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Department of Computer Science  
Rochester Institute of Technology  
Master's Thesis

$$\mathbf{R}(\mathbf{W}_5, \mathbf{K}_5) = \mathbf{27}$$

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April 26, 2004

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## Abstract

The two-color Ramsey number  $R(G, H)$  is defined to be the smallest integer  $n$  such that any graph  $F$  on  $n$  vertices contains either a subgraph isomorphic to  $G$  or the complement of  $F$  contains a subgraph isomorphic to  $H$ . Ramsey numbers serve to quantify many of the existing theorems of Ramsey theory, which looks at large combinatorial objects for certain given smaller combinatorial objects that must be present. In 1989 George R. T. Hendry presented a table of two-color Ramsey numbers  $R(G, H)$  for all pairs of graphs  $G$  and  $H$  having at most five vertices. This table left seven unsolved cases, of which three have since been solved. This thesis eliminates one of the remaining four cases,  $R(W_5, K_5)$ , where a  $K_5$  is the complete graph on five vertices and a  $W_5$  is a wheel of order 5, which can be pictured as a wheel having four spokes or as a cycle of length 4 having all four vertices adjacent to a central vertex. In this thesis we show  $R(W_5, K_5)$  to be equal to 27, utilizing a combinatorial approach with significant computations. Specifically we use a technique developed by McKay and Radziszowski to effectively glue together smaller graphs in an effort to prove exhaustively that no graph having 27 vertices exists that does not contain an independent set on five vertices or a subgraph isomorphic to  $W_5$ . The previous best bounds for this case were  $27 \leq R(W_5, K_5) \leq 29$ .

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# 1 Background

In a mathematical context disorder is really only a matter of scale. It is Ramsey theory that makes a science of studying the properties of order, which must be present to some extent within any combinatorial environment. Ramsey numbers are an important division of Ramsey theory that relate well back to real world scenarios. For instances, if we were to supply a collection of dots and crayons to a child with the one stipulation that each dot must be connected to each other dot with only one line, then we could utilize many of the results within Ramsey numbers to describe the impending ad hoc graphs produced by the child. We can generalize the above example into graph theory by letting the dots represent a collection of vertices, the number of crayons the number of colors within the coloring, and the lines themselves as the edges. In more precise terminology, let  $F$  be the graph produced,  $c$  the number of crayons, and  $n$  the number of dots. Then the Ramsey number  $R_c(G)$  is the smallest number of dots  $n$ , such that any graph  $F$  produced by the child must contain the subgraph  $G$  in some color within the colors,  $c$ . In this thesis we examine solely the case of two-color Ramsey numbers and as such we omit  $c$  henceforth.

There are several other ways to phrase the question of Ramsey numbers, especially in the classical cases, which look only at complete subgraphs. For instance, the Ramsey number  $R(s, t)$  represents the number of people that are required to attend a party so that there is guaranteed either a group of  $s$  who all know each other or a group of  $t$  people who all do not know each other. In this example the act of knowing is assumed to be symmetric but not transitive. Ramsey numbers can also be phrased as a question involving a two-coloring of the edges of a complete  $K_n$  on  $n$  vertices, whereby guaranteeing a monochromatic  $K_s$  subgraph in one color or a monochromatic  $K_t$  subgraph in the other color. In the interest of clarity we settle on the following definition, which deals better with the subgraphs examined in this thesis. Let the Ramsey number  $R(G, H)$  be the smallest integer  $n$  such that for any graph  $F$  on  $n$  vertices, either  $F$  contains a subgraph isomorphic to  $G$  or the complement of  $F$ ,  $\overline{F}$ , contains a subgraph isomorphic to  $H$ .

In this thesis we begin with an overview of the history regarding Ramsey

numbers and to some extent Ramsey theory. Primarily we hope to provide background into the central themes within Ramsey theory and the techniques utilized in determining Ramsey numbers. We conclude with an in-depth discussion of the techniques and results obtained in this thesis for the specific case of  $R(W_5, K_5)$ . Both graphs  $W_5$  and  $K_5$  are illustrated below in Figure I. More specifically, we will show that  $R(W_5, K_5) = 27$ , which we arrive at with a combinatorial approach involving significant computational effort.

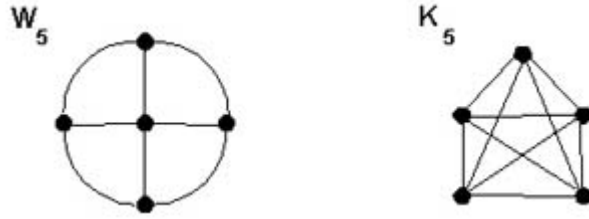


Figure I Graphs  $W_5$  and  $K_5$ .

We only consider graphs without loops and multiple edges, so in other words, we view graphs as a collection of vertices with at most one edge between any two vertices. The number of vertices within a given graph is referred to as the order of that graph. We define the complement of a graph  $G$ ,  $\overline{G}$ , as the swapping of all edges and non-edges within  $G$ . Or said differently, if  $x, y$  are vertices in  $G$ , which are adjacent then  $x, y$  are vertices in  $\overline{G}$  that are nonadjacent. Additionally, let us introduce some essential mathematical notation, which is utilized throughout this thesis.

$|\mathbf{X}|$  represents the cardinality of the set  $\mathbf{X}$ .

$[\mathbf{n}] = \{1, \dots, n\}$  defined for  $n \in \mathbf{N}$ , where  $\mathbf{N}$  represents all the positive integers.

$[\mathbf{x}]^k = \{y : y \subseteq \mathbf{x}, |y| = k\}$ .

## 1.1 Introduction

The underlying principle behind Ramsey theory is accurately captured with the statement that “complete disorder is impossible.” One particularly in-

interesting observation relating to the principles of Ramsey theory was noted by Carl Sagan and later described by Ronald Graham within Paul Hoffman's book "The Man Who Loved Only Numbers." It deals with the TV series *Cosmos*, in which Carl Sagan appealed to Ramsey theory without knowing it.

"Sagan said people often look up and see, say, eight stars that are almost in a straight line. Since the stars lined up, the temptation is to think they were artificially put there, as beacons for an interstellar trade route perhaps. Well Sagan said, if you look at a large enough group of stars, you can see almost anything you want. " [Hof98]

Ramsey theory then deals with the question of how small the universe may be while still guarantying the presence of a given mathematical object.

In more specific terms Ramsey theory can be described as the study of structure under finite decomposition [GR87]. The field has developed into an established branch within combinatorics in a rather short period of time. This is in part due to the relative ease by which many of its problems can be stated and understood. These same questions, while formulated with relative ease, generally prove to be a formidable challenge to solve. As stated by Frank Harary, "[within Ramsey theory] unsolved problems abound ... and additional interesting open questions arise faster than solutions to the existing problems." [Har83]

## 1.2 Origins of Ramsey Theory

The origins of Ramsey theory were as scattered as they were unforeseen. The first true theorem of what would later be called Ramsey theory was proven by Schur in 1916. Schur showed that:

**Theorem 1 (Schur)** *For all  $r$  there exists  $n$  so that given an arbitrary  $r$ -colorization of the elements of a set  $\{1, \dots, n\}$  there exists  $x, y, z$  all the same color, satisfying  $x + y = z$  [Spe83].*

In his original paper Schur, who was motivated by Fermat's last theorem, actually showed that

*For all  $m$ , if  $p$  is prime and sufficiently large the equation*

$$x^m + y^m = z^m$$

*has a nonzero solution in the integers modulo  $p$  [GRS90].*

In the 1920s, Schur developed an interesting conjecture that led to the next major theorem of Ramsey theory. Schur conjectured that:

If the positive integers are divided into two classes, at least one of the classes will contain an arithmetic progression of  $k$  terms, no matter how large the given length  $k$  is [Spe83].

The backdrop for the solution to this conjecture took place over a lunch at the Mathematics Department at University of Hamburg, in 1926 where B. L. van der Waerden supplied the proof. Moreover he showed that:

**Theorem 2 (van der Waerden)** *For all  $r, n$  there exists  $W(r, n)$  such that if the integers from 1 to  $W(r, n)$  are  $r$ -colored there exists a monochromatic arithmetic progression of length  $n$ .*

Graham and Rödl refer to the van der Waerden's theorem in their 1987 survey on Ramsey numbers, "[as] a cornerstone in the edifice of Ramsey theory ." [GR87]

In 1963 A. W. Hales and R. I. Jewett introduced a new theorem whose proof was based on the proof of van der Waerden's theorem and could, although without justice to chronology, cause van der Waerden's theorem to be viewed as a simple corollary. Let  $A$  be a finite set and define a line  $L$  in  $A^n$  to be a set of  $|A|$  points which may be ordered  $\alpha_1, \dots, \alpha_{|A|}$  so that each coordinate of  $L$  is either constant or goes through the elements of  $A$  in order. The Hales–Jewett theorem states that:

**Theorem 3 (Hales-Jewett)** *For all finite sets  $A$  and integers  $r$  there exists  $n$  so that if  $A^n$  is  $r$ -colored arbitrarily there exists a monochromatic line.*



By generalizing van der Waerden’s theorem to points in space, the Hales–Jewett theorem helped introduce a geometric and game-theoretic aspect to Ramsey theory. As stated by Graham, Rothschild and Spencer in [GRS90] that

“in its essence, van der Waerden’s theorem should be regarded, not as a result dealing with integers, but rather as a theorem about finite sequences formed from finite sets. The Hales-Jewett theorem strips van der Waerden’s Theorem of its unessential elements and reveals the heart of Ramsey theory. It provides a focal point from which many results can be derived and acts as a cornerstone for much of the more advanced work. Without this result, Ramsey theory would more properly be called Ramseyian Theorems.”

The proof of van der Waerden’s Theorem made a strong impression on a young mathematician, Richard Rado, who upon hearing a lecture on the theorem set off to shatter its truth. Later recanting, “... on studying the matter more closely I had to admit that the theorem was true and the proof sound. This gave me my start in mathematics... [Spe83].” A short time later Rado become a pupil of Schur and in his dissertation established another central theorem within Ramsey theory. Rado showed in the simplest case that:

**Theorem 4 (Rado)** *A single homogeneous equation  $c_1x_1 + \dots + c_nx_n = 0$  is regular if and only if some nonempty subset of the  $c_i$ s sum to zero.*

A system of equations on variables  $x_1, \dots, x_n$  is said to be regular if for every finite coloration of  $N$  there must exist a monochromatic solution to the system [Spe83].

### 1.3 Frank P. Ramsey (1903-1930)

The eponymous discoverer of Ramsey theory, Frank P. Ramsey did not live to see and could not possibly have foreseen the development of Ramsey theory as a branch of combinatorics. His untimely death at the age of 26 cut short

what would have certainly been an exceptionally brilliant career. Fittingly though, as remarked by Graham, Rothschild, Spencer, "... it seems eminently suitable that this branch of combinatorial analysis be graced with the name of Frank Plumpton Ramsey." [GRS90] Ramsey was from a distinguished Cambridge family and his Father, A. S. Ramsey was a mathematician who served as President of Magdalene College. Ramsey's younger brother took a different path becoming the Archbishop of Canterbury. Frank, however, was an atheist who as poet I. A. Richards later regarded in a radio program about Ramsey,

"He never was a showman at all, not the faintest trace of trying to make a figure of himself. Very modest, gentle, and on the whole he refrained almost entirely from argumentative controversy ... He felt too clear in his own mind, I think to want to refute other people. " [Mel83]

Ramsey graduated as Cambridge's top math student in 1923 where he was a pupil of Bertrand Russell, G. E. Moore, Ludwig Wittgenstein and John Maynard Keynes. As with most that knew him, Moore was particularly impressed with Ramsey, so much so that Moore wrote:

"[Ramsey] combined very exceptional brilliance with very great soundness of judgement in philosophy. He was an extraordinarily clear thinker: no one could avoid more easily than he the sort of confusions of thought to which even the best philosophers are liable, and he was capable of apprehending clearly, and observing consistently, the subtlest distinctions ... I always felt with regard to any subject which we discussed, that he understood it much better than I did, and where (as was often the case) he failed to convince me, I generally thought the probability was that he was right and I wrong and that my failure to agree with him was due to a lack of mental powers on my part. " [GRS90]

Two years after graduating he wrote "The foundations of mathematics" defending Russell's *Principia Mathematica*. At the age of 21, in 1924, he re-

ceived a fellowship at King's College in Cambridge. By 1926 he was a University lecturer in mathematics, a post he remained at until his death in 1930 from complications with abdominal surgery 3 days before his 27th birthday. While Ramsey was a mathematician by study, he tended to practice toward philosophy and write papers dealing with economics and logic. Ramsey wrote only two mathematical economics papers both of which are still often quoted today. Of his paper "A mathematical theory of savings," Keynes said, "[it was] one of the most remarkable contributions to mathematical economics ever made ." [Mel83] As with his famed theorem which bears his name, these works were similarly not widely acclaimed until decades after their publication. D. H. Mellor attributes this to the quality of his work, remarking in [Mel83] that his work

"... was generally highly original and thus hard to appreciate. But the very simplicity and clarity of Ramsey's prose tends to conceal the depth and precision of his thought. Since his writing is so free of jargon, so unacademically light and easy, that one can readily underrate it—until one tries to think through the matter oneself."

Frank P. Ramsey proved his famous theorem in the first eight pages of a 20-page paper, "On a Problem Formal Logic," that appeared in *Proceedings of the London Mathematical Society* in 1930. Ramsey began the proof of his theorem with the infinite version. The finite theorem reads that:

**Theorem 5 (Ramsey)** *For all  $l, r, k$  there exists  $n_0$  so that, for  $n \geq n_0$ , if  $[n]^k$  is  $r$ -colored there exists a monochromatic  $[l]^k$ .*

The original proof is not included here but remains an essential reference within the field of Ramsey theory. Using his theorem, Ramsey goes on to find a decision procedure for a specific class of statements in First Order Logic within his paper. Curiously, it turns out the theorem is much more powerful than Ramsey needed and a much simpler approach can be utilized. The genesis of Ramsey theory was entirely an unexpected consequence of his theorem, with Ramsey primarily concerned with its applications in logic.

In fact, in his famous paper Ramsey went only as far as to acknowledge that his new theorem had independent interest [GRS90]. This is perhaps a commentary on the diminutive stature of combinatorial analysis within the discipline of mathematics in the early part of the 20<sup>th</sup> century.

There are some interesting and ironical circumstances behind the motivations of Ramsey's famous theorem. First, Ramsey had very little interest in combinatorial mathematics, as was evident from his lectures in the Cambridge Mathematics Department, which were almost entirely on the foundations of mathematics, careful to avoid actual mathematics [Mel83]. In fact his famous theorem compromises the entire eight pages of actual mathematics Ramsey ever wrote. Furthermore, the primary motivation for the writing of his paper was an effort to find a general decision procedure for a statements in first order logic [Spe83]. This is not entirely surprising since much of Ramsey's work took place in the late 1920s when there was significant research into this question, which was dubbed the Hilbert program after the famed German mathematician David Hilbert who first posed it in the early 1920s. Interestingly, notice that the question itself assumes that such a procedure exists and that it was only a matter of discovering the method. As such the very question proved over optimistic when in 1931 Kurt Gödel showed that Hilbert's program can never be developed. Thus, it turns out the paper that served to ignite Ramsey theory was an early unsuccessful attempt at trying to solve a problem that Gödel showed one year after Ramsey's death to be unattainable. Hence, Ramsey produced his theorem, which he didn't need, to solve a problem, in the hope of providing a solution to a question that cannot be solved.

## 1.4 Ramsey Theory Rediscovered

Ramsey's theorem was not immediately seized upon until its unanticipated rediscovery in a 1935 paper of Paul Erdős and George Szekeres. At the time Erdős and Szekeres were mathematical students in Budapest, Hungary. It so happened that an associate Esther Klein, later to become Esther Szekeres, made a curious observation about points in a plane. Klein noticed that given any five points in the plane, some four form a convex quadrilateral. Soon the

trio made a general conjecture stating that:

For any  $n$  there exists an  $N$  so that given  $N$  points in the plane some  $n$  form a convex  $n$ -gon.

After Szekeres had given his initial proof, Erdős found an alternative proof utilizing a result now known as the Monotone Subsequence Theorem. It states:

**Theorem 6 (Monotone Subsequence)** *Given a sequence  $l$  of length  $n^2 + 1$  there must exist a monotone subsequence of length  $n + 1$  within  $l$  [ES35].*

This theorem can be quickly derived from Ramsey’s original theorem, although both Erdős and Szekeres were unaware of it at the time. Szekeres later recalled that this paper had a strong influence on the mathematical development of both Szekeres and Erdős, providing Erdős with the initial insight into the vast possibilities open in the new world of combinatorial set theory and combinatorial geometry [Spe83]. Erdős went on to enjoy a brilliant mathematical career, becoming arguably the most notable mathematician of the twentieth century. One of the many contributions he made to the discipline of mathematics was the popularization of Ramsey theory. In fact, in the book “Ramsey theory,” the authors Ronald Graham, Bruce Rothschild and Joel Spencer credit Erdős as the father of modern Ramsey theory.

