \begin{matrix} y_1(t) \\ y_2(t) \end{matrix} \right\} = \left\{ \begin{matrix} y_1(0)e^{\lambda t} \\ y_2(0)e^{\lambda t} \end{matrix} \right\}$$

Since λ_1 λ_2 are both negative, the solutions converge to zero as t→∞ as can be seen in the phase plot. Therefore this solution will be stable. Again since both variables decrease at the same exponential rate the lines on the phase plot show a linear relationship between them.
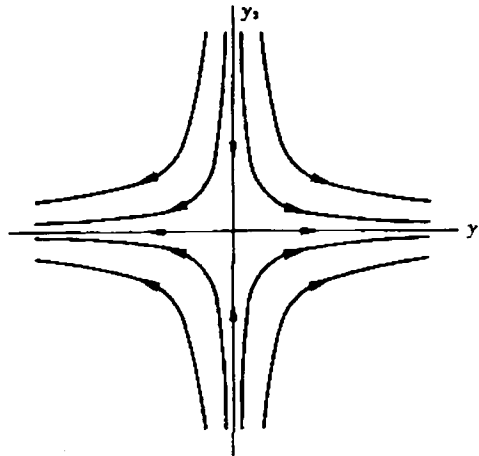
Case iii)

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_2 < 0 < \lambda_1$

λ_1 and λ_2 are the eigenvalues and again the eigenvectors are:

$$\phi_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ and } \phi_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

In this case the phase plot looks like figure 3.5

(Fig. 3.5)

The solution will have the form:

$$\vec{y}(t) = \begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix} = \begin{Bmatrix} y_1(0)e^{\lambda_1 t} \\ y_2(0)e^{\lambda_2 t} \end{Bmatrix}$$

Since λ_1 is positive the system will be unstable. As t→±∞ one or the other of the two variables will also approach infinity.

Case iv)

$$B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$
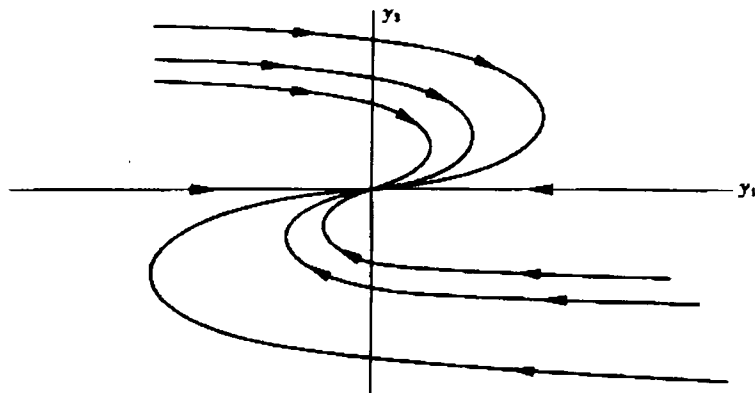
where $\lambda > 0$

or $\lambda < 0$

Again λ represents the repeating eigenvalue (repeats twice). However this time eigenvectors appear as:

$$\phi_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \text{ and } \phi_2 = \begin{Bmatrix} -1 \\ 0 \end{Bmatrix}$$

Here the solution has the form:

$$\vec{y}(t) = \begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix} = \begin{Bmatrix} y_1(0)e^{\lambda t} + y_2(0)te^{\lambda t} \\ y_2 e^{\lambda t} \end{Bmatrix}$$

Negative values of λ will result in stable solutions and results in the phase plot shown in Figure 3.6. As you might expect, positive values of λ will result in unstable solutions. For such a situation the arrows on the phase plot would be reversed.

(Fig. 3.6)

Case v)

$$B = \begin{bmatrix} \sigma & \nu \\ -\nu & \sigma \end{bmatrix}$$

where $\sigma, \nu \neq 0$

and $\sigma > 0$ or $\sigma < 0$

For this case the eigenvalues are much different. Here:

3-8

$$\lambda_1 = \sigma + \nu i$$
$$\text{and}$$
$$\lambda_2 = \sigma - \nu i$$

which are not as readily apparent as in other cases. Also not so obvious are the eigenvectors which are:

$$\phi_1 = \begin{Bmatrix} 1 \\ i \end{Bmatrix} \text{ and } \phi_2 = \begin{Bmatrix} 1 \\ -i \end{Bmatrix}$$

Here the solution has the form:

$$\vec{y}(t) = \begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix} = \begin{Bmatrix} y_1(0)e^{\sigma t}\cos(\nu \cdot t) + y_2(0)e^{\sigma t}\sin(\nu \cdot t) \\ y_1(0)e^{\sigma t}\sin(\nu \cdot t) + y_2(0)e^{\sigma t}\cos(\nu \cdot t) \end{Bmatrix}$$

This case will be unstable when the real part of the eigenvalues (σ) is positive. This is the situation shown in the phase plot shown in figure 3.7. When σ is negative the directions of the arrows on the phase plot would be reversed and the solutions would converge to zero as $t \to \infty$.
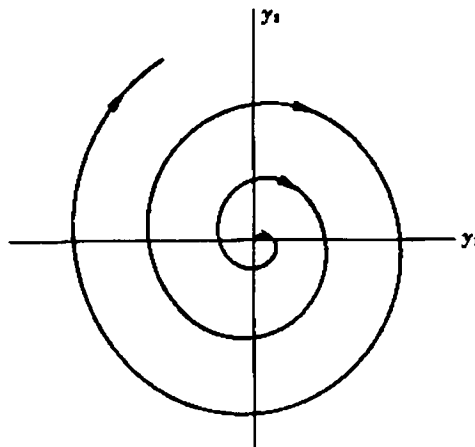
(Fig. 3.7)

Case vi)

$$B = \begin{bmatrix} 0 & \nu \\ -\nu & 0 \end{bmatrix}$$
$$\nu \neq 0$$

In this case the eigenvectors are $\lambda_1 = \nu i$ and $\lambda_2 = -\nu i$. Once again the eigenvectors are:

$$\phi_1 = \begin{Bmatrix} 1 \\ i \end{Bmatrix} \text{ and } \phi_2 = \begin{Bmatrix} 1 \\ -i \end{Bmatrix}$$

The solution of this case has the form:

$$\bar{y}(t) = \begin{Bmatrix} y_1(t) \\ y_2(t) \end{Bmatrix} = \begin{Bmatrix} y_1(0)\cos(v \cdot t) + y_2(0)\sin(v \cdot t) \\ -y_1(0)\sin(v \cdot t) + y_2(0)\cos(v \cdot t) \end{Bmatrix}$$

Since the eigenvalues are purely imaginary the system is stable for any real value of v. The system does not decay down to zero as other stable systems do but its response repeats over and over through the same cycle. Since it is bounded it is then stable.

In this case the phase plot looks like figure 3.8:
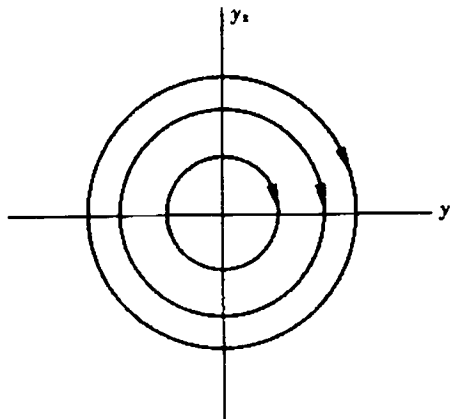
(Fig. 3.8)

CHAPTER 4

4.1 Time Varying Systems

Attention will now be focused on linear systems which have time varying coefficients instead of constant coefficients. The general theory of systems with time varying coefficients is beyond the scope of this investigation. For concrete results, we restrict ourselves to systems with periodically-varying coefficients. Such systems represent an important class of systems regarded as parametrically-forced systems. This definition results from the fact that certain forcing may result in time-varying system parameters.

4.2 Floquet Theory for One Dimensional Systems

Start with a system of the form:

$$\dot{x}(t)=a(t)x(t)$$

which can also be written simply as:

$$\dot{x}=a(t)x \qquad (4\text{-}1)$$

The coefficient $a(t)$ will be periodic, with a period of T, so that $a(t+T)=a(t)$.

The fundamental solution is any solution to (4-1) for which $x(0)=1$.

Examples of various one dimensional linear systems are as follows:

i) For the system $\dot{x}=ax$, 'a' will be a constant. A solution to such a system will be $x(t)=e^{at}$. Since this satisfies the condition $x(0)=1$, this will be a fundamental solution.

ii) For the system $\dot{x}=\cos(t)x$, a solution will be $x(t)=e^{\sin(t)}$. Again this satisfies the condition that $x(0)=1$, so this solution is also a fundamental solution.

iii) For the system $\dot{x}=(1-\sin(t))x$, a solution will be $x(t)=e^{t-1+\cos(t)}$. Once again this does satisfy the condition that $x(0)=1$, therefore this will again be a fundamental solution.

Now let $x(t)$ be any solution. The system given by (4-1) will hold for all t, so:

$$\frac{d}{dt}[x(t+T)]=a(t+T)x(t+T)$$

but since a(t+T)=a(t) it follows that:

$$\frac{d}{dt}[x(t+T)]=a(t)x(t+T)$$

therefore x(t+T) is also a solution of (4-1).

But any solution of (4-1) is a multiple of the fundamental solution. Therefore:

$$x(t+T)=x(t)c$$

where c is constant (and a scalar since this example is a one dimensional system). This follows because a(t) is periodic. Hence, such systems have a very important multiplicative property. This concept can be seen in the sketch below:
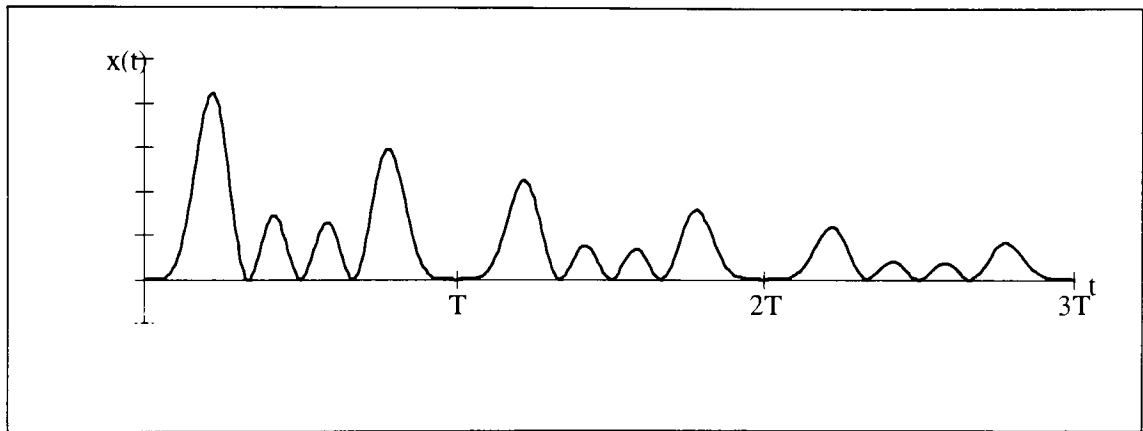
As can be seen in this sketch, values separated by the period, T, differ by a factor of c and values separated by a factor of 2T differ by a factor of c^2.

Stated differently, the solution in any time interval of length T is c times the solution at the corresponding point in the previous interval. Therefore, it is only necessary to solve $\dot{x}=a(t)x$ over a single period $0\leq t\leq T$.

Stated explicitly, $x(t)=x(\bar{t}+nT)=x(\bar{t})\cdot c^n$ for $0\leq \bar{t}\leq T$ where $\bar{t}$ is the location in any given period as shown in the following figure:
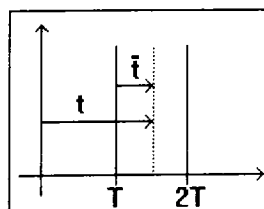
where
$t=1.5T=0.5T+T=\bar{t}+T$

Now if x(t) is a principle fundamental solution then x(0)=1 so x(T)=x(0)·c=c, x(2T)=c^2 ,....., x(nT)=c^n. If x(t) is any (fundamental) solution, but not a principal fundamental solution, then c=$\dfrac{x(T)}{x(0)}$.

The factor c is defined as the characteristic multiplier. Now set c=e^{rT}. Therefore rT=ln(c), and r=ln(c)/T which can be complex when c<0. The number r is referred to as the characteristic exponent.

4.3 Proof of Floquet's Theorem (One-Dimensional)

Let x(t) be any solution of $\dot{x}$=a(t)x, where a(t) is a continuous periodic function with period T. Then there is a periodic function p(t) such that x(t)=p(t)e^{rT}.

Proof:
Construct p(t)=x(t) e^{-rt}. $\dot{p}=\dot{x}\ e^{-rt}-r\cdot x(t)\ e^{-rt}=\dot{x}\ e^{-rt}-r\cdot p$

1) p(t)e^{rt} is a solution of (4-1) as shown here:

$$\frac{d}{dt}(p(t)e^{rt})=\ \dot{p}e^{rt}=r\cdot p\cdot e^{rt}$$

substituting $\dot{p}=\dot{x}\ e^{rt}-r\cdot p$

$$\frac{d}{dt}(p(t)e^{rt})=(\dot{x}\ e^{-rt}-r\cdot p)e^{rt}+r\cdot x$$

$$\frac{d}{dt}(p(t)e^{rt})=\dot{x}-r\cdot p\cdot e^{rt}+r\cdot x$$

substituting $\dot{x}$=a(t)x and p=x·e^{-rt}

$$\frac{d}{dt}(p(t)e^{rt})=a(t)x-rx+rx$$

therefore

$$\frac{d}{dt}(p(t)e^{rt})=a(t)x$$

which verifies that x=p(t)e^{rt} is a solution of $\dot{x}$=ax

2) Now show that p(t) is periodic.

$$p(t+T)=x(t+T)e^{-r(t+T)}$$
$$p(t+T)=c \cdot x(t)e^{-rt}e^{-rT}$$
$$p(t+T)=c \cdot x(t)e^{-rt}c^{-1}$$
$$p(t+T)=x(t)e^{-rt}$$
$$p(t+T)=p(t)$$

therefore p(t) is periodic

Again, the Floquet decomposition is given by $x(t)=p(t)e^{rt}=p(t)\exp\left(\dfrac{\ln(c)}{T}t\right)$

Thus, since p(t) is periodic and continuous, it is bounded. Therefore, stability is determined by the characteristic exponent r.

Recalling that

$$r=\frac{\ln(c)}{T}$$

stability of the system will be determined as follows:

$r<0$ stable
$r>0$ unstable
$r=0$ neutrally stable (periodic/bounded)

Therefore, reiterating Floquet Theory, the solution x(t) of a system such as $\dot{x}=a(t)x$ will be equal to $p(t)e^{rt}$. Here p(t) is a periodic function and e^{rt} will cause x(t) to exponentially decay, grow, or remain the same depending on the value of r. Shown graphically below the function p(t) is periodic:
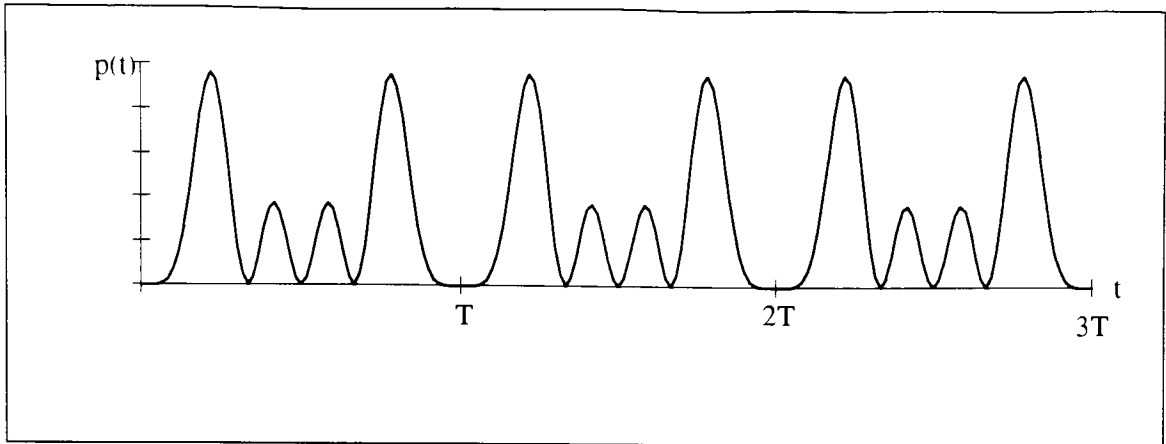
When multiplied by the function e^{rt}:
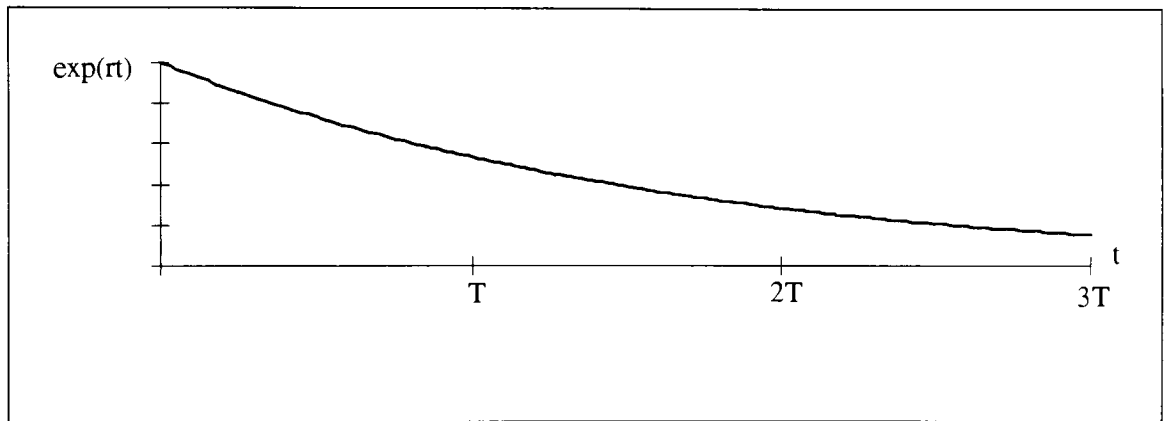
The product is the modulated function $x(t)$:
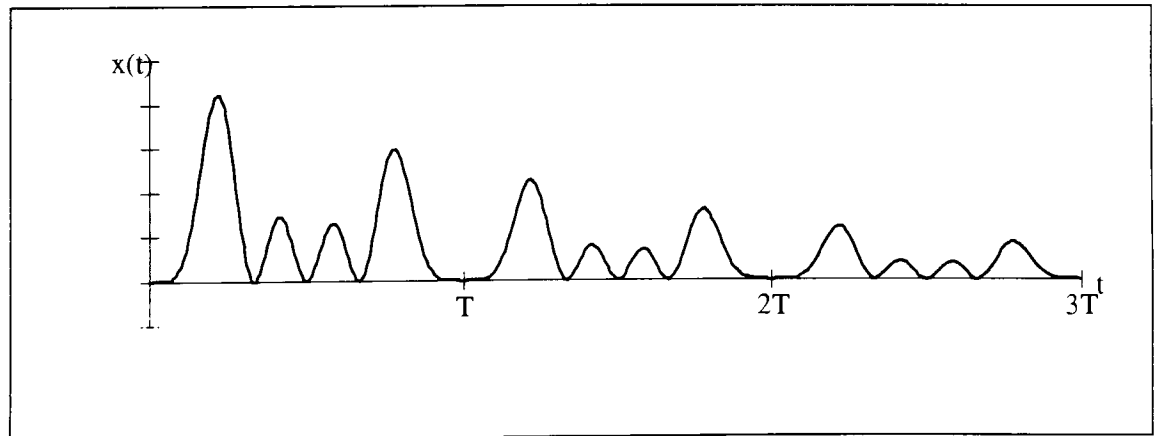
which is now an exponentially decaying response, since r is negative in this case.

4.4 One-Dimensional Examples

As an example of a one dimensional system, let

$$\dot{x} = a(t)x$$

where a(t) is periodic, therefore:

$$a(t+T)=a(t).$$

As shown in chapter 2 the fundamental solution is given by:

$$X(t)=e^{\int a(t)dt}$$

Now x(t) can be rewritten as:

$$x(t)=\frac{e^{\int a(t)dt}}{e^{bt}}e^{bt}$$

or

$$x(t)=[e^{\int a(t)dt-bt}]e^{bt}$$

Furthermore, the characteristic multiplier can be written as:

$$\frac{x(T)}{x(0)} = e^{\int_0^T a(t)dt}$$

so that the characteristic multiplier (eigenvalue) is:

$$c=\exp\int_0^T a(t)dt$$

The corresponding characteristic exponent is:

$$r=\frac{\ln(c)}{T}$$

or

$$r=\frac{1}{T}\int_0^T a(t)dt$$

which is the average value of the variable coefficient a(t), over the period T. Therefore, the stability of $\dot{x}$ =a(t)x is completely determined by the average value:

$$\bar{a} = \frac{1}{T}\int_0^T a(t)dt$$

In summary, solutions to $\dot{x} = a(t)x$ are:

          stable if $\bar{a} < 0$

          periodic if $\bar{a} = 0$

          unstable if $\bar{a} > 0$

As another example using actual functions, consider the following system:

$$\dot{x} = \cos^2(t)x$$

with the solution:

$$x(t) = e^{\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)}$$

Since this is a 1 by 1 system, the coefficient "matrix" A is simply the scalar $\cos^2(t)$ which by definition has a period of T. This is the requirement of the coefficient (matrix). Also the fundamental matrix X(t) is the scalar x(t) as shown above. Therefore x(t+T) must be equal to a scalar multiple of x(t), or x(t+T)=c·x(t). Substituting t+T, where T is equal to π for $\cos^2(t)$, into the above solution will result in:

$$e^{\frac{1}{2}\cos(t)\sin(t)+\frac{t}{2}+\frac{\pi}{2}} = c \cdot e^{\frac{1}{2}\cos(t)\sin(t)+\frac{t}{2}}$$

where c will be equal to $e^{\pi/2}$ or approximately 4.8105 which is the characteristic multiplier. Therefore, the fact that x(t+T) is a scalar multiple of x(t) has been verified.

$$x(t) = e^{\frac{\sin 2t}{4}} \cdot e^{\frac{t}{2}}$$

Also, as expected, x(t) has been expressed as $x(t) = p(t)e^{bt}$, where $p(t+T) = p(t)$.

This can also be illustrated using Van der Pol's equation on the following pages.

4.5 Van der Pol Equation

An important and often-occurring equation in non-linear mechanics is the Van der Pol equation:

$$\ddot{z} + \varepsilon(z^2 - 1)\dot{z} + \omega^2 z = 0 \qquad (4\text{-}2)$$

in which ω and ε are real parameters. This equation has no analytical solution, but it is well known to have a limit cycle, that is, an isolated periodic solution.

The analysis proceeds by writing (4-2) in state variable form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega^2 x_1 + \varepsilon\left(1 - x_1^2\right)x_2 \end{aligned} \qquad (4\text{-}3)$$

In order to analyze the limit cycle, we introduce the polar variables

$$x_1 = \rho \, \sin(\omega\theta)$$
$$\text{and}$$
$$x_2 = \rho\omega \, \cos(\omega\theta)$$

to obtain:

$$\dot{\theta} = 1 - \frac{\varepsilon}{2\omega}\left(1 - \rho^2 \sin^2(\omega\theta)\right) \cdot \sin(2\omega\theta) \qquad (4\text{-}4)$$

$$\dot{\rho} = \varepsilon\left(1 - \rho^2 \sin^2(\omega\theta)\right)\rho\cos^2(\omega\theta) \qquad (4\text{-}5)$$

Let ρ_∞ denote the trajectory of the limit cycle. Then, on the limit cycle, the solution to (4-4) and (4-5) may be expressed as:

$$\rho(t) = \rho_\infty(t)$$

which is a periodic function of period $T = 2\pi/\omega$. Integrating (4-4) with respect to θ and ignoring the higher harmonics results in $\dot{\theta} \approx 1$. Note that this is equivalent to averaging the angular velocity $\dot{\theta}$ over the limit cycle. Substituting $\theta = t$ into (4-5) results in:

$$\dot{\rho} = \varepsilon\left(1 - \rho^2 \sin^2(\omega t)\right) \cdot \rho \cdot \cos^2(\omega t) \qquad (4\text{-}6)$$

Equation (4-6) is still a non-linear equation, but we proceed to analyze the stability of the limit cycle.

Computing the variation of equation (4-6), we obtain:

$$(\delta\rho)^{\bullet} = \varepsilon\left(\delta\rho - 3\rho^2 \cdot \delta\rho \cdot \sin(\omega t)\right) \cdot \cos^2(\omega t)$$

resulting in:

$$(\delta\rho)^{\bullet} = \varepsilon\left(\cos^2(\omega t) - 3\rho^2 \sin(\omega t) \cdot \cos^2(\omega t)\right) \cdot \delta\rho$$

Here the variation $\delta\rho$ represents a perturbation of the limit cycle solution ρ_∞.

Now, if we let

$$\rho = \rho_\infty + \eta$$

and ignore higher order terms, we obtain the perturbation equation

$$\dot{\eta} = \varepsilon\left(\cos^2(\omega t) - 3\rho_\infty^2 \sin^2(\omega t)\cos^2(\omega t)\right) \cdot \eta$$

that is,

$$\dot{\eta} = \varepsilon \cos^2(\omega t)\left(1 - 3\rho_\infty^2 \sin^2(\omega t)\right) \cdot \eta$$

We now have a linear equation in the perturbation, $\eta(t)$, with a variable coefficient

$$a(t) = \varepsilon\cos^2(\omega t)(1 - 3\rho_\infty \sin^2(\omega t))$$

of period $2\pi/\omega$.

As shown in the previous section, the stability of the solution of this equation is governed by the average value:

$$\bar{a} = \varepsilon\left[\frac{1}{2} - \frac{3\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \rho_\infty^2 \sin^2(\omega t)\cos^2(\omega t)dt\right]$$

$$\bar{a} = \frac{\varepsilon}{2} - 3\varepsilon\int_0^1 \rho_\infty^2 \sin^2(2\pi u)\cos^2(2\pi u)du$$

Integrating over the limit cycle

$$\int_0^1 \rho_\infty^2 \sin^2(2\pi u)\cos^2(2\pi u)du = .4484$$

Here, the average-value parameter

$$\bar{a} = -.8452\varepsilon$$

This analysis verifies that for $\varepsilon > 0$, the limit cycle is stable, and nearby solutions converge to the limit cycle at a rate of

$$e^{\bar{a}t} = e^{-.8452\varepsilon t}.$$

A nearby solution for which $x(0)=2.5$ and $\dot{x}(0)=0$ is plotted. It can be clearly seen that the solution converges very quickly to the limit cycle. This plot appears below:
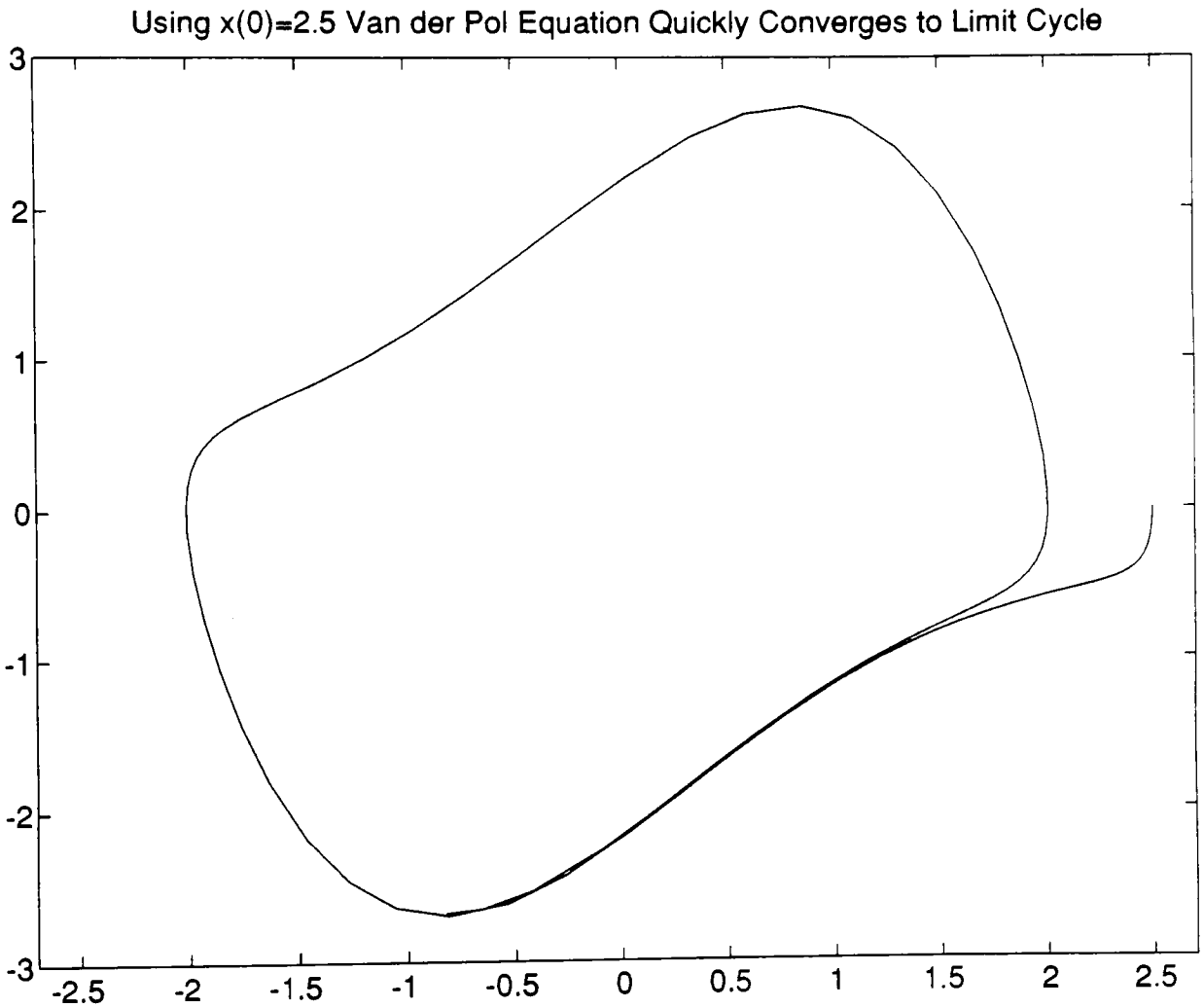

Figure 4-1

Using initial conditions of x(0)=2 and $\dot{x}(0)$=0 the solution a very special case which begins very close to the limit cycle as seen in the following plot:

Limit Cycle for Van der Pol Equation
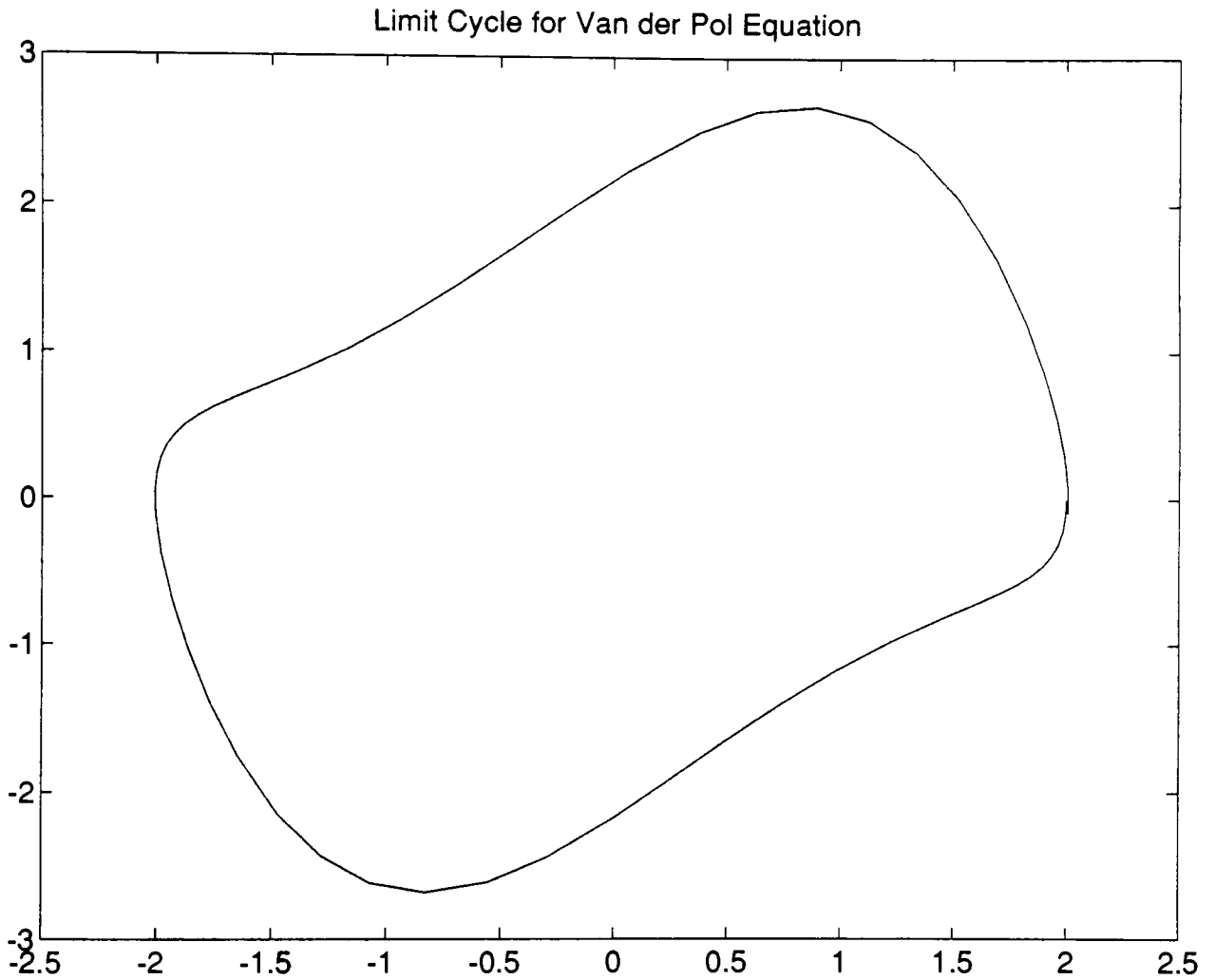
Figure 4-2

Finally a plot of a variety of solutions with different initial values of x(0) is shown below, all of which also converge quickly to the limit cycle as shown before in Figure 4-1.

