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**An Algorithmic Approach for
Multi-color Ramsey Graphs**

by

Wei Li

July 28, 1987

A thesis, submitted to
The Faculty of the School of Computer Science and Technology,
in partial fulfillment of the requirement for the degree of
Master of Science in Computer Science.

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Title of Thesis: An Algorithmic Approach for Multi-color Ramsey Graphs

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ABSTRACT

The classical Ramsey number $R(r_1, r_2, \dots, r_m)$ is defined to be the smallest integer n such that no matter how the edges of K_n are colored with the m colors, 1, 2, 3, \dots , m , there exists some color i such that there is a complete subgraph of size r_i , all of whose edges are of color i .

The problem of determining Ramsey numbers is known to be very difficult and is usually split into two problems, finding upper and lower bounds. Lower bounds can be obtained by the construction of a, so called, Ramsey graph. There are many different methods to construct Ramsey graphs that establish lower bounds. In this thesis mathematical and computational methods are exploited to construct Ramsey graphs.

It was shown that the problem of checking that a graph coloring gives a Ramsey graph is NP-complete. Hence it is almost impossible to find a polynomial time algorithm to construct Ramsey graphs by searching and checking. Consequently, a method such as backtracking with good pruning techniques should be used. Algebraic methods were developed to enable such a backtrack search to be feasible when symmetry is assumed.

With the algorithm developed in this thesis, two new lower bounds were established: $R(3,3,5) \geq 45$ and $R(3,4,4) \geq 55$. Other best known lower bounds were matched, such as $R(3,3,4) \geq 30$. The Ramsey graphs giving these lower bounds were analyzed and their full symmetry groups were determined. In particular it was shown that there are unique cyclic graphs up to isomorphism giving $R(3,3,4) \geq 30$ and $R(3,4,4) \geq 55$, and 13 non-isomorphic cyclic graphs giving $R(3,3,5) \geq 45$.

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NOTATION USED IN THE TEXT

| | |
|--|---|
| B_i | Boolean expression for color i . |
| C_{ij} | j th G-orbits of i -subsets. |
| CNF | conjunctive normal form of Boolean expression. |
| D_n | dihedral group. |
| E | set of edges of graph. |
| $\Gamma(V, E)$ | A graph. |
| $\Gamma[V_1]$ | subgraph of Γ only including the vertex subset V_1 . |
| $GF(p^m)$ | Galois field. |
| I or J | G-orbit. |
| K_n | complete graph of n vertices. |
| P_r | pattern matrix. |
| $R(k, l)$ | two color Ramsey. |
| $R(r_1, r_2, \dots, r_m)$ | m-color Ramsey number. |
| $R(r_1, r_2, \dots, r_m; t)$ | Ramsey number coloring t -subsets. |
| $R(\Gamma_1, \Gamma_2, \dots, \Gamma_m)$ | generalized Ramsey number. |
| $Sym(V)$ | set of all permutations of V . |
| U_i | solution vector for color i . |
| V | set of vertices of graph. |
| \mathbb{Z}_n | integer group with mod n . |
| $d(v)$ | degree of vertex v . |
| $\bar{1}$ | $[1, 1, \dots, 1]^T$. |
| $\alpha(\Gamma)$ | size of largest independent set in Γ . |

1. CHAPTER I: INTRODUCTION

"Most certainly 'Ramsey Theory' is now an established and growing branch of combinatorics. Its results are often easy to state (after they have been found) and difficult to prove; they are beautiful when exact, and colorful". [*Frank Harary*][12]

The notation used in this thesis is in the previous table. The author assumes that the readers of this work have the knowledge of computational complexity, graph theory and algebra. Only a few mathematical concepts were introduced in order to make this thesis succinct. Most of these theorems which can be found in the text books or papers are just stated with references and with their proof omitted. It is important to remark that whenever credit for a theorem is not specifically assigned, the result is original.

1.1. Classical Ramsey Numbers

The *classical Ramsey number* $R(r_1, r_2, \dots, r_m)$ is defined to be the smallest integer n such that no matter how the edges of K_n are colored with m colors, 1, 2, 3, ... m , there exists some i such that there is a complete subgraph K_{r_i} , all of whose edges have color i [2]. It is said to be a multi-color Ramsey number where $m \geq 3$. Ramsey's theorem says that all of these numbers exist [22].

The problem of determining the Ramsey numbers is known to be very difficult [2]. It is usually split into two problems, finding upper bounds and lower bounds.

A general upper bound is obtained by the formula in Theorem 1.

THEOREM 1(Greenwood & Gleason, 1955[11].)

$$R(r_1, r_2, \dots, r_m) \leq 2 + \sum_{i=1}^m (R(r_1, \dots, r_{i-1}, r_i - 1, r_{i+1}, \dots, r_m) - 1)$$

Proof: (omitted).

This bound can be improved in some cases by the next two lemmas.

LEMMA 1(Greenwood & Gleason, 1955[11].) *If $R(k, l-1)$ and $R(k-1, l)$ are both even,*

then

$$R(k, l) \leq R(k, l-1) + R(k-1, l) - 1 \quad (1.3)$$

for any two integers $k \geq 2$ and $l \geq 2$.

It is easy to extend it to the multi-color case and since this thesis only deals with lower bounds of Ramsey numbers, it is stated below without proof.

LEMMA 2. *If $1 + \sum_{i=1}^m (R(r_1, \dots, r_{i-1}, r_i - 1, r_{i+1}, \dots, r_m) - 1)$ is even and there is an i such that*

$R(r_1, r_2, r_i - 1, \dots, r_m)$ is also even then

$$R(r_1, r_2, \dots, r_m) \leq 1 + \sum_{i=1}^m (R(r_1, \dots, r_{i-1}, r_i - 1, r_{i+1}, \dots, r_m) - 1)$$

for $r_i \geq 2$, $1 \leq i \leq m$.

The upper bounds of Theorem 1 is suspected never to be tight for $m \geq 4$ and $r_i \geq 3$.

Lower bounds are usually established by the explicit construction of a coloring of K_n , the complete graph on n vertices, containing no monochromatic complete subgraph K_{r_i} , $1 \leq i \leq m$ in the i th color. A coloring of K_n that established the lower bound on $R(r_1, r_2, \dots, r_m)$ is said to be an $R(r_1, r_2, \dots, r_m)$ Ramsey graph.

In figure 1 are some examples for two-color Ramsey numbers.

The graphs of figure 1 shows that

$$R(3, 3) \geq 6, \quad R(3, 4) \geq 9, \quad R(3, 5) \geq 14, \quad R(4, 4) \geq 18.$$

By the theorem 1, we can get upper bounds for these Ramsey numbers. First

$$R(3, 3) \leq R(3, 2) + R(2, 3) = 3 + 3 = 6$$

and therefore we have $R(3, 3) = 6$. Noting that $R(3, 3)$ and $R(2, 4)$ are both even, we obtain

$$R(3, 4) \leq R(3, 3) + R(2, 4) - 1 = 6 + 4 - 1 = 9$$

So $R(3, 4)$ is 9. Furthermore, we obtain

$$R(3, 5) \leq R(3, 4) + R(2, 5) = 9 + 5 = 14$$

and

$$R(4, 4) \leq R(3, 4) + R(4, 3) = 2R(3, 4) = 18$$

which yield $R(3, 5) = 14$ and $R(4, 4) = 18$.

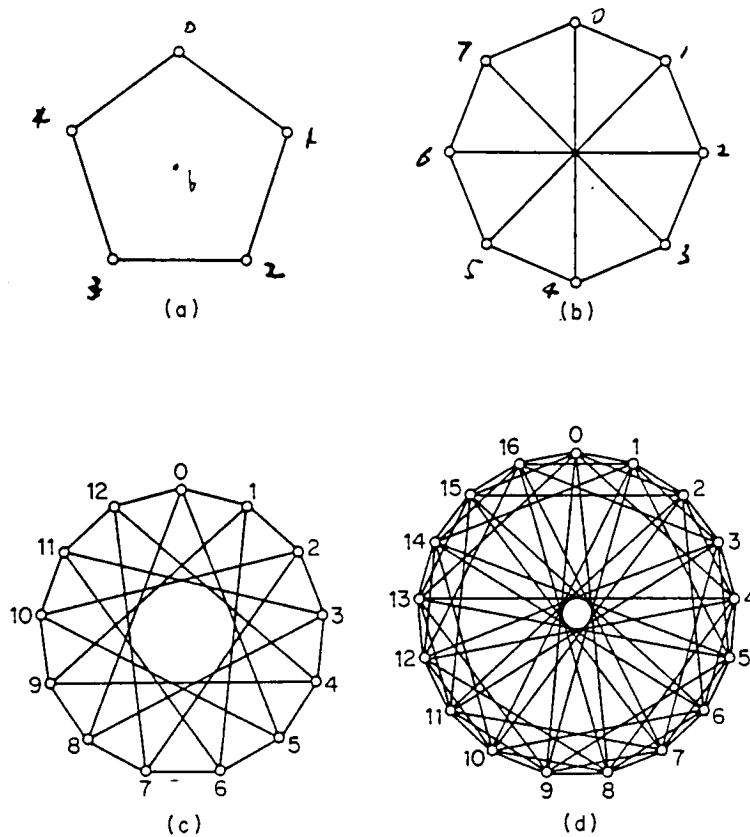


Figure 1: Examples of Ramsey graphs

Only a few exact values and nontrivial bounds are known, and most of them are for $R(k, l)$ the so called *two-color* Ramsey numbers. Table I from [21] contains the exact values and bounds known so far.