Erdős in his mathematical career wrote almost 1500 papers with more than 460 collaborators [Chu97]. This large number of collaborators gave rise to the so called Erdős number, which is the number of papers an author is distanced from a collaboration with Erdős. While Erdős numbers make for an interesting conversation topic among mathematicians, the true legacy of Erdős will forever be captured in the many conjectures and beautiful theorems he left behind. As Chung remarks in [Chu97],

“The main treasure that Paul Erdős has left us is his collection of problems, most of which are still open today. These problems are seeds that Paul showed and watered by giving numerous talks at meetings big and small, near and far. In the past, his problems

have spawned many areas in graph theory and beyond ... Solutions or partial solutions to Erdős problems usually lead to further questions, often in new directions ... Through the problems, the legacy of Paul Erdős continues ....”

Erdős often offered monetary awards for solutions to some of his favorite problems. After his death in 1996 from a heart condition, Chung with help from Ronald Graham, promised to honor any future claims on his problems.

## 2 Ramsey Numbers

Ramsey numbers are extremely helpful research tools within Ramsey theory since they serve to quantify the many existing theorems of Ramsey theory, which can at its extremes become incredibly abstract and in other cases require “necessarily large” graphs. This is evident in the many theorems and general equations within Ramsey theory that either deal with the infinite case or substantially large orders of graphs. Faudree, Rousseau and Schelp remark in [FRS85] on an Erdős and Ulam comment capturing this property of Ramsey theory, which states “the infinite we do immediately, the finite takes a little longer.” This is a modified slogan of the U.S. Army Service Forces in World War II that fits nicely with Ramsey theory.

Recall the pigeon hole principle, which states that if  $m$  pigeons roost in  $n$  holes and  $m > n$  then at least two pigeons must be in the same hole. While the pigeon hole principle is self-evident it serves to adequately set the context for the underlying principle behind Ramsey numbers. Consider that if  $m$  edges are colored with  $n$  colors and  $m > n$  then at least two edges must have the same color. Graph Ramsey numbers then describe the properties of graphs under such conditions. Once again, let the Ramsey number  $R(G, H)$  be the smallest integer  $n$ , such that any graph  $F$  having  $n$  or more vertices must contain a subgraph isomorphic to  $G$ , or the complement  $\overline{F}$  must contain a subgraph isomorphic to  $H$ .

## 2.1 Notation

For convenience we shorten  $R(G, G)$  to just  $R(G)$ ; these numbers are generally referred to as the diagonal numbers. Complete subgraphs are those which contain all possible edges and independent sets are those which have no edges. These graphs are described with the notation  $K_n$  and  $\overline{K}_n$  respectively, where  $n$  is the order of the graph. Also typically the classical cases are referenced as just the size of the subgraphs being examined, hence  $R(4, 3)$  is shorthand for  $R(K_4, K_3)$ . In this thesis we let

$$(G, H) - \text{graphs}$$

denote a family of graphs,  $F$ , not containing a subgraph isomorphic to  $G$  nor a subgraph isomorphic to  $H$  in its complement,  $\overline{F}$ . Next take

$$(G, H; n) - \text{graphs}$$

to be  $(G, H)$ -graphs on  $n$  vertices. Moreover we write

$$(G, H; n; e) - \text{graphs}$$

for  $(G, H; n)$ -graphs having  $e$  edges. Finally the family of  $(G, H; n)$ -graphs having  $n = R(G, H) - 1$  are called critical graphs for the case  $R(G, H)$ .

In addition, the following are a few important properties of Ramsey numbers. We have  $R(2, n) = n$  since any edge yields a  $K_2$ , and with  $n$  vertices if we have no edges then we have a  $\overline{K}_n$ . Hence,  $R(2, n) = n$ . As expected, symmetry holds for all cases

$$R(G, H) = R(H, G).$$

Next let us introduce a concise definition of what it means to be a subgraph. We say a graph  $G$  is a subset or subgraph of a graph  $H$ , which we denote  $G \subset H$ , if every vertex and edge within  $G$  is similarly a vertex and edge within  $H$ . Presented below is a very intuitive property within Ramsey numbers utilizing this concept of subgraphs that proves extremely useful in relating the difficulty of determining values.

$$G' \subset G \text{ and } H' \subset H \Rightarrow R(G', H') \leq R(G, H). \quad (1)$$

Consider the following example illustrating the above relation 1.

$$R(4, 3) = 9 \Rightarrow R(3, 3) \leq 9$$

Suppose we have a graph  $G$ , having 9 vertices or more. It must then contain either a subgraph  $K_4$  or  $\overline{K_3}$  by definition of  $R(4, 3) = 9$ . Clearly  $K_3 \subset K_4$  and as such we have  $R(3, 3) \leq 9$  as desired.

Given the above relation it follows immediately that,

$$R(s, t) \leq \max(R(s), R(t)). \quad (2)$$

There are of course many other graphs of interest for study besides the complete graphs. Some of these graphs are listed below. Note that in a graph context the ‘+’ denotes a concatenation of the two graphs whereby each vertex of each separate graph is connected by an edge to form a new graph. For example,  $K_1 + K_1$  yields a  $K_2$  graph.

$\mathbf{P}_k$ , a simple path on  $k$  vertices

$\mathbf{C}_k$ , a simple cycle on  $k$  vertices

$\mathbf{W}_k = K_1 + C_{k-1}$  is a  $(k - 1)$ -spoked wheel

$\mathbf{B}_k = K_2 + \overline{K_k}$  is the  $k$ -page book of order  $k + 2$ , which can be seen as a graph formed by  $k$  triangular pages sharing one common edge.

$\mathbf{K}_n - \mathbf{e}$  is used to depict a complete graph of order  $n$  missing one edge. So  $K_3 - e$  is a triangle missing an edge.

Observe that there is significant overlap in the subgraph notation with

$$B_1 = C_3 = W_3 = K_3, B_2 = K_4 - e,$$

$$P_3 = K_3 - e \text{ and } W_4 = K_4.$$



## 2.2 General Bounds

There has been considerable research into determining both general equations and general bounds describing many Ramsey number cases. While this research has produced some very interesting results describing cases involving the simpler subgraphs such as paths, trees, and cycles it has unfortunately not met with similar success in the classical cases and more difficult subgraphs. The practical bounds on Ramsey cases are established by some observations regarding the subcases of a given Ramsey number.

A simple upper bound results from the relation

$$R(k, l) \leq R(k, l-1) + R(k-1, l). \quad (3)$$

We give the proof for this relation later in Section 4. In addition, if both  $R(k, l-1)$ ,  $R(k-1, l)$  are even then this relation becomes a strict inequality. Next, by taking a disjoint union of two critical graphs it can be shown that

$$R(k, p) \geq s \text{ and } R(k, q) \geq t \Rightarrow R(k, p+q-1) \geq s+t-1. \quad (4)$$

This has been improved to yield better lower bounds

$$R(k, p+q-1) \geq s+t+k-3 \text{ [Rad02]}. \quad (5)$$

The asymptotic behavior of Ramsey numbers has been the focus of considerable work. In his initial paper Ramsey gave an upper bound of

$$R(n) \leq 2^{n(n-1)/2} \quad (6)$$

and then improved it to

$$R(n) \leq n!. \quad (7)$$

He goes on to say that this value is still much too high [GRS90]. In their 1935 paper Erdős and Szekeres proved the upper bound of

$$R(k, l) \leq \binom{k+l-2}{k-1}. \quad (8)$$

that follows immediately from the recurrence (1). When dealing with diagonal cases where  $k = l$  we get

$$R(k) \leq \binom{2k-1}{k-1}. \quad (9)$$

It would take over 50 years before Rödl [GR87] would provide the first significant improvement on this bound, proving that for suitable positive constants  $c_1$  and  $c_2$

$$R(k, k) \leq c_1 \binom{2k-1}{k-1} / (\log k)^{c_2}. \quad (10)$$

The first interesting lower bound was discovered by Erdős [Erd47] in 1947 who used a probabilistic method to determine

$$R(k) \geq (1 + o(1)) \left( \frac{1}{e * \sqrt{2}} \right) k * 2^{\frac{k}{2}}. \quad (11)$$

Constructing lower bounds with the use of a probabilistic method does not establish an explicit graph. Instead it only provides a proof that such a graph exists. Surprisingly, or rather depressingly, since its discovery this relationship has only been improved by a factor of 2. The current best lower bound was proved by Thomason in 1988 to be

$$R(k) \geq (1 + o(1)) \left( \frac{\sqrt{2}}{e} \right) k * 2^{\frac{k}{2}} \text{ [Tho88]}. \quad (12)$$

## 2.3 Classical Cases

It has only been in the last 20 years or so that there has been any significant effort put into determining the exact values for Ramsey numbers. Previously, the main focus was paid to the asymptotic behavior of Ramsey numbers or diverted toward other parts of Ramsey theory altogether. As described by Graham and Rödl in their 1987 survey [GR87],

“Most of the recent work has focused on far-ranging generalizations of the original concepts, dealing for example, with extensions to  $n$ -dimensional vector spaces, lattices, groups, various transfinite sets, induced and restricted variations ...”

This is somewhat justifiable given the difficulty involved in determining actual values, especially in the classical cases. Because of this the progress on determining actual values had been painfully slow until the recent advent of computer algorithms. Though computers are an incredible tool for combinatorial research, given the difficulty of determining exact Ramsey numbers the technology is not at present sufficiently powerful to determine exact values through any naive approaches. As stated by Faudree, Rousseau and Schelp in [FRS85], “Success in this area (Ramsey numbers) will doubtless require the best mixture of combinatorial technique and computing power. There are not likely to be many shortcuts.”

The  $R(3) = 6$  case is one of the last cases that can be solved by hand with a brief argument. The following constitutes a proof for this case.

**Theorem 7**  $R(3, 3) = 6$

We will show utilizing a brief argument that  $R(3, 3) = 6$  by proving that  $R(3) \geq 6$  and  $R(3) \leq 6$ , respectively. We phrase our proof of the lower bound using people and the act of knowing (which we assume to be a symmetric relationship) for clarity.

$R(3) \geq 6$ : Notice that a  $C_5$  graph has each vertex adjacent to two vertices and independent of two vertices. Hence, it is a  $(3, 3; 5)$ -graph and we therefore have  $R(3) \geq 6$ .

$R(3) \leq 6$ : Assume we have six people: Mary, John, Todd, Jasper, Jeff, and Ryan. Let us fix Mary, notice that Mary must either know 3 or not know 3 of the remaining people. Since the converse is an analogous argument we will assume that she knows Todd, Jeff, and Jasper. Then observe that if any two of Todd, Jeff, or Jasper know each other then we have three people that all mutually know each other. If instead Todd, Jeff, and Jasper did not know each other then we have a group of three people all who do not know each other. Thus, in either case we have a

group of three that all mutually know each other or mutually do not know each other. So, we shown that  $R(3) \leq 6$  as desired.

Thus, we have shown that  $R(3) \geq 6$  and  $R(3) \leq 6$  and therefore have proven that  $R(3) = 6$ , as desired.

This specific case gained temporary fame in 1953 when it was included in the Putnam Examination, on the suggestion of Frank Harary [Har83]. Goodman actually later showed in 1959 that there is guaranteed to be not just one grouping of three in a set of six but in fact two.

To provide an idea of how the problems tend to scale in difficulty, consider that while lower and upper bounds (or perhaps more accurately rough estimates) are known for all diagonal values  $R(n)$ , the last exact value known is  $R(4) = 18$ . This result was produced in 1955 by Greenwood and Gleason who proved that  $R(4) = 18$  [GG55]. Notice that the basic upper bound inequality, (3) is enough to establish an upper bound of 18. Greenwood and Gleason then constructed a graph on 17 vertices that did not contain either  $K_4$  or  $\overline{K_4}$ . To further illustrate the difficulty posed by the classical cases consider that since 1955 when Greenwood and Gleason solved the cases  $R(3, 4) = 9$ ,  $R(3, 5) = 14$ , and  $R(4) = 18$  only 5 other exact values for the classical cases have been determined. The current status of the field becomes even more discouraging when one considers that with the exception of the  $R(4, 5)$  case all other determined cases included the not-so-difficult triangle subgraph  $K_3$ . To paraphrase Faudree, Rousseau and Schelp, in [FRS85] the lack of known values is both a testament to the substantial difficulty posed by the problems and further evidence of the lack of effectiveness in both our knowledge and technique.

Table *I* below illustrates all current non-trivial results known for the classical cases, as catalogued by [Rad02]. Each square in the chart below contains, if one exists, the lower and upper non-trivial bounds for that particular case. Also only the top right triangle of the chart is filled in since symmetry holds for Ramsey numbers.

s / t	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40 43
4		18	25	35 41	49 61	56 84	69 115	80 149
5			43 49	58 87	80 143	95 216	121 316	80 149
6				102 165	111 298	127 495	153 780	177 1171
7					205 540	216 1031	1713	2826
8						282 1870	3583	316 6090
9							565 6588	580 12677
10								798 23556

Table I Known nontrivial values and bounds for classical two color [Rad02]

Currently the case of  $R(5, 5)$  has the bounds of

$$43 \leq R(5, 5) \leq 49$$

which have been known for almost a decade now. Exoo improved the lower bound to 43 in 1989 [Exo89b] and McKay and Radziszowski improved the upper bound to 49 in 1995. Before Exoo improved the lower bound in 1989, the bounds had been stagnant at

$$42 \leq R(5) \leq 55$$

for over 15 years. In [MR97] Radziszowski and McKay along with proving the newest best upper bound of 49, go even farther and give compelling evidence to support a conjecture that  $R(5, 5) = 43$ . They also make the claim that there are only 656  $(5, 5; 42)$  critical graphs for this case. Current computer technology is insufficiently fast to verify this conjecture. In fact, any further progress on this case already seems to be out of reach of current techniques. This follows from the enormous number of graphs on 43 vertices,

which number more than  $10^{200}$ , a number greater than the number of atoms in the universe. As expected the gap between bounds continues to increase the higher the case examined, with the case  $R(6, 6)$  having the current bounds of 102 and 165. Paul Erdős often told a story illustrating the difficulties of the classical cases.

“Imagine that an evil spirit can ask of you anything it wants and if you answer incorrectly, it will destroy humanity. ‘Suppose it decides to ask you the Ramsey party problem for the case of a five-some. Your best tactic, I think, is to get all the mathematicians in the world to drop what they’re doing and work on the problem, the brute-force approach of trying all the specific cases... But if the spirit asks about a sixsome, your best survival strategy would be to attack the spirit before it attacks you [Hof98].’ ”

The last significant classical value determined was  $R(4, 5) = 25$  in 1995 by McKay and Radziszowski [MR95] who utilized a clever computing algorithm to significantly reduce the search space. Next they made use of a network of computers to attack the problem with almost a decade of CPU time. When the result was finally determined it was impressive enough to make it into New York Times Science section. Their technique served as the guide for the specific work of this thesis in determining  $R(W_5, K_5) = 27$  and as such is described in considerable detail in Section 4.

## 2.4 Complexity of Ramsey Problems

We presented above an anecdotal and intuitive discussion regarding the difficulty of determining Ramsey numbers, but naturally no discussion of the difficulty of any mathematical problem is complete without a look into the computational complexity of the problem. In [Bur84] Burr showed that the problem of determining for an integer  $m$  and for arbitrary graphs  $G$  and  $H$ , whether  $R(G, H) < m$ , is an NP-hard problem. This is not a surprise when one considers that determining whether any arbitrary graph contains a clique of order  $k$  is by itself NP-complete. Thus, the naive approach to determining

$R(G, H)$  will require solving an exponential number of NP-complete problems. Burr goes on to state in [Bur87] that it is not likely that the problem of determining arbitrary Ramsey numbers even belongs to NP. Strong evidence for this conjecture is apparent when you consider that the problem exhibits alternating quantifiers, which is a property that often belongs to severely difficult problems.

Is it true that **for all** possible two-coloring of  $K_n$  **there exists** an embedding of  $G$  and  $H$  in  $K_n$ , that **for all** pairs of edges they have the same color.

A more in-depth analysis of the problem reveals even another level of difficulty associated with the problem. Recall the definition of an NP-easy problem, which is a problem that can be solved in polynomial time given the assistance of a constant time subroutine that can solve a fixed NP-complete problem. Unfortunately, it is shown by Burr that general Ramsey numbers are not NP-easy and moreover in the worst case require double exponential time in the order of the graphs of both  $G$  and  $H$  [Bur87]. Building on the works of Burr Schafer [Sch99] determined the computational complexity of a number of specific classes of Ramsey problems. Of particular interest to this paper he showed that the case of determining whether a given graph  $F$  contains a  $W_5$  subgraph or independent set of 5 vertices is in fact  $\Pi_2^P$ -complete ( $\Pi_2^P = coNP^{NP}$ ). This particular case is then a rare natural problem that is complete for the second level of polynomial hierarchy. It is important to note however that the above complexity problem applies only to an instance of determining whether a graph  $F$  is in fact a  $(W_5, K_5)$ -graph.

