Theses

8-1-1991

Thermal stresses in closed spherical shells

Frank W. Keene

Recommended Citation

Thermal Stresses in Closed Spherical Shells

by

Frank W. Keene Jr.

A Thesis Submitted

in

Partial Fulfillment

of the

Requirements for the Degree of

MASTER OF SCIENCE

in

Mechanical Engineering

Approved by:

Professor R. B. Hetnarski (Thesis Advisor)

Professor H. Ghoneim

Professor J. S. Török

Professor C. W. Haines   (Department Head)

DEPARTMENT OF MECHANICAL ENGINEERING

COLLEGE OF ENGINEERING

ROCHESTER INSTITUTE OF TECHNOLOGY

ROCHESTER, NEW YORK

AUGUST 1991

Thermal Stresses in Closed Spherical Shells

I <u>Frank W. Keene Jr.</u> hereby **grant permission** to the Wallace Memorial Library of RIT to reproduce my thesis in whole or in part. Any reproduction will not be for commercial use or profit.

Date _____8/27/91_____

ABSTRACT

The purpose of this work is to discuss thermal stresses in closed spherical shells. This effort is further limited to linear thermoelastic stresses in "thin" shells. The basic concepts associated with three-dimensional continuum mechanics are presented in both direct and general tensor notation. The three-dimensional equations are reduced to the two-dimensional equations of shells under going finite displacements. These are subsequently reduced to those pertaining to spherical shells. A review of the recent literature associated with thermal stresses in spherical shells is included. An appendix is provided which reviews some of the basic elements of general tensors.

ACKNOWLEDGMENT

TABLE OF CONTENTS

1 INTRODUCTION

The subject of thermal stresses in shells is a small part of the much more general subject of mathematical physics. The subject of mathematical physics involves describing physical laws and their various interactions in mathematical terms. Physical laws are assumed to be independent of the frame of reference in which an event takes place, given some a priori assumptions concerning the nature of the actual physical space. The effort of describing physical laws and their various interactions, in mathematical terms, often results in systems of differential equations and sets of equations representing boundary and initial conditions. The equations usually provide insight into the nature of a particular event or set of events. In most cases, assumptions are made regarding the physical laws and their interactions in order to obtain the systems of equations. If the intent is to attempt to solve for various unknown quantities in terms of known quantities, further assumptions are usually required to make the problem more tractable.

The equations and assumptions are often referred to as a theory which is associated with some particular physical phenomena. The assumptions implicit in a particular theory obviously restrict the applicability. The limitations of the theory are usually not obvious from the equations. It is therefore important, from a practical viewpoint, to understand the assumptions used and their consequences. This is best accomplished by following the derivation of the equations, as the simplifying assumptions are being made.

The subjects of differential geometry, tensor analysis, and variational calculus provide the mathematical tools for handling the physical concepts provided by the subjects of thermoelasticity and mechanics of continua. These subjects are used either implicitly or explicitly in most treatments of thermal stresses in shells.

Differential geometry, according to Dubrovin et al. [1], is concerned with the study of the metrical properties of general "smooth" spaces, using the techniques of differential calculus and linear algebra.

Tensors are defined by Sokolnikoff [2] as *abstract objects whose properties are independent of the coordinate system used to describe the objects.* Tensors are represented by a set of functions, which are called tensor components. Whether a set of functions represents a tensor, depends on how the functions transform from one coordinate system to another. If the components of a tensor vanish in one coordinate system, they vanish in all coordinate systems. Tensor analysis is the study of these abstract objects called tensors. Due to the invariant nature of tensors and physical laws, tensor analysis provides an ideal way of studying physical laws and their interactions. If a mathematical description of physical laws or their interactions can be provided in tensor form, it is valid in all frames of reference, given some assumptions regarding our physical space.

Variational calculus according to Washizu [3], is concerned with the study of the stationary value of functionals (functions of functions) with the intent of finding among the group of admissible functions, the one which makes the given functional stationary. The concept of a scalar representing the energy of a body motivates the use of variational methods in problems of continuum mechanics of solids.