The *centered* numbers in the table refer to the exact known values of Ramsey numbers, where a pair of numbers give the best known lower and upper bounds. A *single* number at the *top* of an entry gives the best known lower and upper bounds and there is not known any upper bound better than the one implied by theorem 1.

The only known exact value for m -color Ramsey numbers, with $m \geq 3$ is $R(3, 3, 3) = 17$ [11]. The only known nontrivial bounds on m -color Ramsey numbers with $m \geq 3$ are

Table I

| $k \backslash l$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|------------------|---|----|----------|------------|------------|----------|-----|----------|----------|----------|----------|----------|
| 3 | 6 | 9 | 14 | 18 | 23 | 28 29 | 36 | 39 44 | 46 54 | 49 63 | 58 73 | 63 84 |
| 4 | | 18 | 25 28 | 34 44 | 47 | 52 | 69 | | | | | |
| 5 | | | 42 55 | 57 94 | 76 | 94 | | | | | | |
| 6 | | | | 102 169 | | | | | | | | |
| 7 | | | | | 205 586 | | | | | | | |
| 8 | | | | | | 282 | | | | | | |
| 9 | | | | | | | 565 | | | | | |
| 10 | | | | | | | | 798 | | | | |

$$\begin{aligned}
 51 &\leq R(3,3,3,3) \leq 65 & [3][4] \\
 159 &\leq R(3,3,3,3,3) \leq 322 & [5][26] \\
 128 &\leq R(4,4,4) \leq 254 & [13][7]
 \end{aligned}$$

The proof of $R(3,3,3) = 17$ by Greenwood and Gleason [11] was obtained by considering the Galois field of 2^4 elements. In Figure 2 the subgraph formed by the 16 vertices and the edges of one color is represented. The two subgraphs formed by the 16 vertices and the edges of the other two colors are also isomorphic to this [16].

1.2. Ramsey Theory and Generalized Ramsey Numbers

The original proof of Ramsey's Theorem [22] was given by a man with very little interest in Combinatorial Mathematics. His primary interest was Philosophy. Ramsey proves his theorem in an eight page paper "On a problem of formal logic" [22], which solves a special case of the decision problem for first-order predicate calculus with equality. The irony is that, although Ramsey produced his theorem to help solve this problem it can be solved without it. Moreover, Ramsey only solved this special case as a contribution towards solving the general decision problem, an objective which Godel [9] in effect showed to be unattainable the year after Ramsey died.

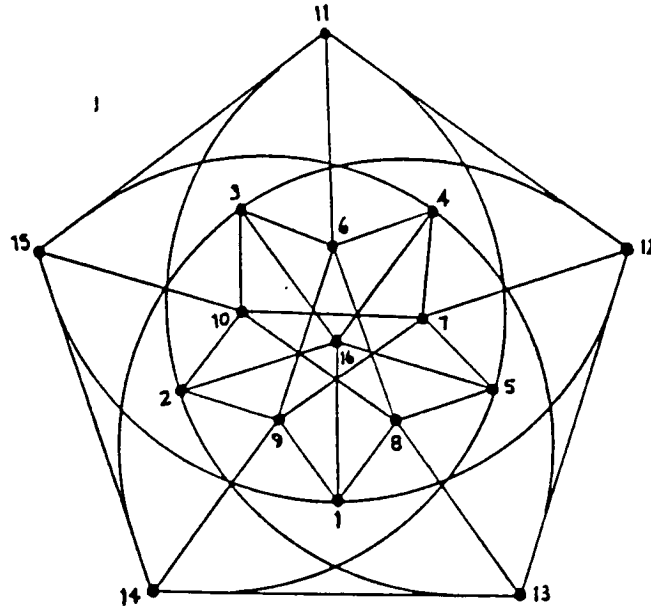


Figure 2: $R(3,3,3) > 16$

Ramsey begins his paper with what is called the infinite version of Ramsey's theorem.

THEOREM 2 (Ramsey, 1930[22].) *Let Γ be an infinite class, and μ and r positive integers; and let all those sub-classes of Γ which have exactly r members, or; as we may say, let all r -combinations of the members of Γ be divided in any manner into μ mutually exclusive classes C_i ($i=1,2,\dots,\mu$), so that every r -combination is a member of one and only one C_i ; then, assuming the Axiom of Selections, Γ must contain an infinite sub-class Δ such that all the r -combinations of the members of Δ belong to the same C_i .*

Ramsey's Theorem, in its full strength, may be rephrased below.

THEOREM 2' (Spencer, 1983[25].) *Let t, m, r_i be positive integers. If N is sufficiently large and if the t -sets of an N -set are colored arbitrarily with m colors then there exists an r_i -set, all of whose t elements subsets are the same color.*

Proof . (see [25])

We now introduce the definitions of generalized Ramsey numbers below.

DEFINITION 1. The Ramsey number $R(r_1, r_2, \dots, r_m; t)$ is defined to be the smallest integer N with the property that whenever S is a set of N elements and we divide the t -element subsets of S into m sets, X_1, X_2, \dots, X_m , then for some i , there is a r_i -element subset of S all of whose t -element subsets are in X_i .

The classical Ramsey number $R(r_1, r_2, \dots, r_m)$ discussed in this thesis is $R(r_1, r_2, \dots, r_m; 2)$

DEFINITION 2. If $\Gamma_1, \Gamma_2, \dots, \Gamma_m$ are graphs, then $R(\Gamma_1, \Gamma_2, \dots, \Gamma_m)$ is the smallest N with the property that every coloring of edges of the complete graph K_N in the m colors $1, 2, \dots, m$ gives rise, for some i , to a subgraph that is isomorphic to Γ_i and is colored all in color i .

1.3. Applications of Ramsey Theory and Numbers

Ramsey theory is interesting. Is it really useful? There are some direct use of the Ramsey numbers found in communications, in information retrieval and in decision making.

1.3.1. Confusion Graphs for Noisy Channels

In communication, the receiver may not be able to receive the same code the sender sent because of the noise in the transmission. In this case we say that the channel is noisy [23].

The *confusion graph* for a noisy channel is defined as a graph whose vertices are elements of a transmission alphabet T and which has an edge between two letters of T iff they can be received as the same letter when sent over the channel.

Given a noisy channel, we would like to choose a *unambiguous code alphabet*, that is a such alphabet of T in which no pair of letter is confusable with another one, for sending messages. This corresponds to choosing an independent set in the confusion graph. Define $\alpha(\Gamma)$ to be the size of the largest independent set in Γ , then the largest unambiguous code alphabet has $\alpha(\Gamma)$ elements.

Figure 3 is an example of a confusion graph. The largest independent graph contains

two vertices, either a, c or a, d .

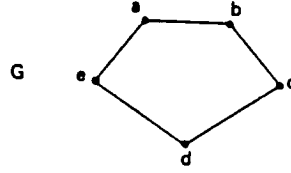


Figure 3: Confusion graph

By using the string of two elements from T , A better unambiguous alphabet can be obtained. For example the strings aa , ac , ca and cc from Figure 3 are nonconfusable to each other.

The confusion graph for the two-letter strings are defined as follow:

There is an edge between vertices xy and uv iff one of the following holds:

- (i) x and u can be confused and y and v can be confused.
- (ii) $x = u$ and y and v can be confused.
- (iii) $y = v$ and x and u can be confused.

The confusion graph for two-letter strings could be represented as $\Gamma \times H$, where the first letter is from Γ and the second from H . The size of the largest independent set of the graph $\Gamma \times H$ could be found by the following theorem.

THEOREM 3(Hedrlin, 1966[23].) If Γ and H are any graphs, then

$$\alpha(\Gamma \times H) \leq R(\alpha(\Gamma)+1, \alpha(H)+1)-1$$

where $R(r_1, r_2)$ is the Ramsey number.