### 3 Generalized Ramsey Numbers

Generalized Ramsey theory does not restrict the graphs in question to the classical case of complete graphs. Not until 1973 did it become recognized as an area for systematic research. In 1973, a combinatorial conference was held at Balatonfüred, Hungary in honor of Paul Erdős. At this conference there were more than two dozen talks devoted to Ramsey theory, including

speakers Ronald Graham, Richard Rado, and Paul Erdős [Spe83]. It is at this conference that a Czechoslovakian mathematician, Jarik Nešetřil, and his student Vojtěch Rödl, were shown a conjecture of Erdős. Erdős had wondered what graphs, other than  $K_6$ , had the property that if their edges were two colored arbitrarily there would necessarily exist a monochromatic triangle. A previous solution existed for the specific case of a two-coloring by Folkman. Surprisingly though this proof did not extend to cases involving more than two colors. Nešetřil and Rödl, while at the conference, produced a proof that made extensive use of Ramsey's theorem [Spe83]. Given the overwhelming difficulties presented by the classical cases it was hoped that perhaps further research into other subgraphs would provide insight into the classical cases.

Thus, the primary motivation for the interest in generalized Ramsey theory for graphs was the difficulties that mathematicians had run into with classical Ramsey theory. Unfortunately though, the research in Generalized Ramsey theory has provided little insight into classical Ramsey theory. Regardless, the efforts have produced a number of interesting results that have been more widespread than initially anticipated. As described by Burr in [Bur87],

“rather than (generalized Ramsey theory for graphs) becoming a collection of intriguing but relatively isolated results, the subject has displayed a remarkable degree of coherence, both in the results obtained and in the methods used to obtain them. A major cause ... of this coherence has been the existence of guiding principles and conjectures that give direction to the subject.”

There has been significant progress in the field of Generalized Ramsey theory since the early 1970s with interesting asymptotic formulas developed as well as insight into various properties influencing Ramsey numbers. Some equations and inequalities have even been developed that provide values for some of the simpler subgraphs such as paths, cycles, and trees. Many of these developments are based on observations regarding properties of the density of graphs. Of primary usefulness being the three measures of sparseness: that of maximum degree, maximum value of the minimum degree throughout all



subgraphs, and the ratio of number of edges to the number of vertices [Bur87]. The results are not included or discussed here given their lack of relevance for our special case of  $R(W_5, K_5)$ .

### 3.1 Results Pertaining to $R(W_5, K_5)$

In 1973 Chvátal and Harary [CH72a] determined  $R(G)$  values for all 10 graphs  $G$  with at most 4 vertices, having no isolated vertices. These graphs are  $K_2$ ,  $P_3$ ,  $K_3$ ,  $2K_2$ ,  $P_4$ ,  $K_{1,3}$ ,  $C_4$ ,  $K_{1,3} + e$ ,  $K_4 - e$  and  $K_4$ . In 1977, extending on the work of Harary and Chvátal, Clancy published in [Cla77] a table of values of  $R(G, H)$  for all but five pairs of graphs  $G$  with at most four vertices, and  $H$  having exactly five vertices. Interestingly, Clancy did not publish the proofs for this table since as he put it,

“...the amount of work involved in writing down the complete proofs seems to be so horrendous that we have decided—at least for the moment—to present just the table summarizing our findings.”

The missing values for the table were the cases of

$$\begin{array}{l} R(K_4 - e, K_5) \quad R(K_4, B_3) \\ R(K_4, W_5) \quad R(K_4, K_5 - e) \\ R(K_4, K_5). \end{array}$$

Later Exoo, Harborth, and Mengersen [EHM88] used both detailed arguments regarding subgraph decomposition, and constructions to show that  $R(K_4, K_5 - e) = 19$ . Bolze and Harborth in 1981 determined  $R(K_4 - e, K_5) = 16$ , and later Hendry showed that  $R(K_4, B_3) = 14$  and  $R(K_4, W_5) = 17$  in 1989. This left only the  $R(K_4, K_5)$  case, which as previously noted was solved in 1995 by Radziszowski and McKay. Faudree, Rousseau and Schelp computed  $R(K_3, H)$  for all connected graphs  $H$  of order six, in 1980. Then in 1983 Burr tabulated all diagonal Ramsey numbers for isolate-free graphs with at most six edges. Next Hendry building on the earlier works published in 1987 a table of values of all diagonal Ramsey numbers for isolate free graphs  $G$  having at most six edges [Hen87]. Most new results in Hendry’s paper were

proved with long arguments referencing a number of lemmas and constructions. Later in 1989, Hendry continued on his earlier work and published a table of values for all graphs having at most five vertices. He presented exact values for all but seven cases,

$$\begin{array}{lll} R(W_5, K_5 - e) & R(B_3, K_5) & R(K_4, K_5) \\ R(K_5 - P_3, K_5) & R(W_5, K_5) & R(K_5 - e, K_5) \\ R(K_5, K_5). \end{array}$$

Hendry chose not to publish his proofs given their length and unfortunately they are no longer available. Yuansheng and Hendry showed that  $R(W_5, K_5 - e) = 17$  in 1995, and Babak, Tse, and Radziszowski proved  $R(B_3, K_5) = 20$  [BRT02] just last year. Babak, Tse, and Radziszowski utilized a similar approach as was used to solve the  $R(K_4, K_5)$  case and as such it provides a nice overlap with this thesis. In fact, in [BRT02], they remark that “ $R(W_5, K_5)$  could be attacked with a method similar utilizing more computational effort.” So with  $R(K_4, K_5)$  also having been solved this left only four open cases from Hendry’s original seven:

$$\begin{array}{ll} 25 \leq R(K_5 - P_3, K_5) \leq 28, & 27 \leq R(W_5, K_5) \leq 29 \\ 30 \leq R(K_5 - e, K_5) \leq 34, & 43 \leq R(K_5, K_5) \leq 49. \end{array}$$

This thesis effectively eliminates the  $R(W_5, K_5)$  case, leaving only three cases open. Of the remaining values the classical diagonal case  $R(K_5, K_5)$  is the most famous and by far presents the greatest difficulty. Progress on the other two can be expected in the near future. The evaluation of  $R(K_5 - P_3, K_5)$  might require as much effort as the result  $R(K_4, K_5) = 25$ , since  $K_4$  is a subgraph of  $K_5 - P_3$ . Determining the exact value of the case  $R(K_5 - e, K_5)$  appears out of reach with present techniques.

## 3.2 Approaches

To prove a lower bound for a given Ramsey number  $R(G, H)$  case it suffices to construct a  $(G, H; n)$ -graph. Such a construction serves to establish a lower bound of  $n + 1$  for the given case. Working with upper bounds is a great deal more difficult since simple constructions will not suffice. Instead,

it must be shown that there cannot be any possible graph on  $n$  vertices, which doesn't contain either  $G$  or  $H$  in the complement.

This section by no means represents an exhaustive review of the approaches utilized in determining various Ramsey number values since the results and proofs are scattered among several papers over a period of almost 50 years. Most early results on Ramsey numbers were determined entirely with arguments based on observations regarding degree constraints within a given graph combined with manually constructing  $(G, H)$ -graphs.

As computer algorithms began to be employed, more complex constructions were made possible, typically attempting to construct cyclic graphs. Simply put cyclic graphs are graphs in which all vertices have the same degree. The degree of a given vertex  $x$  represents the number of vertices adjacent to  $x$ . For instance take the triangle graph, which is cyclic since every vertex has degree 2. An interesting approach for constructing lower bounds involves the use of congruence modulo a given prime  $p$ . Greenwood and Gleason had utilized cubic residues mod 13 and quadratic residues mod 17 to construct cyclic graphs to determine new lower bounds in the course of proving  $R(3, 5) = 14$  and  $R(4, 4) = 18$ . In 1985 James Shearer [She86] used quadratic residues mod a prime  $p$  to improve lower bounds for some classical cases. Most notably he improved the lower bound on  $R(7)$  from 125 to 205. Shearer made use of a computer algorithm written in Fortran requiring several minutes of CPU time. Exoo [Exo89b] found a  $(5, 5; 42)$  graph effectively raising the current best lower bound for  $R(5, 5)$  by one. Exoo began by utilizing a cyclic coloring to determine a  $(5, 5; 41)$ -graph, where the color of an edge depended upon on the numeric difference between its end-vertices, which were numbered with contiguous integers. Exoo then used this graph as input to a simulated annealing algorithm that produced a  $(5, 5; 42)$ -graph. In 1998 [WQHG02] Su Wenlong, Li Qiao, Luo Haipeng and Li Guiqing improved the lower bounds for several Ramsey numbers by constructing cyclic graphs utilizing cubic residues modulo primes of the form  $p = 6m + 1$ . They then produced 16 new lower bounds as follows:

$$\begin{array}{lll}
R(6, 12) \geq 230 & R(5, 15) \geq 242 & R(6, 14) \geq 284 \\
R(6, 15) \geq 374 & R(6, 16) \geq 434 & R(6, 17) \geq 548 \\
R(6, 18) \geq 614 & R(6, 19) \geq 710 & R(6, 20) \geq 878 \\
R(6, 21) \geq 884 & R(7, 19) \geq 908 & R(6, 22) \geq 1070 \\
R(8, 20) \geq 1094 & R(7, 21) \geq 1214 & R(9, 20) \geq 1304 \\
R(8, 21) \geq 1328 & & 
\end{array}$$

Xie, Xu and Radziszowski [XXR02] describe a method for the constructive approach to determining lower bounds on the Ramsey numbers. This method did not require the use of computer algorithms and proceeded on particular observations regarding subgraph decomposition, proving 22 improved lower bounds. Of particular interest they improved the cases,

$$\begin{array}{lll}
R(4, 15) \geq 153 & R(6, 7) \geq 111 & R(6, 11) \geq 253 \\
R(7, 12) \geq 416 & R(8, 13) \geq 635 & 
\end{array}$$

Exoo [Exo89a] utilized a variety of programming techniques to establish new lower bounds for the  $R(4, 7) \geq 49$  and  $R(4, 8) \geq 53$  cases. Also presented by Exoo is a genetic algorithm that has met with mixed success when used for establishing lower bounds. In [Exo88] another interesting computer algorithm was devised by Exoo that supplied general procedures for the edge coloring of complete graphs. This was in an attempt to improve the lower bounds for some small classical cases. As a result several new  $(4, 5)$ -graphs and  $(3, 8)$ -graphs, now known to be critical graphs were determined.

Since working with upper bounds tends to be considerably more difficult, proofs of upper bounds tend to utilize various combinatorial theorems and arguments regarding subgraphs. In [FRS85] Faudree, Rousseau and Schelp, while working on the  $R(K_5 - e)$  case, made the observation that further progress on determining this number could only be accomplished with the knowledge of all critical graphs for the  $(K_4 - e, K_5 - e)$  case. Clancy had already shown in [3] that  $R(K_4 - e, K_5 - e) = 13$  so they required all  $(K_4 - e, K_5 - e; 12)$ -graphs to gain further insight. In [MR94] McKay and Radziszowski made use of various combinatorial lemmas related to properties of the graphs being considered combined with a series of linear program problems to effectively limit the search space to reasonable level that could be

exhaustively searched by computer. This technique allowed them to produce the new upper bounds for classical cases,

$$R(4, 5) \leq 27 \quad R(5, 5) \leq 52 \quad R(4, 6) \leq 43.$$

Each new number represents a decrease of one from the previous bound. McKay and Radziszowski [MR97] established the newest best upper bound of  $R(5, 5) \leq 49$ , using various subgraph counting techniques.

## 4 Problem Decomposition

The space of possible graphs on a given set of  $n$  vertices is quite large with  $2^{\binom{n}{2}}$  graphs having  $n$  vertices. So on 26 vertices there are  $2^{325}$  graphs and recall that the bounds on this case are currently 27 and 29 respectively. Hence given the large orders of graphs examined in this thesis it was infeasible to utilize any naive approach to produce all  $(W_5, K_5)$  critical graphs. By naive, we refer to any approach that requires generating exhaustively graphs on  $n$  vertices. Because of this we utilize a technique of gluing together smaller graphs to construct larger  $(W_5, K_5)$ -graphs. We discuss this method in detail.

Let the neighborhood of a given vertex  $x$  describe the subset of vertices within a given set vertices  $W$  that  $x$  is adjacent too it. Next consider any  $(W_5, K_5; n)$ -graph  $F$ . It follows from  $R(C_4, K_5) = 14$  and  $R(W_5, K_4) = 17$  that every vertex  $v$  in  $F$  must have fewer than 14 vertices in its neighborhood and fewer than 17 vertices that are not in its neighborhood. If not, there must be either a  $W_5$  in  $F$  or  $K_5$  in the complement of  $F$ . Hence every  $(W_5, K_5; n)$ -graph  $F$  can be viewed as a combination of two graphs  $G$  and  $H$ , which are a  $(C_4, K_5; p)$ -graph and  $(W_5, K_4; q)$ -graph, respectively having  $n = p + q + 1$ .  $G$  represents the neighborhood of some vertex  $x$  in  $F$  and  $H$  all vertices not in the neighborhood of  $x$ . Figure II below illustrates this starting point of the gluing algorithm.

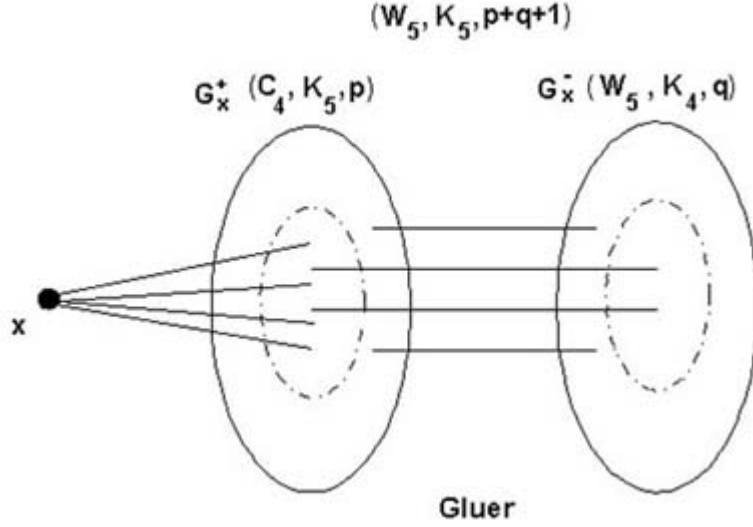


Figure II Basic decomposition behind the gluing algorithm.

Let us introduce some important notation needed in the gluing algorithm we develop in section 4.5. If  $F$  is a graph, let  $VF$  denote the set of vertices in  $F$  and  $EF$  the set of edges in  $F$ . Next, if  $x \in VF$  and  $W \subseteq VF$  then we formalize the neighborhood of  $x$  as follows,

$$N_F(x, W) = \{w \in W \mid xw \in EF\}.$$

The subgraph of  $F$  induced by  $W$  will be denoted by  $F[W]$ . Suppose that  $x$  is a vertex of  $F$ , define the induced subgraphs

$$G_x^+(F) = F[N_F(x, VF)],$$

$$G_x^-(F) = F[VF - N_F(x, VF) - x].$$

Thus given a graph  $F$ ,

**VF** denotes the set of vertices in  $F$ .

**EF** denotes the set of edges in  $F$ .

$\mathbf{N}_F(x, W) = \{w \in W \mid xw \in EF\}$  denotes the formal definition of a neighborhood.

$\mathbf{G}_v^+(\mathbf{F})$  represents the graph induced by the neighborhood of vertex  $v$  in graph  $F$ .

$\mathbf{G}_v^-(\mathbf{F})$  represents the graph induced by the non-adjacent vertices to  $v$  within graph  $F$ .

Now if  $F$  is a  $(W_5, K_5; n)$ -graph, and  $x \in VF$  has degree  $d$ , it is clear that  $G_x^+$  is a  $(C_4, K_5; d)$ -graph and  $G_x^-$  is a  $(W_5, K_4; n - 1 - d)$ -graph. Suppose then that  $G$  is a  $(C_4, K_5)$ -graph and  $H$  is a  $(W_5, K_4)$ -graph. We define  $\mathcal{F}(G, H)$  to be the set of all  $(W_5, K_5)$ -graphs  $F$  such that for some vertex  $x \in VF$ ,  $G_x^+(F) = G$  and  $G_x^-(F) = H$ . From here on we will refer to  $F$  as the current member of  $\mathcal{F}(G, H)$  being considered. Let  $\mathcal{R}(G, H)$  and  $\mathcal{R}(G, H; n)$  denote the set of  $(G, H)$ -graphs and  $(G, H; n)$ -graphs respectively. Table II below illustrates the counts of both  $\mathcal{R}(C_4, K_5; n)$  and  $\mathcal{R}(W_5, K_4; n)$  for  $n \geq 9$ . We describe how they were obtained in Section 4.3.