Thermoelasticity, according to Nowacki [4], is concerned with the theory of the stress and strain fields in an elastic body, resulting from a flow of heat when the temperature and strain fields are coupled. It also involves the theory of thermodynamics.

Mechanics of continua is concerned with the formulation of equations which describe the motion of deformable bodies. It includes the concepts of continuous medium, work and energy, and forces and stresses.

Thermal stresses will be defined as those stresses resulting from a change in the temperature field of a body. This does not imply that a change in the temperature field alone is sufficient to produce thermal stresses. Further, it does not imply in general that the change in the temperature field is entirely a result of the addition of heat energy to the body from an external source.

Problems of continuum mechanics of solids involve, in general, three groups of

relationships which describe the kinematics of a body, the kinetics of body, and the relationship between the first two.

The kinematic relationships are concerned solely with the geometry of a body as a function of time. They consist of a description of the location of all material points of the body relative to a reference frame at two distinct points in time. The difference in position is termed a displacement. In the study of deformable bodies, the concern is primarily with situations in which the shape of the body is different at the two points in time. This interest leads to the need for a definition of a measure of the deformation or straining and to the formulation of strain-diplacement equations. In addition, there are some compatibility requirements (restrictions, constraints) on the relative position of the points of the body after deformation (which limit what are acceptable changes to the geometry).

The second group of relationships is concerned with the balance or conservation laws for mass and energy. These are described as kinetic relationships and are concerned with the concepts of forces, stresses and various forms of energy. They are, in general, dependent on the position and shape of the deformed body.

The first two groups are independent of the nature of the material of which the body is composed. The relationships between the variables in the kinematic relationships and those in the expressions of the conservation laws are called particular laws, or more generally, constitutive relationships. The third group consists of these constitutive relationships which interrelate the first two groups. These are restricted by certain principles. The nature of these relationships are also restricted [5] by certain other general principles such as determinism, local action, and objectivity.

In general, the equations associated with the above three sets of relationships are all nonlinear and therefore not solvable in a closed-form without simplifying assumptions. The increase in computing power and its availability to practicing engineers in the last ten years or so and the expected continued increase in the foreseeable future allows for

3

routine numerical investigation of problems involving a variety of nonlinearities. This seems to drive the research in at least two directions: 1. Ensuring a problem formulation which although "messy" when written out in a component form is reasonably robust when attacked numerically. 2. Continued efforts to develop consistent, sound mathematical formulations of nonlinear behaviors which were not addressed in-depth previously, due to the inability to obtain a closed-form solution.

The kinetic relationships can be divided into two general classes: (a) the loads are time dependant (the body is in a state of dynamic equilibrium) and (b) the loads are independent of time (the body is in a state of static equilibrium). These can be further subdivided by making assumptions concerning the distribution of the loads on the body (concentrated or distributed, symmetric or nonsymmetric). If the loads are time dependant further assumptions regarding the duration of the time interval of the loading can be made (e.g., shock or impulse type load).

One usually begins by making assumptions regarding the kinematic relationships. A variety of classes of problems can be developed by limiting the domain (geometry) that the material points occupy. For example, one can limit the geometry to rectilinear, cylindrical or spherical shapes. Also one can attempt to simplify the equations by making assumptions about the relative magnitudes of the dimensions of the shapes considered. Additional simplifications can be achieved by making assumptions regarding the magnitude of the strains and displacements.

The constitutive equations describe the relationship between the dynamical state of a body and the kinematic state at the same instants of time and possibly the kinematical state of the body's past history. Again, assumptions are usually made to limit the relationships to particular classes of problems.

This treatment is concerned mainly with the subject of thermal stresses in shells and more specifically with linear thermoelastic stresses in closed spherical shells. The shell is assumed to be initially stress free.