Proof. (Omitted)

Γ is the confusion graph of Figure 3. Hence

$$\alpha(\Gamma \times \Gamma) \leq R(3,3)-1 = 5$$

which is larger than $\alpha(\Gamma)$. Thus the size of an unambiguous alphabet is bigger.

1.3.2. Other Applications Using the Generalized Ramsey Numbers

Other applications were found in the design of packet switched network by Boyles and Exoo [23], in information retrieval by Yao [28,29] and in the decision making by Bogart, Rabinovitch and Trotter [1]. These applications use the so called *generalized Ramsey Numbers*. Readers who are interested in these are encouraged to read the works named above.

2. CHAPTER II: THEORETICAL DEVELOPMENT

2.1. Computational Complexity

In this thesis, computational methods are used to search for Ramsey numbers. Thus knowing how long a program is going to run can be useful. It can help in deciding which of several correct algorithms should be used to solve a given problem. The most obvious approach may work perfectly well on small problems but require years or even centuries on large ones. Analysis of algorithm can determine whether an algorithm is practical for the problem size it is intended to run on. If it is not practical, the expense of running a program which can never finish can be avoided and instead one could concentrate on looking for a possible better algorithm.

There is wide agreement that a problem has not been "well-solved" until a polynomial time algorithm is known for it. Unfortunately, the best known algorithms for many problems are exponential in time. These are the so called *inherently intractable* problems. Proving the inherent intractability of a problem can be just as hard as finding efficient algorithms. Although as pointed out by Garey and Johnson that "Even the best theoreticians have been stymied in their attempts to obtain such proofs for commonly encountered hard problems." If such results are available, they can save much fruitless searching for better algorithms.

On the other hand, there are some exponential time algorithms that have practical use, since time complexity is defined as a "worst case" measure. This means that asymptotically only some instances of the problem need that much time. This maybe the case that most problem instances might require far less time than that. For example, branch-and-bound algorithms for knapsack problem have been so successful that many consider it as a well-solved problem, even though these algorithms have exponential time complexity.

One of the most famous open questions is the well known "*P-NP problem*", i.e., the question whether $P=NP$ or $P\neq NP$ holds. The hardest problems in NP are called NP-complete problems. In the short time since its definition in the early 1970's, this term has

come to symbolize the inherent intractability that algorithm designers increasingly face as they seek to solve larger and more complex problems. A wide variety of commonly encountered problems from Mathematics and Computer Science, in such fields as Graph Theory, Game Theory, Number Theory, Operation Research, Logic, Combinatorics, etc are known to be *NP*-complete, and the collection of such problems continues to grow very rapidly.

For example, the *Clique* problem given below is an NP-complete problem. [6]

INSTANCE: A graph $\Gamma = (V, E)$ and a positive $W \leq |V|$.

QUESTION: Does Γ contain a *clique* of size W or more?

Before searching for the multi-color Ramsey numbers, we first try to formalize the problem to see where it could be located in the complexity hierarchy.

Problem 1:

Check Ramsey graph $R(r_1, r_2, \dots, r_m)$ on n vertices with an m -coloring.

INSTANCE: A coloring of the edges of K_n with m colors 1, 2, ..., m , and m positive integers $r_i \leq n$, $1 \leq i \leq m$.

QUESTION: Does K_n contain a clique of size r_i of color i for some i ?

Problem 2.

$R(r_1, r_2, \dots, r_m) \geq n$.

INSTANCE: A complete graph K_n and m positive integers $r_i \leq n$, $1 \leq i \leq m$.

QUESTION: Does there exist a coloring of the edges of K_n with m colors 1, 2, ..., m , that contains no clique of size r_i of color i , for each $1 \leq i \leq m$?

Problem 3.

Construct $R(r_1, r_2, \dots, r_m)$ Ramsey graph.

INSTANCE: Given m positive integers $r_i \leq n$, $1 \leq i \leq m$.

QUESTION: Is n the maximal vertex size on which there exist an m -coloring of the complete graph K_n which contains no clique of size r_i of color i for each i ?

In the exhaustive search approach Problem i is a subproblem of Problem j , where $i < j$. Problem 3 is exactly the multi-color Ramsey problem that we wish to solve. Problem 1 looks similar to the *clique* problem we mentioned before, which is an NP-complete problem. One may believe that the Problem 1 is also an NP-complete problem.

THEOREM 1. Problem 1 is NP-complete.

Proof. It is easy to prove that Problem 1 is in NP since a nondeterministic algorithm need only guess a subset of vertices and check in polynomial time whether that subset induces a monochromatic complete graph.

To show that some NP problem is complete, usually we need to transform a known NP-complete problem to this problem. As mentioned before, *Clique* problem is NP-complete. Consider the two-color case of Problem 1, which is a special case of general Ramsey Graph problem, and also *Clique* problem.

Given any graph $\Gamma_1 = (V, E)$ and a positive integer $W \leq |V|$, we can construct a graph $\Gamma_2 = \Gamma_1 = (V, E)$ and $r_1 = W$ and $r_2 = |V| + 1$. The question for the *clique* problem is if Γ_1 contains a clique of size W or more. The question for problem 1 is if Γ_2 is not a $R(r_1, r_2)$ Ramsey graph. We can see the equivalence of the two problems. If it is true for problem 1, then the answer is yes for the *clique* problem. If the answer is yes for the *clique* problem, then it is true for problem 1. Hence Γ_1 contains a clique of size W or more if and only if Γ_2 is not a $R(W, |V| + 1)$ Ramsey graph.

Hence, problem 1 is an NP-complete problem.

QED

2.2. Pattern Matrices

2.2.1. Permutation Groups and Cyclic Graphs

For notation, definitions and theorems on permutation groups the reader is directed to the book by Wielandt (1964). Here, we introduce some of the notation and concepts relevant to this thesis. If V is a set then $Sym(V)$ denotes the full symmetric group on V . A group G is said to act on a set V if there is a function $V \times G \rightarrow V$ (usually denoted by $(v, g) \mapsto v^g$) such that for all $g, h \in G$ and $v \in V$:

$$v^1 = v \quad \text{and} \quad v^{(gh)} = (v^g)^h.$$

We denote an action by $G|V$. Thus G may be thought of as being mapped homomorphically onto a permutation group on V , and v^g is the image of $v \in V$ under $g \in G$. If $v \in V$ the *stabilizer* in G of v is the subgroup $G_v = \{g \in G : v^g = v\}$ and the *orbit* of $v \in V$ under G is $v^G = \{v^g : g \in G\}$. We note that $|G| = |v^G| * |G_v|$. If $|v^G| = |V|$ then the group action $G|V$ is said to be *transitive*.

A group action $G|V$ induces an action on the collection $\binom{V}{t}$ of t -subsets of V . For if $S \subseteq V$ and $g \in G$ then we define S^g by $S^g = \{v^g : v \in S\}$.

Finally we recall one important combinatorial lemma in permutation groups whose proof may be found in almost any algebra or elementary group theory text.

LEMMA (Cauchy-Frobenius-Burnside). If $Fix(g) = \{v \in V : v^g = v\}$, then the number of distinct orbits of $G|V$ is:

$$\frac{1}{|G|} \sum_{g \in G} |Fix(g)|$$

If $\Gamma = (V, E)$ is a graph and $G \leq Sym(V)$ is such that $e^g \in E$ for every $g \in G$ and edge $e \in E$ then G is said to be an *automorphism group* of the graph $\Gamma(V, E)$. The largest such subgroup of $Sym(V)$ is called the *full automorphism* of $\Gamma(V, E)$.

For even a relatively small number of vertices, an exhaustive computer search for Ramsey graphs is infeasible. For an m -colored graph on n vertices, there are $m^{\binom{n}{2}}$ possible

colorings. However, if certain conditions are imposed on the colorings, then exhaustive searches do become practical for moderate values of m and n .

A graph Γ with vertex set $V = \{0, 1, 2, \dots, n-1\}$ is *cyclic* if the mapping $g : x \rightarrow x + 1$ is an automorphism of Γ , addition performed modulo n .

Note that any cyclic graph Γ must have as an automorphism group at least the dihedral group $D_n = \langle g, h \rangle$, where $h : x \rightarrow -x$, since $g^{(n-u-v)} : \{u, v\} \rightarrow \{-v, -u\}$.

The Ramsey graphs establishing $R(3,3) \geq 6$, $R(3,4) \geq 9$, $R(3,5) \geq 14$ and $R(4,4) \geq 18$ that appear in Figure 1 in section 1.1 are all cyclic graphs.