<b>n</b>	$ \mathcal{R}(C_4, K_5; n) $	$ \mathcal{R}(W_5, K_4; n) $
9	385	15452
10	574	104314
11	457	531892
12	126	1437877
13	1	865055
14		111153
15		2891
16		82

Table II Counts of graphs in  $\mathcal{R}(C_4, K_5; n)$  and  $\mathcal{R}(W_5, K_4; n)$  for  $n \geq 9$ .

So, to combine the vertex  $x$  with graphs  $G$  and  $H$  into  $(W_5, K_5; n)$ -graphs we need to choose the edges between graphs  $G$  and  $H$  so that no  $W_5$  or  $\overline{K_5}$  is introduced. It is computationally infeasible to utilize any naive approach in gluing together  $(C_4, K_5)$ -graphs and  $(W_5, K_4)$ -graphs for larger orders of both graphs. Furthermore, this split can be done in  $n$  ways, one for each vertex in  $F$ . Hence we should avoid any technique that reasons about

individual edges. Instead, we will focus on an approach involving so called intervals of subgraphs or cones. The following approach was developed by McKay and Radziszowski [MR95] for the classical case and is presented with a few necessary adjustments to our specific case.

#### 4.1 The Case of $R(W_5, K_5)$

Recall the upper bound relation, stated below as Theorem 8, that we now provide a proof for below since it provides some insight into the concept behind the gluing algorithm discussed later.

**Theorem 8**  $R(k, l) \leq R(k, l - 1) + R(k - 1, l)$

**Proof:** Let  $n = R(k, l - 1) + R(k - 1, l)$  and next let  $G$  be a  $(k, l; n)$ -graph, hence  $|VG| = n$ . Take one vertex of  $G$ ,  $x$ . We know by the definition of  $G$  that  $|VG_x^+(x)| < R(k - 1, l)$  and  $|VG_x^-(x)| < R(k, l - 1)$  otherwise  $G$  is not an  $(k, l; n)$ -graph. Then we have  $n < |VG_x^+(x)| + |VG_x^-(x)| + |\{x\}|$ , which yields  $n < R(k - 1, l) + R(k, l - 1) + 1$  and finally  $n \leq R(k - 1, l) + R(k, l - 1)$  as desired.

To generalize the above theorem to the case examined in this thesis,  $R(W_5, K_5)$  we must be aware of two cases. Given the particular symmetry in the  $W_5$  graph there are two possible choices to remove a vertex from it. The cleanest is to remove the apex or center, leaving a  $C_4$  graph remaining. On the other hand we could remove one of the outer vertices to leave a  $K_4 - e$  graph. It is important to note that a vertex  $x$  must be adjacent to all vertices of a  $C_4$  to form a  $W_5$  but need only be adjacent to 3 vertices, if it includes the apex, within a  $K_4 - e$  graph to form a  $W_5$ . Because of this we have three cases of particular importance to the decomposition of  $R(W_5, K_5)$  (note the classical case would have only two cases). The follows of these three cases have the values

$$R(C_4, K_5) = 14 \text{ [Hen89a]} \tag{13}$$

$$R(K_4 - e, K_5) = 16 \text{ [BH81]} \tag{14}$$



$$R(W_5, K_4) = 17 \text{ [Hen89a]} \quad (15)$$

By substituting the values above into theorem (8) we arrive at

$$R(W_5, K_5) \leq R(C_4, K_5) + R(W_5, K_4)$$

$$R(W_5, K_5) \leq 14 + 17 \leq 31.$$

We need only consider the  $R(C_4, K_5)$  case for the above theorem since it is the more restrictive case. The current best upper bound for this case is 29, listed by Hendry in his table of values [Hen89a]. Again, unfortunately Hendry did not publish the proof producing this upper bound and it is not currently available.

Next recall the general lower bound equation that

$$R(k, p) \geq s \text{ and } R(k, q) \geq t \Rightarrow R(k, p + q - 1) \geq s + t + k - 3.$$

Applying the analog of this relation to the  $R(W_5, K_5)$  case we get,

$$R(W_5, K_5) \geq 21$$

since  $R(W_5, K_3) = 11$ . The current best lower bound for this case of 27 is also credited to Hendry [Hen89a]. Thus the naive bounds for this case of 22 and 31 have already been improved significantly before this thesis.

## 4.2 Intervals

We define a *cone* to be a subset of vertices within  $H$ . We will consider a *feasible cone* to be a subset of  $VH$  that does not contain a cycle of length 4 or any  $P_3^*$  (Figure III below). Let a  $P_3^*$  be the necessary three vertices of a  $K_4 - e$  that if connected entirely to another vertex in  $VH$  will form a  $W_5$ .

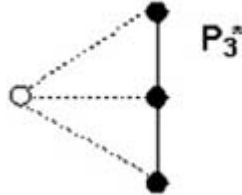


Figure III.

A typical  $(W_5, K_4; n)$ -graph, with  $13 \leq n \leq 16$ , has between 1000 and 2000 *feasible cones* present. Since the gluing computation will utilize a direct backtrack search of depth  $|VG|$  on cone assignments, this is too large a space to begin the gluing computation and further pruning is required. Therefore we will partition the set of feasible cones into well-structured families that can be processed in parallel. Note that if  $B, T$  are *feasible cones* and  $B \subseteq X \subseteq T$  then  $X$  must be a *feasible cone*. An *interval of feasible cones* is a set of *feasible cones* of the form  $\{X | B \subseteq X \subseteq T\}$  for some *feasible cones*  $B \subseteq T$ . This interval, denoted as  $[B, T]$ , will contain  $2^{|T|-|B|}$  feasible cones and we will refer to  $B$  and  $T$  its *bottom* and *top*, respectively. Statistics for the interval decomposition of several  $(W_5, K_4; n)$ -graphs are provided in section 4.6.

### 4.3 Supplementary Graphs

The above calculations require the complete enumeration of all  $(C_4, K_5)$ -graphs and  $(W_5, K_4)$ -graphs. Both sets of graph statistics obtained have been verified to be correct in part by a third party's result and in part by repeated computations with different methods. The statistical breakdown of the  $(C_4, K_5)$ -graphs and  $(W_5, K_4)$ -graphs are provided in Tables *III* and *IV*. The enumeration began by first constructing filters, which for the  $(C_4, K_5)$  case removed all graphs containing either a  $C_4$  or the independent set of five vertices. For the  $(W_5, K_4)$  case all graphs containing either a  $W_5$  or independent set of four vertices were removed. Next using the graph generator **geng** (mentioned later in section 4.8) all possible graphs with 10 vertices or fewer were generated. These graphs were then run through the aforementioned filters providing all  $(W_5, K_4)$ -graphs and  $(C_4, K_5)$ -graphs with 10 vertices or fewer. The subgraph of  $C_4$  was recognized by simply checking the set of vertices to see if any two vertices are adjacent to two distinct vertices. To recognize both  $\overline{K_4}$  and  $\overline{K_5}$ , all  $\overline{K_3}$  graphs were first enumerated and next the entire set of  $\overline{K_3}$  graphs was tested to find if the union of the vertices of any two  $\overline{K_3}$  graphs produces the vertices of a  $\overline{K_5}$  or  $\overline{K_4}$ . Finally to recognize  $W_5$  the previous technique for determining  $C_4$  was used, and all remaining vertices in the graph were checked to attempt to find a vertex, which was adjacent to each vertex in  $C_4$ .

n	1	2	3	4	5	6	7	8	9	10	11	12	13	total
e														
0	1	1	1	1										4
1		1	1	1	1									4
2			1	2	2	1								6
3			1	3	4	4	1							13
4				1	5	7	3	1						17
5					4	11	10	2						27
6					1	11	22	9	1					44
7						4	27	27	4					62
8							17	53	16	1				87
9							5	62	50	5				122
10								31	108	18	1			158
11								5	130	55	3			193
12									66	138	10	1		215
13									10	200	32	1		243
14										126	75	3		204
15										29	129	9		167
16										2	139	15		156
17											59	22		81
18											9	33		42
19												25		25
20												14		14
21												3		3
22														0
23														0
24													1	1
total:	1	2	4	8	17	38	85	190	385	574	457	126	1	1888

Table III Statistics of the number of  $(C_4, K_5; n; e)$ -graphs  
agreed completely with [RT02]

n	13	14	15	16
e				
26	1			
27	4			
28	26			
29	130			
30	701	1		
31	3321	1		
32	13707	1		
33	45166	7		
34	111742	22		
35	192964	114		
36	221640	435		
37	164623	1660		
38	79134	4948		
39	25036	11696		
40	5668	20315		
41	1001	25783	2	
42	164	22922	8	
43	23	14428	44	
44	4	6377	138	
45		1981	357	
46		398	590	
47		59	722	
48		4	566	1
49		2	317	1
50			108	5
51			32	9
52			6	23
53			1	17
54				15
55				6
56				5
total:	865055	111153	2891	82

Table IV Statistics of  $(W_5, K_4; n; e)$ -graph (Tables XVIII, XIX contain remaining  $(W_5, K_4; n; e)$ -graphs for  $n \leq 12$ )

## 4.4 One-Vertex Extensions

As described above all  $(C_4, K_5; n)$ -graphs and  $(W_5, K_4; n)$ -graphs for  $n \leq 10$  were obtained utilizing the graph generator **geng** and filters. The remaining  $(C_4, K_5)$  graphs, depicted in Table *III*, were obtained using a naive one vertex extension algorithm. This algorithm proceeded by extending in every possible manner, every  $(C_4, K_5; n)$ -graph by one vertex. Then after all isomorphic graphs were removed, the remaining graphs were passed through the  $(C_4, K_5)$  filter to get rid of all graphs containing either a  $C_4$  or  $\overline{K_5}$  subgraph. This naive algorithm could not be utilized in the  $(W_5, K_4)$  case given the large number of graphs involved. As a result three new methods of vertex extension were attempted. For concreteness we will refer to the methods as extensions *I*, *II*, and *III* as in subsections 4.4.1, 4.4.2 and 4.4.3 below. Supposing  $F$  is a  $(W_5, K_4; n)$ -graph, we desire to determine all ways to extend this graph by one vertex  $x$  to produce a new set of  $(W_5, K_4; n + 1)$ -graphs. To accomplish this it is necessary to ensure that  $x$  does not cover any  $C_4$  or  $P_3^*$  within  $F$  and ensure that it covers every independent 3-set of  $F$ .

### 4.4.1 Extension *I*

Extension *I* proceeded in the following way for each  $(W_5, K_4; n)$ -graph  $G$ . First it listed all  $C_4$ 's present in the graph as well as all independent triangles. Next it iterated through each possible manner in which a vertex could be added to the graph  $G$  in question. If any combination made the new vertex independent of any independent triangle we could conclude it was undesirable and omit it. If any combination left the new vertex adjacent to all vertices of a  $C_4$  we could similarly conclude that the combination was undesirable and again we omitted it. All graphs that passed both tests were written to file and later all isomorphs were removed. Notice though that these graphs cannot be guaranteed to be  $(W_5, K_4; n + 1)$ -graphs since if a new vertex is adjacent to all three vertices of a  $P_3^*$ , which is in turn connected to some other vertex in  $VH$  forming a  $K_4 - e$ , then a  $W_5$  will be formed. As such the generated graphs were run through the aforementioned filter to obtain a complete set of  $(W_5, K_4; n + 1)$ -graphs.

#### 4.4.2 Extension II

Extension II proceeded analogously to extension I but instead the case that the vertex added is adjacent to all three vertices of a  $P_3^*$  subgraph was considered. If such a situation was recognized we omitted the graph in question. Thus, since all possible forbidden positions where a  $W_5$  or independent 4-set can occur are checked we know that the set of graphs produced comprises a complete set of  $(W_5, K_4; n + 1)$ -graphs. Of course, again the set of graphs obtained required all isomorphs to be removed. Below we have included the pseudo-code for this algorithm.

**INPUT:** List of  $(W_5, K_4; n)$ -graphs

**OUTPUT:** List of  $(W_5, K_4; n + 1)$ -graphs comprising the resulting extension of every inputted graph in every possible manner

**for each** input graph  $F$  **do:**

**let**  $fgraphs$  be a list enumerating all  $C_4, P_3^*$ , and  $K_4$  subgraphs within  $F$

**for each** cone of vertices  $c$  in  $F$  **do:**

**for each** subgraph  $sg$  in  $fgraphs$  **do:**

**if**  $sg$  is a  $C_4$  or  $P_3^*$  **then**

**if**  $c$  covers all vertices of  $sg$  **then** ignore  $c$

**endif**

**If**  $sg$  is a  $\overline{K_4}$  **then**

**if**  $c$  fails to cover a vertex of  $sg$  **then** ignore  $c$

**endif**

**endfor**

**if**  $c$  is not ignored **then**

Output the resulting graph of adding a vertex to  $F$  with the connections stipulated by  $c$

**endif**

**endfor**

**endfor**

#### 4.4.3 Extension *III*

Extension *III* is borrowed from the  $R(4, 5)$  [MR95] paper utilizing the concept of cone intervals. It is presented below with the necessary adjustments for this case. This method proved orders of magnitude faster than the prior two methods and was the only one of three methods that was able to effectively enumerate all  $(W_5, K_4; n)$ -graphs. Because of this the algorithm was also slightly modified for use with  $(W_5, K_5)$ -graphs, the only difference is in considering independent 4-sets instead of 3-sets.

The algorithm proceeds by first enumerating all forbidden subgraphs of  $C_4$ ,  $P_3^*$ , and  $\overline{K_3}$  within  $F$ . It then iterates through this list and if in the current interval being considered it is the case that a  $C_4$  or  $P_3^*$  subgraph is present in the bottom, or no vertices within a given  $\overline{K_3}$  are present in the top, the interval is removed. Otherwise the interval is split into multiple intervals as stipulated below. The process begins by considering the entire lattice of cones as the initial interval and concludes by outputting a list of acceptable intervals within the graph  $F$ . This list of intervals can then be readily utilized to extend  $F$  into a new set of  $(W_5, K_4; n + 1)$ -graphs.

**INPUT:** A  $(W_5, K_4; n)$ -graph  $F$

**OUTPUT:** A list of acceptable intervals within  $F$  forming  $(W_5, K_4; n + 1)$  with a new vertex

**let**  $fgraphs$  be a list enumerating all  $C_4$ ,  $P_3^*$ , and  $K_4$  subgraphs within  $G$

$I := \{[\emptyset, VF]\}$

**for each** subgraph  $sg$  in  $fgraphs$  **do:**

**if**  $sg$  is a  $C_4$  or  $P_3^*$  **then**

**for each**  $[B, T] \in I$  such that  $sg \subseteq T$  **do:**

```

    if  $sg \subseteq B$  then
        Delete  $[B, T]$  from  $I$ .
    else
        Replace  $[B, T]$  by  $[B \cup \{y_1, \dots, y_{j-1}\}, T - \{y_j\}]$ 
        for  $j = 1, \dots, k$ , where  $sg - B = \{y_1, \dots, y_k\}$ .
    endif
endfor
else [if  $sg$  is an independent 3-set]
    for each  $[B, T] \in I$  such that  $sg \cap B = \emptyset$  do:
        if  $sg \cap T = \emptyset$  then
            Delete  $[B, T]$  from  $I$ .
        else
            Replace  $[B, T]$  by  $[B \cup \{y_j\}, T - \{y_1, \dots, y_{j-1}\}]$ 
            for  $j = 1, \dots, k$ , where  $sg \cap T = \{y_1, \dots, y_k\}$ .
        endif
    endfor
endif
endif
endfor

```

## 4.5 Gluing

Recall the definition of a *feasible cone* to be a subset of  $VH$  that does not contain a cycle of length 4 or any  $P_3^*$ , which again is a  $P_3$  subgraph that is connected entirely to another vertex in  $VH$  and thus forms a  $K_4 - e$ . Consider then that we must have  $N_F(v, VH)$  as a *feasible cone* for every  $v \in VG$  otherwise a  $W_5$  will be present in the induced graph. We denote the *feasible cone* associated with a given vertex  $v$  as  $C_v$ , hence  $N_F(v, VH) = C_v$ . For clarity assume that both the vertices and subgraphs of  $G$  are labeled with contiguous integers  $0, 1, 2, \dots$  in the order induced from the labeling of  $G$ . The problem is then to choose *feasible cones*  $C_0, C_1, \dots$ , for each vertex of



$G$  so as to avoid any  $W_5$  and independent sets of order 5 in  $F$ . The following illustrates all positions where such subgraphs can appear.