There are many ways of presenting a treatment of this type. At one extreme, the pertinent equations expressed in terms of spherical coordinates for some assumed simple material and loading and boundary conditions could be presented. At the other extreme, one could attempt to discuss in some detail all or at least many of the basic concepts involved in continuum mechanics, tensor analysis, differential geometry and any number of other related subjects and how eventually one is led to the subject of thermal stresses in spherical shells. We attempt to maintain a course closer to the former than the latter, however, the equations and concepts required are presented in reasonably general form and then refined for the limitations mentioned above.

In addition, to discuss thermal stresses without mentioning some of the more esoteric issues which are often avoided due to the simplifying assumptions required, does an injustice to the amount of work invested in attempting to bring them into clearer focus. Examples of these are the principles or axioms associated with the constitutive relationships, the issue of the second law of thermodynamics in problem formulation, the application of Cosserat surface theory to shell problems, and the infinite speed of propagation of thermal disturbances in the classical formulations.

Familiarity with vector and general tensor analysis and notation is assumed. Lower-case Latin indices are assumed to have the range of 1,2,3 while lowercase Greek indices are assumed to have the range of 1,2, unless otherwise noted. When the term continuous function is used, it is further assumed that the functions are of class C^n, where n is one more than the highest order derivative required in the formulation or derivation. When the term neighboring (neighborhood) is used, it is assumed that distances can be represented by differential elements. The basic concepts associated with three-dimensional continuum mechanics are presented in both direct and general tensor notation. The three-dimensional equations are reduced to the two-dimensional equations of shells under going finite displacements. These are subsequently reduced to those pertaining to spherical shells.

This work is organized as follows:

- Chapter 2 provides an overview of the notations used. Additional information regarding some of the basic elements of general tensor analysis is provided in Appendix A.
- Chapter 3 provides a brief historical review of the subject of thermal stresses in shells.
- Chapters 4-7 contain a review the main elements of kinematics of continuous media and the principles associated with curves and surfaces, in general tensor form.
- Chapter 8 reviews the main elements of the kinematics of continuous media in direct notation.
- Chapters 9-13 consider the kinetic and constitutive relationships in direct notation.
- Chapters 14-16 consider the general equations for shells.
- Chapters 17–18 discuss spherical shells and the reduction of the general equations to the spherical shell equations.
- Chapter 19 provides a review of recent literature associated with thermal stresses in spherical shells.
- Chapter 20 contains conclusions associated with this work.

We briefly review the main elements of continuum mechanics in three dimensions because, although a reduction in the total number of scalar equations involved is generally achieved by reducing the dimension of the body of interest from three to two, the main problems associated with a full three–dimensional treatment still have to be dealt with along with the problems or limitations imposed by the reduction to the two dimensional theory. In addition, it is sometimes necessary to reintroduce the three-dimensional theory in order to treat various boundary conditions.

2 NOTATIONS

Due to the fact that a variety of notations for the same operations appear in the literature, we define the notations used here, which are those used by Malvern [6]. When vector or tensor quantities are written out in terms of their components and base vectors it is assumed that they are defined in a general curvilinear coordinate system.

Boldface small and capital letters represent vectors and second-order tensors respectively, unless otherwise noted. Second-order tensors will be assumed to be linear vector functions. In the following $\mathbf{v}$ and $\mathbf{w}$ are arbitrary vectors.

- The magnitude of a vector $\mathbf{a}$ will be denoted by $|\mathbf{a}|$.

- Scalar (inner or dot product) of two vectors $\mathbf{a}$ and $\mathbf{b}$ will be denoted by $\mathbf{a} \cdot \mathbf{b}$.

- Vector or cross product of two vectors $\mathbf{a}$ and $\mathbf{b}$ will be denoted by $\mathbf{a} \times \mathbf{b}$.

- Tensor or open (dyad) product of two vectors $\mathbf{a}$ and $\mathbf{b}$ which is a second order tensor will be denoted by $\mathbf{ab}$, and defined by $(\mathbf{ab}) \cdot \mathbf{v} = \mathbf{a}(\mathbf{b} \cdot \mathbf{v})$. Higher order tensor (polyad) products are similarly written (e.g., $\mathbf{abc}$ and $\mathbf{abcd}$ for third and fourth order tensors respectively.)