Using the Cauchy Frobenius-Burnside lemma, it is easy to show that D_n acting on the edges of K_n has exactly $\left\lfloor \frac{n}{2} \right\rfloor$ orbits. Note that if we define the distance function $dist(e) = \min\{|i-j|, |j-i|\}$, then two edges e_1 and e_2 of K_n belong to the same orbit if and only if $dist(e_1) = dist(e_2)$. Thus an orbit of edges is completely determined by a single number k , where k is the difference of the pairs in the orbit. For example, a cyclic 3-color Ramsey graph can be completely specified by sets of distances, say *Red*, *Green* and *Blue*. That is $k \in Red$ means every edge e with $dist(e) = k$ is colored red.

EXAMPLE 1. For the Ramsey graph in Figure 1(b) in section 1.1, the edge orbits are

$$C_{21} = \{\{0,1\}, \{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,7\}, \{7,0\}\}$$

$$C_{22} = \{\{0,2\}, \{1,3\}, \{2,4\}, \{3,5\}, \{4,6\}, \{5,7\}, \{6,0\}, \{7,1\}\}$$

$$C_{23} = \{\{0,3\}, \{1,4\}, \{2,5\}, \{3,6\}, \{4,7\}, \{5,0\}, \{6,1\}, \{7,2\}\}$$

$$C_{24} = \{\{0,4\}, \{1,5\}, \{2,6\}, \{3,7\}\}$$

The edge orbits C_{21} , C_{22} , C_{23} and C_{24} correspond to distance 1, 2, 3 and 4 respectively. To construct a two-color Ramsey graph, the edge orbits are partitioned into two sets *Red* and *Blue*. If we set *Red* = $\{1,4\}$ and *Blue* = $\{2,3\}$, an $R(3,4)$ Ramsey graph is obtained.

Due to the cyclic property of the dihedral group, the following notation is introduced in order to simplified the orbit representation.

The vector $(i_0, i_1, \dots, i_{k-1})$ denotes $\{(j_0, j_1, \dots, j_{k-1}) | i_s = (j_{(s+1) \bmod k} - j_s) \bmod n, 0 \leq s < k\} \cup$

$\{(j_0, j_1, \dots, j_{k-1}) | i_s = (j_{(s-1) \bmod k} - j_s) \bmod n, 0 \leq s < k\}$, which is the orbit of $\langle x \rightarrow x+1, x \rightarrow -x \rangle$, the dihedral group on \mathbb{Z}_n . Thus, for example, $C_{23} = (3,5)$ in Example 1.

2.2.2. Pattern Matrix

The *Pattern Matrix* P_r , which is a variation of the incidence matrix as applied in the theory of t-designs [15,18,19] is defined as follows:

Let V be a set of n elements, G a subgroup of the symmetric group of permutations of V , $G \leq \text{Sym}(V)$, and integer r , $2 < r < n$. The *Pattern Matrix* P_r belonging to the group G is defined as follows:

- (a) the rows of P_r are indexed by the G -orbits of 2-subsets of V ;
- (b) the columns of P_r are indexed by the G -orbits of r -subsets of V ;
- (c) $P_r[I, J] = 1$ if there is $F_i \in I$ and $F_j \in J$ such that $F_i \subseteq F_j$ and is 0 otherwise.

EXAMPLE 2. Let $n=8$, $V=\mathbb{Z}_8$, $G=D_8$, and $r=3$. Then the G -orbits of 2-subsets are

$$C_{21} = (1,7), C_{22} = (2,6), C_{23} = (3,5), \text{ and } C_{24} = (4,4).$$

The G -orbits of 3-subsets are

$$C_{31} = (1,1,6), C_{32} = (1,2,5), C_{33} = (1,3,4), C_{34} = (2,2,4), \text{ and } C_{35} = (2,3,3).$$

The G -orbits of 4-subsets are

$$C_{41} = (1,1,1,5), C_{42} = (1,1,2,4), C_{43} = (1,1,3,3), C_{44} = (1,2,2,3), \\ C_{45} = (1,2,1,4), C_{46} = (1,2,3,2), C_{47} = (1,3,1,3), \text{ and } C_{48} = (2,2,2,2).$$

The resulting pattern matrices P_3 and P_4 appear in Table I and Table II.

If V is thought of as the vertex set of the complete graph K_n , then the pattern matrix P_{r_i} describes the incidence between the orbits under G of edges and complete subgraphs of size r_i . Thus $P_{r_i}[I, J] = 1$ means that every K_{r_i} in orbit J contains at least one edge in orbit I . Hence, if we are to avoid the inclusion of a monochromatic K_{r_i} of color i , then the rows

Table I
Matrix P_3

| P_3 | C_{31} | C_{32} | C_{33} | C_{34} | C_{35} |
|----------|----------|----------|----------|----------|----------|
| C_{21} | 1 | 1 | 1 | 0 | 0 |
| C_{22} | 1 | 1 | 0 | 1 | 1 |
| C_{23} | 0 | 1 | 1 | 0 | 1 |
| C_{24} | 0 | 0 | 1 | 1 | 0 |

Table II
Matrix P_4

| P_4 | C_{41} | C_{42} | C_{43} | C_{44} | C_{45} | C_{46} | C_{47} | C_{48} |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| C_{21} | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| C_{22} | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| C_{23} | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| C_{24} | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |

corresponding to the chosen edge orbits of color i must be chosen so that no column of all 1's appears among them.

The next theorem follows immediately from the above discussion.

THEOREM 2. *There is a bijection between the m -color Ramsey graphs Γ with vertex set V , having $G \leq \text{Sym}(V)$ as an automorphism group, and the $(0,1)$ -vectors $U_i, 1 \leq i \leq m$ indexed by the G -orbits of 2-subsets of V , solving simultaneously the inequalities:*

$$(\overline{U_i} \cdot P_{r_i})[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_{r_i}, 1 \leq i \leq m. \quad (1)$$

$$\sum_{i=1}^m U_i = \overline{1}, \text{ where } \overline{1} = [1, 1, \dots, 1]^T. \quad (2)$$

$$U_i \cdot U_j = 0, \text{ for } 1 \leq i < j \leq m, \quad (3)$$

where $P_{r_i}, 1 \leq i \leq m$ are pattern matrices belonging to group G .

In plain language, the equations in theorem 2 can be interpreted as follows: (1) says that the coloring will contain no monochromatic K_{r_i} in the i th color for each i . (2) ensures that every edge is colored, and (3) guarantees no edge is colored twice.

EXAMPLE 3. To search for a cyclic $(3,3,4)$ -Ramsey graph, we need to consider the pattern matrices P_3 , P_3 and P_4 for colors *red*, *green* and *blue* respectively. These pattern matrices are given in the Example 2.

Let U_r be the vector for red color, U_g for green, U_b for blue. The equations and inequalities we have to solve are

$$(\overline{U_r} \cdot P_3)[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_3, \quad (1.1)$$

$$(\overline{U_g} \cdot P_3)[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_3, \quad (1.2)$$

$$(\overline{U_b} \cdot P_4)[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_4, \quad (1.3)$$

$$U_r + U_g + U_b = \overline{1}, \quad (2.1)$$

$$U_r \cdot U_g = 0, \quad (3.1)$$

$$U_r \cdot U_b = 0, \quad (3.2)$$

$$U_g \cdot U_b = 0, \quad (3.3)$$

The solution is

$$U_r = (1, 0, 1, 0)$$

$$U_g = (0, 1, 0, 0)$$

$$U_b = (0, 0, 0, 1)$$

There is a cyclic (3,3,4)–Ramsey graph on 8 vertices.

2.2.3. Pattern Matrix Reduction

The pattern matrices for large n are still too large for computer search. The absorption law $a \star (a + b) = a$ of Boolean algebra can be used to reduce the sizes of pattern matrices, making computer search possible.

Let $G \leq \text{Sym}(V)$, with each color i , $1 \leq i \leq m$ and each orbit I_j , of pairs associate a Boolean variable x_{ij} . The assignment of true to x_{ij} will mean that j th orbit of edges is assigned color i . Also in order to have no monochromatic K_{r_i} -subset of color i , we associate with each column h of the pattern matrix P_{r_i} of G , the clause c_{ih} given by $c_{ih} = \sum_j (\overline{x_{ij}} \cdot P_{r_i}[j, h] = 1)$. Whence, if $B_i = \prod_h c_{ih}$, then B_i is satisfied if and only if there is no monochromatic r_i -subset of color i .

EXAMPLE 4. The pattern matrix P_3 in Example 2, labeled as in the Table III.

The Boolean expression B for the pattern matrix P_3 is

Table III
Labeled Matrix P_3

| P_3 | C_{31} | C_{32} | C_{33} | C_{34} | C_{35} |
|----------|----------|----------|----------|----------|----------|
| x_{11} | 1 | 1 | 1 | 0 | 0 |
| x_{12} | 1 | 1 | 0 | 1 | 1 |
| x_{13} | 0 | 1 | 1 | 0 | 1 |
| x_{14} | 0 | 0 | 1 | 1 | 0 |

$$B = (\overline{x_{11} + x_{12}})(\overline{x_{11} + x_{12} + x_{13}})(\overline{x_{11} + x_{13} + x_{14}})(\overline{x_{12} + x_{14}})(\overline{x_{12} + x_{13}}).$$

If $B_i = \prod_h c_{ih}$ is the Boolean expression for P_{r_i} , then $B_i' = \prod_{h \in H'} c_{ih}$ where H' are the

clauses remaining after applying the absorption property, is an equivalent Boolean expression.