$D_{2A}$ : Two vertices  $v, w \in VG$  have  $C_v \cap C_w$  that covers some  $P_3$  in  $H$ .

$D_{2B}$ : Two vertices  $v, w \in EG$  have  $C_v$  covering some  $P_3 \in H$  with  $C_w$  covering the two vertices of local degree 1 within that  $P_3$ .

$D_{2C}$ : Two vertices  $v, w \in EG$  have  $C_v \cup C_w$  covering some  $K_3$  of  $H$  and  $C_v, C_w$  each covering at least two vertices of the  $K_3$ .

$D_{3A}$ : Three vertices  $u, v, w \in VG$  that form a  $P_3$  graph have  $C_u \cap C_v \cap C_w$  that cover a vertex of  $H$ .

$D_{3B}$ : Three vertices  $u, v, w \in VG$  that form a  $K_3$  graph have  $C_v$  that covers a  $K_2$  in  $H$  with  $C_u, C_w$  each covering a distinct vertex of that  $K_2$ .

$E_t$ : For some independent set  $w_0, \dots, w_{t-1}$  of  $G$ , there is an independent set of order  $5 - t$  in  $H$  that is completely missed by  $C_{w_0} \cup C_{w_1} \cup \dots C_{w_{t-1}}$  ( $t = 2, 3, 4$ ).

Suppose  $n = |VG|$ . If  $C_0, \dots, C_{n-1}$  are feasible cones, then  $F(G, H; C_0, \dots, C_{n-1})$  denotes the graph  $F$  with vertex  $x$  such that  $G_x^+(F) = G$ ,  $G_x^-(F) = H$ , and  $C_i = N_F(i, VH)$  for  $0 \leq i \leq n - 1$ .  $F$  is a  $(W_5, K_5)$ -graph if and only if all forbidden conditions above are avoided. Also note that if  $I_0, I_1, \dots, I_{n-1}$  are intervals, then  $\mathcal{F}(G, H; I_0, \dots, I_{n-1})$  represents the set of all  $(W_5, K_5)$ -graphs  $F(G, H; C_0, \dots, C_{n-1})$  such that  $C_i \in I_i$  for  $0 \leq i \leq n - 1$ . This structure will then form the basis of the gluing algorithm for this case, which will proceed by setting up intervals that can be shortened and collapsed given the occurrence of forbidden subgraphs.

Given the graph  $H$ , we will define three functions  $H_1, H_2, H_3 : 2^{VH} \rightarrow 2^{VH}$ . Now, for  $X \subseteq VH$  let

$$H_1(X) = \{w \in VH \mid vw \in EH \text{ for some } v \in X\}$$

$$H_2(X) = \{w \in VH \mid vw \notin EH \text{ for some } v \notin X\}$$

$$H_3(X) = \{w \in VH \mid \{u, v, w\} \text{ is an independent 3-set of } H \text{ for some } u, v \notin X\}$$

The above functions can be computed in a straightforward manner and stored to save computing time. Given them, we can define *collapsing rules* that apply to sequences  $I_0, \dots, I_{m-1}$  of intervals. The rules will depend on the graphs  $G$  and  $H$ . In every case, an interval is either replaced by an interval contained in it, or the special event FAIL occurs. In the rules illustrated below we take  $u, v, w, z \in VG$  and  $r, s, t \in VH$  to provide concrete examples. Next suppose  $I_i = [B_i, T_i]$  for each  $i$ , and define collapsing rules (a) – (h) as follows:

- (a) Suppose  $\{u, v\} \notin EG$ , where  $u, v$  are distinct vertices of  $G$   
**if**  $H_3(T_u \cup T_v) \not\subset T_u \cup T_v$  **then** FAIL  
**else**  $B_u := B_u \cup (H_3(T_u \cup T_v) - T_v)$

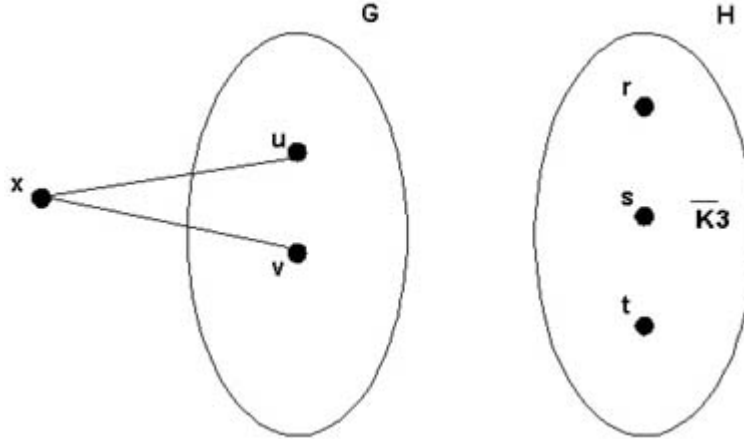


Figure IVa. Illustrates the fail case for rule (a) and represents the forbidden occurrence  $E_2$  that must be avoided. This results since two vertices of  $G$  are independent of an independent 3-set within  $H$ .

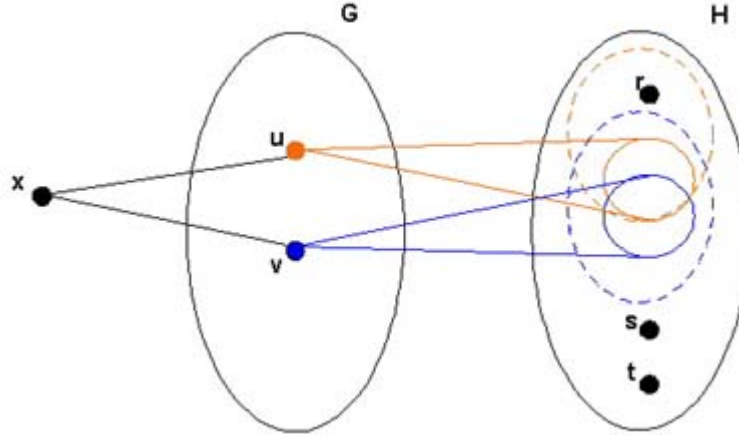


Figure IVb. The collapsing case for rule (a). The tops of intervals  $I_u, I_v$  fail to cover an independent two set  $\{s, t\}$  within  $G$  that in turn forms a independent 3-set with vertex  $r$ . So the collapsing rule (a) is applied to grow the bottom of interval  $I_u$  so that an independent set of five vertices is avoided.

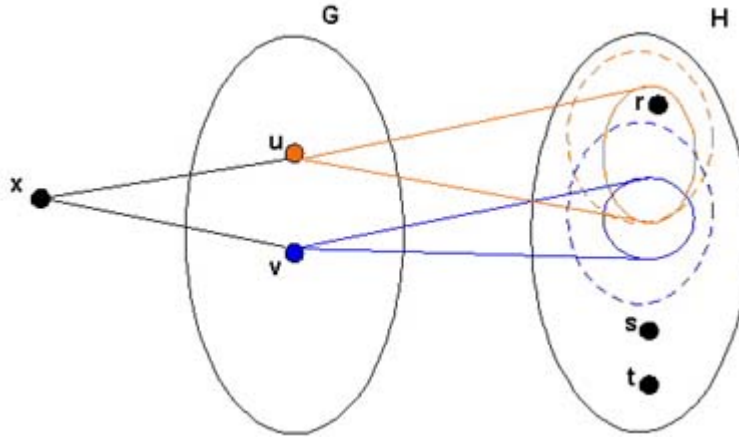


Figure IVc. Result of applying collapsing case for rule (a) to the intervals  $I_u, I_v$ . Notice that the bottom of interval  $I_u$  is now larger to include vertex  $r$  in  $H$

- (b) Suppose  $\{u, v, w\}$  is an independent 3-set of  $G$   
**if**  $H_2(T_u \cup T_v \cup T_w) \not\subset T_u \cup T_v \cup T_w$  **then** FAIL  
**else**  $B_u := B_u \cup (H_2(T_u \cup T_v \cup T_w) - (T_v \cup T_w))$

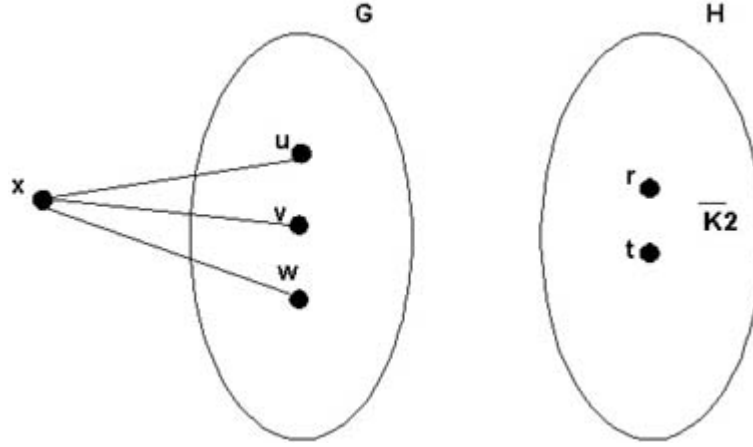


Figure Va. Illustrates the fail case for rule (b) and represents the forbidden occurrence  $E_3$  that must be avoided. This results since three vertices of  $G$  are independent of an independent 2-set within  $H$ .

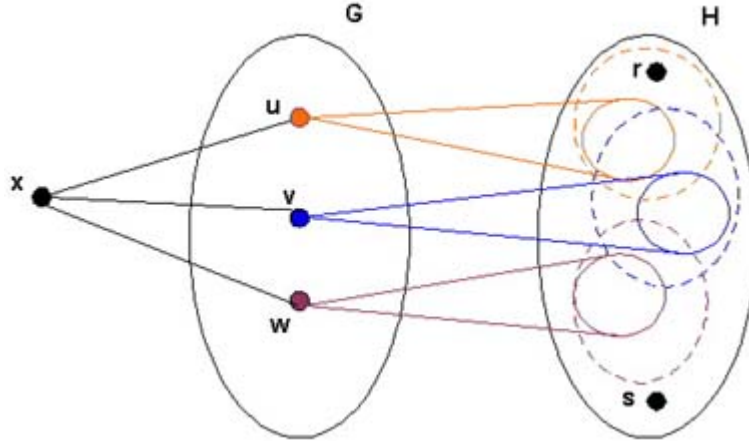


Figure Vb. The collapsing case for rule (b), where the tops of intervals  $I_u, I_v, I_w$  fail to entirely cover an independent two set  $\{r, s\}$  within  $H$ . So the collapsing rule (b) is applied to grow the bottom of interval  $I_u$  so that an independent set of five vertices is avoided.

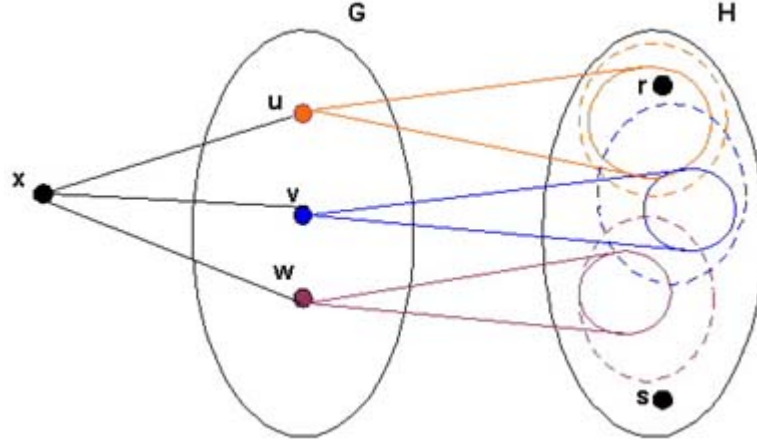


Figure Vc. Result of applied collapsing case for rule (b) to the intervals  $I_u, I_v, I_w$ . Notice that the bottom of interval  $I_u$  is now larger to include vertex  $r$ .

- (c) Suppose  $\{u, v, w, z\}$  is an independent 4-set of  $G$   
**if**  $T_u \cup T_v \cup T_w \cup T_z \neq VH$  **then** FAIL  
**else**  $B_u := B_u \cup (VH - (T_v \cup T_w \cup T_z))$

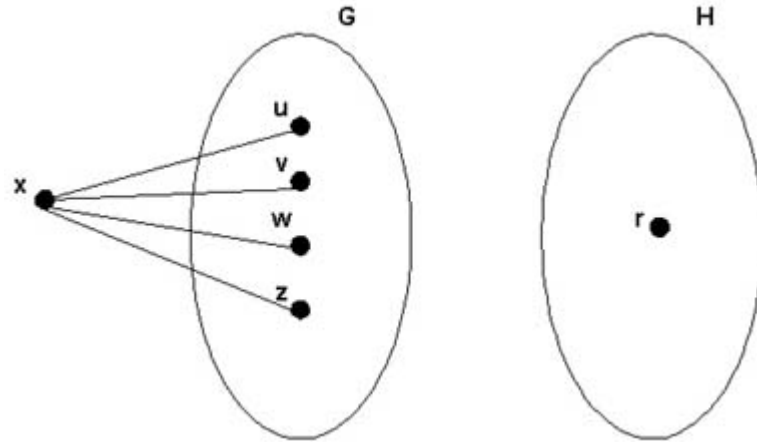


Figure VIa. The above depicts the fail case for rule (c) and illustrates the forbidden occurrence  $E_4$  that must be avoided. Four vertices of  $G$  are independent of a vertex within  $H$  and hence form an independent

set of five vertices.

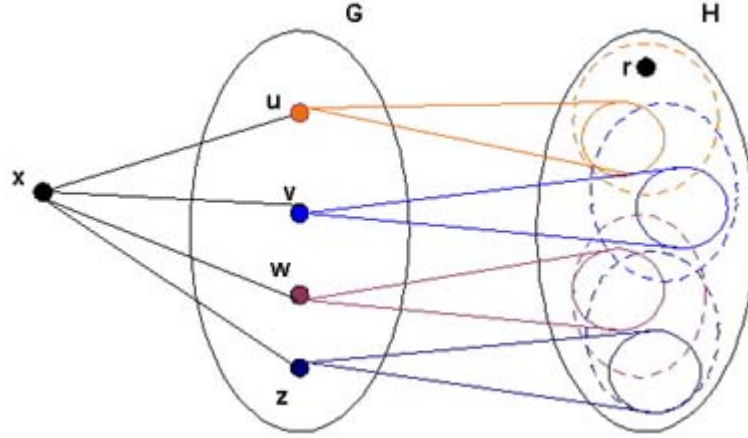


Figure VIb. The collapsing case for rule (c) simply adds the vertex  $r$  to the bottom of interval  $I_u$ .

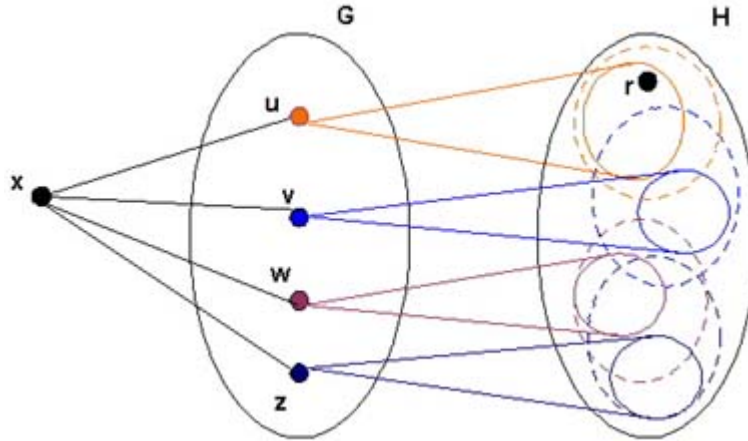


Figure VIc. Result of applying collapsing case for rule (c) to the intervals  $I_u, I_v, I_w, I_z$ . Notice that the bottom of interval  $I_u$  now contains vertex  $r$  in  $H$

- (d) Suppose  $\{u, v\} \in VG$ , where  $u, v$  are distinct vertices of  $G$   
**if**  $B_u \cap B_v \cap H_1(B_u \cap B_v)$  covers a  $P_3$  then FAIL  
**For each**  $K_2 : k \in B_u \cap B_v \cap H_1(B_u \cap B_v)$  do:  
 $T_u := T_u - (B_v \cap H_1(k) - k)$

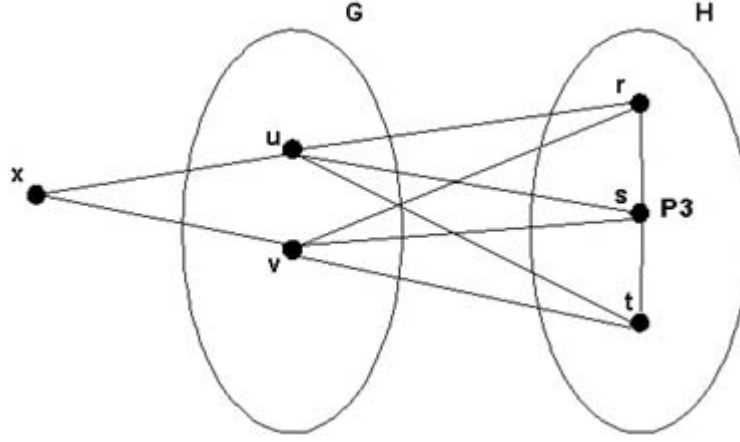


Figure VIIa. Portrays the fail case for rule (d) and represents the forbidden occurrence  $D_{2A}$ . In this instance the subgraph  $W_5$  is formed with the apex being vertex  $s$  within  $H$ .