- The transpose of a tensor $\mathbf{T}$, written as $\mathbf{T}^T$ and defined by

$$\mathbf{v} \cdot (\mathbf{T} \cdot \mathbf{w}) = \left(\mathbf{T}^T \cdot \mathbf{v} \right) \cdot \mathbf{w} \qquad 2.1$$

for all vectors $\mathbf{v}$ and $\mathbf{w}$, where if $\mathbf{T} = \mathbf{ab}$, then $\mathbf{T}^T = \mathbf{ba}$.

- If a tensor $\mathbf{T}$ is symmetric then $\mathbf{T} = \mathbf{T}^T$.

- If a tensor $\mathbf{T}$ is skew symmetric then $\mathbf{T} = -\mathbf{T}^T$.

- The trace of a tensor $\mathbf{T}$ written as $\text{tr}\mathbf{T}$ where if $\mathbf{T} = \mathbf{ab}$, then $\text{tr}\mathbf{T} = \mathbf{a} \cdot \mathbf{b}$.

- The operational product of a vector $\mathbf{u}$ and a tensor $\mathbf{T}$ which yields a vector $\mathbf{v}$ will be written as

$$\mathbf{v} = \mathbf{u} \cdot \mathbf{T} = \mathbf{T}^T \cdot \mathbf{u} \qquad 2.2$$

- Tensor or open product of two second-order tensors **T** and **U**, written as **T·U**, is defined by

$$(\mathbf{T \cdot U}) \cdot \mathbf{v} = \mathbf{T} \cdot (\mathbf{U \cdot v}) \qquad 2.3$$

- The inverse (when it exists) of a second-order tensor **T** is denoted by $\mathbf{T}^{-1}$ and defined by

$$\mathbf{T} \cdot \mathbf{T}^{-1} = \mathbf{T}^{-1} \cdot \mathbf{T} = 1 \qquad 2.4$$

The relationship between $\mathbf{T}^{-1}$ and **T** is

$$\mathbf{T}^{-1} = \frac{(\mathbf{T}^{\mp})^T}{\det \mathbf{T}} \qquad 2.5$$

where $\mathbf{T}^{\mp}$ is the tensor of cofactors of **T** and is referred to as the adjugate of **T** [5].

- We denote the operation of a fourth-order tensor on a second-order tensor which results is a second order tensor [5] as H[D], where **H** and **D** are fourth-order and second-order tensors, respectively. If $\mathbf{H} = \mathbf{abcd}$ and $\mathbf{D} = \mathbf{ef}$, then

$$\mathbf{H}[\mathbf{D}] = (\mathbf{d} \cdot \mathbf{e})(\mathbf{c} \cdot \mathbf{f})\mathbf{ab} \qquad 2.6$$

- The two forms of the scalar (inner or double dot) product of two second-order tensors written as **T** : **U** and **T** · ·**U** defined by

$$\mathbf{T} : \mathbf{U} = \mathrm{tr}\left(\mathbf{T \cdot U}^T\right) = \mathrm{tr}\left(\mathbf{T}^T \cdot \mathbf{U}\right) \qquad 2.7$$

and

$$\mathbf{T} \cdot \cdot \mathbf{U} = \mathrm{tr}(\mathbf{T \cdot U}) \qquad 2.8$$

or

$$\mathbf{T} \cdot \cdot \mathbf{U} = \mathbf{T}^T : \mathbf{U} = \mathbf{T} : \mathbf{U}^T \qquad 2.9$$

8

where if either of the two tensors are symmetric, the two forms of the scalar product are equivalent. If $\mathbf{T} = \mathbf{ab}$ and $\mathbf{U} = \mathbf{cd}$ then

$$\mathbf{T} : \mathbf{U} = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) \qquad 2.10$$

and

$$\mathbf{T} \cdot \cdot \mathbf{U} = (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \qquad 2.11$$