Hence, a reduced pattern matrix P_{r_i}' can be defined to reflex the incidence of Boolean variables and clauses in B_i' .

EXAMPLE 5. The Boolean expression B for the pattern matrix P_3 above can be reduced by the absorption property to

$$B = (\overline{x_{11} + x_{12}})(\overline{x_{11} + x_{13} + x_{14}})(\overline{x_{12} + x_{14}})(\overline{x_{12} + x_{13}}).$$

Hence the reduced pattern matrix P_3' of P_3 is given in Table IV and is the one we would actually use for computer search.

Table IV
Matrix P_3'

| P_3 | C_{31} | C_{33} | C_{34} | C_{35} |
|----------|----------|----------|----------|----------|
| x_{11} | 1 | 1 | 0 | 0 |
| x_{12} | 1 | 0 | 1 | 1 |
| x_{13} | 0 | 1 | 0 | 1 |
| x_{14} | 0 | 1 | 1 | 0 |

Similarly the reduced pattern matrix P_4' is given in Table V.

Note that the number of columns of P_3' has $4/5 = 80\%$ of the original number of columns in P_3 . For pattern matrix P_4 , the number of columns in P_4' represents a reduction of $3/8 = 37.5\%$. Although this reduction may not be significant for small matrices, for big

Table V
Matrix P_4'

| P_4 | C_{41} | C_{45} | C_{48} |
|----------|----------|----------|----------|
| x_{31} | 1 | 1 | 0 |
| x_{32} | 1 | 0 | 1 |
| x_{33} | 1 | 1 | 0 |
| x_{34} | 0 | 1 | 1 |

matrices it does make a difference. For example, the pattern matrix P_5 for $R(3,3,5)$ on 44 points with dihedral group D_{44} has 12,446 columns. After applying the absorption property, only 1395 columns remain.

From the above discussion, Theorem 2 in the chapter 3 can be rephrased as Theorem 4 below.

THEOREM 4. *There is a bijection between the m -color $R(r_1, r_2, \dots, r_m)$ Ramsey graphs Γ with vertex set V , having $G \leq \text{Sym}(V)$ as an automorphism group, and the $(0,1)$ -vectors $U_i, 1 \leq i \leq m$ indexed by the G -orbits of 2-subsets of V , solving simultaneously the inequalities:*

$$(\overline{U_i} \cdot P_{r_i})[J] > 0 \text{ for all } G\text{-orbits } J \text{ labeling a column of } P_{r_i}, 1 \leq i \leq m. \quad (1)$$

$$\sum_{i=1}^m U_i = \overline{1}, \text{ where } \overline{1} = [1, 1, \dots, 1]^T. \quad (2)$$

$$U_i \cdot U_j = 0, \text{ for } 1 \leq i < j \leq m, \quad (3)$$

where $P_{r_i}, 1 \leq i \leq m$ are reduced pattern matrices (by Boolean absorption laws) belonging to group G .

2.2.4. A Example of Searching for 3-color Ramsey Graphs

We consider the problem of searching for cyclic $R(3,4,4)$ Ramsey-graph on 14 vertices. The goal is to find a coloring of *red*, *green* and *blue* on the edges of the complete graph K_{14} such that there are no red triangles, no green K_4 's, and no blue K_4 's.

The pattern matrix for color $r = \text{red}$ is described in Table VI.

The Boolean expression corresponding to P_{red} is

Table VI
Pattern Matrix P_{red}

| | | | | | |
|----------|---|---|---|---|---|
| x_{r1} | 1 | 1 | 1 | 0 | 0 |
| x_{r2} | 1 | 1 | 0 | 1 | 1 |
| x_{r3} | 0 | 1 | 1 | 0 | 1 |
| x_{r4} | 0 | 0 | 1 | 1 | 0 |

$$B_{red} = (\overline{x_{r1} + x_{r2}})(\overline{x_{r1} + x_{r2} + x_{r3}})(\overline{x_{r1} + x_{r3} + x_{r4}})(\overline{x_{r2} + x_{r4}})(\overline{x_{r2} + x_{r3}})$$

$$= (\overline{x_{r1} + x_{r2}})(\overline{x_{r1} + x_{r3} + x_{r4}})(\overline{x_{r2} + x_{r4}})(\overline{x_{r2} + x_{r3}}),$$

and the reduced pattern matrix of P_{red} appears in Table VII.

Table VII
Reduced Pattern Matrix P_{red}'

| | | | | |
|----------|---|---|---|---|
| x_{r1} | 1 | 1 | 0 | 0 |
| x_{r2} | 1 | 0 | 1 | 1 |
| x_{r3} | 0 | 1 | 0 | 1 |
| x_{r4} | 0 | 1 | 1 | 0 |

The pattern matrix for color $g=green$ is described in Table VIII.

Table VIII
Pattern Matrix P_{green}

| | | | | | | | | |
|----------|---|---|---|---|---|---|---|---|
| x_{g1} | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| x_{g2} | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| x_{g3} | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 |
| x_{g4} | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |

The Boolean expression for P_{green} is

$$B_{green} = (\overline{x_{g1} + x_{g2} + x_{g3}})(\overline{x_{g1} + x_{g2} + x_{g3} + x_{g4}})(\overline{x_{g1} + x_{g2} + x_{g3} + x_{g4}})(\overline{x_{g1} + x_{g2} + x_{g3} + x_{g4}})(\overline{x_{g1} + x_{g3} + x_{g4}})$$

$$(\overline{x_{g1} + x_{g2} + x_{g3} + x_{g4}})(\overline{x_{g1} + x_{g2} + x_{g3}})(\overline{x_{g2} + x_{g4}})$$

$$= (\overline{x_{g1} + x_{g2} + x_{g3}})(\overline{x_{g1} + x_{g3} + x_{g4}})(\overline{x_{g2} + x_{g4}}).$$

The reduced pattern matrix of P_{green} is given in Table IX.

The pattern matrix P_{blue} and Boolean expression B_{blue} for color $b=blue$ are the same as those for color green, since we are looking for some graph without either green or blue K_4 .

Table IX
Reduced Pattern Matrix P_{green}

| | | | |
|----------|---|---|---|
| x_{g1} | 1 | 1 | 0 |
| x_{g2} | 1 | 0 | 1 |
| x_{g3} | 1 | 1 | 0 |
| x_{g4} | 0 | 1 | 1 |

Hence the reduced matrix is as in Table X, which is the same as Table IX.

Table X
Reduced Pattern Matrix P_{blue}

| | | | |
|----------|---|---|---|
| x_{b1} | 1 | 1 | 0 |
| x_{b2} | 1 | 0 | 1 |
| x_{b3} | 1 | 1 | 0 |
| x_{b4} | 0 | 1 | 1 |

By using the algorithm described in the chapter 5 or simply by checking since these matrices are small, the following solution is found:

$$U_r = (x_{r1}, x_{r2}, x_{r3}, x_{r4}) = (1, 0, 0, 1)$$

$$U_g = (x_{g1}, x_{g2}, x_{g3}, x_{g4}) = (0, 1, 0, 0)$$

$$U_b = (x_{b1}, x_{b2}, x_{b3}, x_{b4}) = (0, 0, 1, 0)$$

It is easy to check if the solution satisfies the conditions in the Theorem 2.

3. CHAPTER III: DATA STRUCTURES AND SEARCH ALGORITHM

The computer programs were written in the C programming language on a Pyramid Technologies 90X computer running *Unix**. By taking advantage of bit operation, memory and time requirements were reduced.

3.1. Data Structure

Since the pattern matrices are 0-1 matrices, the bit operations in the programming language C make it possible to store and manipulate a matrix as an array of integers. Assuming there are no more than 32 orbits of edges, we represent each column of P , by a single 32 bit integer.

EXAMPLE 1. Consider the pattern matrix P_3' .

Table I
Matrix P_3'

| | | | |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 |

The data structure for this matrix would be

unsigned int mat[4];

where

$$mat[0] = (3)_{10} = (11)_2$$

$$mat[1] = (13)_{10} = (1101)_2$$

$$mat[2] = (10)_{10} = (1010)_2$$

$$mat[3] = (6)_{10} = (110)_2$$

In order to speed up the bit operations, the array *mask* was used, where $mask[i] = 2^i$. Thus, to check if the pattern matrix P has a 1 in entry $P[i,j]$, we only have to check if $mat[j] \& mask[i]$ is true.