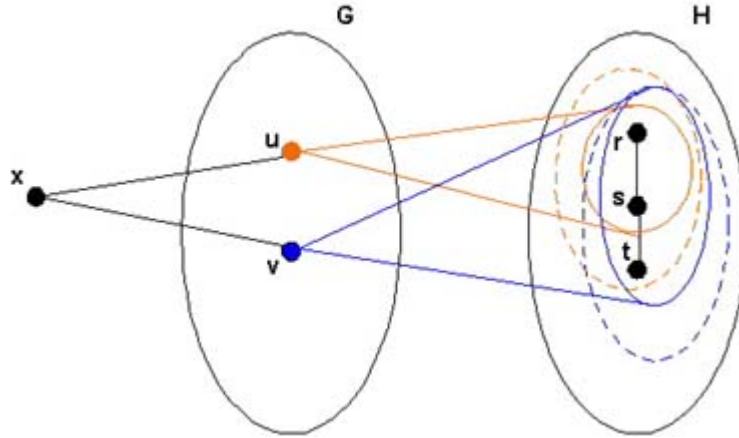


Figure VIIb. Illustrates the collapsing case for rule (d), since the bottoms of intervals  $I_u, I_v$  cover a  $K_2$ ,  $\{r, s\}$ , which is in turn part of a  $P_3$ ,  $\{r, s, t\}$ . As a result the top of interval  $I_u$  is shrunk to avoid covering

the entire  $P_3$ , which would introduce the forbidden occurrence  $D_{2A}$  from above.

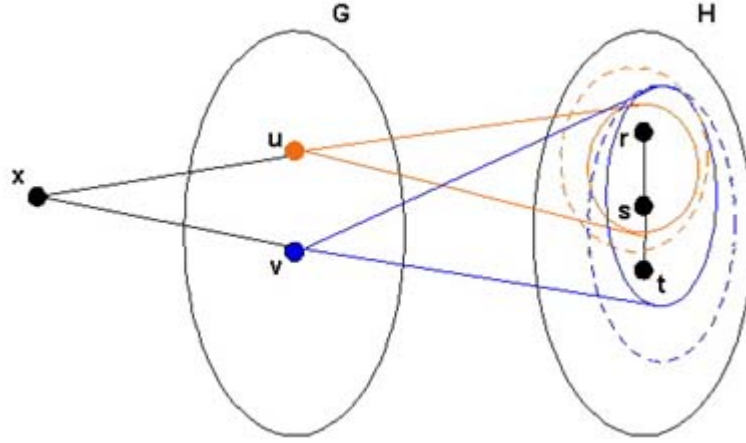


Figure VIIc. Result of applying collapsing case for rule (d).

- (e) Suppose  $\{u, v\} \in EG$ , where  $u, v$  are distinct vertices of  $G$

**For each**  $P_3$ :  $p \in H$  covered by  $B_v$  do:

**if**  $B_u$  covers the two vertices within  $p$  having degree 1 then FAIL

**else if**  $B_u$  covers one vertex,  $w$ , within  $p$  having local degree 1  
then  $T_u := T_u - w$

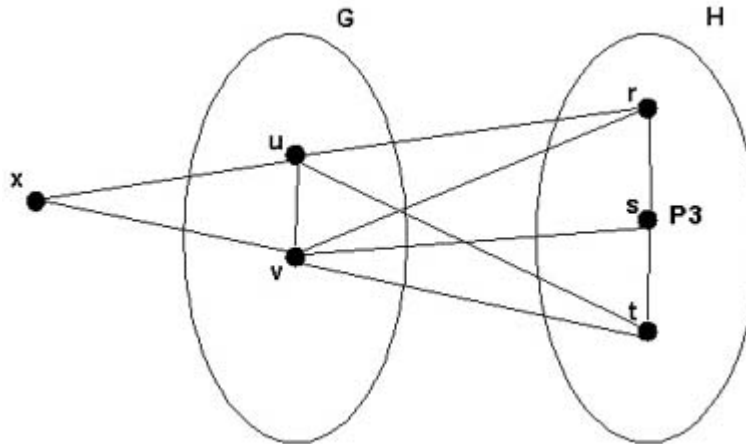


Figure VIIIa. The fail case for rule (e), where a  $K_2$  ( $\{u, v\}$ ) has the



necessary edges between a  $P_3$  ( $\{r, s, t\}$ ) within  $H$  to yield the forbidden case  $D_{2B}$ .

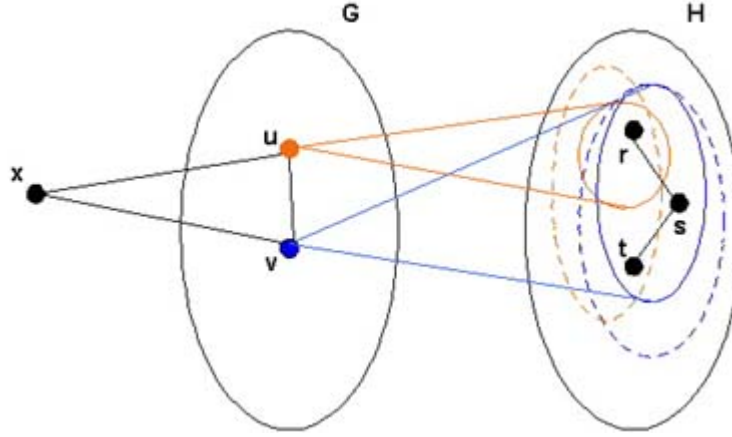


Figure VIIIb. The collapsing case for rule (e) where again the tops of intervals  $I_u, I_v$  are covering the forbidden case  $D_{2B}$  and need to be shrunk accordingly.

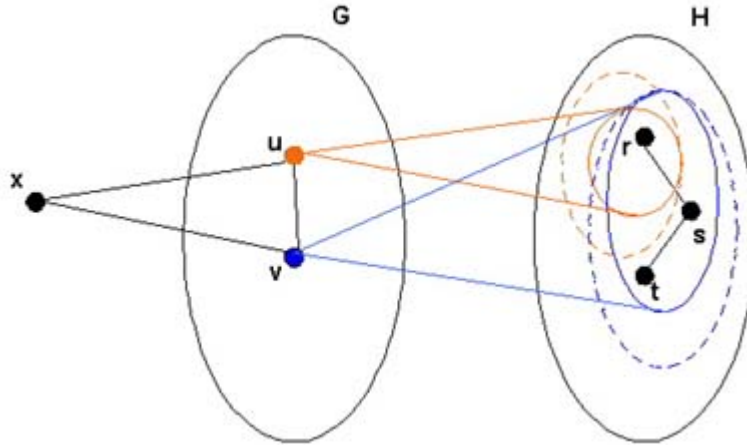


Figure VIIIc. The result of applying the collapsing case for rule (e) to the top of the interval  $I_u$  and shrinking it necessarily so that it now only covers one vertex within the  $P_3$  present in the graph  $H$ .

(f) Suppose  $\{u, v\} \in EG$  and  $u, v$  are distinct vertices of  $G$

**For each**  $K_3 : k \in H$  covered by  $B_u \cup B_v$  do:

**if**  $|B_u \cap k| > 1$  and  $|B_v \cap k| > 1$  then FAIL

**else if**  $|B_u \cap k| = 1$  and  $|B_v \cap k| > 1$  then  $T_u := T_u - (k - B_u)$

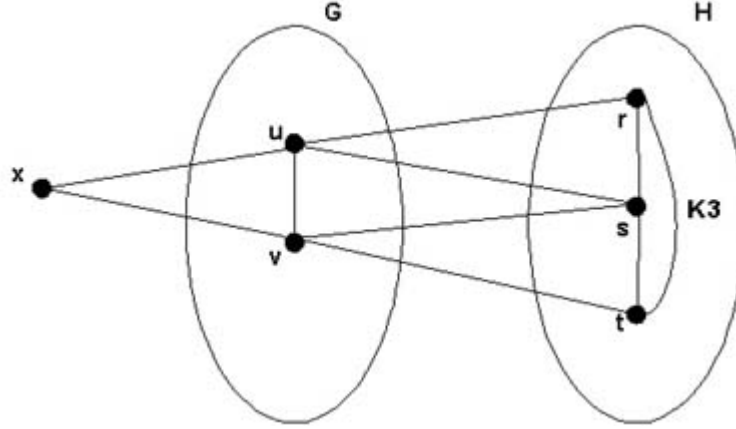


Figure IXa. This represents the fail case for rule (f), which is the forbidden subgraph  $D_{2C}$ . In this instance a  $K_2$  within  $G$  has each vertex  $\{u, v\}$  adjacent with two distinct vertices within a  $K_3$  ( $\{r, s, t\}$ ) in  $H$ .

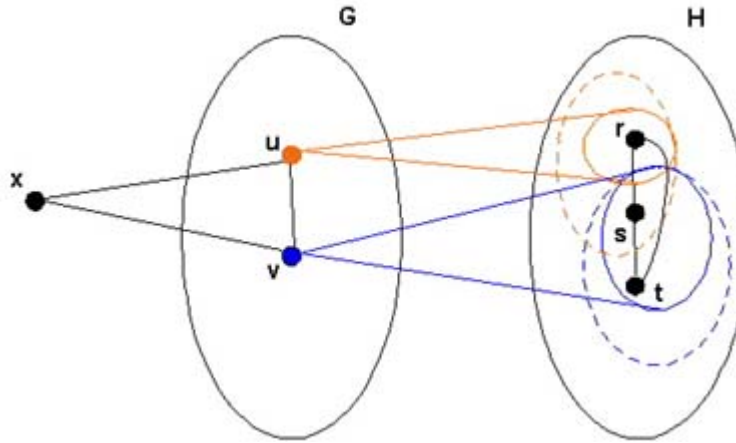


Figure IXb. The collapsing case for rule (f) again requires that we decrease the interval  $I_u$  to remove a vertex within a subgraph of  $H$ . For this case we must remove the necessary vertex within a  $K_3$ , vertex  $s$ , from interval  $I_u$ .

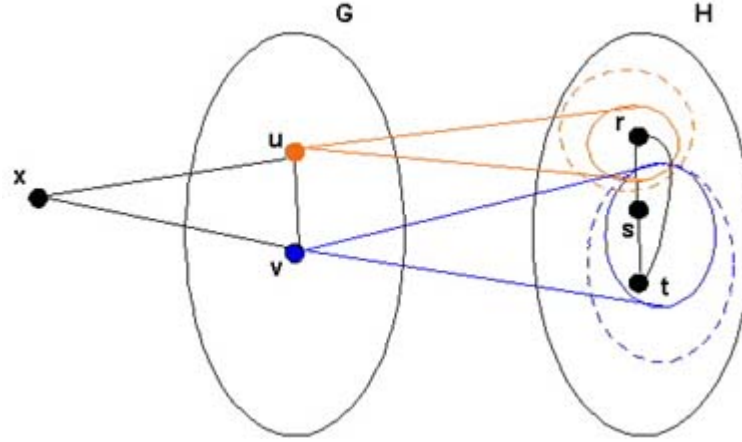


Figure IXc. Illustrates the result of shrinking interval  $I_u$  as instructed by rule (f) of the collapsing cases.

- (g) Suppose  $\{u, v, w\}$  forms a  $P_3$  in  $G$   
**if**  $B_u \cap B_v \cap B_w \neq \emptyset$  then FAIL.  
**else**  $T_u := T_u - (B_v \cap B_w)$

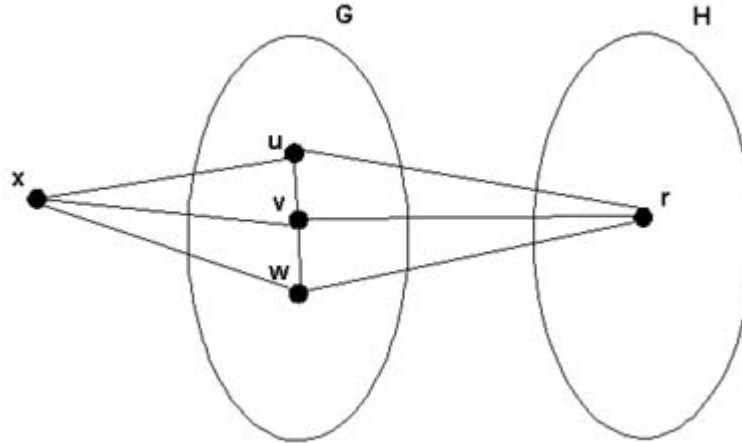


Figure Xa. Fail case for rule (g). This depicts forbidden subgraph  $D_{3A}$ , with vertex  $v$  as the apex of the resulting  $W_5$  graph. So we must ensure that no vertex in  $H$  is adjacent to all vertices of a  $P_3$  in  $G$ .

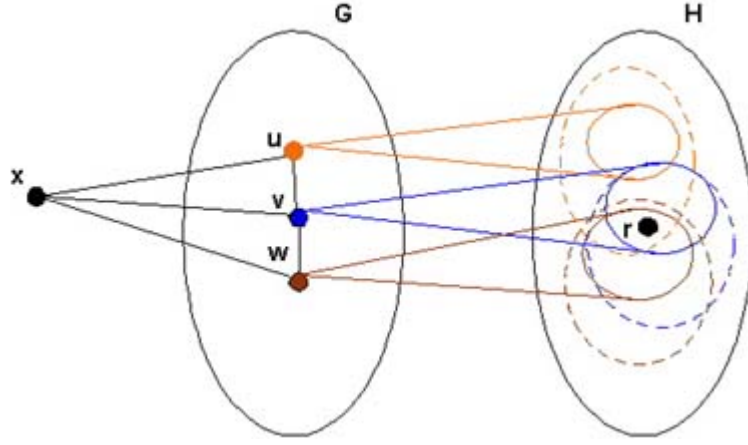


Figure Xb. The collapsing case for rule (g) simply removes all vertices from interval  $I_u$  that are contained in both the bottoms of intervals  $I_v, I_w$ .

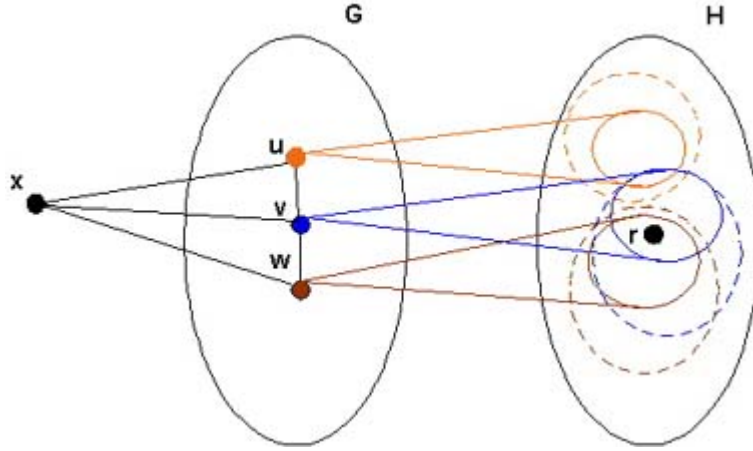


Figure Xc. Result of applied collapsing case for rule (g). Notice that the top of interval  $I_u$  was reduced to remove vertex  $r$ .

- (h) Suppose  $\{u, v, w\}$  forms a  $K_3$  in  $G$   
**if**  $H_1(B_w \cap B_v) \cap B_v \cap B_u \neq \emptyset$  then FAIL.  
**else**  $T_u := T_u - (H_1(B_w \cap B_v) \cap B_v - B_u)$

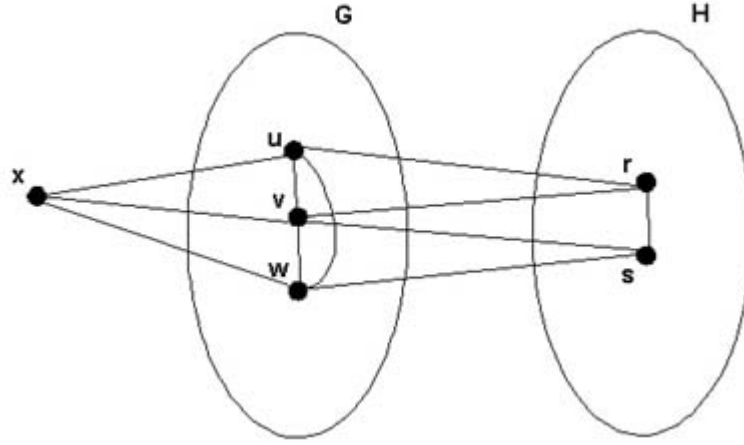


Figure XIa. Fail case for rule (h). This figure illustrates the forbidden subgraph  $D_{3B}$ , which results from a  $K_3$  in  $G$  having the necessary edges between a  $K_2$  within  $H$ .