Various Solutions of the Van der Pol Equation
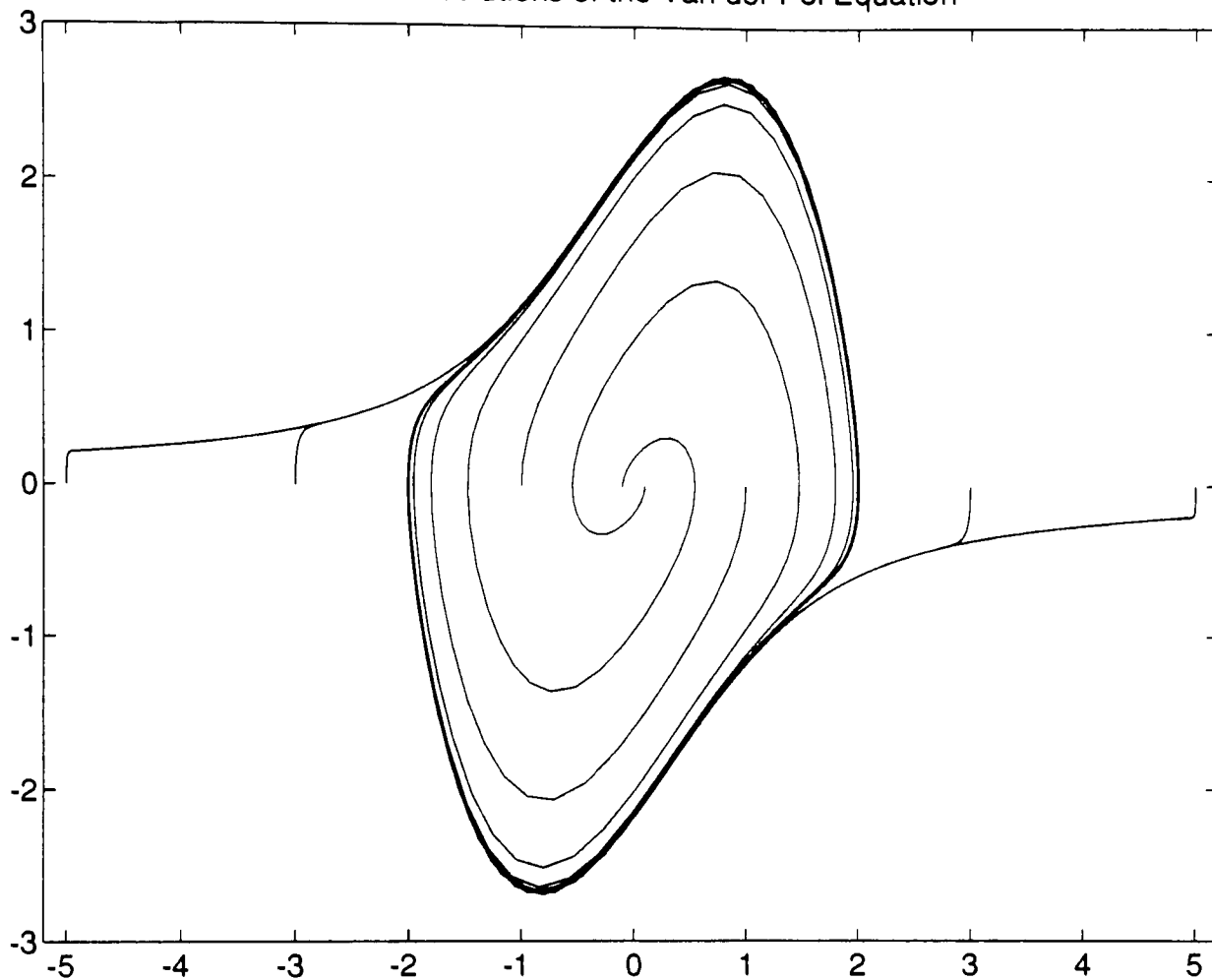
Figure 4-3

4.6 Floquet Theory for Systems

Now let's take a look at a system of the form:

$$\dot{\vec{y}} = A(t)\vec{y} \qquad (4\text{-}7)$$

where $A(t)$ is a continuous periodic n by n matrix of period T. Due to the periodic nature of the matrix, $A(t+T)=A(t)$.

Now there exists a set of n linearly independent solutions $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ which all satisfy (4-7). Arranging these vectors as columns of a matrix denoted as X(t) results in a matrix of the form $X(t)=[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n]$. X(t) will now be a fundamental solution matrix of (4-7), therefore:

$$\frac{d}{dt}[X]=[A(t)]X$$

Since (4-7) is true for all time, it is true that:

$$\frac{d}{dt}[X(t+T)]=A(t+T)[X(t+T)]$$

Since A is periodic, A(t+T)=A(t). Therefore this equation can be rewritten as:

$$\frac{d}{dt}[X(t+T)]=A(t)[X(t+T)]$$

So X(t+T) must also be a fundamental solution matrix.

From the general theory of differential equations it is true that:

$$X(t+T)=X(t)C \tag{4-8}$$

where C is a constant nonsingular matrix. In other words, a constant multiple of a solution of a linear system is also a solution.

Now, from linear algebra, if C is nonsingular, there is a matrix $R=\frac{1}{T}\log(C)$ such that $e^{TR}=C$, where T is a scalar and R is a matrix.

We can assume that X(T) is the principle fundamental solution, in which:

$$X(0)=\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$$

Since X(t) is a fundamental matrix, det[X(t)]≠0
Then from (4-8), $X(T)=e^{TR}$ as was explained in chapter 2.

4.7 Floquet's Theorem for n-Dimensions

Let A(t) be a continuous periodic n by n matrix of period T for which A(t+T)=A(t) and X(t) is a fundamental solution matrix of (4-7). There is a periodic matrix P(t) of period T and a constant matrix R such that $X(t)=P(t)e^{tR}$.

Corollary: By setting $\bar{y}=[P(t)]\bar{u}$, (4-7) is transformed to a constant coefficient system:

$$\dot{\bar{u}}=R\bar{u} \qquad\qquad (4-9)$$

The solution of (4-9) is $\bar{u}(t)=e^{tR}\bar{u}_0$. However, it should be noted that these results require complete knowledge of the matrices P(t) and R.

Restricting this discussion to 2-dimensions, there are 3 generic forms that the matrix R can have. In equation (4-8), C is a constant matrix, so C is similar to one of 3 generic matrices denoted as G. That is, there is a change of basis matrix Q such that:

$$Q^{-1}CQ = G$$
$$\text{or}$$
$$C=QGQ^{-1}$$

Furthermore, $\log(C)=Q[Log[G]]Q^{-1}$.

Case 1:
If C has distinct eigenvalues λ_1 and λ_2 then:

$$G=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

then

$$Log[G]=\begin{bmatrix} \log\lambda_1 & 0 \\ 0 & \log\lambda_2 \end{bmatrix}$$

If either of λ_1 and λ_2 is negative, its log is not real. However:

$$Log[G^2]=\begin{bmatrix} \log(\lambda_1)^2 & 0 \\ 0 & \log(\lambda_2)^2 \end{bmatrix}$$

is real. Thus,

$$R = \frac{1}{T}\log C = Q\begin{bmatrix} \dfrac{\log\lambda_1}{T} & 0 \\ 0 & \dfrac{\log\lambda_2}{T} \end{bmatrix}Q^{-}$$

so:

$$e^{tR} = e^{tQ\begin{bmatrix} \frac{\log\lambda_1}{T} & 0 \\ 0 & \frac{\log\lambda_2}{T} \end{bmatrix}Q^{-1}}$$

$$e^{tR} = Qe^{t\begin{bmatrix} \frac{\log\lambda_1}{T} & 0 \\ 0 & \frac{\log\lambda_2}{T} \end{bmatrix}}Q^{-1}$$

$$e^{tR} = Q\begin{bmatrix} e^{t\frac{\log\lambda_1}{T}} & 0 \\ 0 & e^{t\frac{\log\lambda_2}{T}} \end{bmatrix}Q^{-1}$$

Case 2:

If C has equal eigenvalues, $\lambda_1 = \lambda_2 = \lambda$ (and $\lambda \neq 0$), then:

$$G = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

then,

$$\log G = \begin{bmatrix} \log\lambda & \dfrac{1}{\lambda} \\ 0 & \log\lambda \end{bmatrix}$$

Thus,

$$R = \frac{1}{T}\log C = Q\begin{bmatrix} \dfrac{\log\lambda}{T} & \dfrac{1}{\lambda T} \\ 0 & \dfrac{\log\lambda}{T} \end{bmatrix}Q^{-1}$$

so,

$$e^{tR}=e^{Qt\begin{bmatrix}\frac{\log\lambda}{T} & \frac{i}{\lambda T}\\ 0 & \frac{\log\lambda}{T}\end{bmatrix}Q^{-1}}$$

$$e^{tR}=Qe^{\begin{bmatrix}t\frac{\log\lambda}{T} & t\frac{i}{\lambda T}\\ 0 & t\frac{\log\lambda}{T}\end{bmatrix}}Q^{-1}$$

$$e^{tR}=Q\left\{\begin{bmatrix}e^{t\frac{\log\lambda}{T}} & 0\\ 0 & e^{t\frac{\log\lambda}{T}}\end{bmatrix}\cdot\begin{bmatrix}1 & \frac{t}{\lambda T}\\ 0 & 1\end{bmatrix}\right\}Q^{-i}$$

Case 3:
If C has complex eigenvalues:

$$\lambda_i=\alpha+i\beta$$
$$\lambda_2=\alpha-I\beta$$

then,

$$G=\begin{bmatrix}\alpha & -\beta\\ \beta & \alpha\end{bmatrix}$$

and,

$$\log G=\begin{bmatrix}\log(\alpha^2+\beta^2)^{1/2} & -\tan^{-i}\left(\frac{\beta}{\alpha}\right)\\ \tan^{-i}\left(\frac{\beta}{\alpha}\right) & \log(\alpha^2+\beta^2)^{1/2}\end{bmatrix}$$

thus,

$$R=\frac{1}{T}\log C=Q\begin{bmatrix}\dfrac{\log(\alpha^2+\beta^2)^{1/2}}{T} & -\dfrac{\tan^{-i}\left(\frac{\beta}{\alpha}\right)}{T}\\ \dfrac{\tan^{-i}\left(\frac{\beta}{\alpha}\right)}{T} & \dfrac{\log(\alpha^2+\beta^2)^{1/2}}{T}\end{bmatrix}$$

so,

$$e^{tR}=e^{Qt\begin{bmatrix}\dfrac{\log(\alpha^2+\beta^2)^{1/2}}{T} & -\dfrac{\tan^{-1}\left(\dfrac{\beta}{\alpha}\right)}{T}\\[4mm] \dfrac{\tan^{-1}\left(\dfrac{\beta}{\alpha}\right)}{T} & \dfrac{\log(\alpha^2+\beta^2)^{1/2}}{T}\end{bmatrix}Q^{-1}}$$

$$e^{tR}=Qe^{\begin{bmatrix}t\dfrac{\log(\alpha^2+\beta^2)^{1/2}}{T} & -t\dfrac{\tan^{-1}\left(\dfrac{\beta}{\alpha}\right)}{T}\\[4mm] t\dfrac{\tan^{-1}\left(\dfrac{\beta}{\alpha}\right)}{T} & t\dfrac{\log(\alpha^2+\beta^2)^{1/2}}{T}\end{bmatrix}}Q^{-1}$$

The eigenvalues of the matrix $C=e^{TR}$ are defined as characteristic multipliers. Furthermore, the eigenvalues of the matrix

$$R=1/T \log(C)$$

are known as characteristic exponents. It should be noted that characteristic exponents are the generalization of eigenvalues of constant-coefficient systems.

Based on Floquet's Theorem, the stability of a system with periodic coefficients is governed by the characteristic exponents. This follows from the fact that:

$$X(t)=P(t)e^{tR}$$

where P(t) is periodic and continuous, and hence, bounded.

4.8 Hill's Equation

We can now move on to the study of Hill's Equation which is:

$$\ddot{y}+p(t)y = 0$$

again with a periodic coefficient p(t) resulting in the fact that p(t+T)=p(t). This equation can be changed into a set of two coupled first order equations by setting $x_1=y$ and $x_2=\dot{y}$. The resulting system will appear as:

$$\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\end{Bmatrix}=\begin{bmatrix}0 & 1\\ -p(t) & 0\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix}$$

Let X be the fundamental matrix satisfying X(0)=I. In other words,

$$X(t)=\begin{bmatrix}y_1(t) & y_2(t)\\ \dot{y}_1(t) & \dot{y}_2(t)\end{bmatrix}$$

where $y_1(t)$ and $y_2(t)$ are solutions to Hill's Equation which satisfy:

$$y_1(0)=1 \qquad y_2(0)=0$$
$$\dot{y}_1(0)=0 \qquad \dot{y}_2(0)=1$$
$$\text{or}$$
$$X(0)=I$$

In the previous chapter we saw that $X(t+T)=X(t)C$. Therefore for $t=0$ we know that:

$$C=X(T)=\begin{bmatrix} y_1(T) & y_2(T) \\ \dot{y}_1(T) & \dot{y}_2(T) \end{bmatrix}$$

The characteristic multipliers of this matrix would be the roots of the equation

$$\det[C-\lambda I]=\det[X(T)-\lambda I]=0$$

Written explicitly this equation is:

$$\lambda^2 - (y_1(T)+\dot{y}_2(T))\lambda + \dot{y}_2(T)y_1(T) - y_2(T)\dot{y}_1(T) = 0$$
$$\text{or}$$
$$\lambda^2 - (y_1(T)+\dot{y}_2(T))\lambda + \det(C) = 0$$

If A and X are n by n matrices and $\dot{X}=AX$ then it is true that $\dfrac{d}{dt}|X|=\text{tr}(A)|X|$. For Hill's equation $\text{tr}(A)=0$, where tr is the trace of a matrix, or the sum of the diagonal elements. Therefore for $\dfrac{d}{dt}|X|=\text{tr}(A)|X|$ to be true, $|X|$ must be a constant. However, for this to be the case the equation $|X(t+T)| = |X(t)| \cdot |C|$ implies that $|C|$ must equal 1.

Going back to the characteristic equation above and substituting $|C|=1$ we get the result:

$$\lambda^2 - (y_1(T)+\dot{y}_2(T))\lambda + 1 = 0$$

This is of the form $\lambda^2+Z\lambda+1=0$, which can be factored into $(\lambda-\lambda_1)(\lambda-\lambda_2)=0$ which could then be multiplied back out into the form $\lambda^2 +-(\lambda_1+\lambda_2)+\lambda_1\lambda_2 =0$. Therefore with λ_1 and λ_2 as the roots of this equation (the characteristic factors), they must satisfy $\lambda_1\lambda_2 =1$. Since the characteristic exponents are defined by $\lambda_1=e^{Tr_1}$ and $\lambda_2 =e^{Tr_2}$ then $1=\lambda_1\lambda_2 =e^{T(r_1+r_2)}$. Using the complete definition of the logarithm for a complex number z, $\ln z=\ln r+i(\theta + 2\pi n)$, it can easily be found that $\omega(r_1+r_2)=2\pi i n$ where n is an integer. Therefore $r_1+r_2=\dfrac{2\pi}{T}n$.

Remembering the condition $\lambda_1\lambda_2=1$ there are the following results:

(1) If $\lambda_1\neq\lambda_2$ then Hills' Equation has two linearly independent solutions, these can be expressed as:

$$y_1(t)=e^{r_1t}f_1(t)$$
$$y_2(t)=e^{r_2t}f_2(t)$$

where $r_1=r_2$ and $f_i(t)$ has period T.

(2) If $\lambda_1=\lambda_2=1$ then Hill's Equation has a solution of period T.

(3) If $\lambda_1=\lambda_2=-1$ then Hill's Equation has a solution of period 2T.

Proof of the first result, was shown earlier in this chapter for a one dimensional system. The same result will apply here since the system is composed of two (coupled) one dimensional equations.

4.8 The Mathieu Equation Form of Hill's Equation

A special form of Hill's Equation occurs when p(t)=a+b cos(t). This is known as the Mathieu Equation and has the following form:

$$\ddot{\theta} + (a + b\cos(t))\theta = 0$$

This type of equation would be encountered in a parametrically excited pendulum as shown in the following figure:
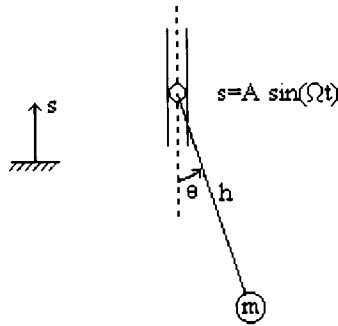
The equation of motion can be determined using Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

Where L=T-V, T represents kinetic energy, and V represents potential energy. Additionally, the system is assumed to be conservative (no friction is present).

The resulting equation of motion is:

$$\ddot{\theta} + \left(\frac{g}{h} - \frac{A\Omega^2}{h}\cos(\Omega\tau)\right)\theta = 0 \qquad (4\text{-}10)$$

now make the substitution that

$$t = \Omega\tau$$

therefore

$$dt = \Omega d\tau$$

and

$$\frac{d}{d\tau} = \Omega \frac{d}{dt}$$

results in:

$$\Omega^2 \frac{d^2\theta}{du^2} + \left(\frac{g}{h} + \frac{A\Omega^2}{h} \cos(t) \right)\theta = 0$$

$$\frac{d^2\theta}{dt^2} + \left(\frac{g}{h\Omega^2} + \frac{A}{h} \cos(t) \right)\theta = 0$$

now let:

$$a = \frac{g}{h\Omega^2}$$
$$\text{and}$$
$$b = \frac{A}{h}$$

Substitution yields:

$$\ddot{\theta} + (a + b\cos(t))\theta = 0$$

which is the Mathieu Equation as stated earlier. This is the form of Hill's Equation which we will analyze in the next chapter.

CHAPTER 5

5.1 Analysis of Matheiu's Equation

Now we turn our attention to the equation of motion for the parametrically excited pendulum derived in the last chapter which has the following form

$$\ddot{\theta} + (a + b\cos(t))\theta = 0 \qquad (5\text{-}1)$$

This equation can be transformed into two coupled first order equations by assigning Assign the variables $q_1 = \theta$, $q_2 = \dot{\theta}$, leading to the system of first order equations

$$\dot{q}_1 = q_2$$
$$\dot{q}_2 = -(a + b\cos(t))q_1$$

Let the matrix

$$\Phi(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) \end{bmatrix} \qquad (5\text{-}2)$$

represent the principal fundamental solution of (5-2). This means that

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Note, the principal fundamental solutions will be denoted as ϕ_i instead of q_i which represents any arbitrary solution. By the previous section, and since the period of the coefficients of (5-1) is $T=2\pi$, the characteristic multipliers are the eigenvalues λ_i of:

$$\Phi(2\pi) = \begin{bmatrix} \phi_1(2\pi) & \phi_2(2\pi) \\ \dot{\phi}_1(2\pi) & \dot{\phi}_2(2\pi) \end{bmatrix}$$

Furthermore, the characteristic exponents are

$$r_i = \frac{1}{2\pi} \ln \lambda_i \qquad (5\text{-}3)$$

In equation (5-3), the complex-valued logarithm is assumed. Consequently, the characteristic multipliers satisfy the characteristic equation:

$$\lambda^2 - \text{trace}[\Phi(2\pi)] \cdot \lambda + \det[\Phi(2\pi)] = 0 \qquad (5\text{-}4)$$

By Abel's formula,

$$\det[\Phi(t)] = \det[\Phi(0)] \cdot \exp\left[\int_0^t \text{trace}(A)dt\right] \tag{5-5}$$

Hence:

$$\det[\Phi(t)] = \det[\Phi(0)]$$
$$= 1$$

Therefore, the characteristic equation (5-4) is:

$$\lambda^2 - \left[\phi_1(2\pi) + \dot{\phi}_2(2\pi)\right]\lambda + 1 = 0 \tag{5-6}$$

Unfortunately, we can only numerically solve for the fundamental solutions $\phi_1(t)$ and $\phi_2(t)$.

The system:

$$\ddot{\phi} + (a + b\cos(t))\phi = 0$$

must be integrated twice, once with initial conditions:

$$\phi_1(0) = 1, \ \dot{\phi}_1(0) = 0$$

and once with initial conditions:

$$\phi_2(0) = 0, \ \dot{\phi}_2(0) = 1$$

The characteristic equation (5-6) is obtained by integrating these initial conditions up to the final time $t = 2\pi$, to determine the trace of $[\Phi(2\pi)]$. By solving the characteristic equation (5-4), for the characteristic multipliers λ, the stability of the solutions are determined with respect to the parameters a and b. These parameters, of course, relate back to the physical parameters $g/h\Omega^2$ and A/h respectively as shown in Chapter 4.

In the following, let:

$$S(a, b) = \text{trace}[\Phi(2\pi)]$$
$$= \phi_1(2\pi) + \dot{\phi}_2(2\pi)$$

The characteristic equation becomes:

$$\lambda^2 - S(a, b)\lambda + 1 = 0$$

hence the characteristic multipliers are:

$$\lambda_{1,2} = \frac{S}{2} \pm \frac{1}{2}\sqrt{S^2 - 4}$$

The characteristic exponents are:

$$r_{1,2} = \frac{1}{2\pi}\ln\left[\frac{S}{2} \pm \frac{1}{2}\sqrt{S^2 - 4}\right]$$

5.2 Illustrative Examples

Three cases will be discussed in detail, with different values of a and b depending on the values of S(a,b) where S(a,b) is the stability surface.

The stability of the Mathieu Equation depends on S(a,b). The areas in which the value of S(a,b) results in a stable system can be sketched as in the following Strutt diagram. The hatched regions are areas for which the equation will be stable.
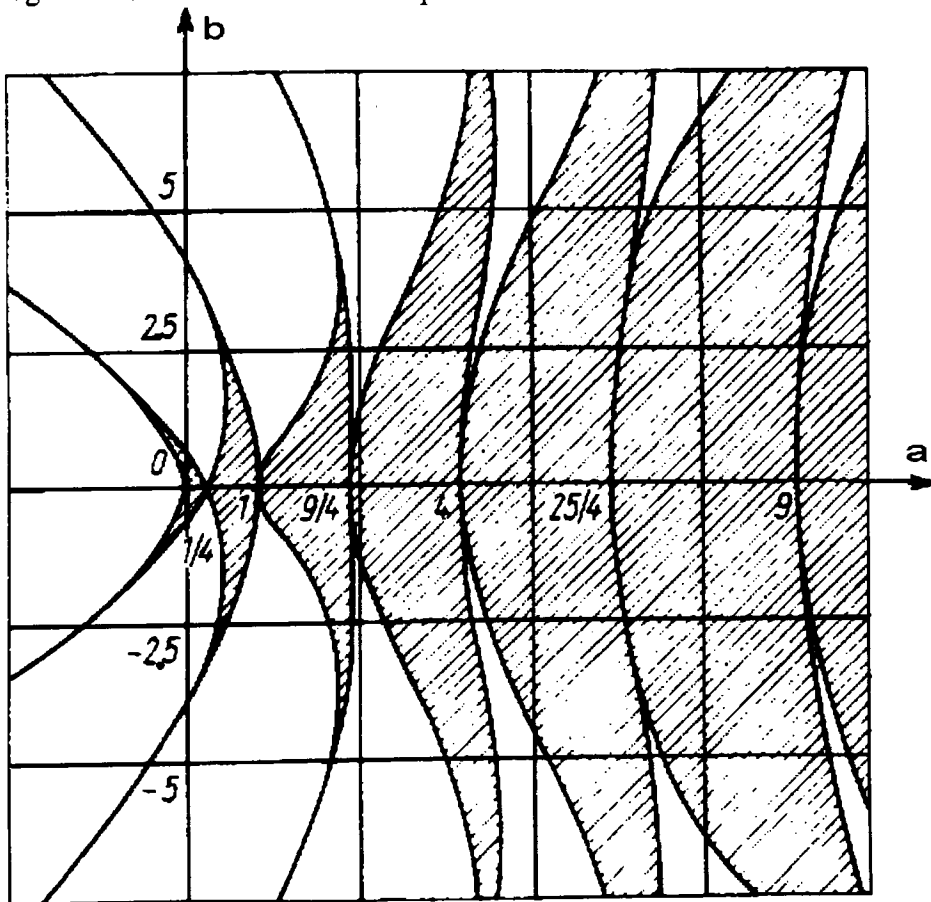
Strutt Diagram -- Figure 5-1

This surface is plotted out in three dimensions using the collection of MATLAB programs related to RUNPTS.