*Unix is a Trademark of Bell Laboratories.

The array *zero* is defined to assist in clearing some specific bits of P_k , where $zero[i] = \sim mask[i]$. To clear the entry $P[i,j]$, we only have to do $mat[j] = mat[j] \& zero[i]$.

3.2. Memory Allocation

Each color is associated with a C programming language structure as defined below.

```
#define COLORS 3
#define EDGES 32
struct color {
    unsigned *mat;
    int *bits;
    int set[EDGES];
    int cols;
    int mem;
    int key;
} *c[COLORS];
```

Static memory allocation is used, i.e a piece of memory was allocated for the matrix of each color before recursion starts. Each segment of memory was constructed as a stack. Each time when the recursion routine goes to next level, the new matrix is saved on the top of the stack. When the recursion routine comes back to a higher level, it pops the matrix from the stack, maintaining the current matrix environment.

The advantage of the static memory allocation is that it takes much less time than the dynamic memory allocation, especially for the large computation. The disadvantage is that the size of the static memory has to be changed if the size of the matrix or the depth of recursion has been changed.

3.3. Pruning Strategy

The facts presented in the proceeding chapters enabled the size of the pattern matrices to be reduced and makes computer search possible. As mentioned before, the search problem remains to be very hard. Whence algorithm must be carefully designed in order to com-

plete the search in reasonable time.

The problem as shown in Chapter 2 is still NP-complete. Hence backtrack algorithm was applied with appropriate pruning strategies. The search tree for a (r_1, r_2, \dots, r_m) Ramsey-graph on n vertices is a m -ary tree with a height of $\left\lfloor \frac{n}{2} \right\rfloor$. The number of leaves in such a tree is $m^{\lfloor n/2 \rfloor}$. This is a huge number even for moderate values of m and n . In a 3-color Ramsey graph, the tree is a 3-ary. For example, for $R(3,4,4) \geq 55$ case, the size of the full tree is $3^{27} = 7,625,597,484,987$.

Since we want to find not only one solution but all solutions, the whole tree must be traversed. Thus, to make a fast algorithm the tree should be pruned as much as possible. The rule for skipping or pruning nodes is that on certain nodes, if there already exist a monochromatic complete graph K_{r_i} with color i , then all children of those nodes are pruned.

We say that the root of a search tree is at level 0 and the children of a node at level i are at level $i+1$. Define the function $f(i) = \frac{m^{n-i+1} - 1}{m - 1}$. Then $f(i)$ is the number of nodes in a subtree of an m -ary tree at level i . It is a decreasing function. This means that the earlier we prune a branch, the more nodes are eliminated from the search.

EXAMPLE 2. Search for $R(3,3,3)$ Ramsey graph on 14 vertices.

After applying absorption laws to the original pattern matrices, we have the three reduced matrices in the Table II.

Table II
Pattern Matrices P_{red} , P_{green} and P_{blue}

| No. | P_{red} | P_{green} | P_{blue} |
|-----|-------------------------|-------------------------|-------------------------|
| 0 | 1 0 0 0 0 0 1 0 1 0 0 1 | 1 0 0 0 0 0 1 0 1 0 0 1 | 1 0 0 0 0 0 1 0 1 0 0 1 |
| 1 | 1 1 0 1 0 0 0 1 0 0 1 0 | 1 1 0 1 0 0 0 1 0 0 1 0 | 1 1 0 1 0 0 0 1 0 0 1 0 |
| 2 | 0 0 0 0 1 0 1 1 0 1 0 0 | 0 0 0 0 1 0 1 1 0 1 0 0 | 0 0 0 0 1 0 1 1 0 1 0 0 |
| 3 | 0 1 1 0 0 1 1 0 0 1 0 0 | 0 1 1 0 0 1 1 0 0 1 0 0 | 0 1 1 0 0 1 1 0 0 1 0 0 |
| 4 | 0 0 1 0 0 0 0 1 1 0 1 0 | 0 0 1 0 0 0 0 1 1 0 1 0 | 0 0 1 0 0 0 0 1 1 0 1 0 |
| 5 | 0 0 0 1 1 1 0 0 1 0 0 1 | 0 0 0 1 1 1 0 0 1 0 0 1 | 0 0 0 1 1 1 0 0 1 0 0 1 |
| 6 | 0 0 0 0 0 0 0 0 0 1 1 1 | 0 0 0 0 0 0 0 0 0 1 1 1 | 0 0 0 0 0 0 0 0 0 1 1 1 |

Define the initial level be level 0. At level 0, we assigned color *red* to the edge orbit C_0 . To obtain the matrices at level $i+1$, if the edge orbit i is colored c , then row i of all pattern matrices is deleted and each column j in the other matrices P_d are deleted, where $P_d[i,j]=1$, $d \neq c$. Thus in the example, recursion continues to the level 1 and the current pattern matrices considered for this level are in Table III.

Table III
Sub-pattern Matrices P_{red} , P_{green} and P_{blue}

| No. | P_{red} | P_{green} | P_{blue} |
|-----|-------------------------|-----------------|-----------------|
| 1 | 1 1 0 1 0 0 0 1 0 0 1 0 | 1 0 1 0 0 1 0 1 | 1 0 1 0 0 1 0 1 |
| 2 | 0 0 0 0 1 0 1 1 0 1 0 0 | 0 0 0 1 0 1 1 0 | 0 0 0 1 0 1 1 0 |
| 3 | 0 1 1 0 0 1 1 0 0 1 0 0 | 1 1 0 0 1 0 1 0 | 1 1 0 0 1 0 1 0 |
| 4 | 0 0 1 0 0 0 0 1 1 0 1 0 | 0 1 0 0 0 1 0 1 | 0 1 0 0 0 1 0 1 |
| 5 | 0 0 0 1 1 1 0 0 1 0 0 1 | 0 0 1 1 1 0 0 0 | 0 0 1 1 1 0 0 0 |
| 6 | 0 0 0 0 0 0 0 0 0 1 1 1 | 0 0 0 0 0 0 1 1 | 0 0 0 0 0 0 1 1 |

THEOREM 1. After the i th orbit of pairs is assigned color c , then in considering the rest of uncolored orbits we can delete row i of all pattern matrices and each column j of the other matrices, where the $[i,j]$ entry is 1. The new matrices are called the *sub-pattern matrix* of the original matrices.

Proof. Suppose the i th orbit is colored c . Then if the $[i,j]$ -th entry of pattern matrix P_d , $d \neq c$, is 1, then it is impossible to have a monochromatic complete graph of color d in the orbit labeling column j . Hence column j can be deleted.

QED

DEFINITION 1. The *weight* of a column in the pattern matrix is the number of non-zero's in the column.

The column in a sub-pattern matrix is a part of some column in the pattern matrix.

THEOREM 2. If there is a column with weight 0 in any sub-pattern matrix, then the partial coloring can not be extended to a Ramsey coloring.

Proof. Suppose in the sub-pattern matrix for coloring r -subsets with color c , column j

has 0 in all entries. From the process of getting the sub-pattern matrix, if column j is a weight 0 column in the sub-pattern matrix, then all orbits with 1 in this column of the original matrix have been colored color c . Hence the graph contains monochromatic r -subset in the j th orbit.

QED

EXAMPLE 2.(cont.) Assign color *red* to the class C_1 . Then the sub-pattern matrices become as in Table IV. The first column of P_{red} is a weight 0 column. Both class C_0 and C_1 have been colored *red*. Thus they yield a red triangle in the first orbit. Consequently, it is impossible to construct a Ramsey graph based on this partial coloring.

Table IV
Sub-pattern Matrices P_{red} , P_{green} and P_{blue}

| No. | P_{red} | P_{green} | P_{blue} |
|-----|-------------------------|-------------|------------|
| 2 | 0 0 0 0 1 0 1 1 0 1 0 0 | 0 1 0 1 | 0 1 0 1 |
| 3 | 0 1 1 0 0 1 1 0 0 1 0 0 | 1 0 1 1 | 1 0 1 1 |
| 4 | 0 0 1 0 0 0 0 1 1 0 1 0 | 1 0 0 0 | 1 0 0 0 |
| 5 | 0 0 0 1 1 1 0 0 1 0 0 1 | 0 1 1 0 | 0 1 1 0 |
| 6 | 0 0 0 0 0 0 0 0 0 1 1 1 | 0 0 0 1 | 0 0 0 1 |

We know that the algorithm is faster if we can do pruning earlier. At the stage of level 1 in EXAMPLE 2, we have the choice the next orbit C_i to be colored, $1 \leq i \leq 6$, $i \neq 1$. No branch at this stage can be pruned. If orbit C_1 is colored *red*, by Theorem 2 above, it can not be extended to a Ramsey coloring. So a subtree can be cut from the search tree at level 1.