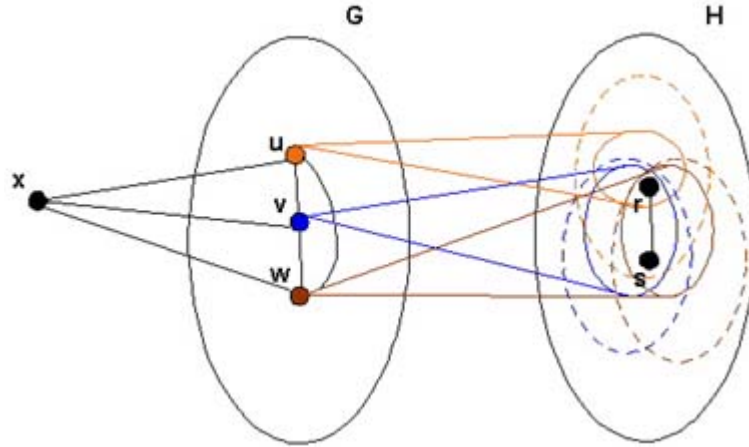


Figure XIb. The collapsing case for rule (h) needs to remove the a vertex within the  $K_2$  from the top of interval  $I_u$ .

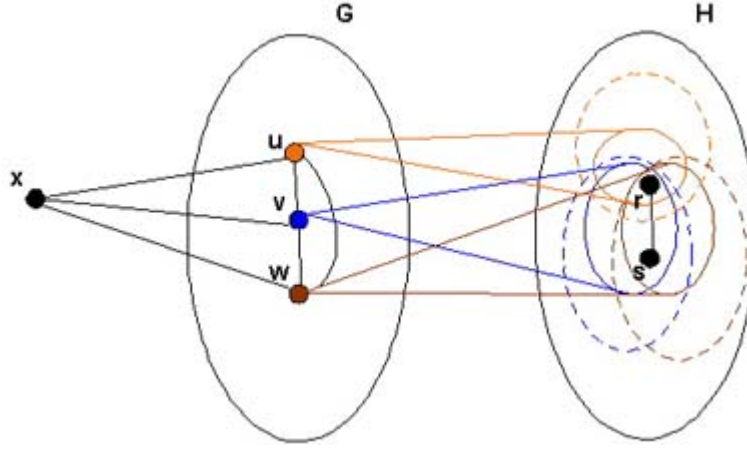


Figure XIc. Illustrates the application of the collapsing case for rule (h) where the top of interval  $I_u$  has been reduced to eliminate vertex  $s$ .

Notice that all of these rules are symmetric in nature, with the exception of the rules (e), (f) and (h). The resulting shrinking of the intervals within the symmetric rules can be applied to all vertices being considered. The non-symmetric rules are special cases and in contrast can only be applied to the vertex  $u$  being examined. These non-symmetric rules are seldom utilized in practice but were included for completeness.

The above rules differ from the classical case [MR95] in a few important ways. First of all they are more complicated and less clean in that the intervals can only be shrunk in certain cases. This follows from the structure of the subgraph  $W_5$ . To make the collapsing rules more effective we wanted to consider intervals with small dimensions. So any collection of intervals that successfully passes through the applications of all the rules is split into smaller pieces and again passed through the rules so that we may more accurately check the relationships between the intervals. The split is a simple halving of one of the intervals currently associated with a given vertex. This process continues until either a fail condition is met or all intervals being considered have dimension 0.

## 4.6 Interval Setup

The division of the entire lattice of possible cones into intervals proved to be a more straightforward endeavor than initially thought. The *feasible cone* intervals were constructed by first compiling a list of all maximum feasible cones present within a given  $(W_5, K_4)$ -graph. This is easily achievable by checking all possible cones and since there are only some 65000 possible cones in the most extreme case, this can readily be done. Next determining the list of maxima was a simple undertaking since a *feasible cone* is maximum if and only if adding any vertex to its set of vertices makes it infeasible. In this manner a list of maximum *feasible cones* was easily created. Pseudo code for this procedure is given below.

**INPUT:** List  $\mathbf{C_4P_3}$  of all  $C_4$  and  $P_3^*$  subgraphs to test feasibility

**OUTPUT:** *max\_fcones*, the list of maximum feasible cones within the lattice

```

for each cone  $C$  within the entire lattice of cones do:
    if  $C$  is feasible then
        if for each vertex  $v$  not in feasible cone  $C$ ,  $(C \cup v)$  is not
        feasible then
            add  $C$  to max_fcones
        endif
    endif
endfor

```

**Return** list of maxima, *max\_fcones*

The set of intervals, denoted as  $I$ , was then created with a direct recursive algorithm on the list of maximum feasible cones, *max\_fcones*. When only one maximum remains the base case will either throw it away or add an interval to  $I$ . The one remaining maximum cone ( $c$ ) is discarded if its dimension is smaller than the necessary minimum cone size needed to produce a vertex of possible minimum degree in the resulting graph  $F$ . If  $c$  is equal

to *MINCONESIZE* then a simple interval  $[c, c]$  is added to  $I$ , otherwise if  $c$  is greater than *MINCONESIZE* an interval  $[cbottom, c]$  is created with the current lattice bottom *cbottom* and  $c$ . The value of *MINCONESIZE* is determined by taking one less than the difference between  $|VG|$  and the maximum degree of  $G$ . This value is the smallest possible cone size that can be assigned as  $N_f(x)$  for any vertex  $x \in G$ . This condition is added to the algorithm because we want to restrict any resulting graphs  $F$  formed by the gluing to having minimum degree equal to the order of graph  $G$  ( $|VG|$ ). In this manner we both further restrict the space of individual gluings and avoid producing duplicate graphs over gluings utilizing graphs  $G$  of different orders. The recursive step splits the list of maxima on one vertex determined by a simple heuristic, which could have just as easily been chosen at random. The heuristic simply checks which vertex within the current lattice provides the greatest separation within the remaining list of maxima. Simply stated the heuristic provides a vertex  $v$  to separate the list of maxima by, moving all feasible cones without  $v$  into a list *maxl1* and all those containing  $v$  into another *maxl2*. The heuristic used in this thesis was modified from one utilized by McKay and Radzisowski in [MR95]. The split was then two-fold with the first case removing  $v$  from the current top of the lattice *ctop* and recursed on *maxl1*. The other case included  $v$  in the current bottom *cbottom* and recursed on *maxl2*. Again pseudo code is illustrated below as a reference.

**INPUT:** *max\_fcones* a list of maximum feasible cones,  
*cbottom* the current bottom cone being considered,  
*ctop* the current top cone being considered

**OUTPUT:**  $I$ , a collection of intervals that covers all feasible cones within the interval  $[cbottom, ctop]$

**NOTE:**  $MINCONESIZE = |VG| - MAXDEGREE(G) - 1$

**if** only one feasible cone  $c$  remains in the list *max\_fcones* **then**  
    **if** ( $|c| < MINCONESIZE$ ) **then**  
        Ignore and **Return**



```

if ( $|c| > MINCONESIZE$ ) then
    Add interval  $[cbottom, c]$  to collection  $I$ 
    Return
if ( $|c| = MINCONESIZE$ ) then
    Add interval  $[c, c]$  to collection  $I$ 
    Return

Determine the vertex  $v$  within the lattice that provides the most
    balanced division within the list of maxima.

Divide the list of maxima into two lists  $maxl1$  and  $maxl2$  with  $maxl1$ 
    including all feasible cones without  $v$  has a member and  $maxl2$ 
    including all feasible cones having  $v$  has a member

if  $maxl1$  is not empty then
    Recurse with
         $max\_fcones = maxl1$ 
         $cbottom = cbottom$ 
         $ctop = ctop - x$ 

if  $maxl2$  is not empty then
    Recurse with
         $max\_fcones = maxl2$ 
         $cbottom = cbottom + x$ 
         $ctop = ctop$ 

```

Using this technique the several thousand *feasible cones* within a typical  $(W_5, K_4)$ -graph can be expressed within a few hundred intervals. The following tables depict the numbers of *feasible cones* and intervals that are typically found in various  $(W_5, K_4; n; e)$ -graphs.

	<b>Total</b>		<b>Interval Dimensions</b>							
<b>Graph</b>	<b>Cones</b>	<b>Intervals</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>
$(W_5, K_4; 16; 52)$	1578	175	10	12	48	63	33	8	1	
$(W_5, K_4; 15; 47)$	1083	134	13	25	39	36	14	5	1	1
$(W_5, K_4; 14; 41)$	1059	116	13	23	36	23	12	5	3	1
$(W_5, K_4; 13; 36)$	666	107	20	29	25	21	6	5	1	
$(W_5, K_4; 12; 31)$	570	59	6	12	17	15	4	3	1	1

Table V Cone and Interval breakdown for typical  $(W_5, K_4; n; e)$ -graphs.

<b>topsize / bottomsize</b>	<b>6</b>	<b>5</b>	<b>4</b>	<b>3</b>	<b>2</b>	<b>1</b>	<b>0</b>
0	1						
1	8	2					
2	31	23	3				
3	40	25	6				
4	20	6	10				

Table VI Interval breakdown for typical  $(W_5, K_4; 16; 52)$ -graphs.

<b>topsize / bottomsize</b>	<b>7</b>	<b>6</b>	<b>5</b>	<b>4</b>	<b>3</b>	<b>2</b>	<b>1</b>	<b>0</b>
0	1							
1	1	4	2	1				
2	1	12	24	10				
3		11	26	18				
4		3	7	12				
5			1					

Table VII Interval breakdown for typical  $(W_5, K_4; 15; 47)$ -graphs.

<b>topsize / bottomsize</b>	7	6	5	4	3	2	1	0
0	1							
1	3	4						
2	1	12	17	2				
3		6	31	9				
4		3	14	10				
5			3					

Table VIII Interval breakdown for typical  $(W_5, K_4; 14; 41)$ –graphs.

<b>topsize / bottomsize</b>	6	5	4	3	2	1	0
0	1						
1	5	2					
2	4	21	5				
3		20	24	1			
4		5	19				

Table IX Interval breakdown for typical  $(W_5, K_4; 13; 36)$ –graphs.

<b>topsize / bottomsize</b>	7	6	5	4	3	2	1	0
0	1							
1	1	3	1					
2		3	15	2				
3			15	4				
4			8	6				

Table X Interval breakdown for typical  $(W_5, K_4; 12; 31)$ –graphs.

## 4.7 Methods of Speedup

In more specific terms, the typical  $(W_5, K_4; 16)$ –graph has between 1000 and 2000 feasible cones and this can be written as 50 – 175 disjoint intervals of cones. We had originally expected that the number of intervals for such a graph would be considerably smaller given the restrictive nature of the

definition of a feasible cone for this case. Fortunately this number could be significantly reduced through an examination of the necessary degrees of the graphs being considered. By examining the maximum degree of the  $(C_4, K_5)$ -graph that will be glued and the minimum degree of the possible resulting  $(W_5, K_5)$ -graph. For this reason  $(C_4, K_5; n)$ -graphs were grouped by maximum degree. This restriction is enforced with the *MINCONESIZE* parameter in the above pseudo-code. Recall that it removes from consideration all maximum feasible cones of size one smaller than the difference of the maximum degree of the current  $(C_4, K_5)$ -graph and the minimum expected degree of the  $(W_5, K_5)$ -graph. We can take this step since any maxima failing this test will only serve to construct an interval whose top is too small to provide the proper minimum degree within the constructed  $(W_5, K_5)$ -graph. We can further expand this concept to each individual vertex within  $G$  by only applying the collapsing rules if the following criteria are satisfied by all vertices being considered. First, the degree of the vertex and the dimension of the bottom of the interval must be smaller than the maximum possible degree, 13, of a  $(W_5, K_5; n)$ -graph. If not, we would be allowing an association that will clearly fail, since no vertex within a  $(W_5, K_5)$ -graph can have degree less than 14. Next the degree of the vertex and the dimension of the top of the interval must be larger than the minimum possible degree of the  $(W_5, K_5; n)$ -graph being considered. This is the same restraint we utilize when filtering *feasible cones*. Consider that after performing the gluing of all  $(C_4, K_5; 9)$  and  $(W_5, K_4; 16)$ -graphs we have all possible  $(W_5, K_5; 26)$ -graphs having minimum degree 9, so we no longer need to consider  $(W_5, K_5; 26)$ -graphs having minimum degree 9 in any future 26 vertex gluing. If we were to remove such a restriction we would produce copies of the  $(W_5, K_5; 26)$ -graphs having minimum degree 9 in further gluing operations. In practice we still collect copies of graphs having smaller minimum degree than desired but this results from vertices in  $H$  and would require more effort to restrict then it warrants.

In the interest of efficiency all functions utilized by the collapsing rules were precomputed and indexed in arrays by the integer value that the cone in question represented. These functions include all  $H$  functions that became

single dimension arrays having size 65536 each. Additionally, all  $K_3$  and  $P_3$  graphs were similarly tabulated for each possible cone value and were collected in two two-dimensional arrays of size 65536 by 250. In this manner we can avoid a significant number of calls to the same functions for the same cone values.

Our original implementation relied on performing the collapsing on all possible permutations of intervals to the vertices of a given  $(C_4, K_5; n)$ -graph. This proved a completely infeasible approach given the enormous number of possible permutations. The next improved effort used a direct backtrack search within the vertices of a given  $(C_4, K_5; n)$  graph and provided orders of magnitude improvement since it eliminated any application of the collapsing rules for an interval combination that we already knew was invalid. For example, if we had four vertices, numbered 0, 1, 2, 3 for clarity, and four intervals to consider,  $A, B, C$ , and  $D$ , then we have 256 possible interval combinations. Observe though that if we know the application of the rules fails with the associations of vertex 0 with interval  $A$  and vertex 1 with interval  $B$  then we no longer need to consider all remaining combinations that include these two associations. In other words, there is no reason to even consider any associations for vertices 3 and 4 with those associations for vertices 0 and 1. As a result we can eliminate 16 future interval combinations (all those having  $1 \Rightarrow A$  and  $2 \Rightarrow B$ ). Below high-level pseudo code of the collapsing functions is provided.

**INPUT:**  $n$ , the order of graph  $G$  being considered.

An association of the vertices of a graph  $G(v_0, v_1, \dots, v_i)$  with intervals of the form  $[B_0, T_0], [B_1, T_1], \dots, [B_i, T_i]$ .

**OUTPUT:** All graphs resulting from extending the association to  $i + 1$  vertices and surviving the application of all collapsing rules.

**NOTE:**  $\deg_G(v)$  - Degree of vertex  $v$  within graph  $G$ .

$ORDER(C)$  - The order of cone  $C$ .

**let**  $I$  be all acceptable intervals within the lattice of cones.

**for** each interval  $[B, T] \in I$  **do**:

**let** vertex  $v_{i+1}$  be assigned interval  $[B, T]$

    If  $v_{i+1} := [B, T]$  meets the degree restrictions of  $F$  with

$\deg_G(v_i) + ORDER(B) \geq n$

$\deg_G(v_i) + ORDER(T) \leq 13$  **then**

**apply** the collapsing rules to the intervals assigned to all vertices  
             $v_0, \dots, v_{i+1}$  of  $G$

**if** the intervals successful pass the application of the rules **then**

**if** all vertices are assigned an interval, ie.  $i + 1 = n$  **then**

                sub-divide intervals and re-apply collapsing rules until  
                no further subdivision is possible. Output any resulting  
                interval combinations.

**else**

**recurse** on next vertex within  $G$ , with associations

$v_0 := [B_0, T_0], \dots, v_{i+1} := [B, T]$

**endif**

**else**

**maintain** associations

$v_0 := [B_0, T_0], \dots, v_i := [B_i, T_i]$  and **loop**

**endif**

**endif**

## 4.8 Further Improvements

There are some areas of improvement not utilized in this work that can still be employed. They were not chosen to be included since the speed of the computations was already acceptable. Consider that the gluing required to determine all  $(W_5, K_5; 26)$ -graphs barely took a week of CPU time to complete. Further speedup can be achieved by a special arrangement of the vertices of the graph  $G$  being considered by the gluing. The algorithm behind this approach has considerable complexity and was utilized in the computations of [MR95]. Next we could also have restricted the minimum size of the bottom of any interval being considered. This follows since the previous technique is only concerned with a listing of all maximum *feasible cones* and makes no attempt to ensure any correctness regarding the bottom of the lattice being considered. Clearly it will be the case much more often than not that any *feasible cone* containing only a few vertices will be unacceptable for not introducing an independent set of size 5. As such in practice the bottoms of intervals are initially raised abruptly. No attempt was made to remedy this situation since the division within the lattice would lead to many more intervals, so it is not clear that it would even work.