- The three principal invariants of a second-order tensor $\mathbf{T}$ will be written as I_T, II_T, and III_T, where

$$\mathrm{I}_T = \mathrm{tr}\mathbf{T} \qquad 2.12$$

$$\mathrm{II}_T = \frac{1}{2}(\mathbf{T} : \mathbf{T} - \mathrm{I}_T^2) \qquad 2.13$$

$$\mathrm{III}_T = \det\mathbf{T} \qquad 2.14$$

- When a tensor is written out in component form (e.g., $\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j = T_i^{\cdot j}\mathbf{g}^i\mathbf{g}_j$) the first index on the component belongs with the first vector of the dyad.

- If a vector $\mathbf{v}$ is defined in terms of the covariant and contravariant bases by $\mathbf{v} = v^i\mathbf{g}_i = v_i\mathbf{g}^i$, then the following formulas for differentiation apply, where the x^i are the coordinates and the double vertical bars denote covariant differentiation with respect to the metric of the three dimensional space associated with the base vectors:

$$\frac{\partial \mathbf{v}}{\partial x^n} = v^m{}_{\|n}\mathbf{g}_m = \left[v^m{}_{,n} + \left\{ {m \atop sn} \right\} v^s \right]\mathbf{g}_m \qquad 2.15$$

$$d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial x^n}dx^n = v^m{}_{\|n}\,dx^n\mathbf{g}_m = \delta v^m\mathbf{g}_m \qquad 2.16$$

9

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial x^n}\frac{dx^n}{dt} = v^m\big\|_n\frac{dx^n}{dt}\mathbf{g}_m = \frac{\delta v^m}{\delta t}\mathbf{g}_m \qquad\qquad 2.17$$

- The physical components of a vector or tensor will be indicated by enclosing the indices in angled brackets. For example the vector **v** when expressed in terms of its physical components has the form

$$\mathbf{v} = v^{\langle k\rangle}\frac{\mathbf{g}_k}{|\mathbf{g}_k|} = v_{\langle k\rangle}\frac{\mathbf{g}^k}{|\mathbf{g}^k|} \qquad\qquad 2.18$$

- When it is necessary to distinguish between symmetric and skew-symmetric tensor components of a tensor parentheses will be used for the former and square brackets for the latter. For example

$$T^{ij} = T^{(ij)} + T^{[ij]} \qquad\qquad 2.19$$

- The two forms of the gradient (del or nabla) operator will be denoted by

$$\overleftarrow{\nabla} = \frac{\overleftarrow{\partial}}{\partial x^n}\mathbf{g}^n \qquad\qquad 2.20$$

$$\overrightarrow{\nabla} = \mathbf{g}^n\frac{\partial}{\partial x^n} \qquad\qquad 2.21$$

- The gradient of a scalar ϕ is given by

$$\text{grad }\phi = \phi\overleftarrow{\nabla} = \overrightarrow{\nabla}\phi = \frac{\partial\phi}{\partial x^n}\mathbf{g}^n \qquad\qquad 2.22$$

10

- The gradient of a vector field **v** is the tensor **T** defined by

$$\frac{d\mathbf{v}}{ds} = \mathbf{u} \cdot \mathbf{T} = \mathbf{T}^T \cdot \mathbf{u} \qquad 2.23$$

$$\frac{d\mathbf{v}}{ds} = \mathbf{u} \cdot \overrightarrow{\nabla} \mathbf{v} = \mathbf{v} \overleftarrow{\nabla} \cdot \mathbf{u} \qquad 2.24$$

where **u** is a unit vector and $\frac{d\mathbf{v}}{ds}$ is the rate of change of **v** with respect to distance in the direction of **u** and denoted by

$$\mathbf{v}\overleftarrow{\nabla} = \frac{\partial \mathbf{v}}{\partial x^n} \mathbf{g}^n \qquad 2.25$$

$$\overrightarrow{\nabla} \mathbf{v} = \mathbf{v}\overleftarrow{\nabla}^T = \mathbf{g}^n \frac{\partial \mathbf{v}}{\partial x^n} \qquad 2.26$$