In viewing of the above discussion, we adopt the following coloring strategy.

COLORING STRATEGY. Choose to color orbit i , if $P_c[i, j]$ is 1 and column j is the column with the least weight of all pattern matrices.

EXAMPLE 2.(cont.) Using the above coloring strategy, at the level 1, class 1 is chosen to be colored next. If it is colored the orbit 1 red, there will be weight 0 column in the subma-

trix of P_{red} , which will not lead to a Ramsey coloring. Thus we try green for orbit 1 and continue to color the rest of the orbits. This leads to the solution: $C_0(\text{red})$, $C_1(\text{green})$, $C_2(\text{green})$, $C_3(\text{red})$, $C_4(\text{blue})$, $C_5(\text{blue})$, and $C_6(\text{don't care})$. The class C_6 can be colored arbitrarily that is no matter how this is colored it is always a Ramsey graph.

The Ramsey number $R(3,3,3)$ is known to be 17 (see Chapter 1) and a Ramsey graph on 16 vertices was constructed there. This graph however is not cyclic. The largest cyclic $R(3,3,3)$ Ramsey graph is on 14 vertices. Hence, in general cyclic Ramsey graphs may not give the best lower bounds, although in some cases they do give tight lower bounds.

4. CHAPTER IV: RESULTS AND ANALYSIS OF THE NEW RAMSEY GRAPHS

4.1. New Lower Bounds of some 3-color Ramsey Numbers

Using the algorithm described in chapter 4, we have constructed the three 3-color cyclic graphs given in Table I. These are the maximal cyclic Ramsey graphs which can found, that is these lower bounds can only be improved by non-cyclic Ramsey graphs.

Table I

| New Bound | n | Edge Orbits (Differences) | | |
|--------------------|----|---------------------------|-----------------------------|---------------------------|
| | | <i>Red</i> | <i>Green</i> | <i>Blue</i> |
| $R(3,3,4) \geq 30$ | 29 | 1 4 10 12 | 2 5 6 14 | 3 7 8 9 11 13 |
| $R(3,3,5) \geq 45$ | 44 | 1 4 9 12 15 22 | 2 3 10 14 18 19 | 5 6 7 8 11 13 16 17 20 21 |
| $R(3,4,4) \geq 55$ | 54 | 1 4 9 15 20 22 27 | 7 8 13 14 16 17 18 19 23 26 | 2 3 5 6 10 11 12 21 24 25 |

These three graphs give the three new lower bounds on three 3-color Ramsey numbers below,

$$R(3,3,4) \geq 30; \quad R(3,3,5) \geq 45; \quad R(3,4,4) \geq 55.$$

The lower bound $R(3,3,4) \geq 30$ was independently found by J. G. Kalbfleisch at the University of Waterloo in his doctoral dissertation[15], but does not appear in the literature. The other two lower bounds are apparently new.

4.2. Primitivity

A subset V_1 of V is called a *block of imprimitivity* (b.i.) of transitive group action $G|V$, if for each $g \in G$ the set V_1^g either coincides with V_1 or is disjoint from V_1 . Obviously V and the singleton subsets are b.i.'s and these are called the trivial blocks of the group action. A transitive group action $G|V$ is said to be *imprimitive* if it has at least one nontrivial b.i. V_1 , otherwise it is primitive. In particular it is easy to see that a transitive group action $G|V$ is primitive whenever $|V|=p$ is a prime. In this case G is isomorphic to a subgroup of $AF(p) = \{x \rightarrow \alpha x + \beta : \alpha, \beta \in \mathbb{Z}_p, \alpha \neq 0\}$, or G is 2-transitive.

4.3. Analysis of the Graphs

In searching for an $R(3,3,4)$ Ramsey graph on 29, the pattern matrix P_3 had 126 columns and the pattern matrix P_4 had 819 columns. After applying the absorption laws, there are only 56 columns in P_3 and 63 columns in P_4 remained. Also due to the early pruning strategy, only 157 leaves in the search tree were really visited. The search leads to 14 cyclic Ramsey graphs and these appear in Table II.

Table II

| 14 cyclic $R(3,3,4)$ Ramsey graphs on K_{29} | | | |
|--|------------|------------|----------------|
| No. | Red | Green | Blue |
| 1 | 6 11 13 14 | 3 7 8 9 | 1 2 4 5 10 12 |
| 2 | 3 7 8 9 | 4 10 11 13 | 1 2 5 6 12 14 |
| 3 | 4 10 11 13 | 2 5 8 9 | 1 3 6 7 12 14 |
| 4 | 1 3 7 12 | 6 11 13 14 | 2 4 5 8 9 10 |
| 5 | 2 5 8 9 | 1 4 10 12 | 3 6 7 11 13 14 |
| 6 | 1 4 10 12 | 2 5 6 14 | 3 7 8 9 11 13 |
| 7 | 2 5 6 14 | 1 3 7 12 | 4 8 9 10 11 13 |
| 8 | 3 7 8 9 | 6 11 13 14 | 1 2 4 5 10 12 |
| 9 | 4 10 11 13 | 3 7 8 9 | 1 2 5 6 12 14 |
| 10 | 2 5 8 9 | 4 10 11 13 | 1 3 6 7 11 14 |
| 11 | 6 11 13 14 | 1 3 7 12 | 2 4 5 8 9 10 |
| 12 | 1 4 10 12 | 2 5 8 9 | 3 6 7 11 13 14 |
| 13 | 2 5 6 14 | 1 4 10 12 | 3 7 8 9 11 13 |
| 14 | 1 3 7 12 | 2 5 6 14 | 4 8 9 10 11 13 |

THEOREM 1. *There are, up to isomorphism, only two cyclic Ramsey $R(3,3,4)$ graphs on 29 vertices. Furthermore, the full automorphism group of each is $G = \langle x \rightarrow x + 1, x \rightarrow \omega^7 x \rangle$ where $\omega \in \mathbb{Z}_{29}$ is a primitive root of unity.*

Proof. Let (V, Γ) be a cyclic $R(3,3,4)$ Ramsey graph. Then $G|V$ is transitive and since $|V| = 29$ is prime G acts primitively. Moreover, G cannot be 2-transitive for then there is only one orbit of edges and 3-color ramsey graphs require at least three. Whence, G must be one of the 6 transitive subgroups of $\text{AF}(29) = \{x \rightarrow \alpha x + \beta : \alpha, \beta \in \mathbb{Z}_{29}, \alpha \neq 0\}$. These 6 subgroups are $H_d = \langle x \rightarrow x + 1, x \rightarrow \omega^d x \rangle$ where $d | 28$ and ω is a primitive root modulo 29. Whence, H_d is an automorphism group of a cyclic $R(3,3,4)$ Ramsey graph if and only if

multiplication by ω^d preserves the coloring. A complete list of all cyclic $R(3,3,4)$ Ramsey graphs was generated by the algorithm in this thesis and is given in Table II. It is easy to check that multiplication by -1 and by ω^7 preserves each and every one of these 14 colorings. Also, it can be seen that the other multiplications permute the 14 colorings into two orbits $\Delta_1 = \{1,2,3,4,5,6,7\}$ and $\Delta_2 = \{8,9,10,11,12,13,14\}$. Thus up to isomorphism there are only two cyclic $R(3,3,4)$ Ramsey graph as claimed.

QED

THEOREM 2. *There is a unique, up to isomorphism and interchange of colors, cyclic $R(3,3,4)$ Ramsey graph on 29 vertices. Furthermore, its full automorphism group is $G = \langle x \rightarrow x + 1, x \rightarrow \omega^7 x \rangle$ where $\omega \in \mathbb{Z}_{29}$ is a primitive root of unity.*

Proof. We note that the mapping induce on the colorings by interchanging the colors Red and Green swaps rows in Table II according to the permutation $(1,8)(2,9)(3,10)(4,11)(5,12)(6,13)(7,14)$. Thus without fixing colors, there is a unique cyclic $R(3,3,4)$ Ramsey graph as claimed.

QED

From theorem 2, solution No. 1 is isomorphic to solution No. 8 if interchanging colors *red* and *green* is allowed.

In searching for $R(3,3,5)$ Ramsey graph on 44, the pattern matrix P_3 has 161 columns and the pattern matrix P_4 has 12446 columns. After applying the absorption laws, there are only 141 columns in P_3 and 1395 columns in P_4 . For the Ramsey graphs of $R(3,3,5)$ on K_{44} , 260 cyclic graphs were found with the algorithm presented in this thesis. These 260 graphs may be obtained from the 13 graphs listed in Table III by multiplying modulo 44 by numbers α relatively prime to 44 and/or interchanging red and green edges.