The following figure is one such plot for the area where 0<a<2.5 and 0<b<.25:
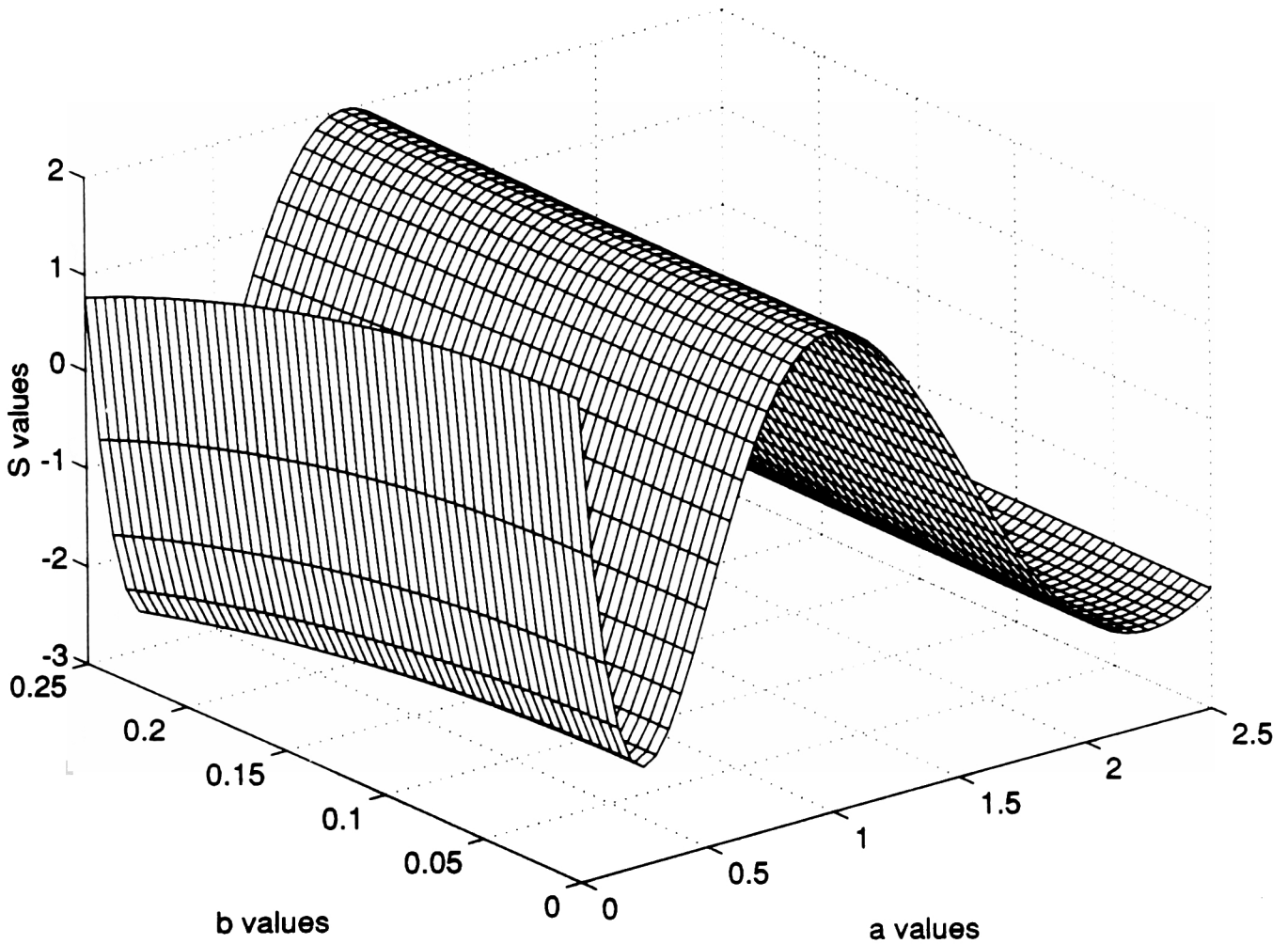
Figure 5-2

Note, as can be seen in Figure 5-1, the stability surface will be symmetric about the a-axis therefore to save considerable computing time, the surface is only plotted in the first quadrant.

Another plot of the stability surface appears in the next figure, this time with 0<a<2.5 and 0<b<2.5:
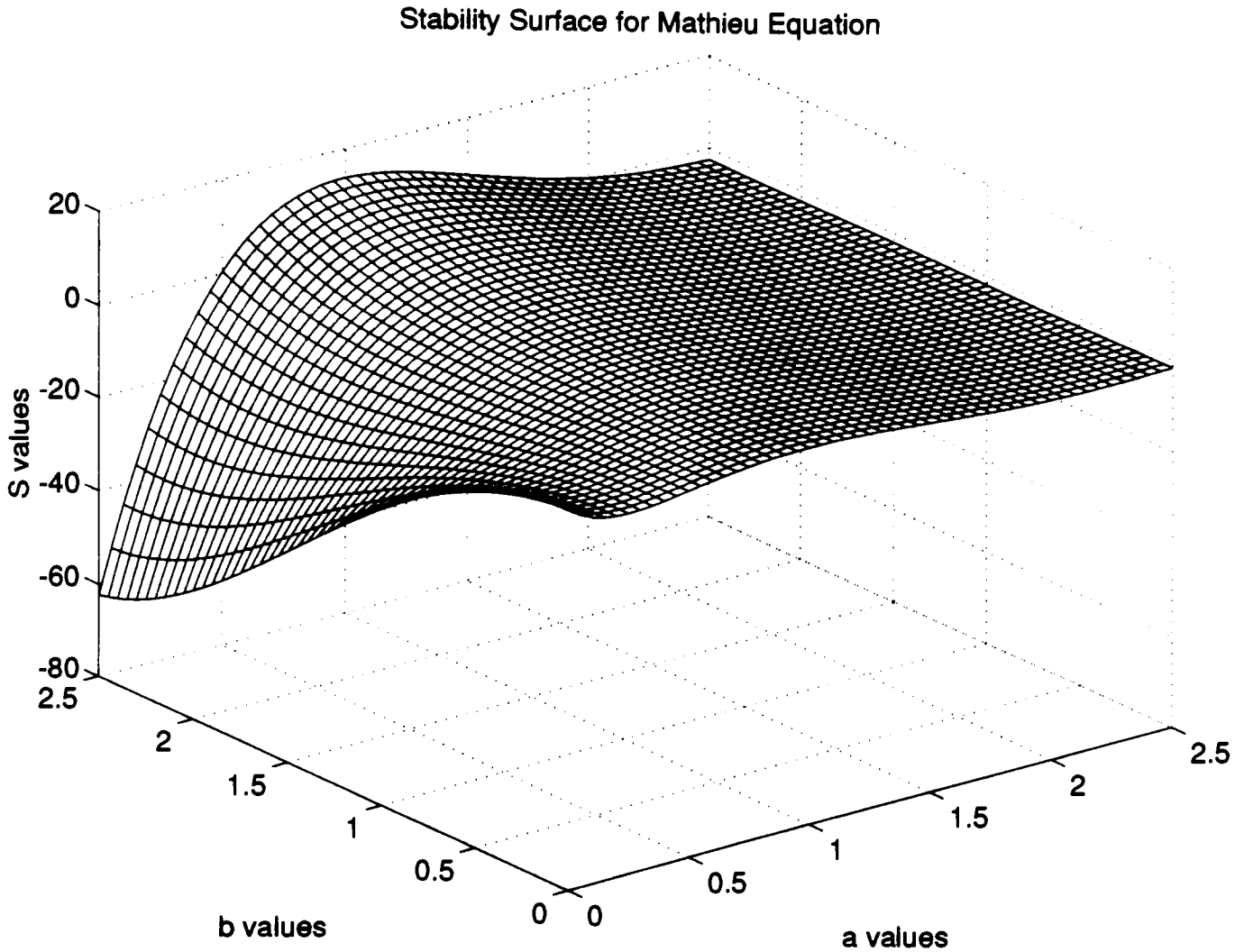
Figure 5-3

The large magnitudes for larger values of 'b' make the details along the a-axis harder to see than in Figure 5-2 but the larger magnitudes encountered away from the a-axis are made apparent.

The Runge-Kutta algorithm used to numerically solve for the fundamental solution at each point on these stability surface plots is a built-in function on MATLAB accessed through the command ode45. As implied by the name, the algorithm uses fourth and fifth order Runge-Kutta formulas. Accuracy of the solution is determined by the variable 'tol' within MATLAB. The default value for 'tol', and the value used for all stability surface plots in this thesis, was 1.e-6, or 0.000001. Therefore, the surface plots are accurate to within plus or minus 0.000002 since S is the sum of the two fundamental solutions.

The unstable regions in Figure 5-1 can be shown in the following plot. It was made using the MATLAB program called DOTS. DOTS finds the value of S(a,b) and if its magnitude is less than 2.0 that point is marked as stable, otherwise it is marked as unstable. This particular region is mostly stable, as can be seen in the plot. A larger region with more significant unstable regions will be shown on the following page.
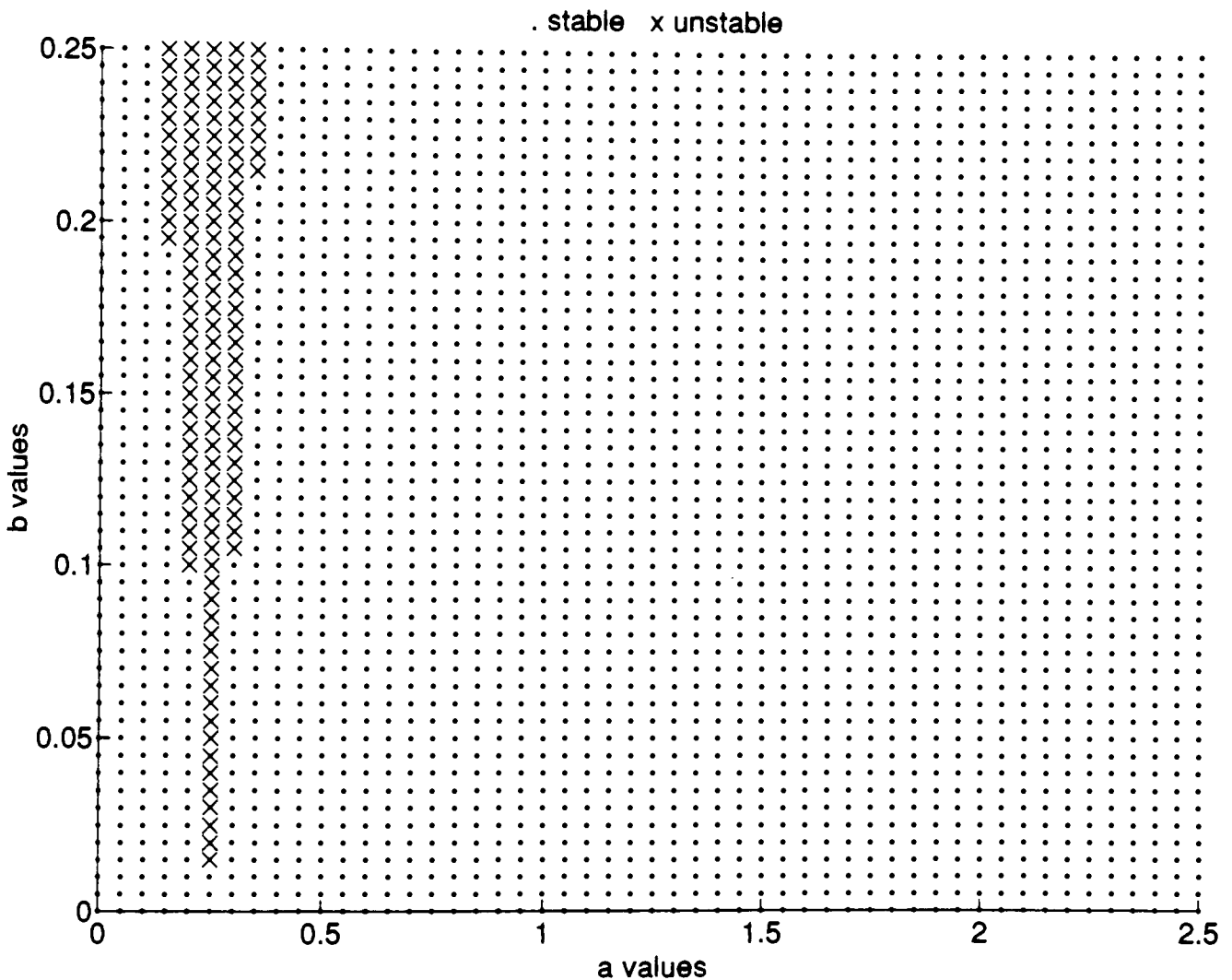
Figure 5-4
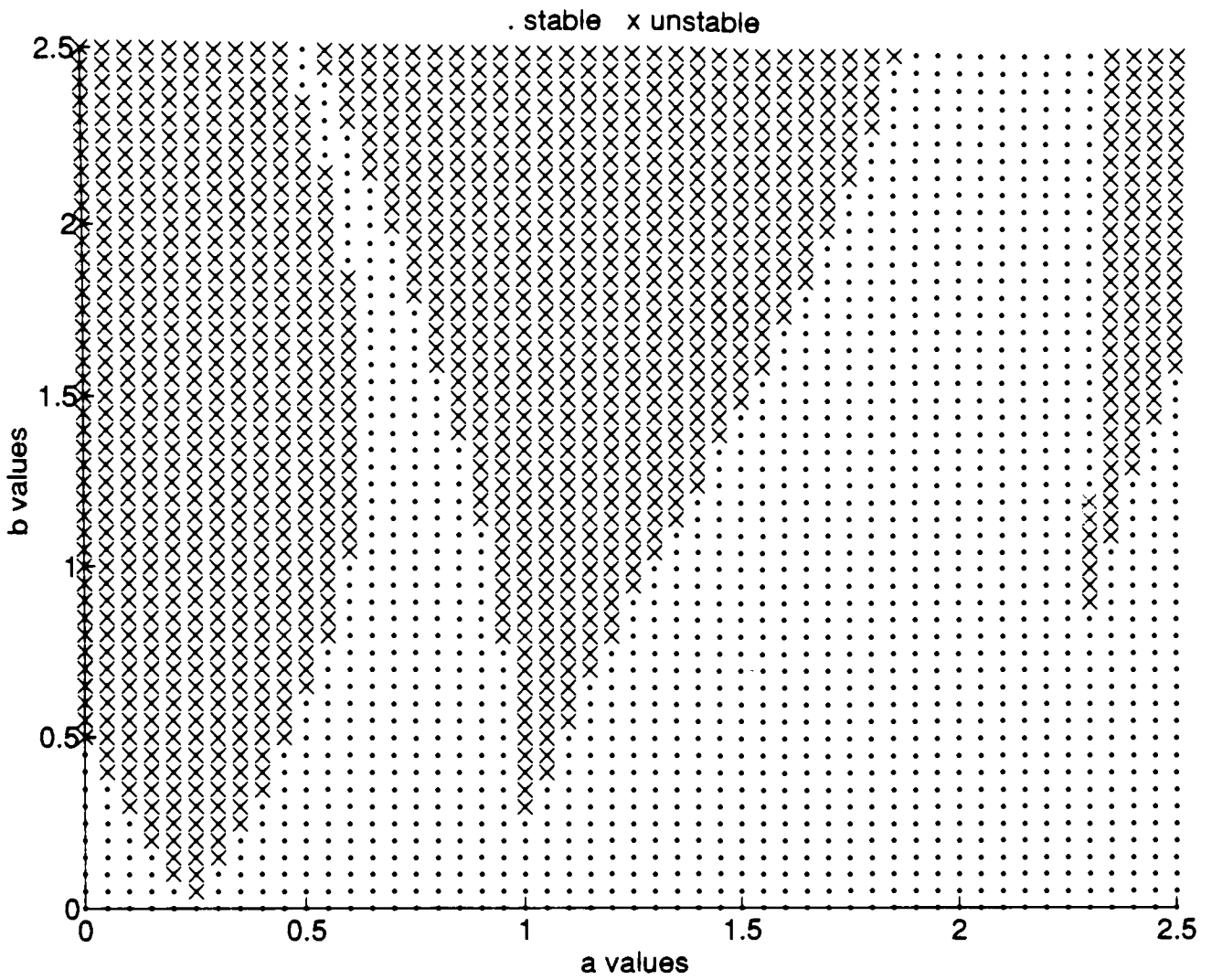
Figure 5-5

This final plot is starting to resemble the Strutt diagram (Figure 5-2). Although it would be theoretically possible to reconstruct the entire plot using the MATLAB programs used here, the solutions are extremely calculation intensive and would require enormous amounts of time and computing power to calculate with any appreciable resolution.

Note that when $b = 0$:

$$S(a,0) = 2\cos\left(2\pi\sqrt{a}\right).$$

This case corresponds to the equation:

$$\ddot{\theta} + a\theta = 0$$

which has fundamental solutions

$$\phi_1(t) = \cos\sqrt{a}\,t$$

and

$$\phi_2(t) = \frac{1}{\sqrt{a}}\sin\sqrt{a}\,t$$

resulting in the principal fundamental matrix:

$$[\Phi(t)] = \begin{bmatrix} \cos\sqrt{a}\,t & \dfrac{1}{\sqrt{a}}\sin\sqrt{a}\,t \\ -\sqrt{a}\sin\sqrt{a}\,t & \cos\sqrt{a}\,t \end{bmatrix}$$

Thus, the characteristic multipliers are the eigenvalues of:

$$[\Phi(2\pi)] = \begin{bmatrix} \cos\left(\sqrt{a}\,2\pi\right) & \dfrac{1}{\sqrt{a}}\sin\left(\sqrt{a}\,2\pi\right) \\ -\sqrt{a}\sin\left(\sqrt{a}\,2\pi\right) & \cos\left(\sqrt{a}\,2\pi\right) \end{bmatrix}$$

The characteristic equation is:

$$\lambda^2 - 2\cos\left(2\pi\sqrt{a}\right)\lambda + 1 = 0$$

with solution

$$\lambda = \cos 2\pi\sqrt{a} \pm i\sin 2\pi\sqrt{a}$$

In order to understand the more general case, $b \neq 0$, the above can also be realized by the following:

We start with:

$$[\Phi(2\pi)] = \exp(2\pi R)$$

where:

$$R = \frac{1}{2\pi} \ln[\Phi(2\pi)]$$

Next, we obtain the modal matrix, Q, which has the eigenvectors of $[\Phi(2\pi)]$ as its columns.

Thus:

$$[\Phi(2\pi)] = Q \cdot G \cdot Q^{-1}$$

where:

$$G = \begin{bmatrix} \cos(2\pi\sqrt{a}) & -\sin(2\pi\sqrt{a}) \\ \sin(2\pi\sqrt{a}) & \cos(2\pi\sqrt{a}) \end{bmatrix}$$

The eigenvalues of G are the characteristic multipliers:

$$\lambda = \exp(\pm i2\pi\sqrt{a})$$

To compute the characteristic exponents, we note that:

$$\ln[\Phi(2\pi)] = Q\ln[G]Q^{-1}$$

Which results in:

$$R = \frac{1}{2\pi} Q\ln[G]Q^{-1}$$

The characteristic exponents are given by:

$$r = \frac{1}{2\pi}(\text{eigenvalues of } \ln[G])$$

So that:

$$r = \pm i\sqrt{a}$$

By making a slicing plane through the stability surface where b = 0 the curve, S(a,b) which appears in Figure 5-4 can be seen. When increasing b to .25 and then to .5, the stability surface expands outward. For small values of 'a' it expands downward, outside of the stable range between -2 and positive 2. These additional slices through the stability surface can also be seen in Figure 5-4.
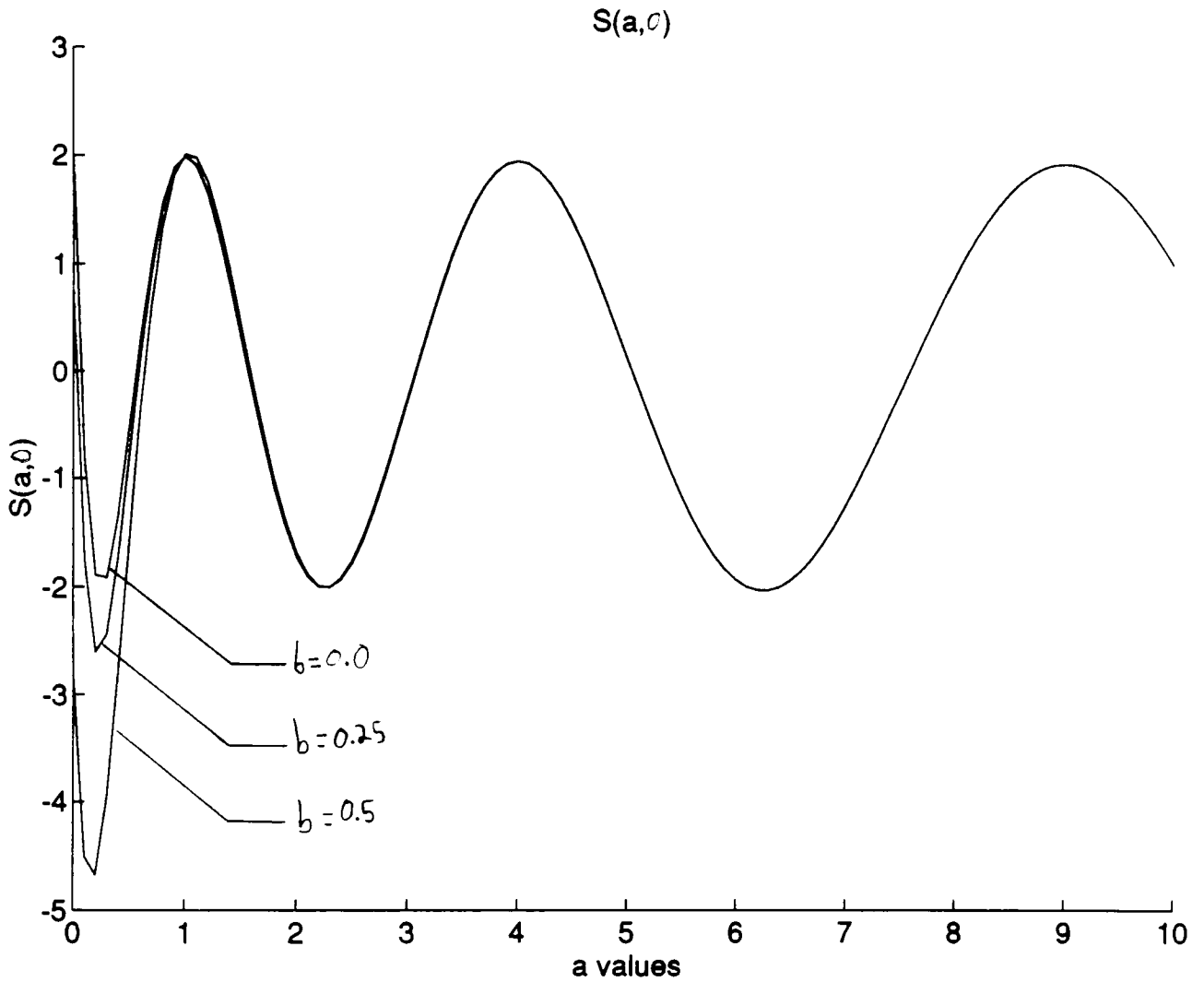
Figure 5-6

Setting a = 0, thus looking along the b-axis, the values of S(0,b) which appears in Figure 5-5 below. The stable range is in the band -2 < S < 2 which is marked on the graph. As can be seen, the system is only stable for a very restrictive number of values of b when a = 0.
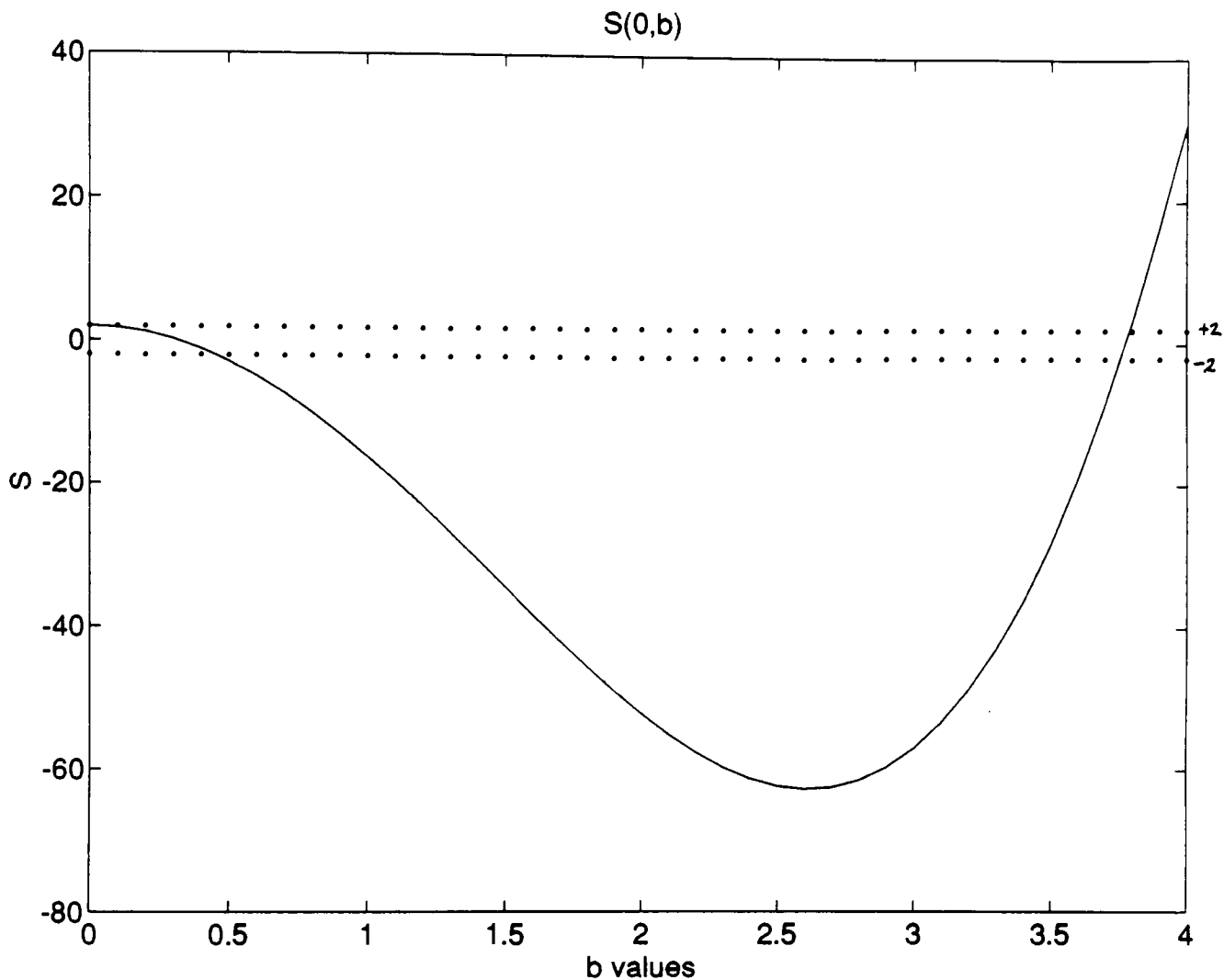
Figure 5-7

To emphasize how unstable the system becomes, the value of Log(| S(0,b) |) has been plotted up to b = 15. This appears in Figure 5-6.
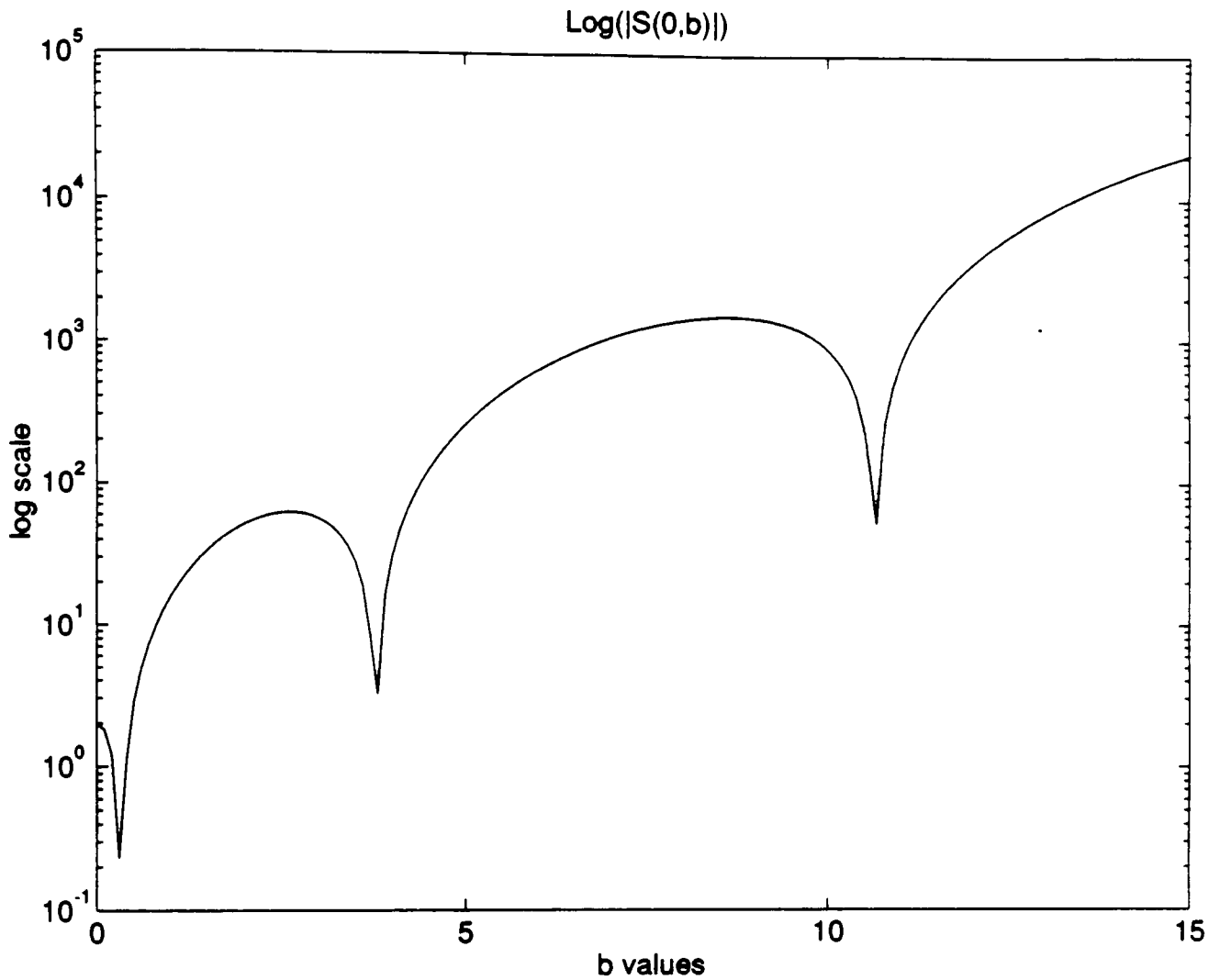
Figure 5-8

5.3 General Cases

Recall the characteristic equation:

$$\lambda^2 - S(a,b)\lambda + 1 = 0$$

and the eigenvalues:

$$\lambda_{1,2} = \frac{S}{2} \pm \frac{1}{2}\sqrt{S^2 - 4}$$

Depending on the value of S, the solution will fall into one of three different cases.