- The gradient of a tensor **T** is likewise

$$\mathbf{T}\overleftarrow{\nabla} = \frac{\partial \mathbf{T}}{\partial x^n} \mathbf{g}^n \qquad 2.27$$

$$\overrightarrow{\nabla} \mathbf{T} = \mathbf{g}^n \frac{\partial \mathbf{T}}{\partial x^n} \qquad 2.28$$

- The divergence of a vector, which results in a scalar

$$\text{div } \mathbf{v} = \mathbf{v} \cdot \overleftarrow{\nabla} = \overrightarrow{\nabla} \cdot \mathbf{v} = v^m|_m \qquad 2.29$$

$$\text{div } \mathbf{v} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, v^i \right) \qquad 2.30$$

- The two forms of the divergence of a tensor **T** which results in a vector

$$\mathbf{T} \cdot \overleftarrow{\nabla} \qquad , \qquad \overrightarrow{\nabla} \cdot \mathbf{T} = T^{ns}|_n \mathbf{g}_s \qquad 2.31$$

11

where

$$\vec{\nabla} \cdot \mathbf{T} = \mathbf{T}^T \cdot \overleftarrow{\nabla} \qquad \text{2.32}$$

- The Laplacian of a scalar ϕ is written as $\nabla^2 \phi = \text{div}\left(\vec{\nabla}\phi\right)$, and is given by

$$\nabla^2 \phi = g^{ij}\left(\frac{\partial^2 \phi}{\partial x^i \partial x^j} + \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} \frac{\partial \phi}{\partial x^k}\right) \qquad \text{2.33}$$

$$\nabla^2 \phi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\,g^{ij}\frac{\partial \phi}{\partial x^j}\right) \quad . \qquad \text{2.34}$$

When it is necessary to distinguish with respect to which basis the operator is referred, a subscript will be used and/or the indices will have upper or lowercase letters, e.g.,

$$\mathbf{u}\overleftarrow{\nabla}_x = \frac{\partial \mathbf{u}}{\partial x^n}\mathbf{g}^n \qquad \text{2.35}$$

$$\mathbf{u}\overleftarrow{\nabla}_X = \frac{\partial \mathbf{u}}{\partial X^N}\mathbf{G}^N \qquad \text{2.36}$$

$$\frac{\partial \mathbf{g}_m}{\partial x^n} = \left\{ \begin{matrix} s \\ mn \end{matrix} \right\}\mathbf{g}_s \qquad \text{2.37}$$

$$\frac{\partial \mathbf{G}_M}{\partial X^N} = \left\{ \begin{matrix} S \\ MN \end{matrix} \right\}\mathbf{G}_S \qquad \text{2.38}$$

- The scalar triple product of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ will be written as $[\mathbf{a},\mathbf{b},\mathbf{c}]$ where

$$[\mathbf{a},\mathbf{b},\mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \qquad \text{2.39}$$

- The general permutation symbols

$$\varepsilon_{ijk} = [\mathbf{g}_i,\mathbf{g}_j,\mathbf{g}_k] = \sqrt{g}\,e_{ijk} \qquad \text{2.40}$$

$$\varepsilon^{ijk} = \left[\mathbf{g}^i, \mathbf{g}^j, \mathbf{g}^k\right] = \frac{1}{\sqrt{g}}\, e^{ijk} \qquad\qquad 2.41$$

where

$$[\mathbf{g}_i, \mathbf{g}_j, \mathbf{g}_k,]^2 = g = \det[g_{ij}] \qquad\qquad 2.42$$

and

$$\left[\mathbf{g}^i, \mathbf{g}^j, \mathbf{g}^k,\right]^2 = g^{-1} = \det\left[g^{ij}\right] \qquad\qquad 2.43$$

and the alternator symbols are given by

$$e^{ijk} = e_{ijk} = \begin{cases} 0, & \text{when any two indices are equal} \\ +1, & \text{when } i,j,k \text{ are a cyclic permutation of } 1,2,3 \\ -1, & \text{when } i,j,k \text{ are a non}-\text{cyclic permutation of } 1,2,3 \end{cases} \qquad 2.44$$