## 4.9 Software Package

We made extensive use of the `gtools` package made available by Brendan McKay [McK03]. It is a package, which can be used to generate and check the isomorphism of graphs. Isomorphic graphs are equivalent in a mathematical context and in the course of this thesis we do not wish to consider the same graph on multiple occasions. The `gtools` package is written in a highly portable subset of C, so few compatibility issues arose. The package effectively abstracts away the storing and manipulating of individual graphs and given this software package we were able to work only with graphs that were not isomorphic to each other. The primary functionality of this package is provided by the `nauty` program. `Nauty` can efficiently generate the canonical labeling of graphs. Once the canonical labeling have been determined, graph isomorphisms can be determined, since two graphs are isomorphic if and only if

their canonical labeling are the same [McK03]. Informally, canonical labeling is a mathematical function that relabels the vertices of a graph [McK03].

Of specific use to our efforts in enumerating critical graphs were programs, built on top of `nauty`, contained in the `gtools` software package. One such program `geng` is a powerful program for generating all possible graphs on a small number of vertices. `Geng` can only be reasonably utilized to produce graphs having fewer than 12 vertices given the enormous number of possible graphs (see Table XI). Because of this an approach using one vertex extensions is needed to create graphs containing larger numbers of vertices. The programs `showg` and `pickg` were extremely useful for viewing and selecting graphs with given properties. Another program `shortg` is a program, which will remove all isomorphic graphs from a given graph file. All graphs were stored utilizing a compact notation known as `graph6` format. `Graph6` format writes only the upper right triangle of the adjacency matrix as a bit vector  $x$  of length  $n(n-1)/2$ , having the following ordering:  $(0, 1), (0, 2), (1, 2), (0, 3), (1, 3), (2, 3), \dots, (n-1, n)$  [McK03]. The format also includes the size of the graph as the first byte of its output. The bit vector is then broken into 6 bit blocks, next 63 is added to each 6 bit block and the result is stored as a series of bytes.



Vertices	# Graphs	Time (sec)
1	1	0.00
2	2	0.00
3	4	0.00
4	11	0.00
5	34	0.00
6	156	0.01
7	1044	0.03
8	12346	0.42
9	274668	10.45
10	12005168	450.85

Table XI Geng statistics on generating all graphs up to 10 vertices. Running time for **geng** program on a Sun Ultra 5/10 UPA/PCI (UltraSPARC-IIi 440MHz) @ 110.0 MHz.

Once the entire algorithm had been properly prepared and the gluing jobs necessarily subdivided the **autoson** tool was utilized to run the jobs over a LAN network of computers. This tool helps to manage the scheduling of multiple jobs over a network. Each job required approximately 3 hours of computations and a few hundred jobs were needed all together. As much as possible jobs were separated based on the maximum degree in the set of  $(C_4, K_5)$ -graphs being considered for the reasons stated earlier in section 4.7. As many as 50 computers were utilized at any given time during this computation and the specific measurements are attached as a reference (see section 5).

## 4.10 Implementation Specifics

The implementation of the gluer was done entirely in  $C$  for both speed considerations and compatibility with the **nauty** software package. The following files make up the entire gluer and are available as a reference at <http://www.cs.rit.edu/~jjs9804>.

Files	Description
glue.c	Main program, handles flow of events
partition.c	Contains necessary functions to partition and set up the intervals
collaspe.c	Contains necessary functions to perform the collapsing of intervals
globals.c	Contains all necessary global variables
globals.h	Contains the necessary global definitions
tools.c	Contains any other necessary functions in the gluer
times.h	Contains necessary timing functions
prototypes.h	Contains the definitions for all functions utilized by the gluer
includes.h	Maintains all the include files for the gluer

The remaining files utilized in this thesis are listed below and again are made readily available at <http://www.cs.rit.edu/~jjs9804>.

Files	Description
C4K5.c	Program to filter out all non $(C_4, K_5)$ –graphs
W5K4.c	Program to filter out all non $(W_5, K_4)$ –graphs
W5K5.c	Program to filter out all non $(W_5, K_5)$ –graphs
W5K4-ext.c	Extension III for $(W_5, K_4)$ –graphs
W5K5-ext.c	Extension III for $(W_5, K_5)$ –graphs
vertex-rem.c	Vertex removal for $(W_5, K_5)$ –graphs

## 5 Results

The approach in this thesis is two fold. Primarily we developed a gluing algorithm in an attempt to glue together  $(C_4, K_5)$ –graphs and  $(W_5, K_4)$ –graphs to construct a complete set of  $(W_5, K_5; 26)$ –graphs. This required enumerating all  $(C_4, K_5; p)$  and  $(W_5, K_4; q)$  graphs such that  $p + q + 1 = 26$ . Upon completion of the gluing procedure we utilized a one vertex extension algorithm to determine if any of the  $(W_5, K_5; 26)$ –graphs could be extended to  $(W_5, K_5; 27)$ –graphs. The second aspect of this thesis was the attempted direct construction, with use of the gluing procedure, of  $(W_5, K_5)$ –graphs on 27, 28, or 29 vertices. If no such graphs could be constructed we would have further evidence of the correctness of the prior calculations.

After grouping all  $(C_4, K_5; n)$ -graphs by maximum degree we began with the gluing process by first performing all gluings that could possibly result in a  $(W_5, K_5; 30)$ -graph (see Table *XII*). This meant considering all  $(C_4, K_5; 13)$ -graphs and  $(W_5, K_4; 16)$ -graphs. This gluing operation was quickly completed taking only 17 seconds. Next we sought to eliminate the possibility of any  $(W_5, K_5; 29)$ -graphs (see Table *XIII*),  $(W_5, K_5; 28)$ -graphs (see Table *XIV*), and  $(W_5, K_5; 27)$ -graphs (see Table *XV*). No graphs of these types were found and the statistics of the gluing procedure are provided below. Of particular interest is that none of these computations required the multi-tasking approach utilized for exhaustive enumeration of all  $(W_5, K_5; n)$ -graphs.

$(W_5, K_5; 30)$ gluing				
Graphs G	Graphs H	# Gluings	# Graphs Found	Time
$(C_4, K_5; 13)$	$(W_5, K_4; 16)$	82	0	17s
<b>Total:</b>		82	0	17s

Table XII Statistics for  $(W_5, K_5; 30)$  gluing

$(W_5, K_5; 29)$ gluing				
Graphs G	Graphs H	# Gluings	# Graphs Found	Time
$(C_4, K_5; 13)$	$(W_5, K_4; 15)$	2891	0	3m
$(C_4, K_5; 12)$	$(W_5, K_4; 16)$	10332	0	5m
<b>Total:</b>		13223	0	8m

Table XIII Statistics for  $(W_5, K_5; 29)$  gluing

$(W_5, K_5; 28)$ gluing				
Graphs G	Graphs H	# Gluings	# Graphs Found	Time
$(C_4, K_5; 13)$	$(W_5, K_4; 14)$	111153	0	5h:49m
$(C_4, K_5; 12)$	$(W_5, K_4; 15)$	364266	0	2h:19m
$(C_4, K_5; 11)$	$(W_5, K_4; 16)$	37474	0	6m:28s
<b>Total:</b>		512893	0	8h:14m

Table XIV Statistics for  $(W_5, K_5; 28)$  gluing

$(W_5, K_5; 27)$ gluing				
Graphs G	Graphs H	# Gluings	# Graphs Found	Time
$(C_4, K_5; 13)$	$(W_5, K_4; 13)$	865055	0	1d: 3h
$(C_4, K_5; 12)$	$(W_5, K_4; 14)$	14005278	0	1d: 4h
$(C_4, K_5; 11)$	$(W_5, K_4; 15)$	1321187	0	2h: 12m
$(C_4, K_5; 10)$	$(W_5, K_4; 16)$	574	0	32m: 52s
<b>Total:</b>		16238588	0	2d: 10h

Table XV Statistics for  $(W_5, K_5; 27)$  gluing

These calculations provided a proof that  $R(W_5, K_5) \leq 27$  since no  $(W_5, K_5; 27)$ -graphs were found, implying  $R(W_5, K_5) = 27$ . We proceeded with the gluing operation necessary to enumerate all possible  $(W_5, K_5; 26)$ -graphs to show by another path that  $R(W_5, K_5) = 27$ . This computation took only around a week of CPU time producing only one  $(W_5, K_5; 26)$ -graph that was cyclic of degree 9 having distances 1, 5, 8, 12, 13 (see Table XVI). The adjacency matrix of this graph, with appropriate labeling of the composite graphs  $G$  and  $H$  is included in Figure XIII below. Next we utilized the aforementioned one vertex extension algorithm on the one  $(W_5, K_5; 26)$ -graph and as expected it could not be extended to a  $(W_5, K_5; 27)$ -graph. This provided independent evidence that no  $(W_5, K_5; 27)$ -graph exists.

$(W_5, K_5; 26)$ gluing				
Graphs G	Graphs H	# Gluings	# Graphs Found	Time
$(C_4, K_5; 13)$	$(W_5, K_4; 12)$	1437877	0	15h: 53m
$(C_4, K_5; 12)$	$(W_5, K_4; 13)$	108996930	0	4d: 2h
$(C_4, K_5; 11)$	$(W_5, K_4; 14)$	50796921	0	1d: 17h
$(C_4, K_5; 10)$	$(W_5, K_4; 15)$	1659434	0	17h: 37m
$(C_4, K_5; 9)$	$(W_5, K_4; 16)$	31570	1	1h: 48m
<b>Total:</b>		162922732	1	7d: 6h

Table XVI Statistics for  $(W_5, K_5; 26)$  gluing

<b>V</b>	0111111111	0000000000000000
	<b>G</b>	<b>Gluer</b>
	1010000001	1100000011110000
	1100000001	1100000011110000
	1000100001	0000110011000011
<b>G</b>	1001000001	0000110011000011
	1000001001	0000001100111100
	1000010001	0000001100111100
	1000000011	0011000000001111
	1000000101	0011000000001111
	1111111110	0000000000000000
		<b>H</b>
	0110000000	0100111100000011
	0110000000	1000111100000011
	0000000110	0001111100110000
	0000000110	0010111100110000
	0001100000	1111010000001100
	0001100000	1111100000001100
	0000011000	1111000111000000
	0000011000	<b>H</b> 1111001011000000
	0111100000	0000001101001100
	0111100000	0000001110001100
	0110011000	0011000000010011
	0110011000	0011000000100011
	0000011110	0000110011000100
	0000011110	0000110011001000
	0001100110	1100000000110001
	0001100110	1100000000110010

Figure *XII* Adjacency matrix for the only  $(W_5, K_5; 26)$  critical graph. It is cyclic of degree 9 with distances 1, 5, 8, 12, 13.

To provide further evidence to support our claim that our programs are correct and  $R(W_5, K_5) = 27$ . We next used an algorithm to remove a vertex in every possible way from the one  $(W_5, K_5; 26)$  critical graph. In this manner we can repeatedly apply the algorithm to arrive at a collection of some  $(W_5, K_5; n)$ -graphs, where  $n < 26$ . Then we can take the one vertex extension algorithm and extend the new collection of graphs back to  $(W_5, K_5; 26)$ -graphs. If our collection of critical graphs is exhaustive we should arrive at precisely the same graph when applying our one vertex extensions. We proceeded by repeatedly applying the one vertex take away algorithm until a collection of  $(W_5, K_5; 21)$ -graphs was obtained and then applied the one vertex extension algorithm on this new collection until another set of  $(W_5, K_5; 26)$ -graphs was obtained. Given the enormous divergence in the number of graphs it was only reasonably possible to retrace our steps starting at 21 vertices, although we took the removal deeper in the interests of curiosity. As expected the same  $(W_5, K_5; 26)$  critical graph was discovered with this approach. The results (illustrated below) of applying the vertex removal algorithm to the unique  $(W_5, K_5; 26)$ -graph were verified independently by Stanisław. This was particularly important since the results of applying the vertex removal algorithm were surprising given that generally there is a much more significant divergence in the numbers observed between the down and up phases of the algorithm.

Order	# graphs down	# graphs up
26	1	1
25	1	6
24	4	3765
23	10	62890
22	40	115758
21	103	15674
20	264	-
19	546	-

Table XVII One vertex take away results.

## 5.1 Collaboration with Kung-Kuen Tse

The statistics for the  $(C_4, K_5; n)$ -graphs had previously been computed by Radziszowski and Tse [RT02] and this provided clear evidence as to the correctness of our collection of  $(C_4, K_5; n)$ -graphs. Tse also independently provided the same collection of  $(W_5, K_4; n)$ -graphs that completely agreed with the set determined within this thesis. Hence, we could reasonably conclude the correctness of our collection of  $(W_5, K_4; n)$ -graphs.

In the interest of obtaining further evidence as to the correctness of the gluing method, we sought to obtain independent confirmation of the results of several identical gluings. If we could achieve agreement on several different gluings, each producing a few thousand graphs, then we could with reasonably certainty conclude that our processes were indeed correct. Once again, Tse kindly provided this collaboration by performing the same gluing processes utilizing a gluing algorithm developed completely independent of the efforts within this thesis. The first gluing attempted was to glue together all  $(C_4, K_5; 7)$  and  $(W_5, K_4; 16)$  graphs. This required attempting 6970 separate gluings (85  $(C_4, K_5; 7)$ -graphs and 82  $(W_5, K_4; 16)$ -graphs), taking around 45 minutes to complete. Importantly, this initial collaboration resulted in a disagreement with Tse discovering 17421  $(W_5, K_5; 24)$ -graphs of minimum degree 7 and the original gluing algorithm in this thesis determining only 14823. Upon further analysis a significant oversight was discovered in the implementation of the gluing algorithm. This oversight including an oversimplification of the reduction within rule (d) of the collapsing rules and an implementation mistake regarding the division of cone intervals. After the necessary changes had been made the identical set of 17421  $(W_5, K_5; 24)$ -graphs was obtained, agreeing completely with Tse. Next we performed the gluing of all 190  $(C_4, K_5; 8)$ -graphs and all 2891  $(W_5, K_4; 15)$ -graphs into the complete set of  $(W_5, K_5; 24)$ -graphs having minimum degree 8. We both arrived an identical collection of 1768  $(W_5, K_5; 24)$  having minimum degree 8. For completeness we ran a third collaboration by attempting the gluing of all 38  $(C_4, K_5; 6)$ -graphs and all 82  $(W_5, K_4; 16)$ -graphs. Once again we agreed completely arriving at the same set of 869853  $(W_5, K_5; 23)$ -graphs all having minimum degree 6. As expected in the gluings of particular importance

we similarly agreed completely with neither of us finding any such graphs  $(W_5, K_5; n)$  for  $27 \leq n \leq 30$ . Finally, and most importantly, we agreed on the finding of one unique  $(W_5, K_5; 26)$ -graph, being cyclic of degree 9.

## 6 Additional Tables

n	1	2	3	4	5	6	7	8
e								
0	1	1	1					
1		1	1	1				
2			1	2	1			
3			1	3	3	1		
4				2	5	2		
5				1	6	7	1	
6				1	6	15	4	
7					4	20	13	1
8					1	21	32	4
9					1	17	65	14
10						9	93	49
11						2	95	139
12							67	317
13							25	519
14							5	590
15							1	429
16								185
17								48
18								9
19								2
20								1
total:	1	2	4	10	26	94	401	2307

Table XVIII  $(W_5, K_4; n; e)$ -graph statistics,  $n \leq 8$ .



n	9	10	11	12
e				
9	1			
10	2			
11	9			
12	35	1		
13	129	3		
14	429	13		
15	1223	51	1	
16	2619	201	2	
17	3932	742	10	
18	3826	2583	39	1
19	2270	7572	157	1
20	785	17010	598	4
21	162	26486	2275	14
22	26	26612	8100	54
23	3	16174	25518	192
24	1	5596	64329	737
25		1087	117555	2700
26		151	143045	9702
27		25	108332	32019
28		5	47815	91650
29		2	11908	207011
30			1853	337844
31			290	367179
32			49	251305
33			13	104167
34			2	26853
35			1	5080
36				1048
37				230
38				65
39				14
40				5
41				1
42				1
total:	15452	104314	531892	1437877

Table XIX  $(W_5, K_4; n; e)$ -graph statistics,  $n \geq 9$ .

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