Cases:

Case 1) $|S(a,b)| < 2$

Again, the eigenvalues, or characteristic multipliers are:

$$\lambda_{1,2} = \frac{S}{2} \pm i\frac{1}{2}\sqrt{4 - S^2}$$
$$= e^{\pm i\alpha}$$

and the characteristic exponents are:

$$r_{1,2} = \frac{1}{2\pi}(\pm i\alpha)$$
$$= \frac{\pm i\alpha}{2\pi}$$

Therefore,

$$\ln[G] = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}$$

and the resulting Q matrix is:

$$Q = \begin{bmatrix} -\dfrac{2\pi}{\alpha} & 0 \\ 0 & 1 \end{bmatrix}$$

$$Q^{-1} = \begin{bmatrix} -\dfrac{\alpha}{2\pi} & 0 \\ 0 & 1 \end{bmatrix}$$

Using the relation:

$$R = \frac{1}{2\pi} Q \ln[G] Q^{-1}$$

the matrix R is found to be:

$$R = \begin{bmatrix} 0 & 1 \\ -\dfrac{\alpha^2}{4\pi^2} & 0 \end{bmatrix}$$

And then by manipulation:

$$e^{tR} = e^{\begin{bmatrix} 0 & t \\ \frac{\alpha^2 t}{4\pi^2} & 0 \end{bmatrix}}$$

$$e^{tR} = \begin{bmatrix} \cos\dfrac{\alpha t}{2\pi} & \dfrac{2\pi}{\alpha}\sin\dfrac{\alpha t}{2\pi} \\ -\dfrac{\alpha}{2\pi}\sin\dfrac{\alpha t}{2\pi} & \cos\dfrac{\alpha t}{2\pi} \end{bmatrix}$$

Note that

$$e^{tR} = \begin{bmatrix} \cos\dfrac{\alpha}{2\pi}t & \dfrac{2\pi}{\alpha}\sin\dfrac{\alpha}{2\pi}t \\ -\dfrac{\alpha}{2\pi}\sin\dfrac{\alpha}{2\pi}t & \cos\dfrac{\alpha}{2\pi}t \end{bmatrix}$$

is the principal fundamental solution of the system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\dfrac{\alpha^2}{4\pi^2} & 0 \end{bmatrix} x$$

or

$$\ddot{z} + \frac{\alpha^2}{4\pi^2} z = 0$$

In this case the period of these solutions is $T_\alpha = \dfrac{4\pi^2}{\alpha}$

The principal fundamental solution of the original system is

$$[\Phi(t)] = [P(t)]\exp(tR)$$

where $[P(t)]$ is periodic of period $T = 2\pi$. Thus, we only find a periodic solution if $\dfrac{\alpha}{2\pi}$ is rational.

In particular, if $S = 0$, $\alpha = \dfrac{\pi}{2}$ and $r_{1,2} = \pm\dfrac{i}{4}$, and we obtain a period-4 solution. That is, the solution has period 8π!

Case 2) $S(a,b) = 2$

The roots of the characteristic equation are

$$\lambda_1 = \lambda_2 = 1$$

For the case of equal eigenvalues, $[\Phi(2\pi)]$ is similar to either

$$G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } G = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

In the former case,

$$R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $[\Phi(t)] = [P(t)]$ is periodic with period 2π.

In the latter case,

$$R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$\exp(tR) = \frac{1}{2\pi} Q \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} Q^{-1}$$

Hence the system has at least one periodic solution, as long as the initial condition is a multiple of the sole eigenvector of $[\Phi(2\pi)]$, that is, a multiple of the first column of the modal matrix Q.

Case 3) $|S(a,b)| > 2$

The roots of the characteristic equation

$$\lambda^2 - S(a,b)\lambda + 1 = 0$$

are both real. But since the product of the roots is

$$\lambda_1 \cdot \lambda_2 = 1$$

This implies that at least one of them has magnitude strictly greater than one.

Here

$$G = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

and

$$\ln G = \begin{bmatrix} \ln \lambda_1 & 0 \\ 0 & \ln \lambda_2 \end{bmatrix}$$

Thus the characteristic exponents are

$$r_{1,2} = \frac{1}{2\pi} \ln \lambda_i$$

which may be complex-valued, but at least one of them will have a positive real part.

This implies that the solutions will generally be unbounded.

5.4 Numerical Examples

A series of sample plots were created using MATLAB to demonstrate how stability depends on the values of a and b and how these patterns correspond to the Strutt diagram and the stability surfaces which were looked at earlier.

These plots are included in the Appendix. One can observe that 'a' and 'b' for stable and unstable cases corresponds with the stable and unstable regions in the Strutt diagram.

5.5 Effects of Damping

Having understood how the parameters of the Mathieu equation affect its stability, we now add damping to the system. The modified equation then becomes

$$\ddot{\theta} + 2c\dot{\theta} + (a + b\cos(t))\theta = 0$$

Here the damping coefficient is related to the original system parameters by

$$c = \frac{\zeta}{\Omega}\sqrt{\frac{g}{h}}$$

where ζ is the viscous damping ratio.

In state variable form, we have

$$\dot{q}_1 = q_2$$

$$\dot{q}_2 = -(a + b\cos t)q_1 - 2cq_2$$

or

$$\dot{\vec{q}} = \begin{bmatrix} 0 & 1 \\ -(a + b\cos t) & -2c \end{bmatrix}\vec{q}$$

The analysis now proceeds as before. First, a principal fundamental matrix is computed. As previously, this matrix is denoted by $\Phi(t)$.

One significant difference is that

$$\text{trace}[A(t)] = -2c$$

Recall that Abel's theorem states that

$$\det[\Phi(t)] = \det[\Phi(0)] \cdot \exp\left[\int_0^t \mathrm{tr}(A)dt\right]$$

Hence

$$\det[\Phi(2\pi)] = \exp(-4\pi c)$$

Therefore, the characteristic multipliers, which are the eigenvalues of $\Phi(2\pi)$, are the roots of

$$\lambda^2 - S(a,b)\lambda + \exp(-4\pi c) = 0$$

As in the undamped case, the stability of the solutions depends on the magnitude of the characteristic multipliers λ_1 and λ_2. In particular, if

$$|\lambda_i| > 1$$

then the solutions are unstable. The characteristic exponents are given by

$$r_i = \frac{1}{2\pi} \mathrm{Log}\,\lambda_i$$

Thus, the system is stable, if and only if, the real parts of r_i are negative.

Returning to the characteristic multiplier equation

$$\lambda^2 - S(a,b)\lambda + \exp(-4\pi c) = 0$$

it is easy to show that $|\lambda| \leq 1$, provided that

$$|S(a,b)| \leq 2\cosh(2\pi c) \cdot e^{-2\pi c}$$

or

$$\frac{e^{2\pi c}}{\cosh(2\pi c)} \cdot |S(a,b)| \leq 2$$

Note that this equation is a direct generalization of the undamped case.

Once again, by Floquet's Theorem, the general solution is given by

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = [P(t)] \cdot \exp[tR] \cdot \begin{bmatrix} q_1(0) \\ q_2(0) \end{bmatrix}$$

where

$$R = \frac{1}{2\pi} \text{Log}(\Phi(2\pi))$$

and P(t) is a continuous and 2π-periodic matrix.

To see the effect that light damping has on the shape of the stability surfaces, the following stability surface plots have been constructed using the MATLAB programs related to RUNPTS. The cases plotted here include damping constants of 0, 0.1, 0.2, 0.3, 0.4 and 0.5. As can easily be seen, the larger the damping the flatter the stability surface, thus the larger the stable range of 'a' and 'b'. This is intuitive since a more heavily damped system would be expected to be more stable.
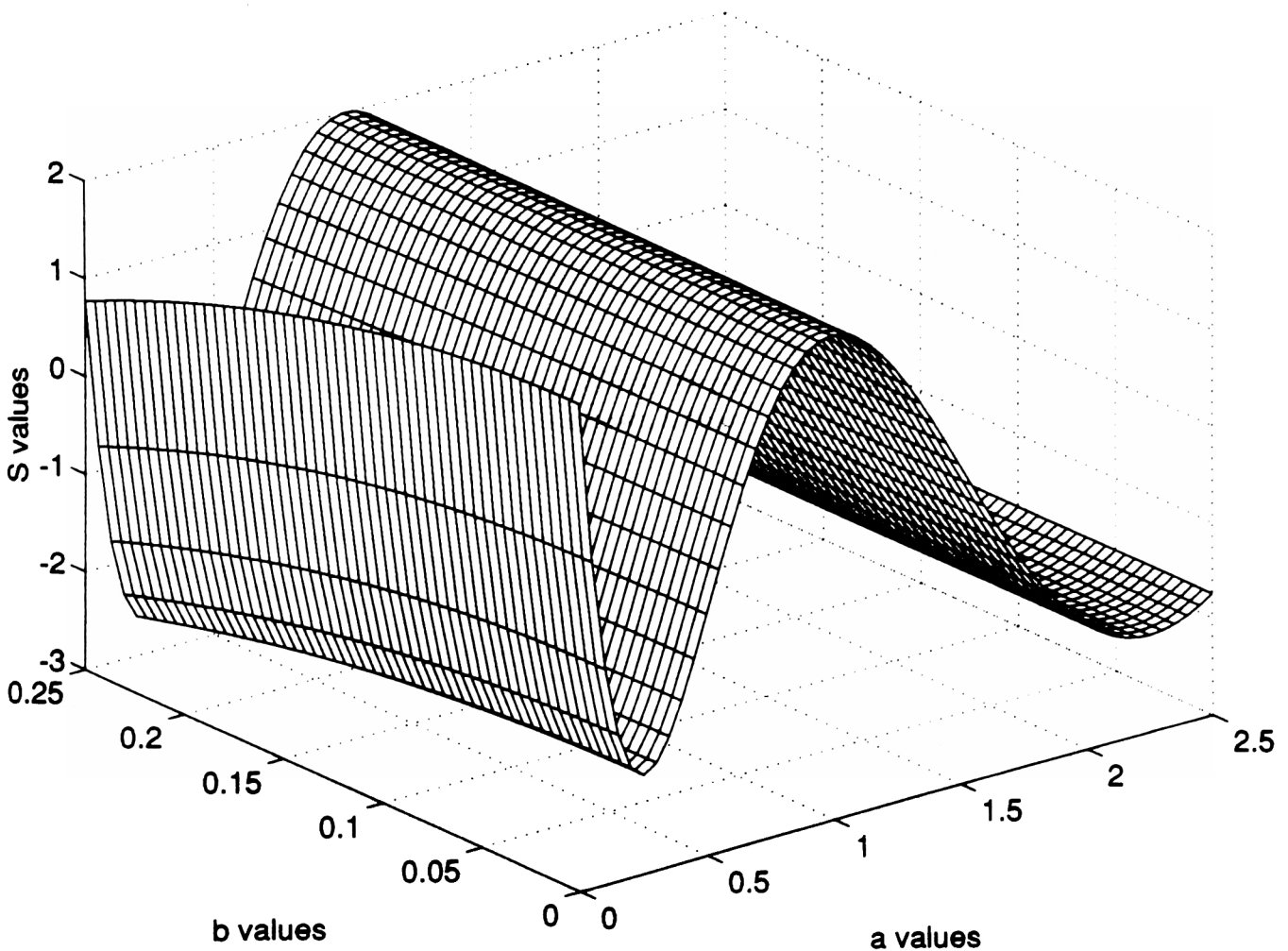
Figure 5-9

Stability Surface for Damped Mathieu Equation (c=.1)
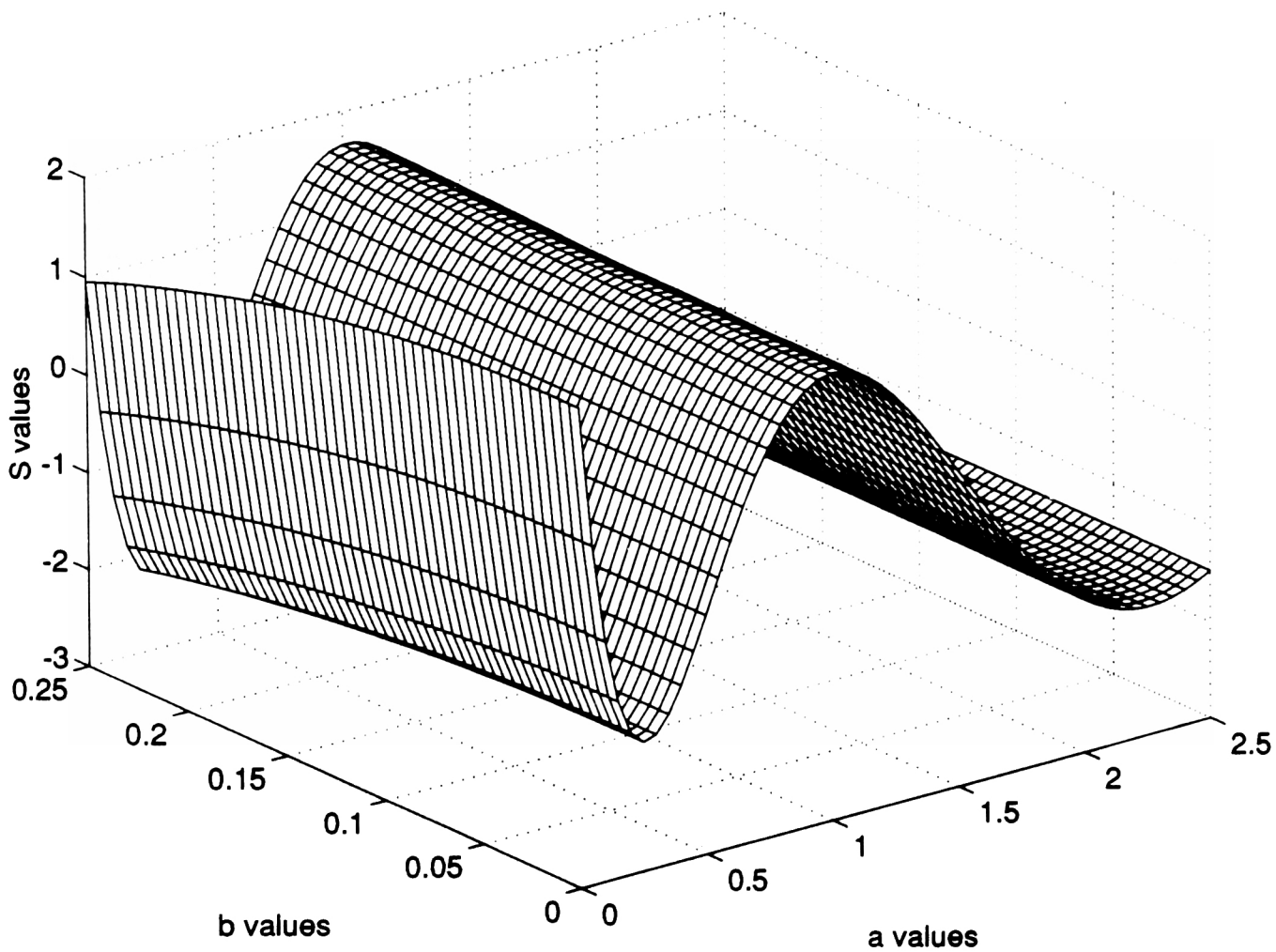
Figure 5-10

Stability Surface for Damped Mathieu Equation (c=.2)
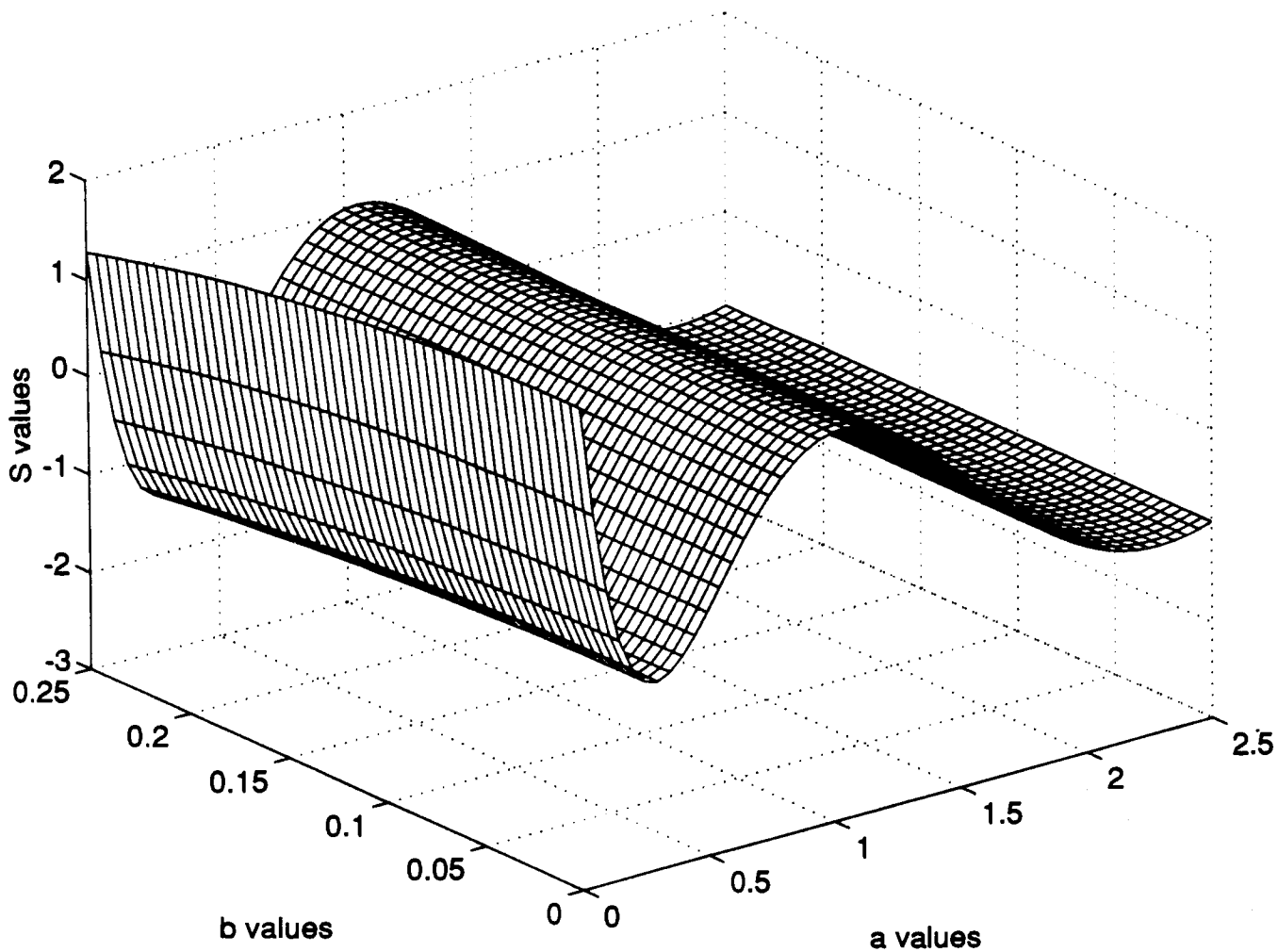
Figure 5-11

Stability Surface for Damped Mathieu Equation (c=.3)
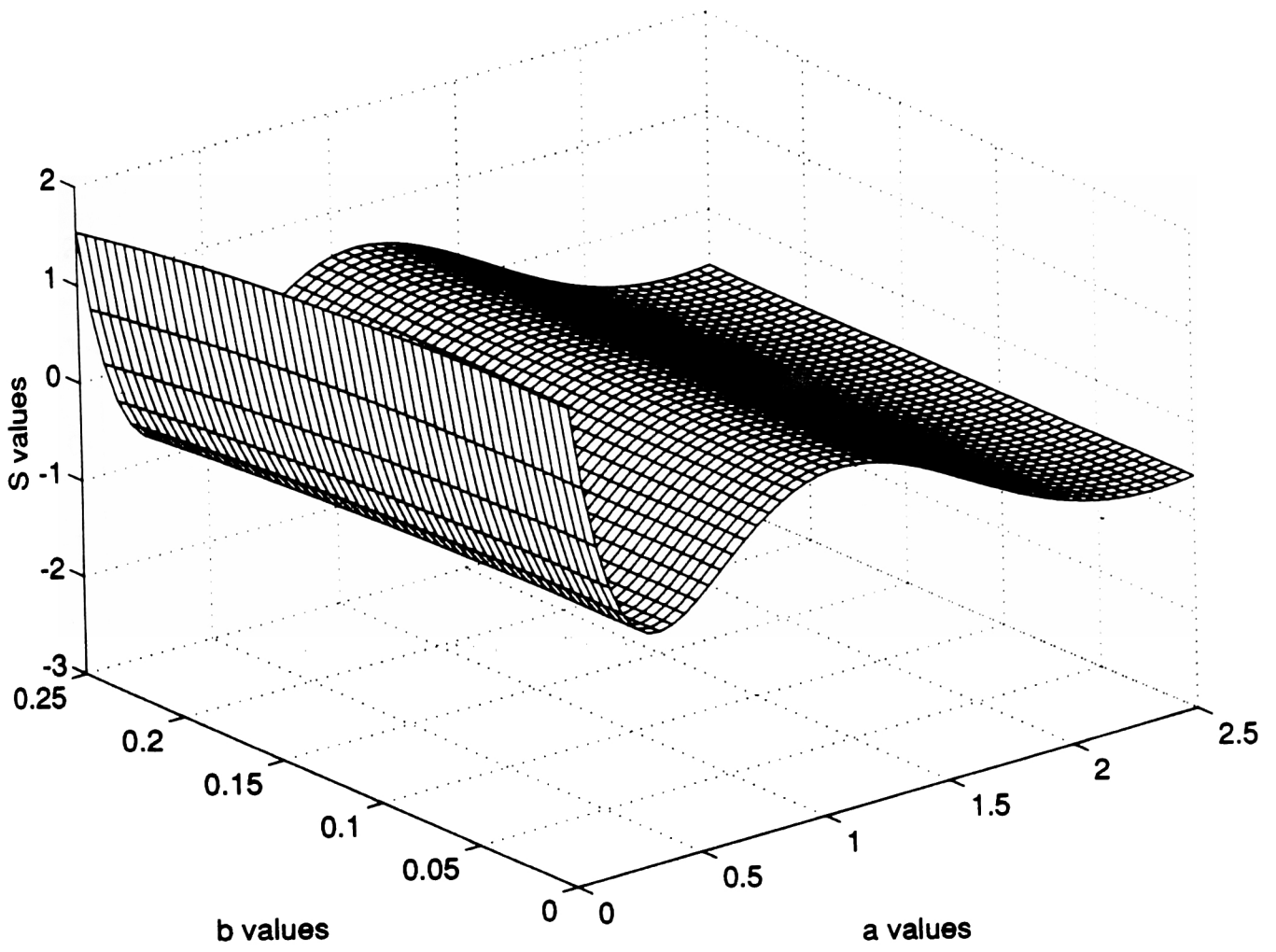
Figure 5-12

Stability Surface for Damped Mathieu Equation (c=.4)
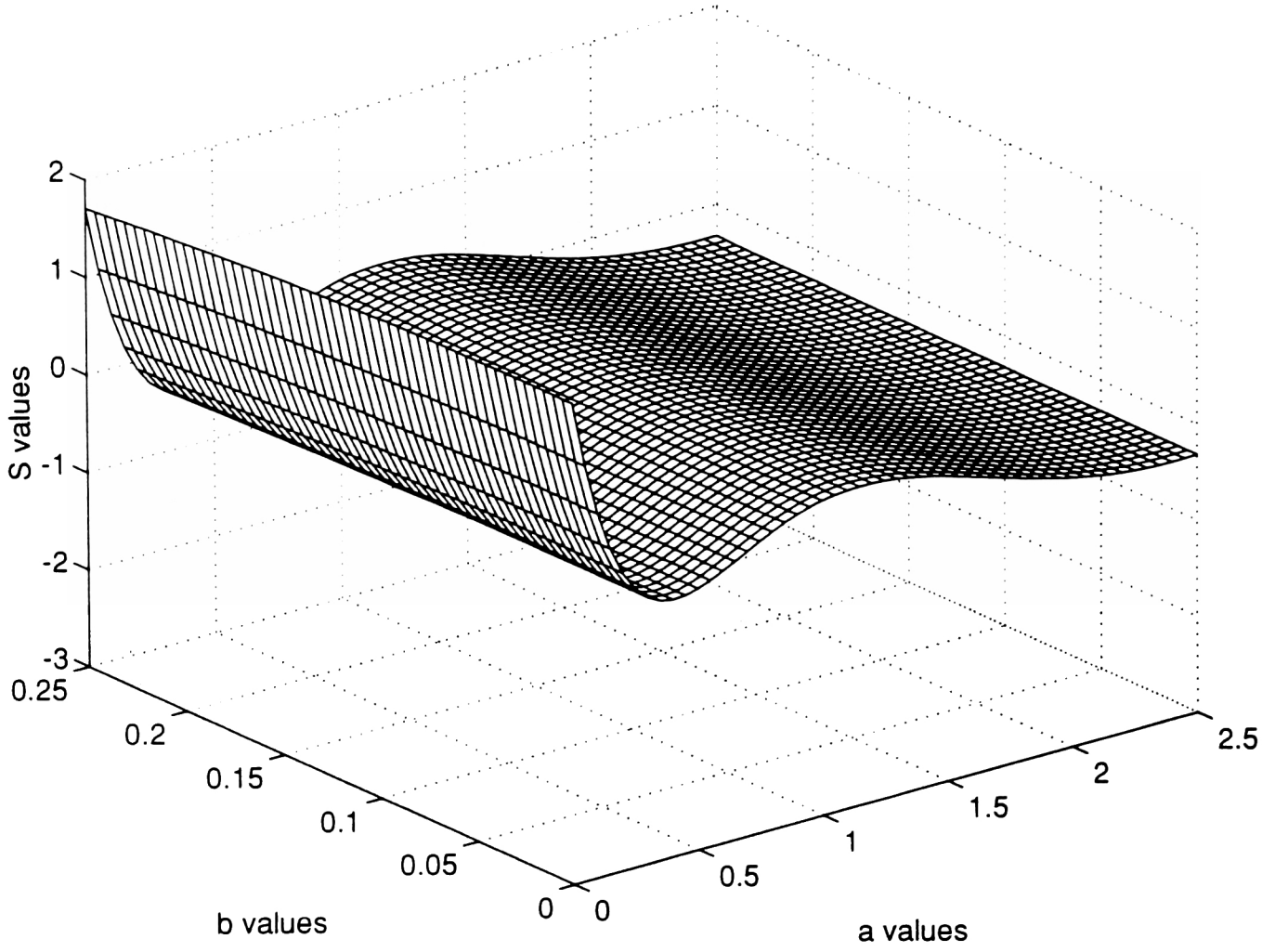
Figure 5-13
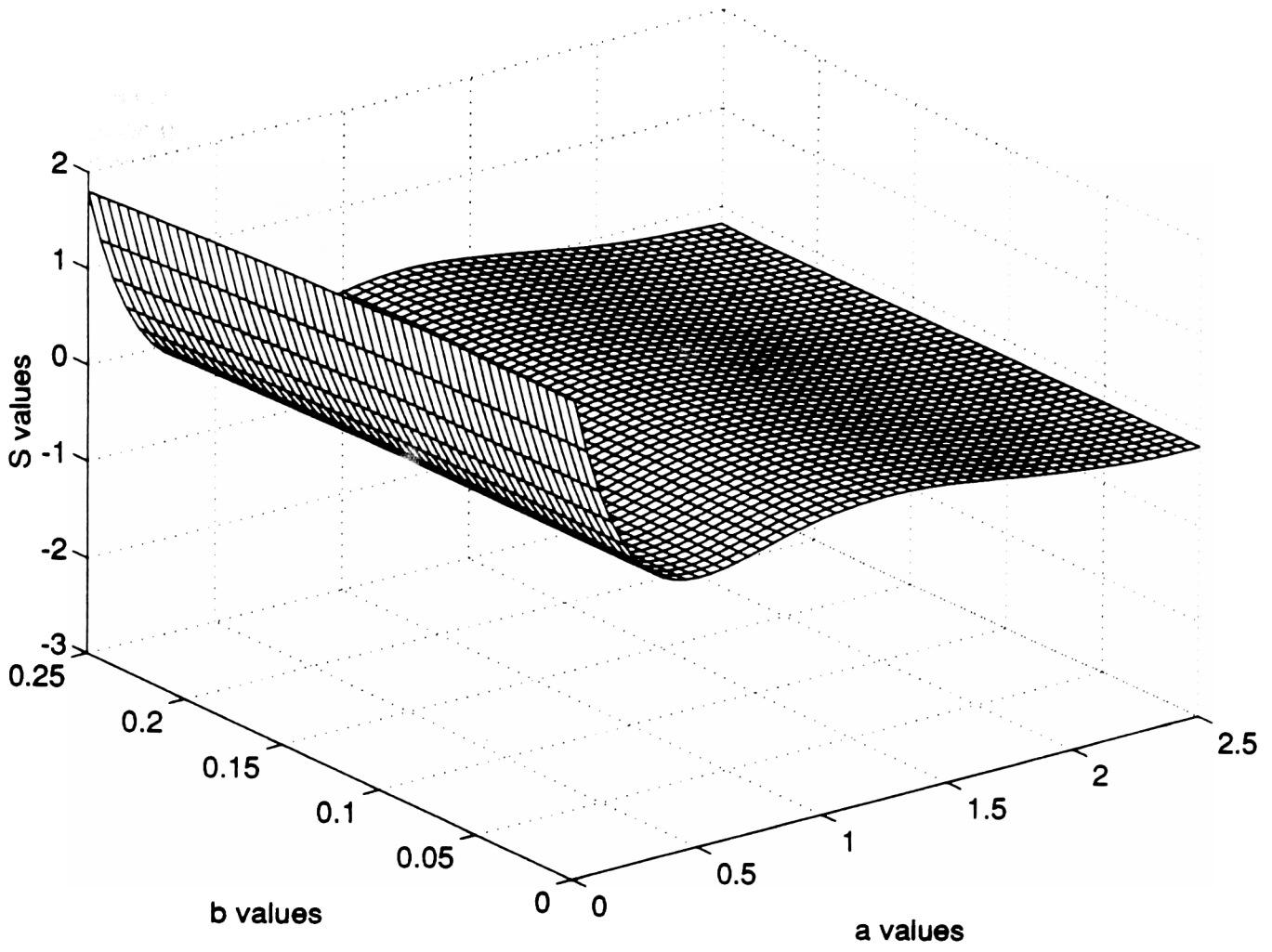
Figure 5-14

Another MATLAB program called DOTS was utilized to gain a clearer picture of the effect of light damping on our system. The results are a two-dimensional view of the a-b plane in which each node in an 11 by 11 grid was analyzed for stability. This criterion was that the absolute value of the value of the stability surface at that point must be less than 2.0. As expected, as the damping is raised, the size of the stable areas in increased. Although the resolution on these plots is limited, the trend is unmistakable.