THEOREM 3. *There are 13, up to isomorphism and interchange of colors, cyclic $R(3,3,5)$*

Table III

| 13 non-isomorphic cyclic $R(3,3,5)$ Ramsey graphs on K_{44} | | | |
|---|-----------------|-----------------|---------------------------------|
| No. | Red | Green | Blue |
| 1 | 1 4 9 12 15 22 | 2 3 10 14 18 19 | 5 6 7 8 11 13 16 17 20 21 |
| 2 | 1 4 12 15 22 | 2 3 10 14 18 19 | 5 6 7 8 9 11 13 16 17 20 21 |
| 3 | 1 4 10 16 21 | 2 5 6 15 18 22 | 3 7 8 9 11 12 13 14 17 19 20 |
| 4 | 1 3 11 16 18 20 | 2 6 9 14 17 22 | 4 5 7 8 10 12 13 15 19 21 |
| 5 | 1 3 11 16 18 20 | 2 6 9 14 17 | 4 5 7 8 10 12 13 15 19 21 22 |
| 6 | 1 3 10 14 18 | 2 5 8 11 12 15 | 4 6 7 9 13 16 17 19 20 21 22 |
| 7 | 1 3 7 11 15 20 | 2 8 9 12 13 | 4 5 6 10 14 16 17 18 19 21 22 |
| 8 | 1 3 10 14 18 22 | 2 8 9 12 13 | 4 5 6 7 11 15 16 17 19 20 21 |
| 9 | 1 3 10 14 22 | 2 8 9 12 13 | 4 5 6 7 11 15 16 17 18 19 20 21 |
| 10 | 1 3 8 12 17 22 | 2 9 10 13 14 | 4 5 6 7 11 15 16 18 19 20 21 |
| 11 | 1 3 8 13 15 19 | 4 7 12 21 22 | 2 5 6 9 10 11 14 16 17 18 20 |
| 12 | 1 3 8 15 19 | 4 7 12 13 21 22 | 2 5 6 9 10 11 14 16 17 18 20 |
| 13 | 1 3 8 15 19 | 4 7 12 21 22 | 2 5 6 9 10 11 13 14 16 17 18 20 |

Ramsey graph on 44 vertices.

Proof. To show these 13 graphs are non-isomorphic, we have the following observation. Fix the vertex 0 of the graphs, we call the subgraphs which are induced by the vertices adjacent to vertex 0 by red edges, the *red subgraphs*. The red subgraphs of Ramsey graph 2 and 9 both have 9 vertices. The red subgraphs of Ramsey graph 3, 6, 12 and 13 have 10 vertices. The red subgraphs of Ramsey graph 1, 8 and 10 have 11 vertices. The red subgraphs of Ramsey graph 4, 5, 7 and 11 have 12 vertices. By also considering the number of green edges in the green subgraphs, we can conclude that all 13 Ramsey graphs are non-isomorphic except for possibly 8 and 10. In the red subgraph of graph 8, there are however vertices with green degree 2, while graph 10 has no such vertices. Consequently, they are also non-isomorphic. Hence there are 13 non-isomorphic $R(3,3,5)$ Ramsey graphs.

QED.

In searching for $R(3,4,4)$ Ramsey graph on 54, the pattern matrix P_3 has 243 columns and the pattern matrix P_4 has 1807 columns. After applying the absorption laws, there are only 196 columns in P_3 and 950 columns in P_4 . For the Ramsey graphs of $R(3,4,4)$ on K_{54} , the algorithm presented in chapter 4 found 18 solutions. These 18 solutions are listed in

Table IV.

Table IV

| 18 cyclic $R(3,4,4)$ Ramsey graphs on K_{54} | | | |
|--|---------------------|------------------------------|------------------------------|
| No. | Red | Green | Blue |
| 1 | 1 4 9 15 20 22 27 | 7 8 13 14 16 17 18 19 23 26 | 2 3 5 6 10 11 12 21 24 25 |
| 2 | 2 5 8 9 20 21 27 | 7 11 13 14 16 18 19 22 23 26 | 1 3 4 6 10 12 15 17 24 25 |
| 3 | 2 9 10 13 16 21 27 | 4 5 7 8 14 17 18 20 23 25 | 1 3 6 11 12 15 19 22 24 26 |
| 4 | 2 9 14 15 19 22 27 | 1 4 5 8 10 17 18 20 23 25 | 3 6 7 11 12 13 16 21 24 26 |
| 5 | 3 4 9 10 11 26 27 | 7 8 14 16 17 18 19 20 23 25 | 1 2 5 6 12 13 15 21 22 24 |
| 6 | 3 7 8 9 22 26 27 | 1 2 4 5 10 11 17 18 20 25 | 6 12 13 14 15 16 19 21 23 24 |
| 7 | 3 8 9 10 14 25 27 | 1 2 7 11 13 16 18 19 22 26 | 4 5 6 12 15 17 20 21 23 24 |
| 8 | 4 9 14 15 16 17 27 | 1 2 5 10 11 13 18 19 22 26 | 3 6 7 8 12 20 21 23 24 25 |
| 9 | 9 16 20 21 23 26 27 | 1 2 4 5 10 11 13 18 22 25 | 3 6 7 8 12 14 15 17 19 24 |
| 10 | 1 4 9 15 20 22 27 | 2 3 5 6 10 11 12 21 24 25 | 7 8 13 14 16 17 18 19 23 26 |
| 11 | 2 5 8 9 20 21 27 | 1 3 4 6 10 12 15 17 24 25 | 7 11 13 14 16 18 19 22 23 26 |
| 12 | 2 9 10 13 16 21 27 | 1 3 6 11 12 15 19 22 24 26 | 4 5 7 8 14 17 18 20 23 25 |
| 13 | 2 9 14 15 19 22 27 | 3 6 7 11 12 13 16 21 24 26 | 1 3 4 8 10 17 18 20 23 25 |
| 14 | 3 4 9 10 11 25 27 | 1 2 5 6 12 13 15 21 22 24 | 7 8 14 16 17 18 19 20 23 25 |
| 15 | 3 7 8 9 22 26 27 | 6 12 13 14 15 16 19 21 23 24 | 1 2 4 5 10 11 17 18 20 25 |
| 16 | 3 8 9 10 14 25 27 | 4 5 6 12 15 17 20 21 23 24 | 1 2 7 11 13 16 18 19 22 26 |
| 17 | 4 9 14 15 16 17 27 | 3 6 7 8 12 20 21 23 24 25 | 1 2 5 10 11 13 18 19 22 26 |
| 18 | 9 16 20 21 23 26 27 | 3 6 7 8 12 14 15 17 19 24 | 1 2 4 5 10 11 13 18 24 25 |

Similarly to the proof of theorem 2 we have the following theorem.

THEOREM 4. *There are two, up to isomorphism, cyclic $R(3,4,4)$ Ramsey graph on 54 vertices. They are listed in Table V.*

Table V

| two non-isomorphic cyclic $R(3,4,4)$ Ramsey graphs on K_{54} | | | |
|--|-------------------|-----------------------------|-----------------------------|
| No. | Red | Green | Blue |
| 1 | 1 4 9 15 20 22 27 | 7 8 13 14 16 17 18 19 23 26 | 2 3 5 6 10 11 12 21 24 25 |
| 10 | 1 4 9 15 20 22 27 | 2 3 5 6 10 11 12 21 24 25 | 7 8 13 14 16 17 18 19 23 26 |

Also, it is again easy to see that interchanging green and blue swaps these two graphs.

THEOREM 5. *There is a unique, up to isomorphism and interchange of colors, cyclic $R(3,4,4)$ Ramsey graph on 54 vertices.*

5. CHAPTER V: LIMITATIONS AND EXTENSIONS

The first limit of the search program is that since we choose an unsigned integer to store a column of the matrix, the length of the column in the pattern matrix is restricted to be less than 32 because an integer in programming language C has 32 bits long. This restriction is not hard to avoid since we can pair two integers or concatenate several integers (using an array) to get a longer column.

The long column means increasing the number of rows. From the analysis of the search algorithm, we see that its complexity is an exponential function of the number of rows. Hence increasing the number of the rows will increase the time complexity dramatically. Thus the complexity of the algorithm will again become a bottleneck to solving the problem. The other way is to find a heuristic algorithm to do the search. The trade-off here is that you may not be able to find the solution even if it exists.

Another direction in which this work could be followed is that automorphism groups other than the dihedral group could be studied. This requires experience in choosing the right group.

The algorithm presented here can be easily extended to search for 4-color Ramsey numbers, and such a version of the algorithm was implemented. But since more matrices are involved, this implementation suffered memory shortage and was not able to improve any of the known lower bounds.

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