Figure 5-15

$C = 0.1$
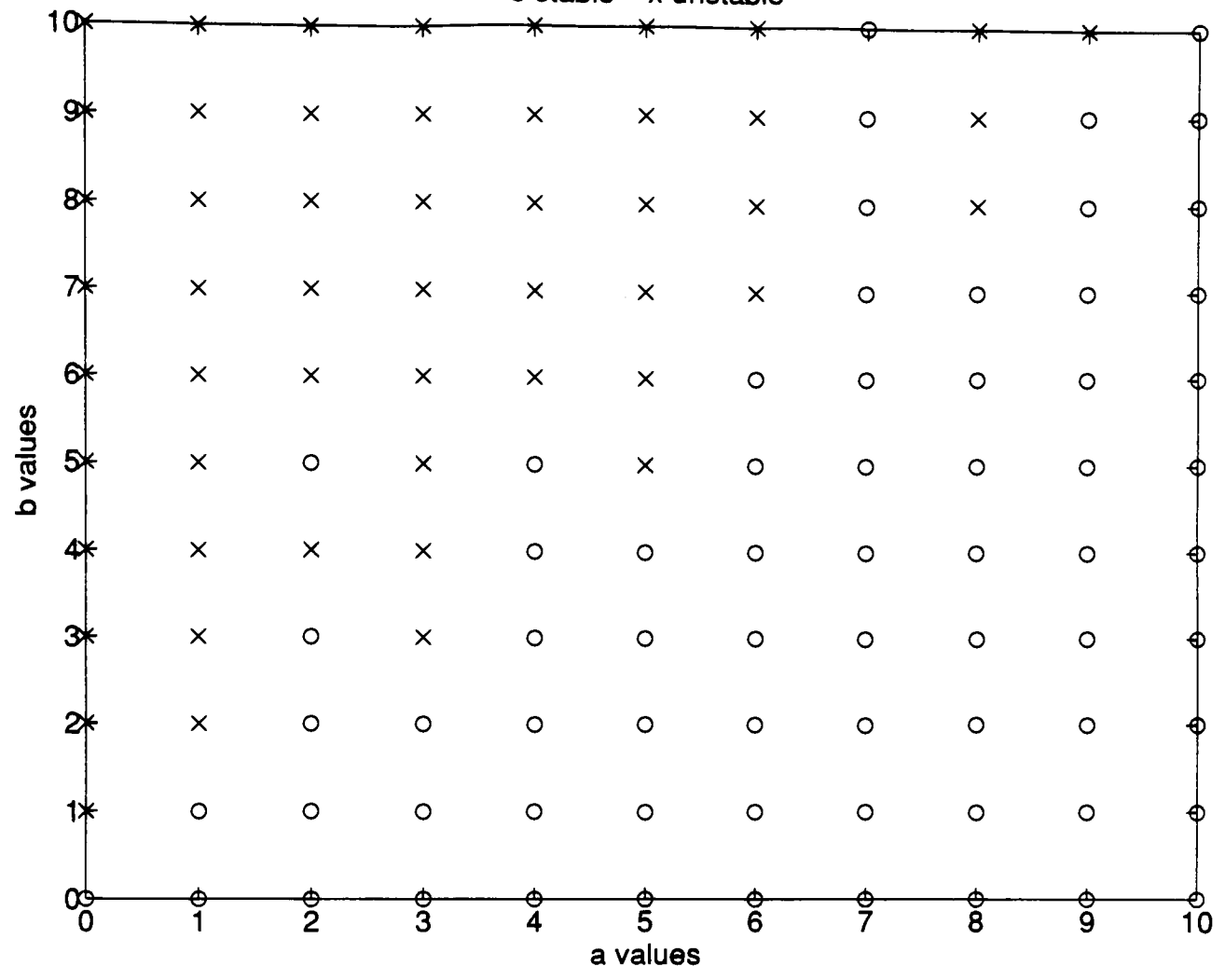
o stable   x unstable

Figure 5-16
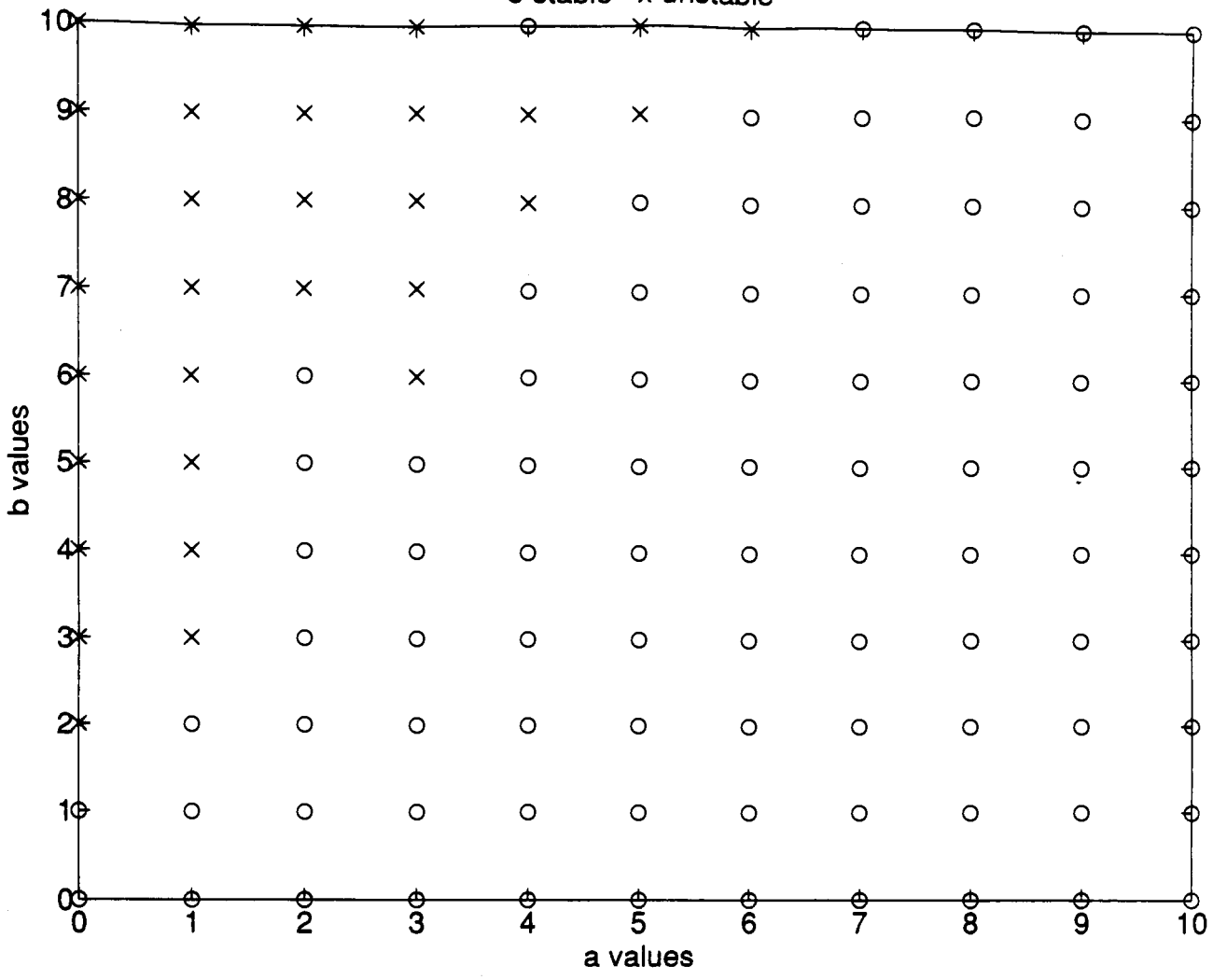
Figure 5-17

The changes in the shape of the stability surface can also be seen in the cross sectional cuts made through the b=0 plane for a few cases of damping. This plot is also included to demonstrate this effect. The maximum limits of the Stability Surface at b=0 is greatly reduced as damping is increased. This effect along the a-axis is representative of the effect that damping has on the entire stability surface.
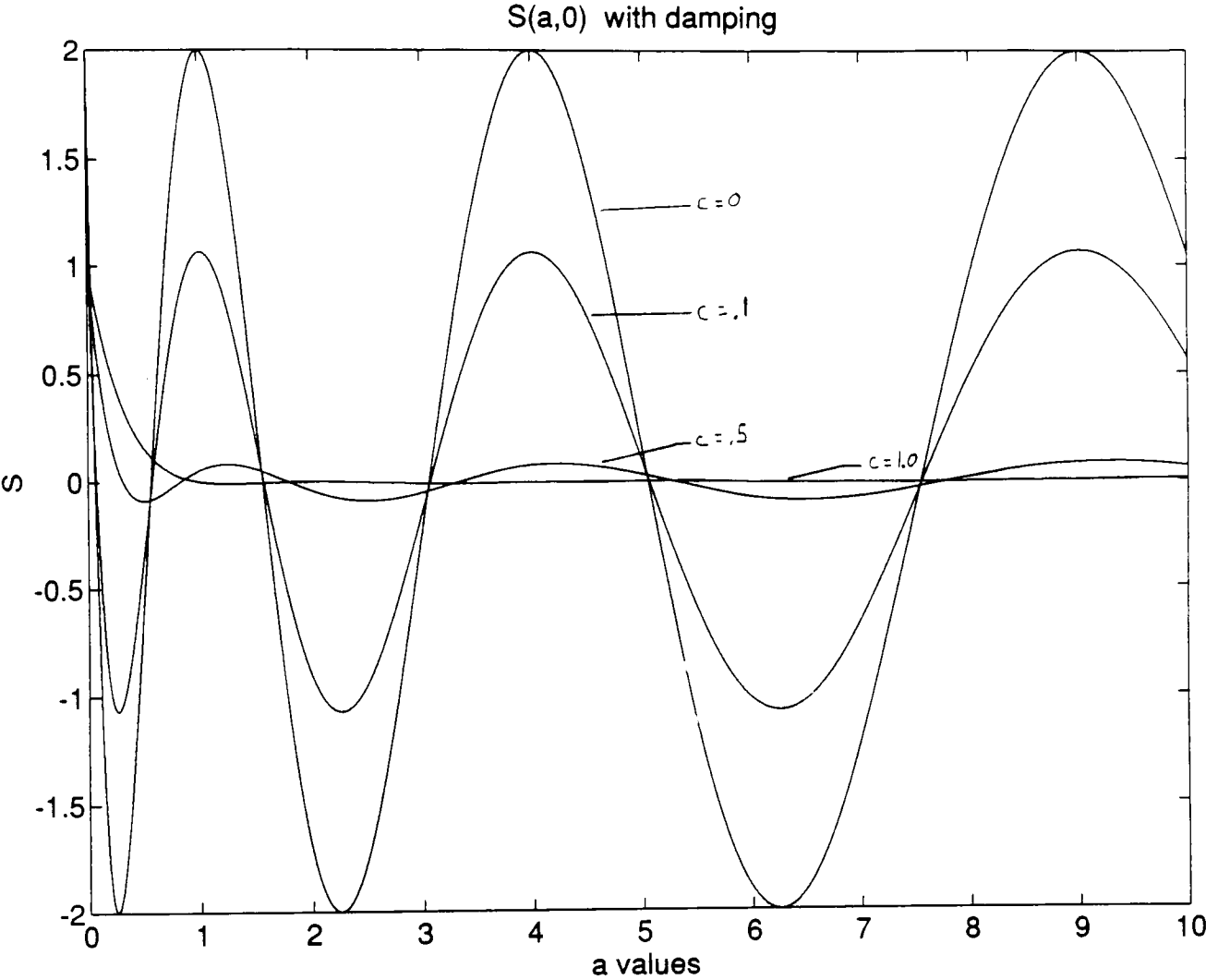
Figure 5-18

The following outlines the procedure required to analyze the stability of linear systems with periodic coefficients.

Let $\dot{\bar{q}} = A(t)\bar{q}$ be defined, where A(t) is continuous and has period T.

1.   Determine the principal fundamental matrix $[\Phi(t)]$. If $[X(t)]$ is any fundamental matrix, then $[\Phi(t)] = [X(t)][X(0)]^{-1}$

2.   Determine the eigensystem

$$Q^{-1}[\Phi(T)]Q = G$$

where G is in canonical form. The characteristic multipliers, λ_i, are the eigenvalues of $\Phi(2\pi)$ and G.

3.   The characteristic exponents are given by

$$r_i = \frac{1}{T}\text{Log}\,\lambda_i$$

which are the eigenvalues of

$$R = \frac{1}{T}Q[\log G]Q^{-1}$$

as well as the eigenvalues of $\frac{1}{T}\log G$.

4.   By the Floquet Decomposition Theorem,

$$\Phi(t) = [P(t)]\exp(tR)$$

in which $[P(t)]$ is a periodic matrix.

Solutions are stable if $|\lambda_i| < 1$ or $\text{Re}(r_i) < 0$,
        unstable if $|\lambda_i| > 1$ or $\text{Re}(r_i) > 0$.

Solutions are marginally stable, or periodic, if
        $|\lambda_i| = 1$ or $\text{Re}(r_i) = 0$.

Chapter 6

Conclusions and Recommendations

A physically-realizable dynamic system must satisfy the criterion of stability for it to be observed in nature. Stability is an extremely important system specification. If a system is unstable, then it is useless, and possibly even dangerous. The concept of stability has been discussed from the point of view of linear systems.

The stability of linear constant coefficient systems is governed by the eigenvalues of the coefficient matrix. Such systems are stable as long as the real parts of the eigenvalues are negative. Although this notion is simple conceptually, the question of stability is still a formidable one if the system is large. Moreover, if the system parameters are only approximately known, care must be taken to ensure that the eigenvalues are bounded away from the positive half of the complex plane.

The stability of systems with time-varying coefficients is an immense topic in itself. In this investigation, systems with periodically-varying coefficients were analyzed. It was shown that even under such restrictive conditions on the coefficients, the system can give rise to quite a diverse set of responses. No longer are the eigenvalues (which are not constant) the sole determining factor with regard to the stability of the system.

However, an amazingly beautiful set of results known as Floquet Theory allows one to decompose the general response as the matrix product of a periodic part and an exponential-type part. Further, the stability of the solutions is characterized not by the usual eigenvalues, but by certain values known as characteristic multipliers. In principle, such systems can be converted to constant coefficient systems under an appropriate change of variables.

These concepts were illustrated on an example of a parametrically excited pendulum. This problem is a favorite choice for the illustration of Lagrange's equations of motion. Nevertheless, few thorough analyses of the system are available. The stability diagram, known as the Strutt Diagram, is found in the literature. For the first time, however, the actual stability surface has been mapped out in full three dimensions. As was demonstrated, the stability of the system was governed by the values of the stability surface. Perhaps in the near future, an analytical expression for the stability surface can be obtained. This would be a milestone in the analysis of such problems.

More work also needs to be done with respect to coupled systems of parametrically-forced oscillators. Although the theory is complete, more investigation needs to be done with regard to specific examples. This would entail more efficient computation of the characteristic multipliers and in general, the stability surfaces.

References

Brauer, Fred, Nohel, John A. <u>Qualitative Theory of Ordinary Differential Equations</u> W. A. Benjamin, Inc., New York, 1969.

Collatz, Lothar. <u>Differential Equations: An Introduction With Applications</u> John Wiley & Sons, New York, 1986.

Edwards, C.H., Jr. Penney, David E. <u>Elementary Differential Equations with Boundary Value Problems</u> Prentice Hall, Englewood Cliffs, New Jersey, 1989.

Goldberg, Jack L., Schwartz, Arthur, J. <u>Systems of Ordinary Differential Equations: An Introduction</u> Harper & Row Publishers, New York, 1972.

Hostetter. Gene, H., Santina, Mohammed, S. Santina, D'Captio-Montalvo, Paul <u>Analytical, Numerical, and Computational Methods for Science and Engineering</u> Prentice Hall, Englewood Cliffs, New Jersey, 1991.

Nemytskii V. V., Stepanov, V. V. <u>Qualitative Theory of Differential Equations</u> Princeton University Press, Princeton, NJ, 1960.

Newland, D.E. <u>Mechanical Vibration Analysis and Computation</u> Longman Scientific & Technical. Essex, England. co-published with John Wiley & Sons, New York, 1989.

Stoker, J. J. <u>Nonlinear Vibrations in Mechanical and Electrical Systems</u> Interscience Publishers, Inc., New York, 1950.

<u>The Student Edition of MATLAB</u> Prentice Hall, Englewood Cliffs, NJ, 1992.

APPENDIX A

Solution of first order linear systems of the form:

$$\dot{x} = a(t)x + f(t)$$

which can be rearranged into the form:

$$\dot{x} - a(t)x = f(t)$$

using an integrating factor of:

$$e^{-\int a(t)dt}$$

results in:

$$\left(e^{-\int a(t)dt}\right)\dot{x} - a(t)\left(e^{-\int a(t)dt}\right)x = \left(e^{-\int a(t)dt}\right)f(t)$$

which is of the form:

$$\frac{d}{dt}\left[e^{-\int a(t)dt} \cdot x\right] = e^{-\int a(t)dt} \cdot f(t)$$

changing t to u and integrating with respect to u from 0 to t results in:

$$e^{-\int a(u)du} x(u)\Big|_{u=0}^{u=t} = \int_0^t e^{-\int a(u)du} \cdot f(u)du$$

solving for x(t):

$$x(t) = e^{\int a(t)dt}c + e^{\int a(t)dt}\left[\int_0^t e^{-\int a(u)du} \cdot f(u)du\right]$$

Note: By substituting t=0 into this expression, it is found that c=x(0). However, in the example in Chapter 2 which utilizes this integral, the constant c (denoted as c_2) is combined with other constants forming c_3.

A-1

APPENDIX B

In this thesis a commonly occuring operation involved raising e to the power of a matrix, such as e^{At}. Although such an operation can be easily performed using MATLAB, the meaning of such an operation will be explained using an illustrative example.

If we were dealing with a scalar form such as e^{at}, the following series would be the solution we would be looking for:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots + \frac{(at)^n}{n!} + \cdots \qquad \text{(B-1)}$$

therefore in matrix form this equation would be:

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots + \frac{A^n t^n}{n!} + \cdots \qquad \text{(B-2)}$$

This alone explains the meaning of e raised to the power of a matrix. However, this is a very computation intensive method, which would require a very large number of calculations to yield an acceptable level of accuracy.

Therefore, the following alternate procedure can be used to get the desired results.

First, recall that the modal matrix, P, for a set of equations with a coefficient matrix A is composed of the eigenvectors arranged as columns.

Next, note that a matrix A can be diagonalized by premultiplying by the inverse of the modal matrix and postmultiplying by the modal matrix such as:

$$D = P^{-1} A P$$

or alternatly:

$$A = P D P^{-1} \qquad \text{(B-3)}$$

Note, the elements of the diagonal matrix D will be the eigenvalues denoted as λ_i, such that:

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Substituting equation B-3 into equation B-2 results in the following of relations:

$$e^{At} = I + PDP^{-1}t + PD^2P^{-1}\frac{t^2}{2!} + PD^3P^{-1}\frac{t^3}{3!} + \cdots + PD^nP^{-1}\frac{t^n}{n!} + \cdots$$

$$e^{At} = P\left[I + D + \frac{D^2t^2}{2!} + \frac{D^3t^3}{3!} + \cdots + \frac{D^nt^n}{4!} + \cdots\right]P^{-1}$$

Therefore,

$$e^{At} = P\left[\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}t + \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \ddots & \\ & & & \lambda_3^2 \end{bmatrix}\frac{t^2}{2!} + \cdots\right]P^{-1}$$

which can be written as:

$$e^{At} = P\begin{bmatrix} \left(1 + \lambda_1 t + \frac{\lambda_1^2 t^2}{2!} + \cdots + \frac{\lambda_1^n t^n}{n!} + \cdots\right) & & & \\ & \left(1 + \lambda_2 t + \frac{\lambda_2^2 t^2}{2!} + \cdots + \frac{\lambda_2^n t^n}{n!} + \cdots\right) & & \\ & & \ddots & \\ & & & \left(1 + \lambda_n t + \frac{\lambda_n^2 t^2}{2!} + \cdots + \frac{\lambda_n^n t^n}{n!} + \cdots\right) \end{bmatrix}P^{-1}$$

You should recognize the diagonal terms from the scalar case shown earlier in equation B-1. This results in:

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

and finally:

$$e^{At} = P\, e^{Dt}\, P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}P^{-1}$$

which again defines what is meant by raising e to the power of a matrix.

APPENDIX C

After looking at a Strutt diagram as well as the three-dimensional plots of the stability surface the patterns of stable and unstable combinations of coefficients can be seen. To demonstrate this stability and instability for different choices of 'a' and 'b' the following series of plots was produced using MATLAB.

For all plots, the horizontal axis represents increasing time starting from the same initial conditions of $\theta(0)=1$ and $\dot{\theta}(0)=0$. The solid line represents θ and the dashed line represents $\dot{\theta}$. The corresponding values of 'a' and 'b' appear in the title of each plot.

a=0.5 b=0.5
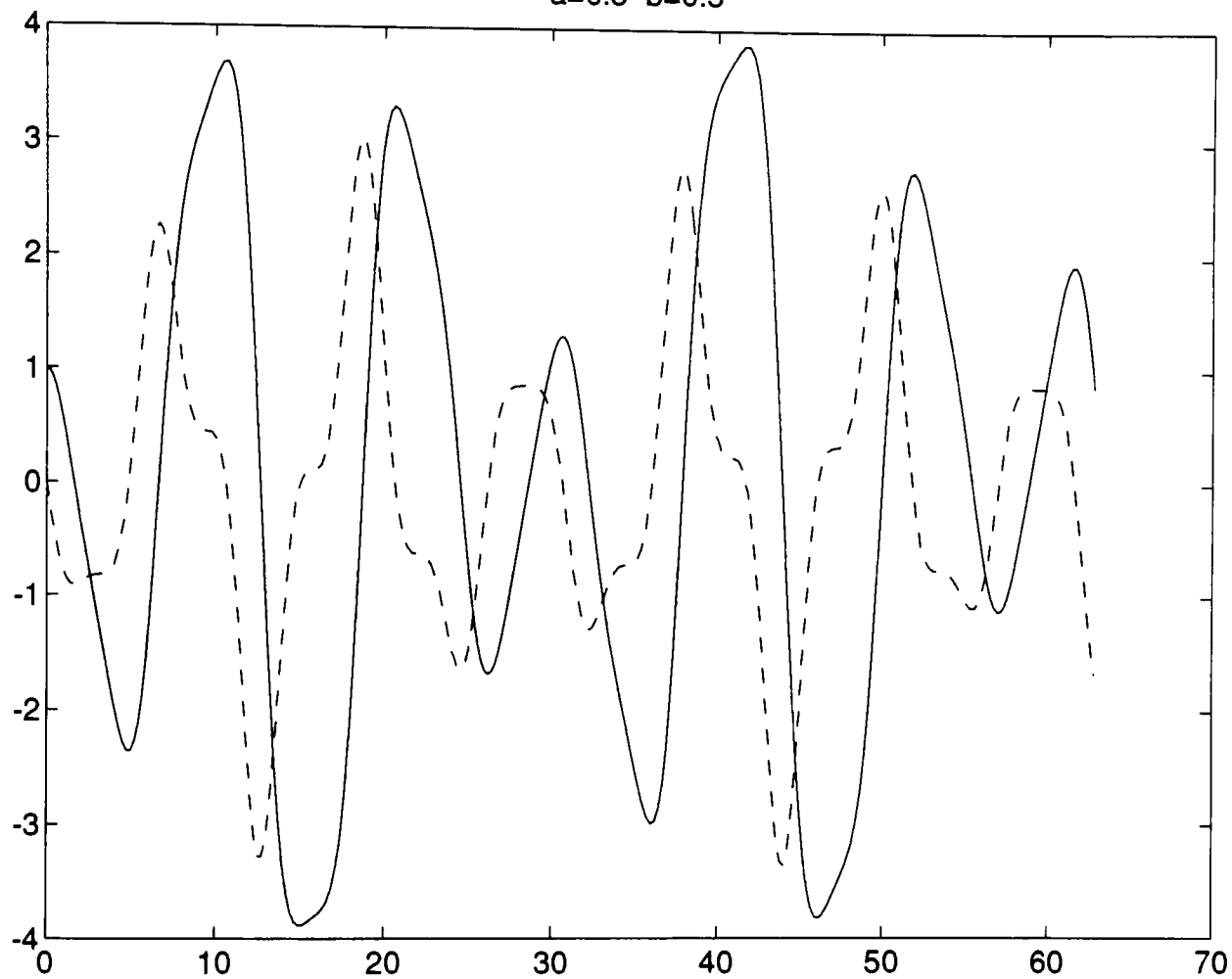
a=1.0 b=0.5
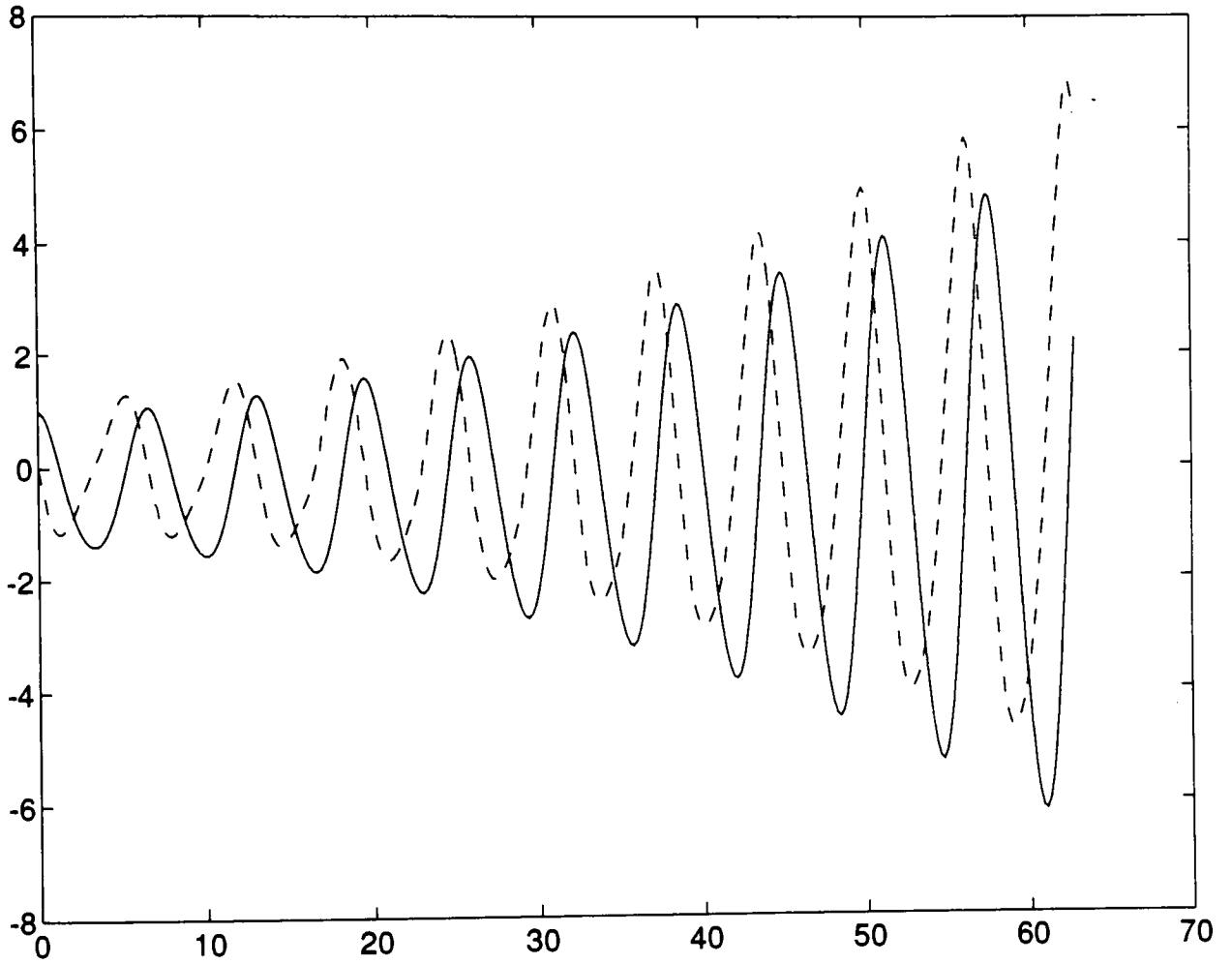
a=1.5 b=0.5
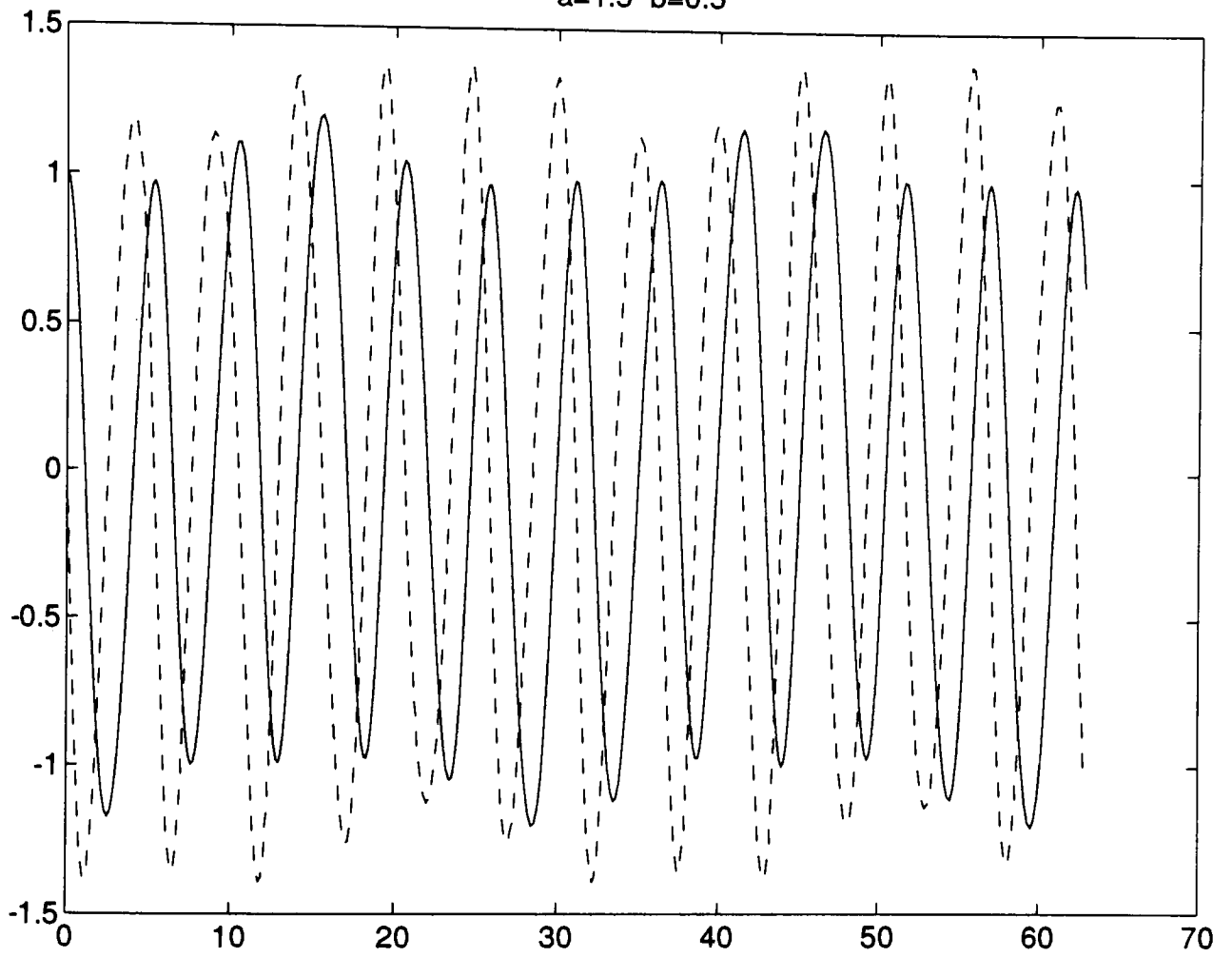
a=2.0 b=0.5
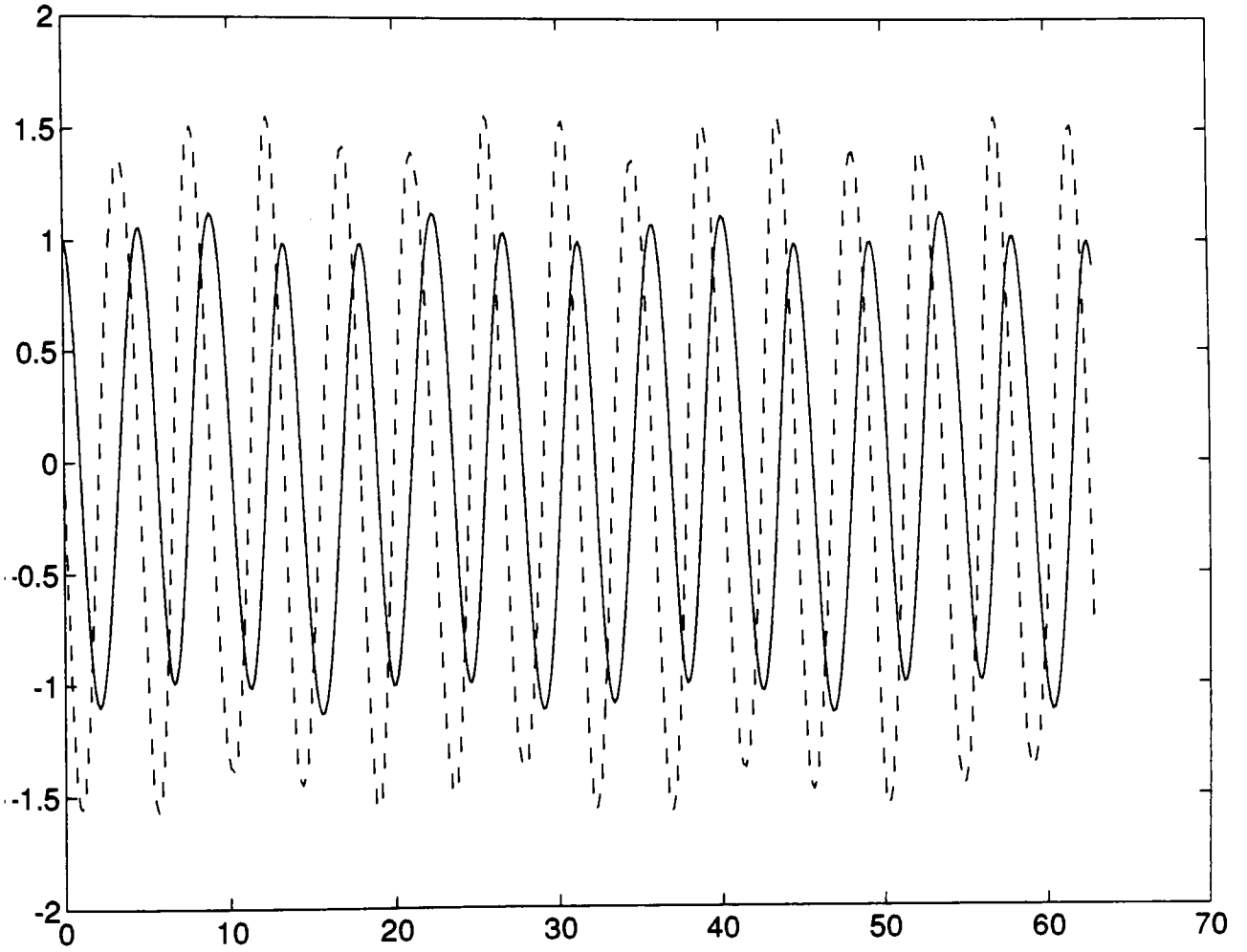
a=2.5 b=0.5
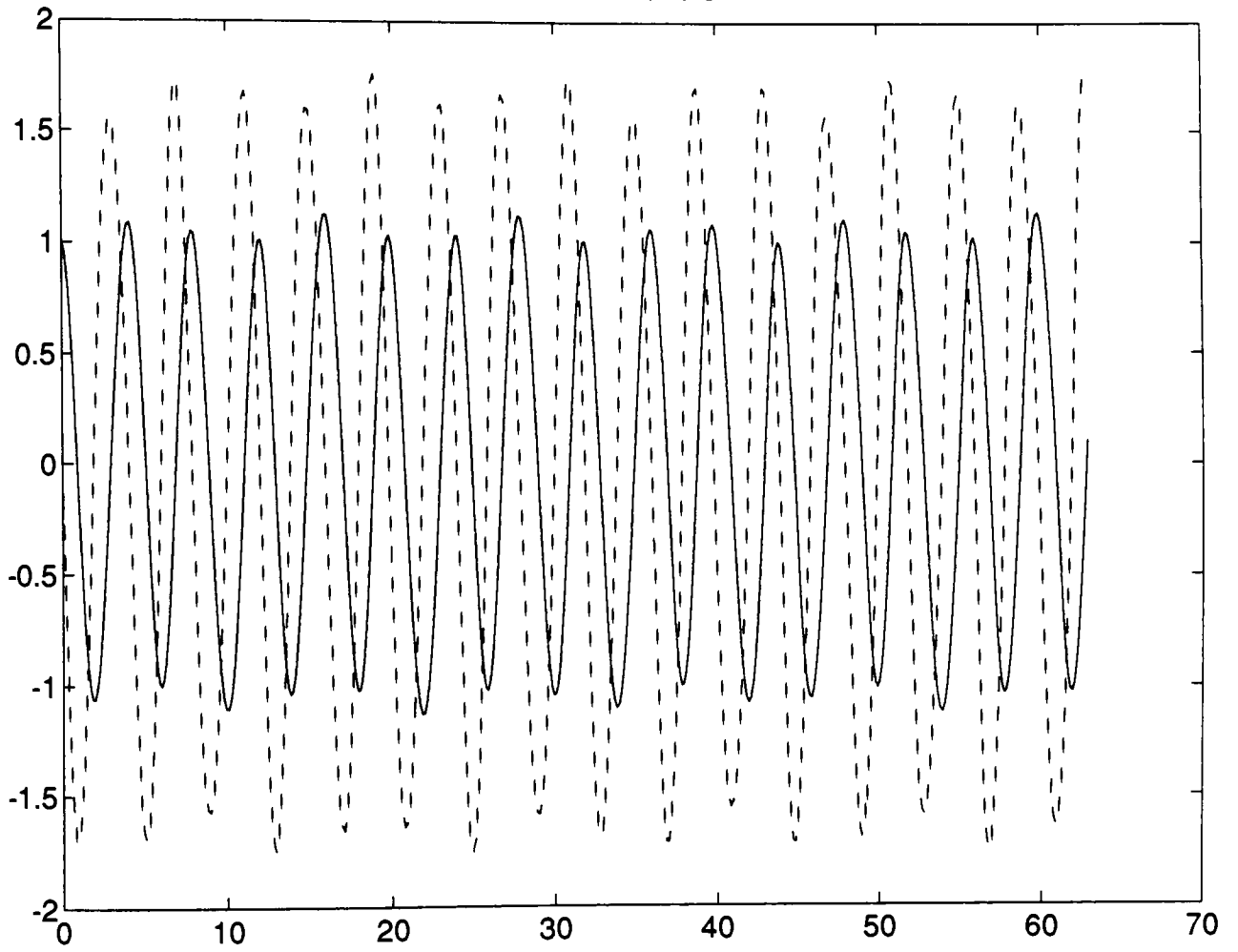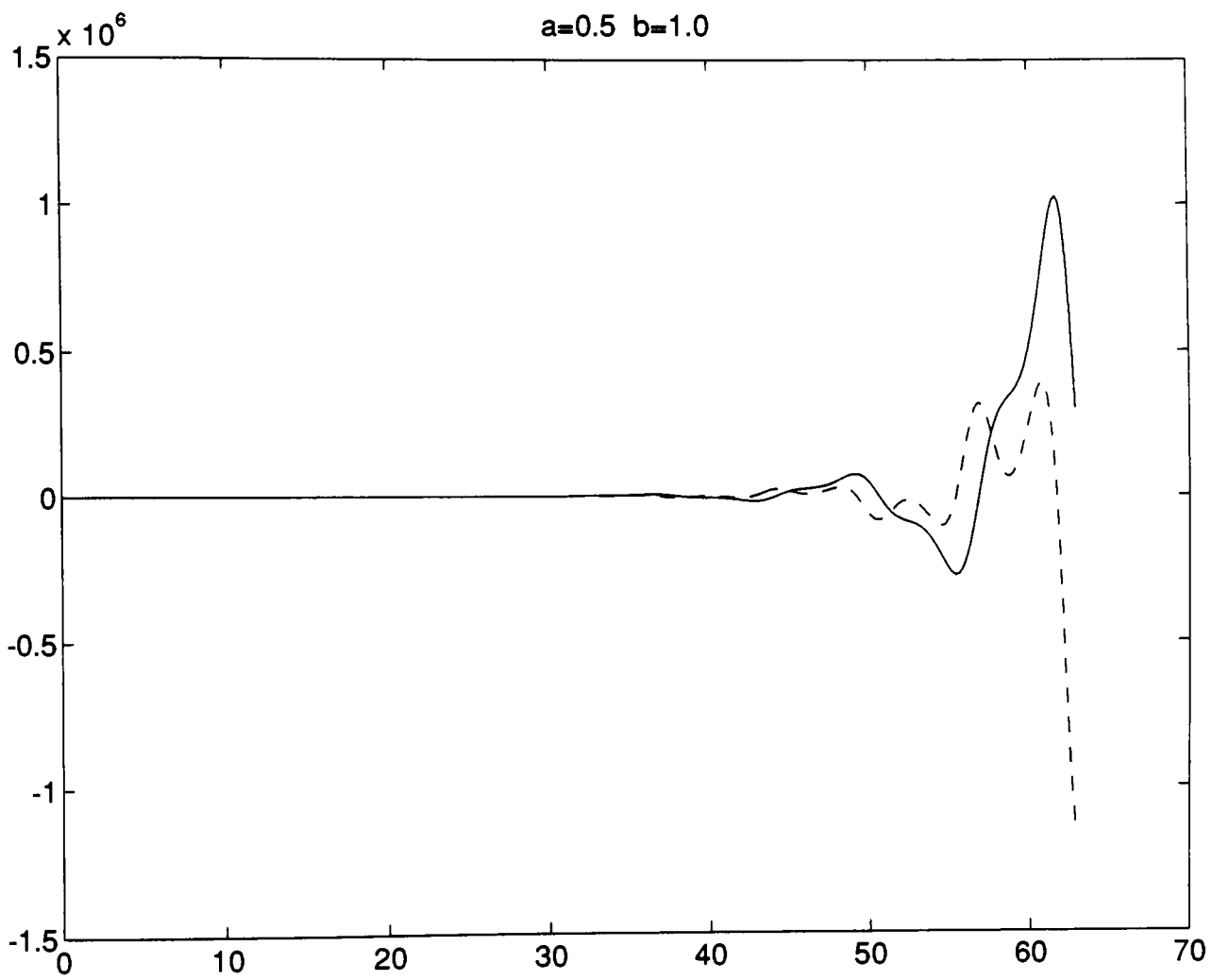
a=0.5 b=1.0
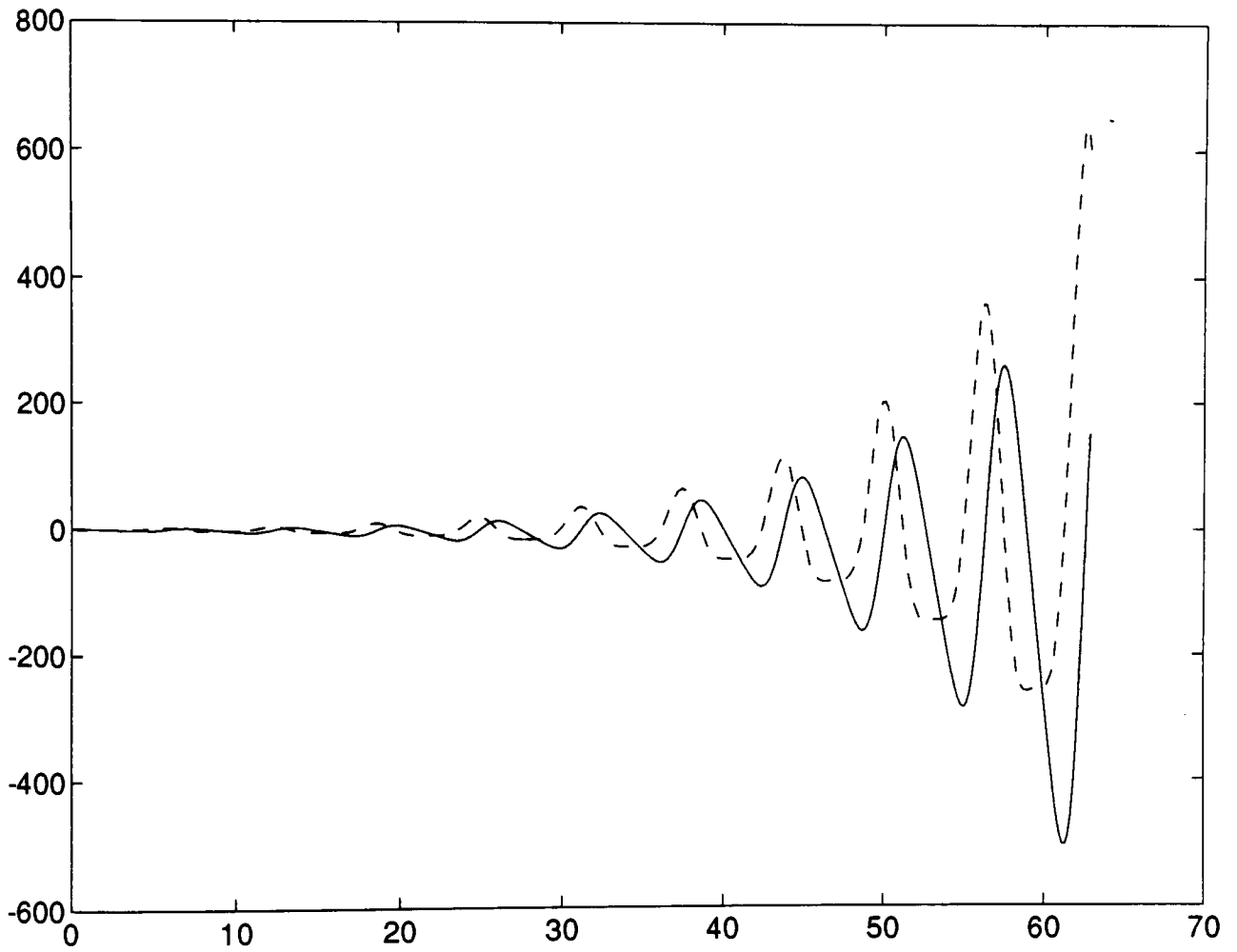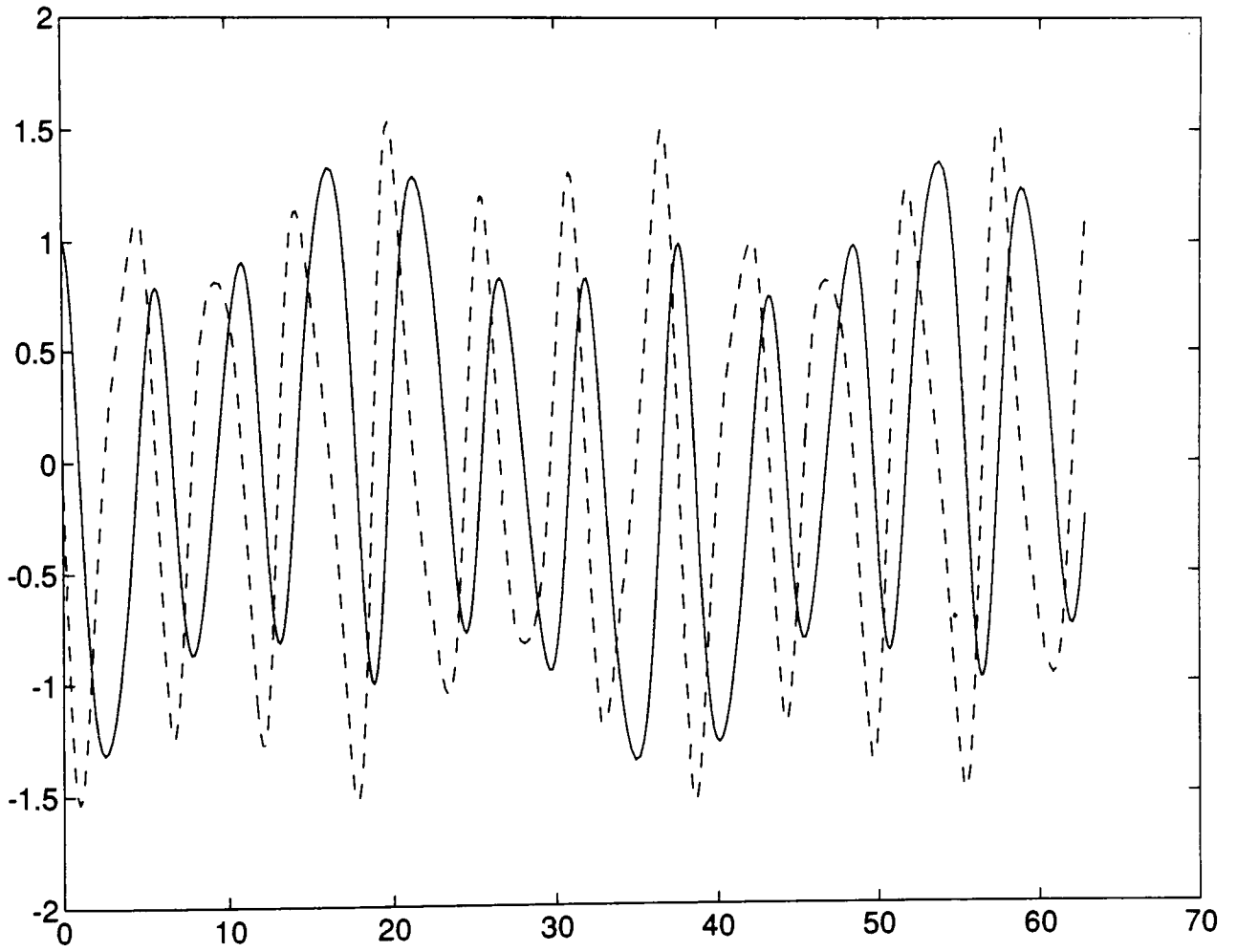
a=1.0 b=1.0

a=1.5 b=1.0

a=2.0 b=1.0
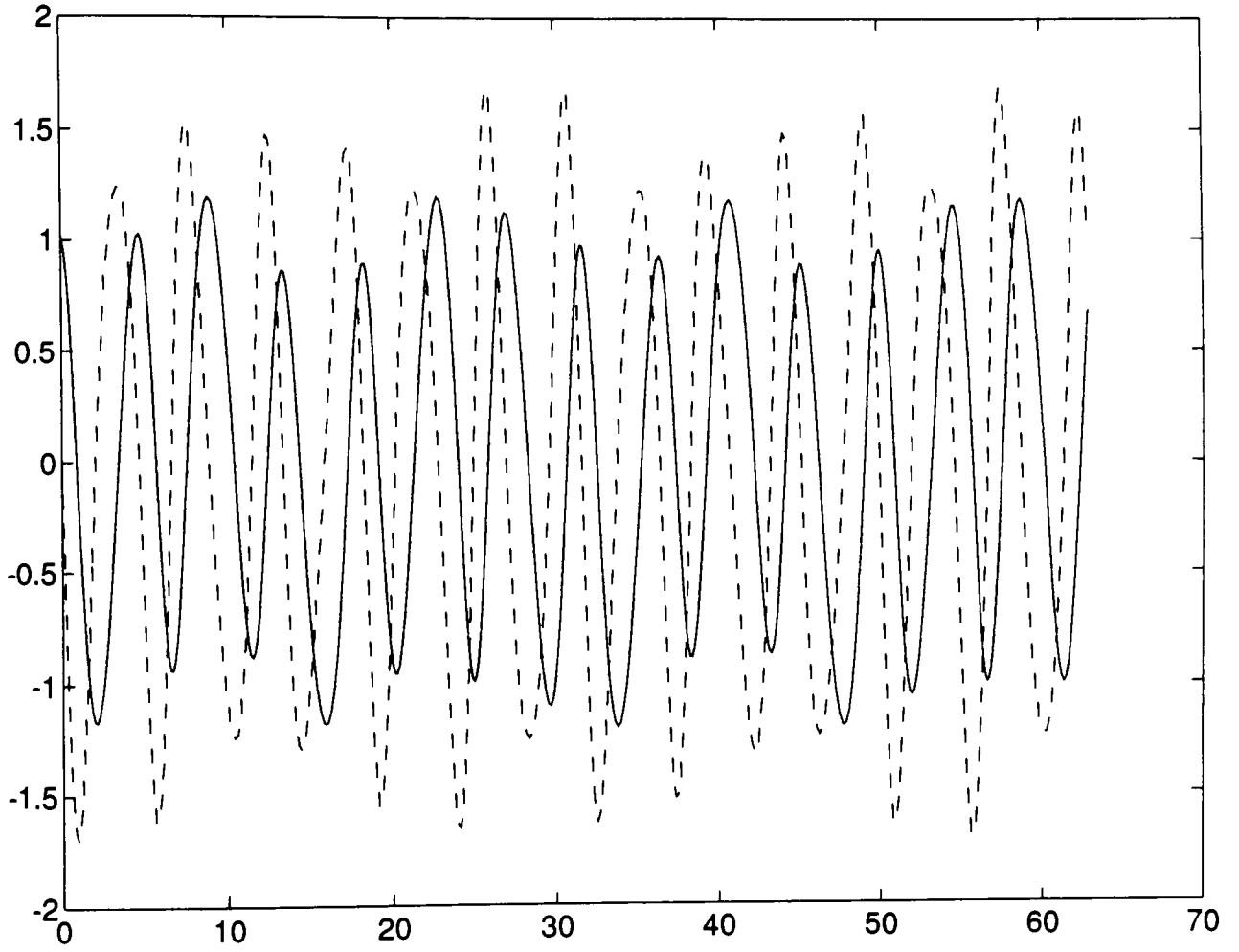
a=2.5 b=1.0
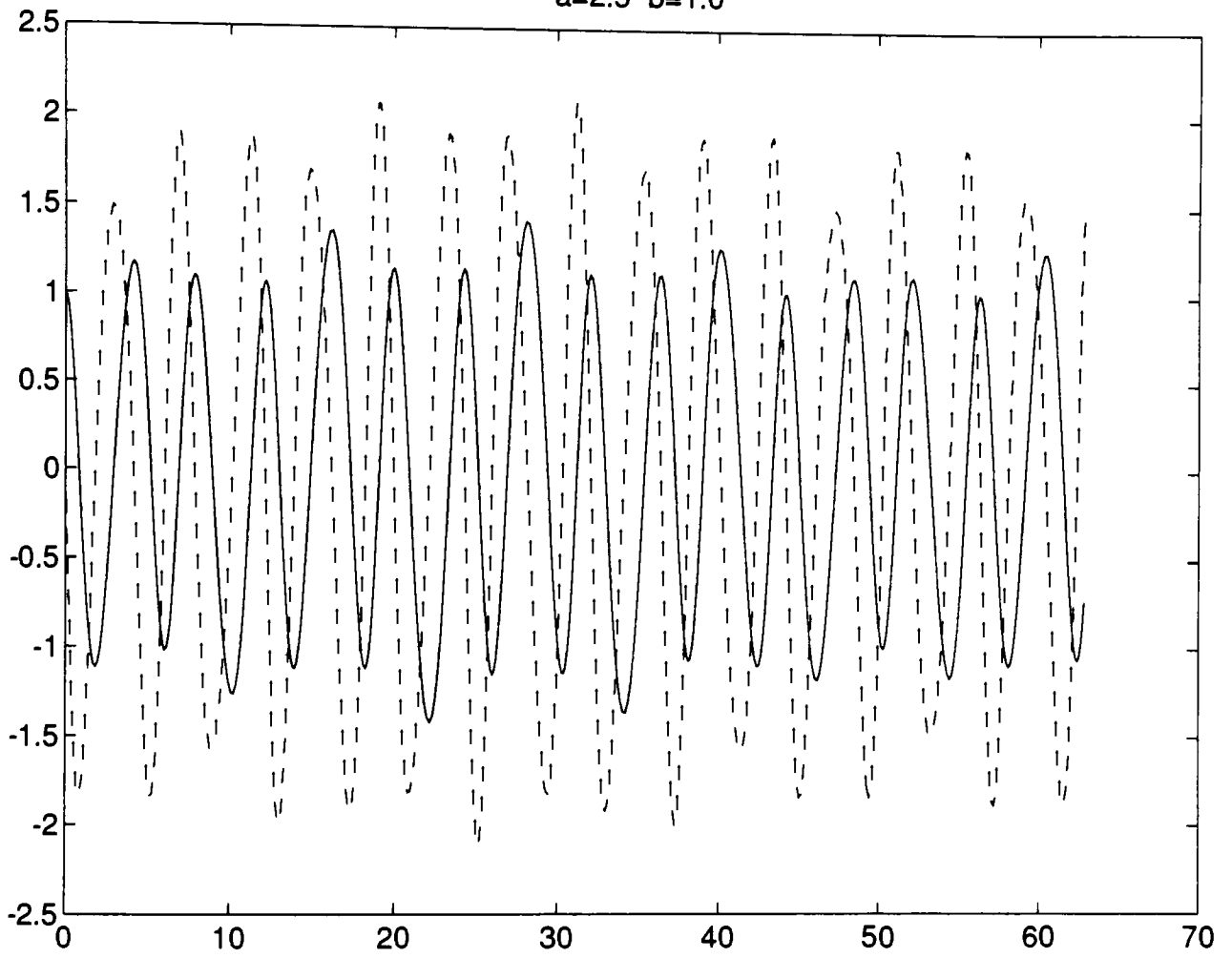
a=0.5 b=1.5
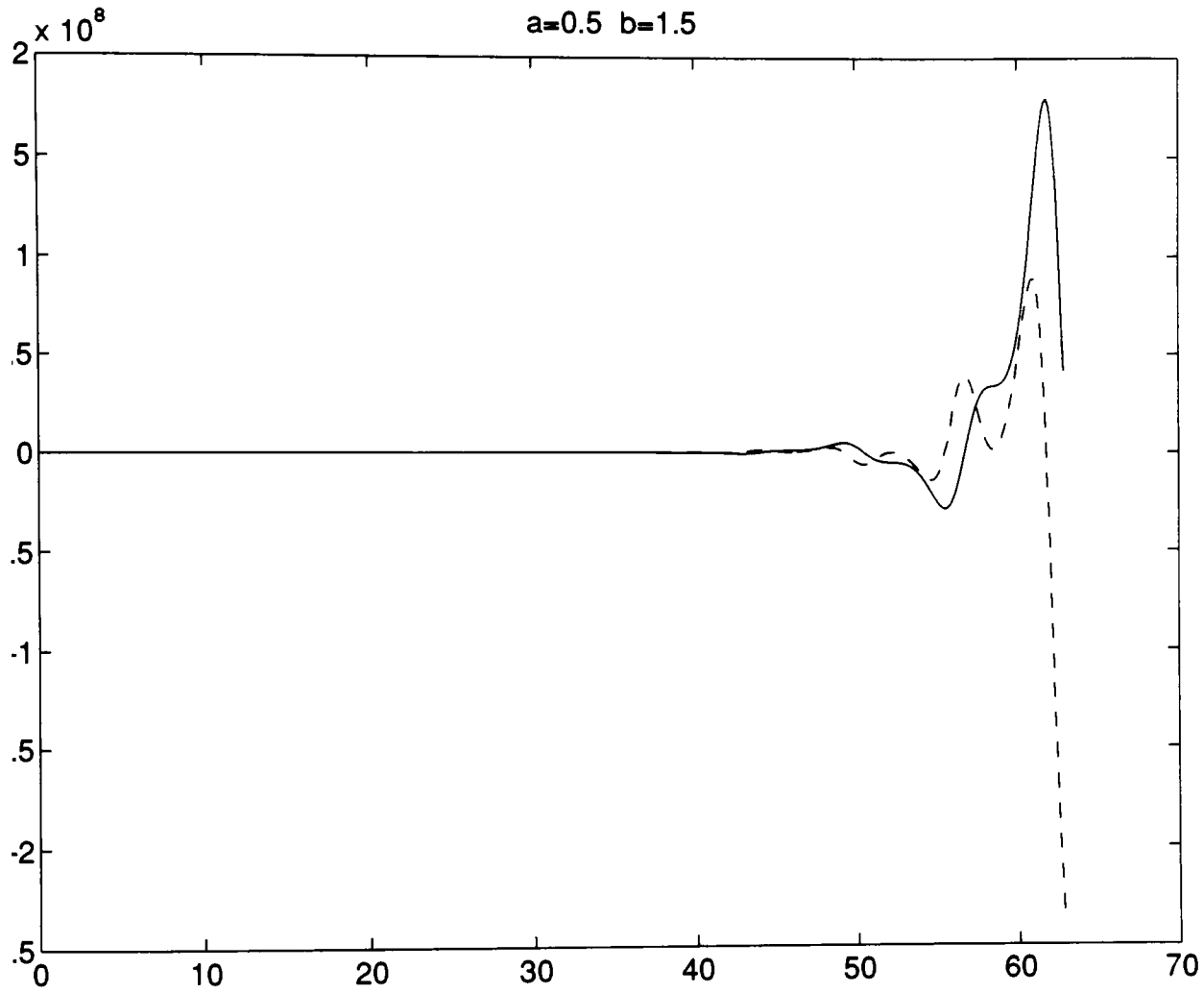
a=1.0 b=1.5
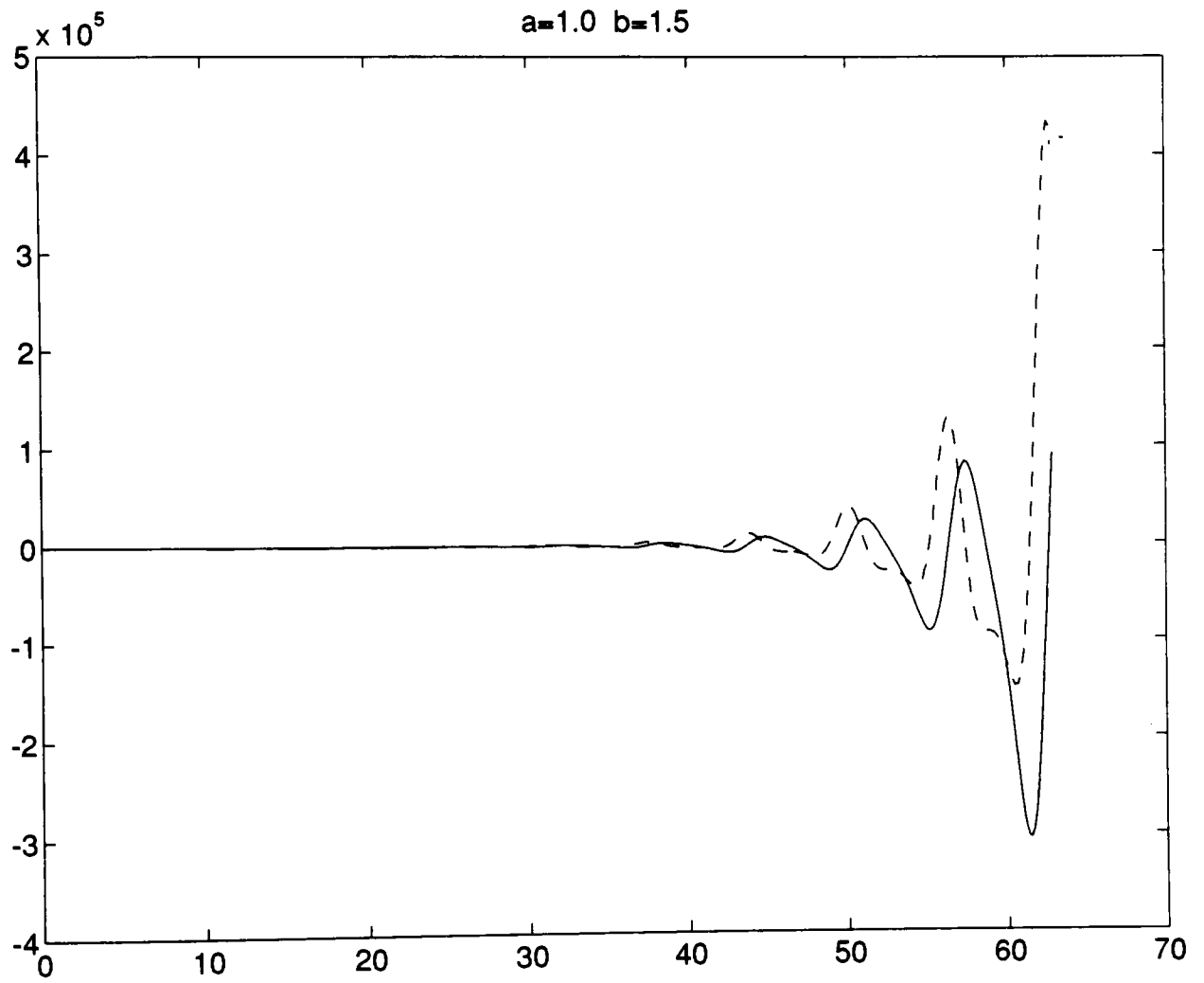
a=1.5 b=1.5

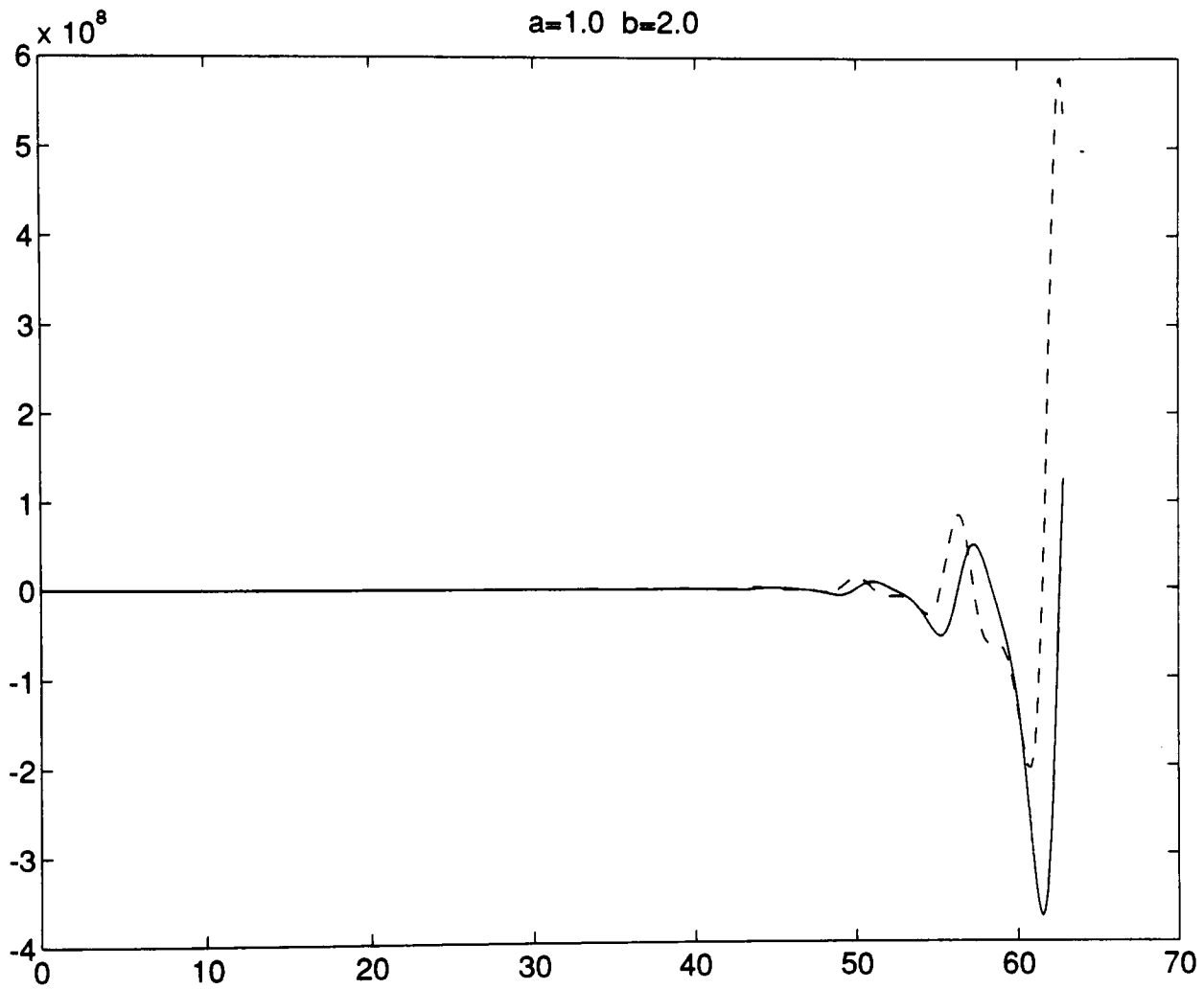a=2.0 b=1.5

a=2.5 b=1.5

a=0.5 b=2.0
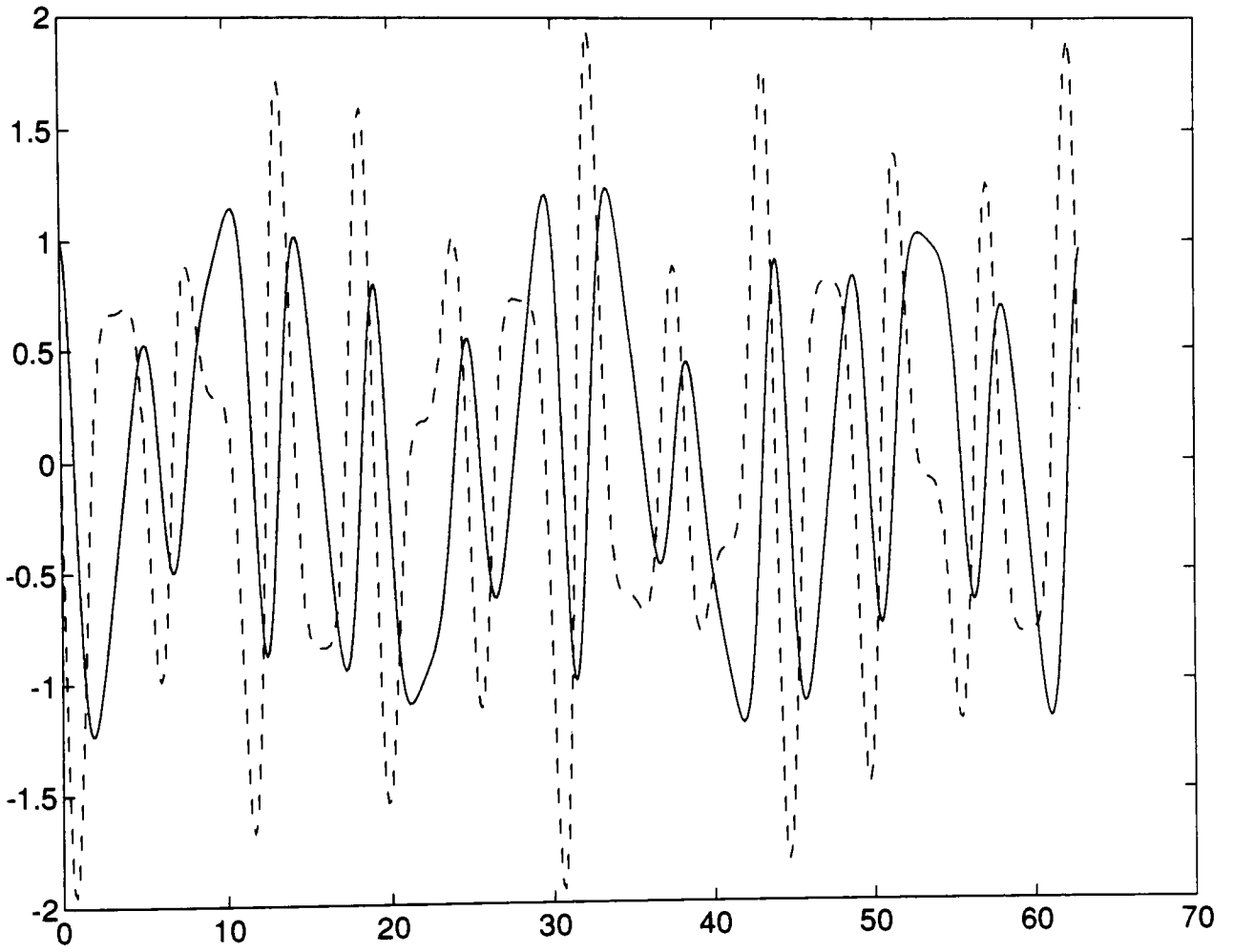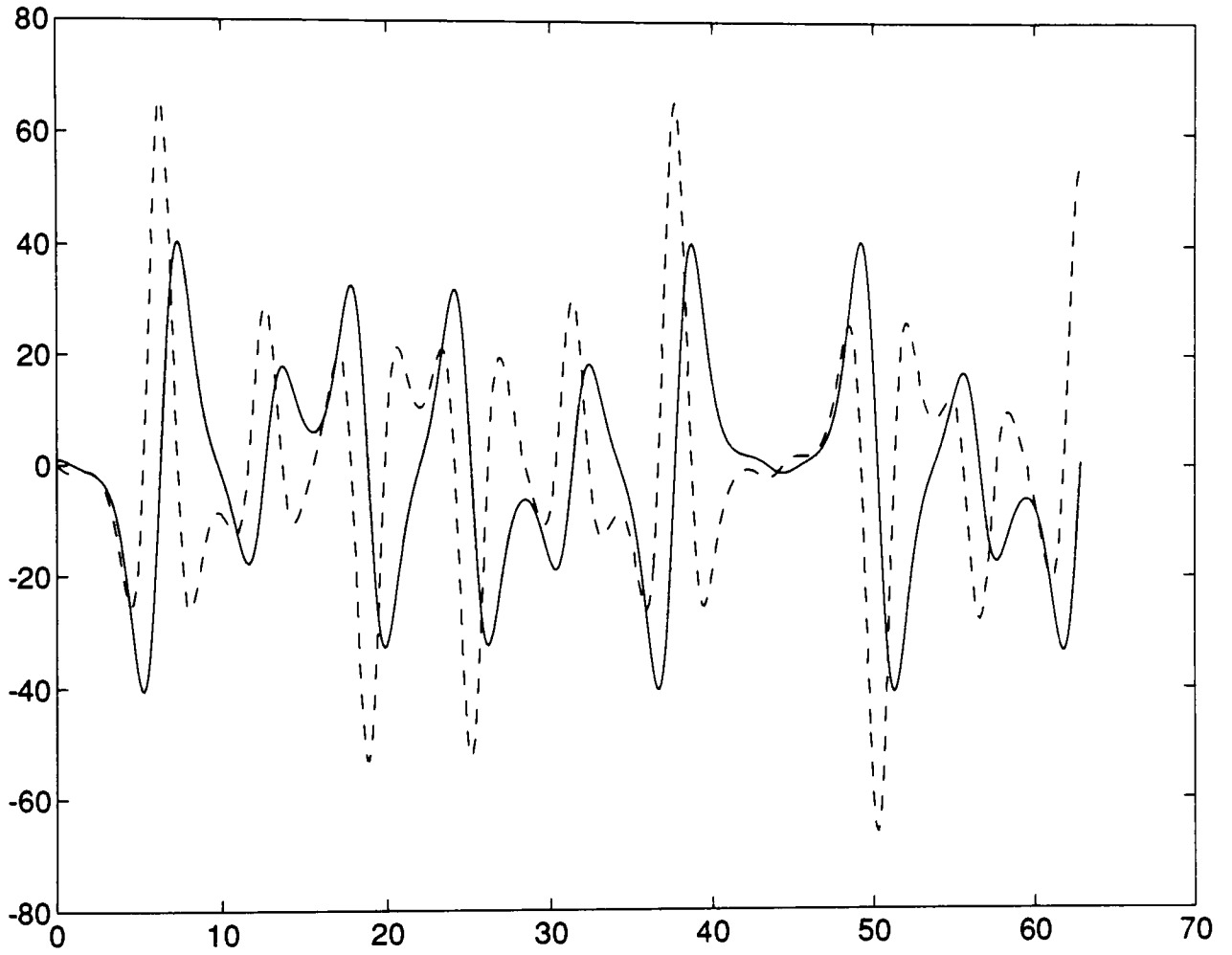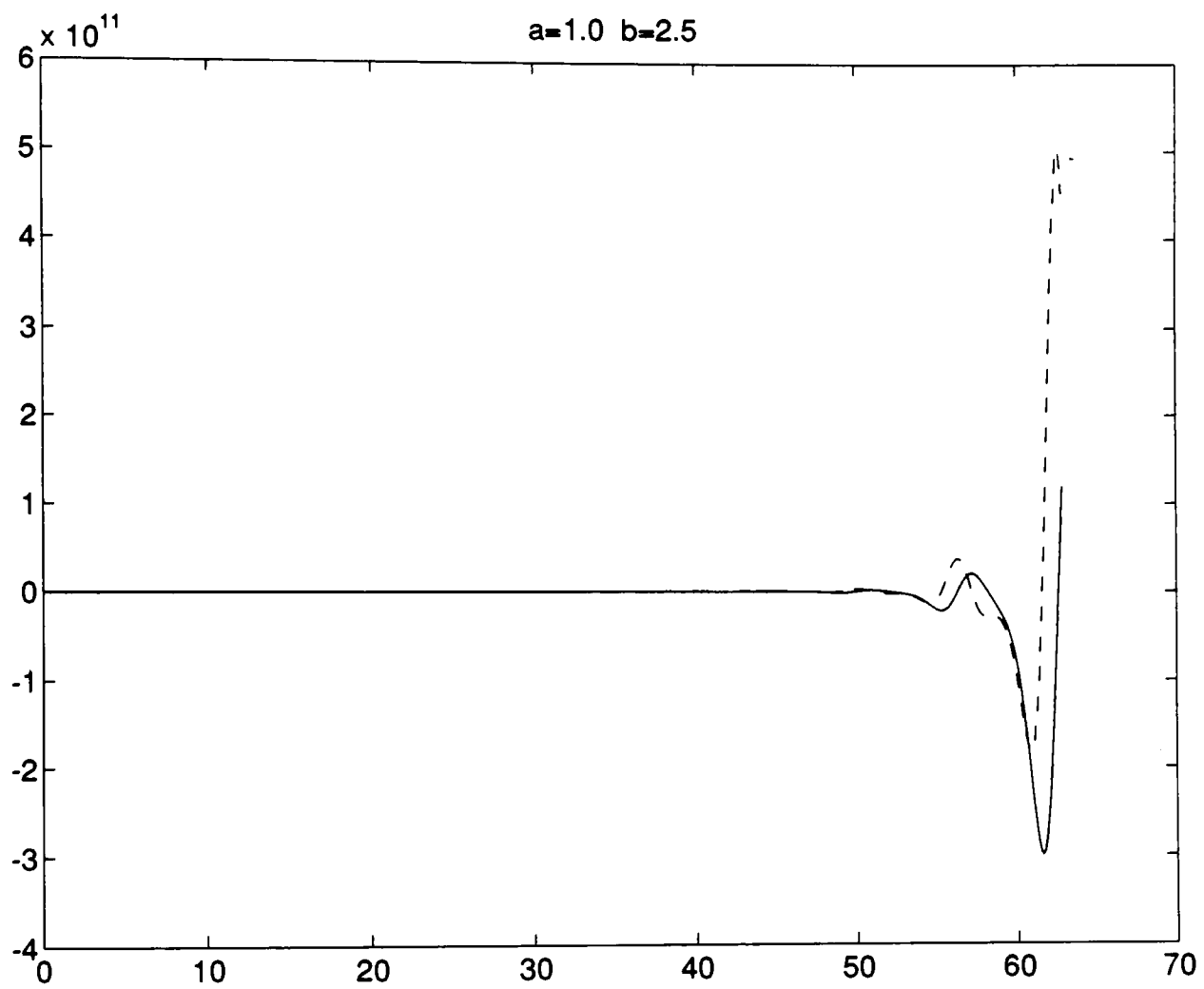
$\times 10^8$
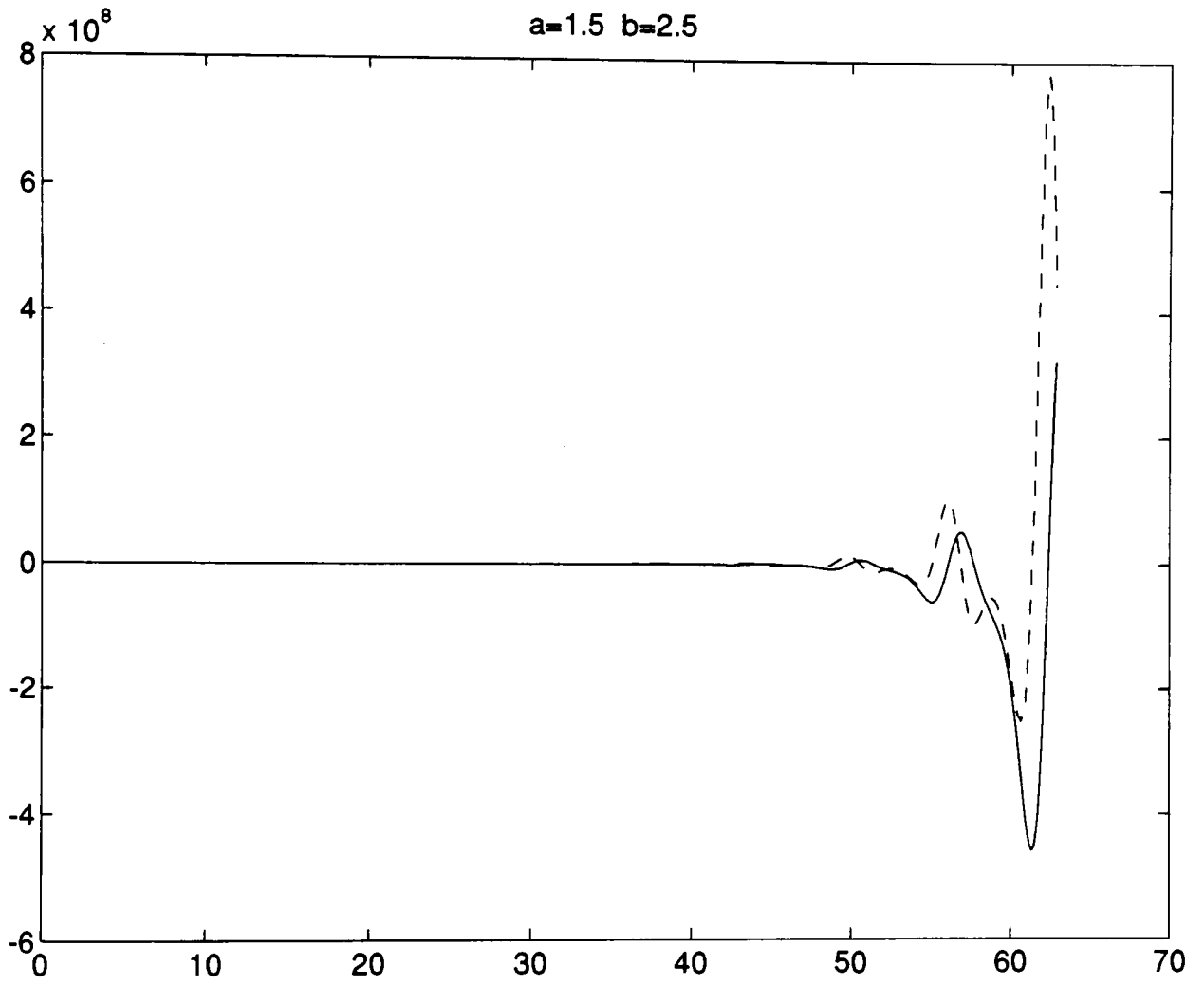
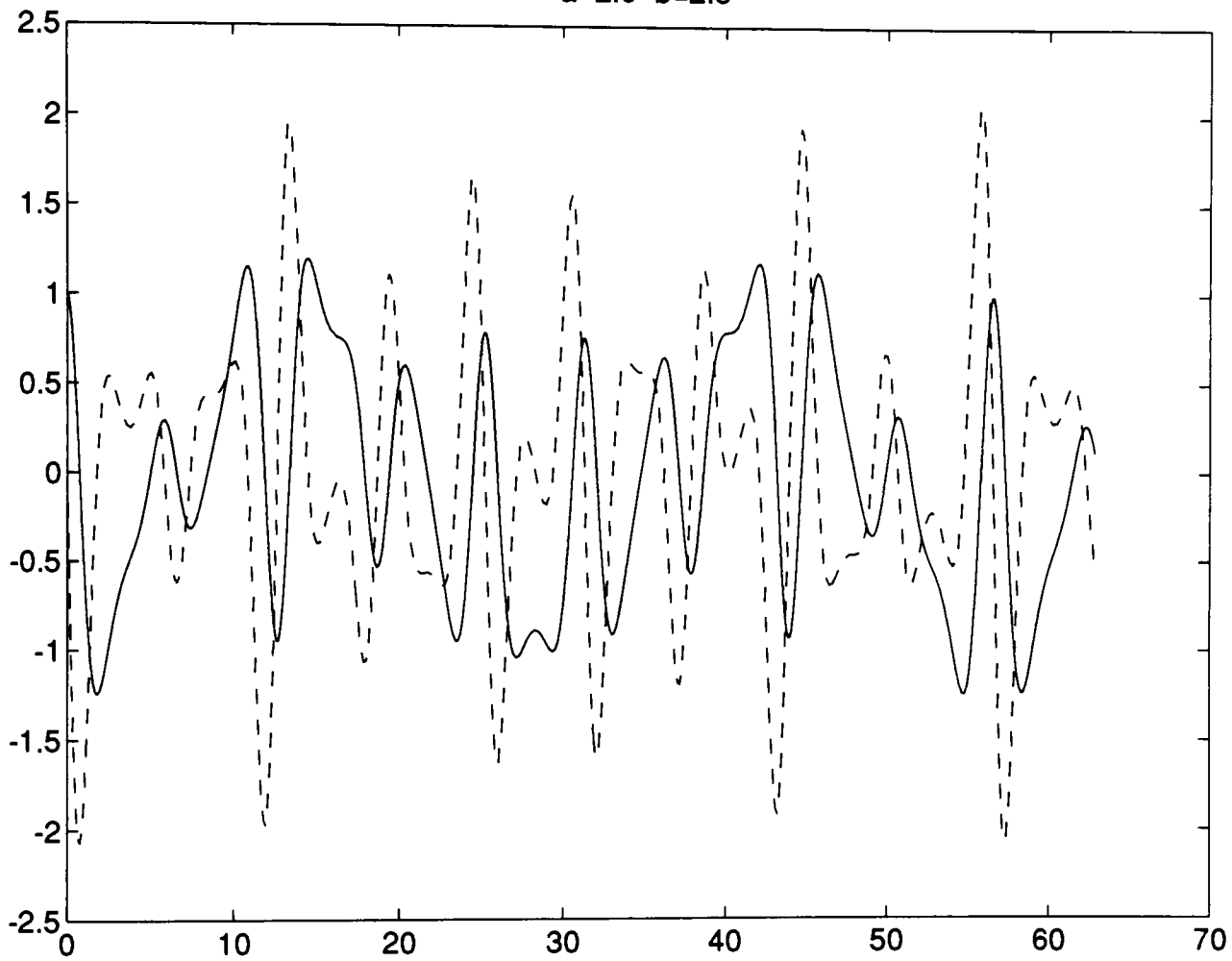a=1.0 b=2.0
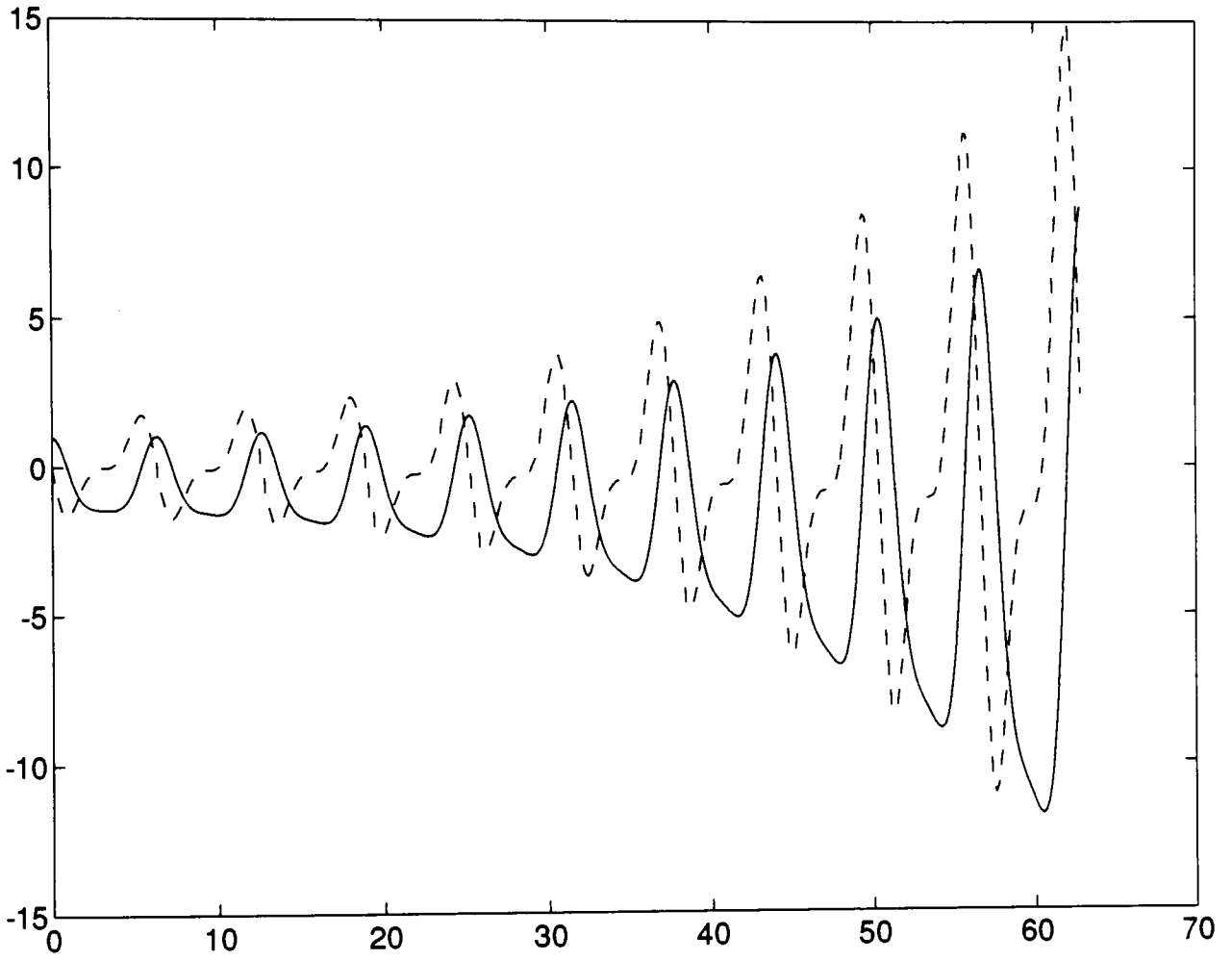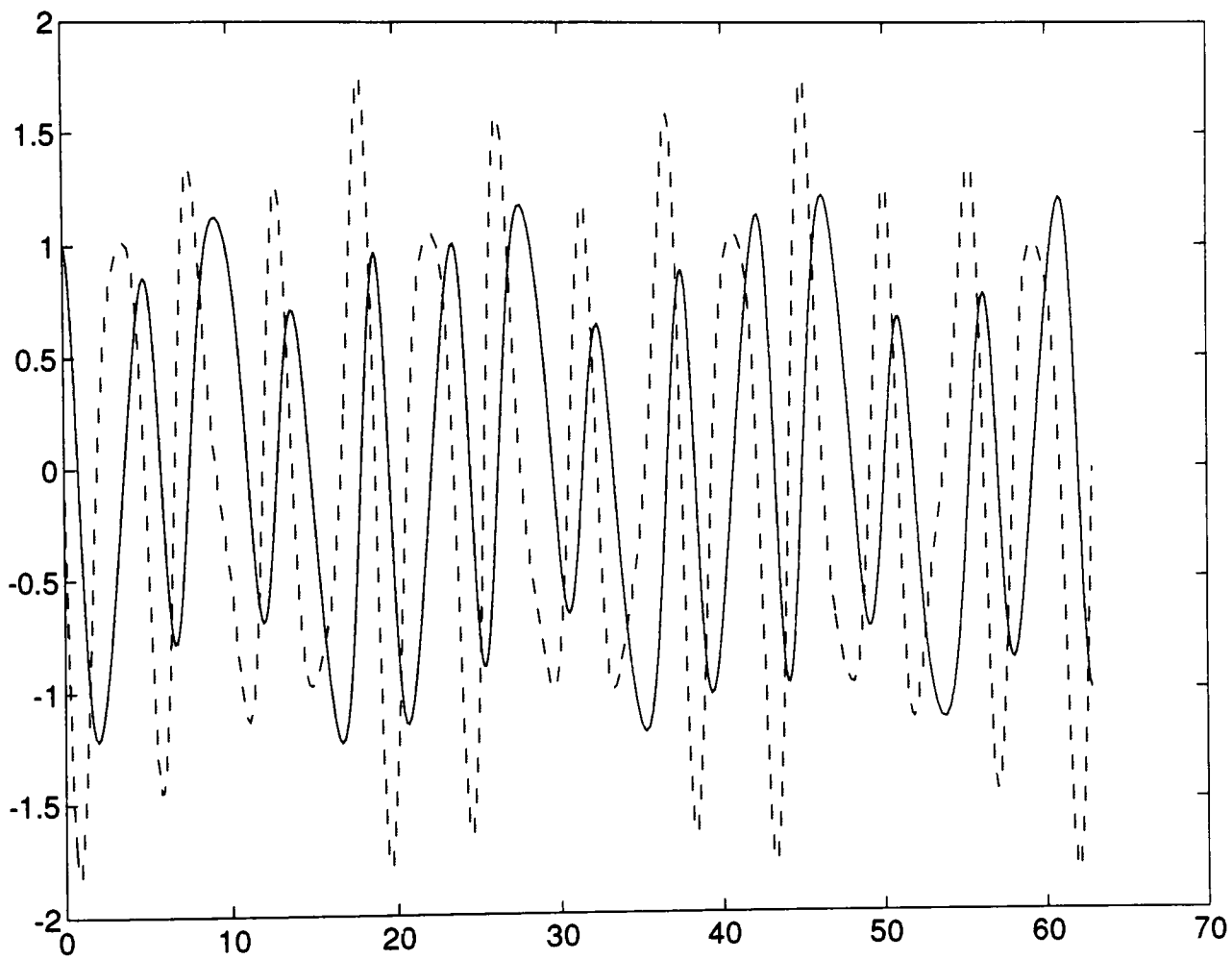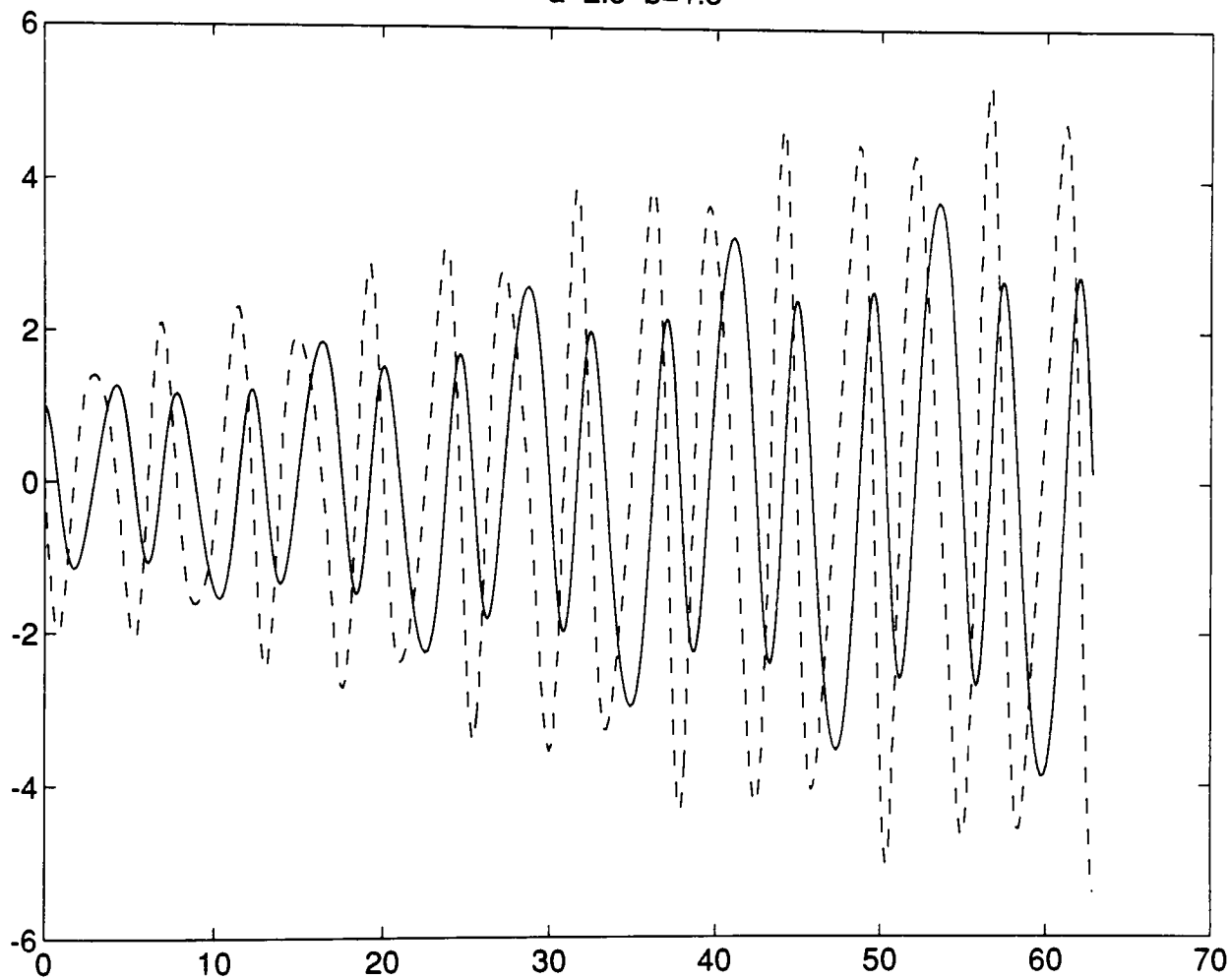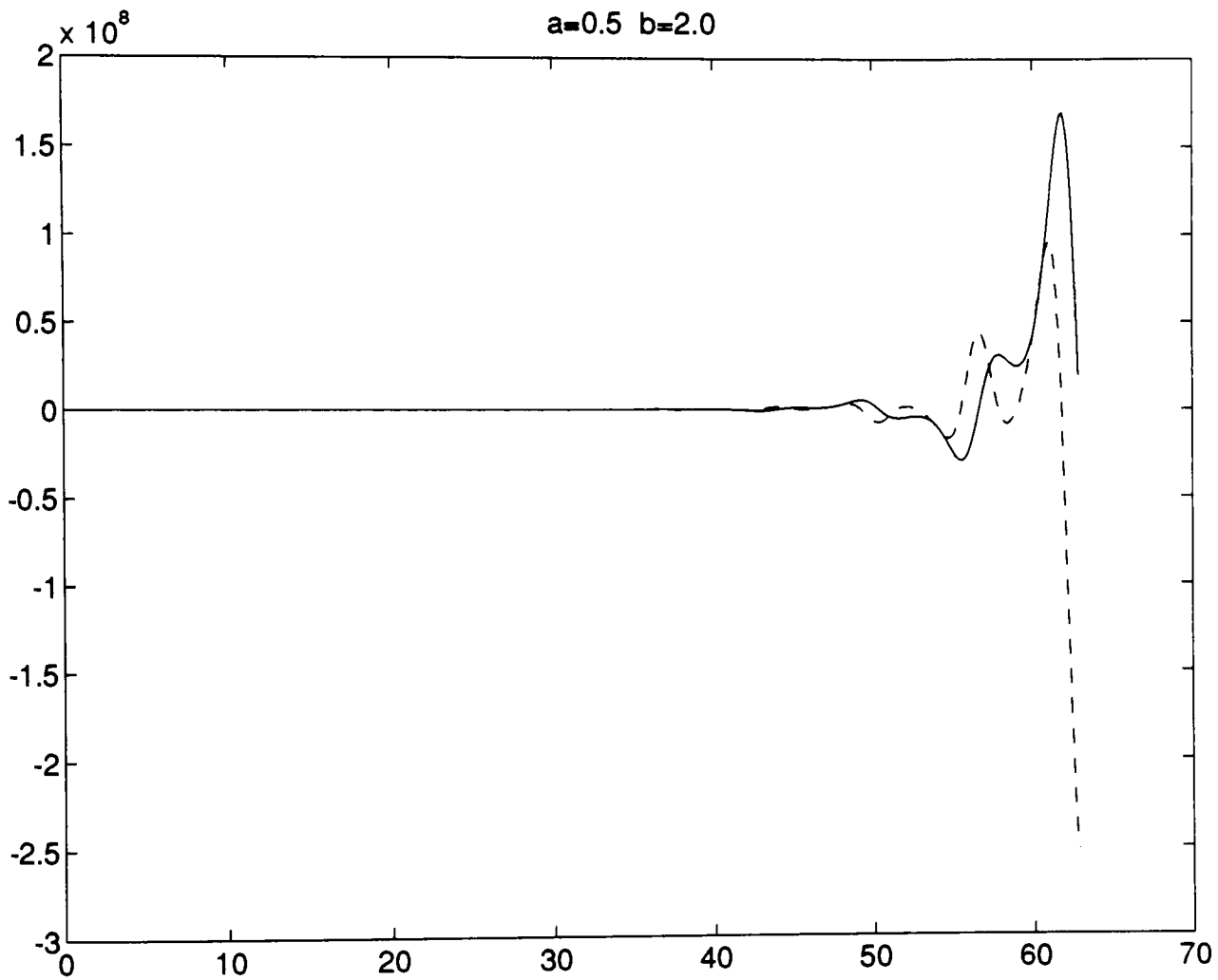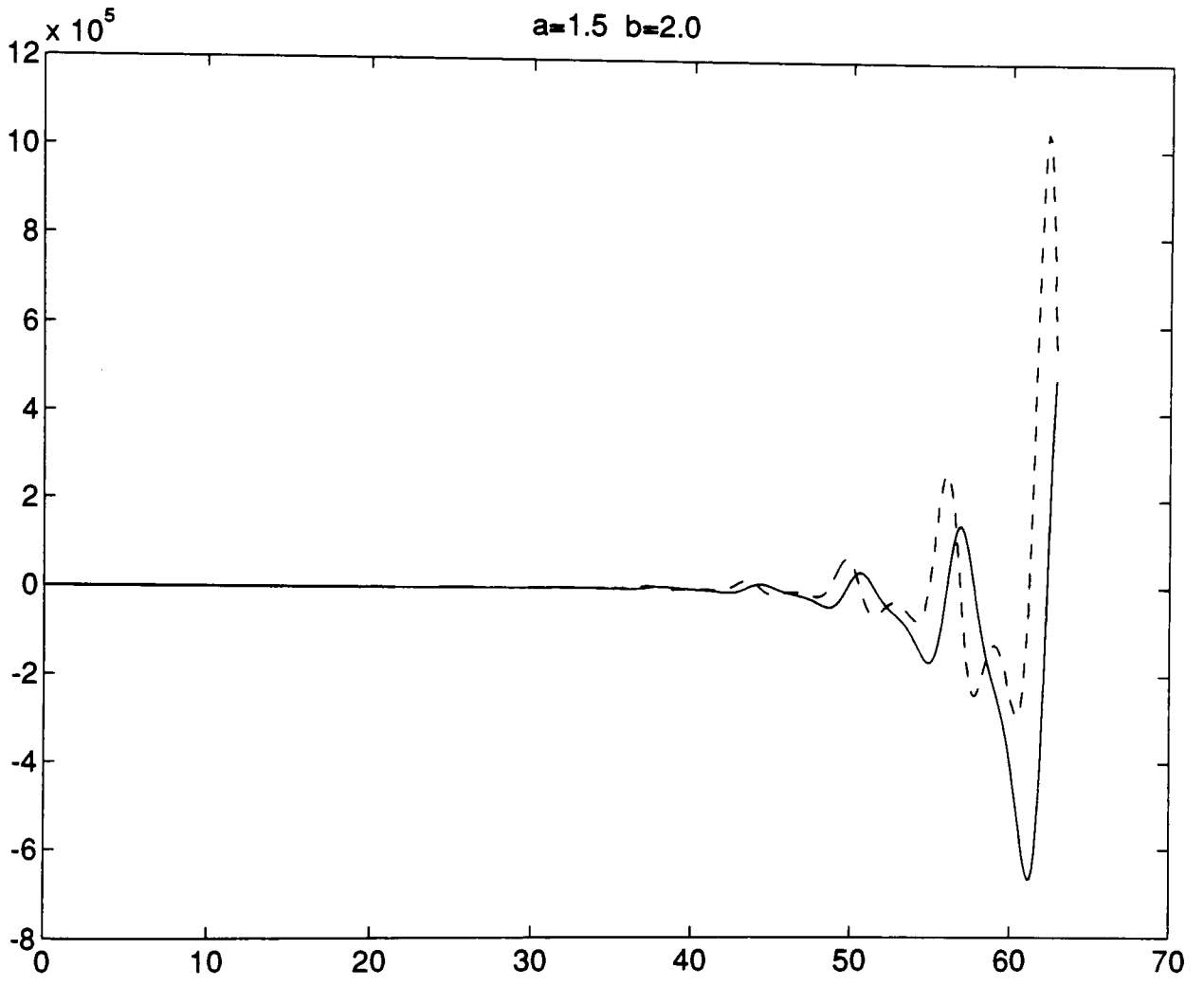
a=2.0 b=2.0
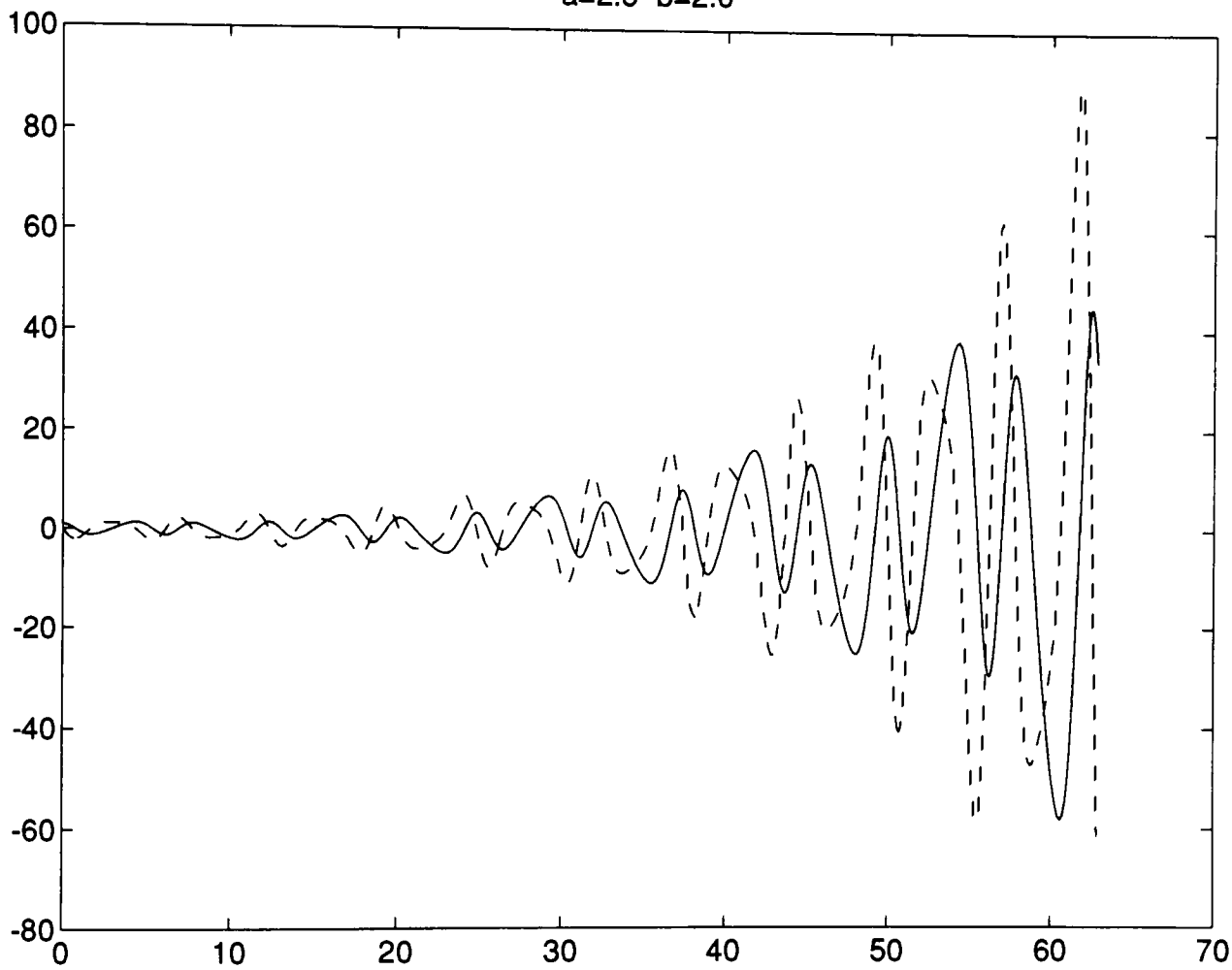
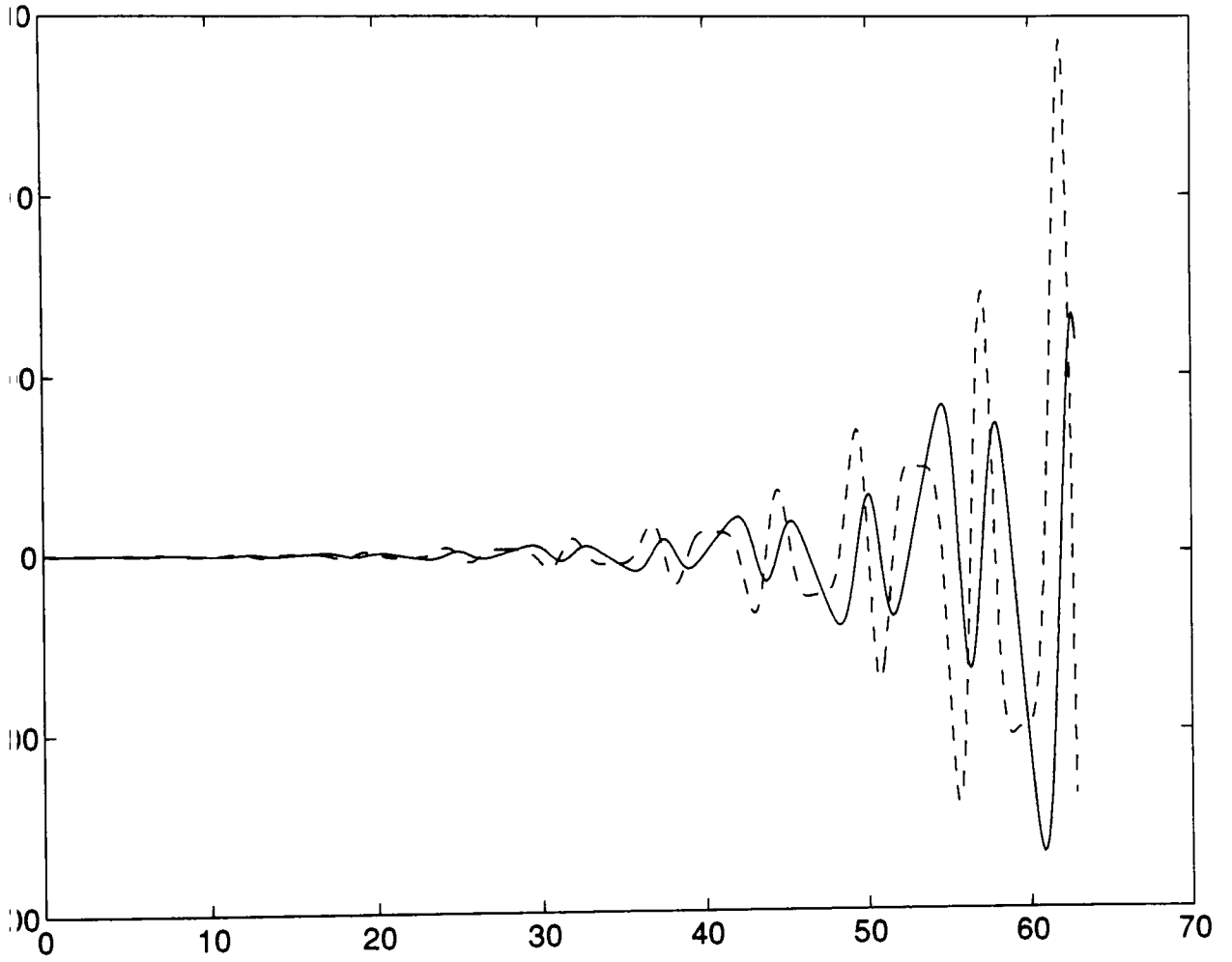a=2.5 b=2.0

a=0.5 b=2.5

a=1.0 b=2.5

a=2.0 b=2.5

a=2.5 b=2.5

APPENDIX D

These are the major MATLAB programs which were used to generate the plots of the stability surfaces for the Mathieu Equation.

The program POINTS2 created the arrays of points , 'hor' and 'ver', at which the value of the stability surface will be solved for.

```
%
%                POINTS2.M
%
% First define amax  - the lower horizontal limit in parameter space
%            amax  - the upper horizontal limit in parameter space
%            bmin  - the lower vertical limit in parameter space
%            bmax  - the upper vertical limit in parameter space
%
%            N  - the GRID SIZE , i.e. number of points in each
%                 direction of the a-b plane.
%
delta_a= (amax-amin)/(N-1)
delta_b= (bmax-bmin)/(N-1)
%
hor = (amin:delta_a:amax)
%
ver = (bmin:delta_b:bmax)
%
clear values
clear S
%
```

The program RUNPTS does the actual calculation of the value of the stability surface at each point in the 'hor' and 'ver' arrays by calling the program RUN.M.

```
%
%                RUNPTS.M
%
%   This program actually determines the stability value
%        S = phi1(2pi) + phi2'(2pi)
%
%   It calls the program RUN.M, which solves for the fundamental
%   solutions, at each of the desired parameter values.
%
```

```
%    f1 and f2 represent the fundamental solutions.
%
values=zeros(1,N);
%
for i = 1:N
     for j = 1:N
          xi=[1,0,hor(i),ver(j)]';
          run
          f1=x(length(t),1);
          xi=[0,1,hor(i),ver(j)]';
          run
          f2=x(length(t),2);
          values(i,j)=f1+f2;
     end
end
%
% Rotate matrix to align horiz and vert
%
S=rot90(values);
%
%    Once the program is run , the " surface "  S(a,b) can be
%    plotted.
```

The program RUN calls the ODE45 program to solve Hill's Equation which is contained within the program HILL. ODE45 numerically solves ordinary differential equations using fourth and fifth order Runge-Kutta formulas with an accuracy of 1.e-6.

```
%
% This program  RUN.M  calls the ODE-solver  ode45
%
%
ti=0.0;
tf=2*pi;
%
[t,x] = ode45('hill',ti,tf,xi);
%
%
%
%
% Note that it calls the defining function program  HILL.M
%
%
```

The program HILL defines Hill's Equation which is used by ODE45.

```
%                    HILL.M
%
%      This program defines Hill's equation  y" +(a +b*q(t))*y = 0
%      to be used in   ode45 . The parameters are able to be
%      inputted ( artificially ) by setting them equal to
%      x(3) = a   and x(4) = b,  as "initial conditions" (clever).
%
function xdot = vce(T,x)
v%   parametric oscillator
xdot(1) = x(2);
xdot(2) = -x(3)*x(1) -x(4)*cos(T)*x(1);
xdot(3) = 0.;
xdot(4) = 0.;

end;
```

The program ICON sets the initial conditions and the constants for the program HILL (listed here again for completeness) which solves the Mathieu equation. The input of 'a' and 'b' is done using a clever method which will be discussed on the next page.

After calculations are done, a plot of θ and $\dot{\theta}$ versus time is made with θ being a solid line and $\dot{\theta}$ being a dashed line. The response of the system as shown using this program will verify if the system is stable or unstable as previously determined.

```
%
%                    icon.m
%
%      This program allows setting initial conds and constants in
%      the Mathieu equation solver  HILL.M .
%
%      x1 = theta,  x2 = theta_dot,  x3= a ,  x4 = b
%      Total time is given by  2*mult*pi ,
%      so MULT =  number of forcing periods
%
%      OPTIONAL :  surf = S(a,b),   cm = [ characteristic multipliers ]
%                           ce = [ characteristic exponents ]
%
clear x
x0=input( ' theta = ' );
y0=input( ' th_dot = ');
x3=input( ' a =  ' );
x4=input( ' b =  ' );
mult= input( ' Enter INTEGER multiple of 2*pi : ');
%
vals=[x0,y0,x3,x4];
%
[t,x]=ode45('hill',0,2*mult*pi,vals');
 x1=x(:,1);
x2=x(:,2);
%
NN=length(t);
 surf=x1(NN)+x2(NN);
%
cm=[surf/2.+sqrt(surf^2-4.)/2., surf/2.-sqrt(surf^2-4.)/2.];
ce=log(cm);
%
 plot(t,x1,t,x2,'--')
%
```

The program HILL defines Hill's Equation which is used by ODE45. The values of 'a' and 'b' are allowed to be changed for each program run by inputting them as the initial conditions x3 and x4 and then setting the derivatives of these variables equal to zero. This makes x3 and x4 constants as is desired.

```
%
%                    HILL.M
%
%      This program defines Hill's equation  y" +(a +b*q(t))*y = 0
%      to be used in  ode45 . The parameters are able to be
%      inputted ( artificially ) by setting them equal to
%      x(3) = a   and x(4) = b,  as "initial conditions" (clever).
%
function xdot = vce(T,x)
%  parametric oscillator
xdot(1) = x(2);
xdot(2) = -x(3)*x(1) -x(4)*cos(T)*x(1);
xdot(3) = 0.;
xdot(4) = 0.;

end;
```