- The Kronecker delta symbols are

$$\delta_{ij} = \delta^{ij} = \delta^i_j = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \qquad\qquad 2.45$$

and the relationship between the alternator symbols and the Kronecker delta symbols is

$$e_{ijk}\, e^{irs} = \delta^r_j \delta^s_k - \delta^s_j \delta^r_k \qquad\qquad 2.46$$

and from [2]

$$e^{ijk}\, e_{rst} = \delta^{ijk}_{rst} = \begin{vmatrix} \delta^i_r & \delta^i_s & \delta^i_t \\ \delta^j_r & \delta^j_s & \delta^j_t \\ \delta^k_r & \delta^k_s & \delta^k_t \end{vmatrix} \qquad\qquad 2.47$$

or in expanded form

$$e^{ijk}\, e_{rst} = \delta^{ijk}_{rst} = \qquad\qquad 2.48$$

$$\delta^i_r(\delta^j_s \delta^k_t - \delta^j_t \delta^k_s) - \delta^i_s(\delta^j_r \delta^k_t - \delta^j_t \delta^k_r) + \delta^i_t(\delta^j_r \delta^k_s - \delta^j_s \delta^k_r) \qquad 2.49$$

- The cross products of the base vectors is

$$\mathbf{g}_i \times \mathbf{g}_j = \pm\varepsilon_{ijk}\mathbf{g}^k \qquad\qquad 2.50$$

$$\mathbf{g}^i \times \mathbf{g}^j = \pm\varepsilon^{ijk}\mathbf{g}_k \qquad\qquad 2.51$$

13

- The cross product of two vectors **a** and **b** in terms of their components is

$$\mathbf{a} \times \mathbf{b} = \pm \varepsilon_{ijk} a^i b^j \mathbf{g}^k = \pm \varepsilon^{ijk} a_i b_j \mathbf{g}_k \qquad 2.52$$

- The curl of the vector **a** is curl $\mathbf{a} = \overrightarrow{\nabla} \times \mathbf{a}$ or

$$\overrightarrow{\nabla} \times \mathbf{a} = \mathbf{g}^i \times \frac{\partial \mathbf{a}}{\partial x^i} = a_{j\|i} (\mathbf{g}^i \times \mathbf{g}^j) = \varepsilon^{ijk} a_{j\|i} \mathbf{g}_k \qquad 2.53$$

- The curl of the tensor **T** is curl $\mathbf{T} = \overrightarrow{\nabla} \times \mathbf{T}$ or

$$\overrightarrow{\nabla} \times \mathbf{T} = \mathbf{g}^k \times \frac{\partial \mathbf{T}}{\partial x^k} = T_{ij\|k} (\mathbf{g}^k \times \mathbf{g}^i) \mathbf{g}^j = T_{ij\|k} \varepsilon^{kil} \mathbf{g}_l \mathbf{g}^j \qquad 2.54$$

or in an alternate form

$$\mathbf{T} \times \overleftarrow{\nabla} = \frac{\partial \mathbf{T}}{\partial x^k} \times \mathbf{g}^k = T_{ij\|k} \mathbf{g}^i (\mathbf{g}^j \times \mathbf{g}^k) = T_{ij\|k} \varepsilon^{jkl} \mathbf{g}^i \mathbf{g}_l \qquad 2.55$$

DIVERGENCE THEOREM: The divergence theorem, also known as Gauss's theorem, relates a surface integral of a vector function over a closed surface with the volume integral over the volume enclosed by the surface (Haines [7], p.179). Let **a** be a continuous vector function and **n** be the unit normal to the surface, then in vector notation

$$\int_S \mathbf{a} \cdot \mathbf{n}\, dS = \int_V div\, \mathbf{a}\, dV \qquad 2.56$$

or in tensor notation

$$\int_S a^i n_i\, dS = \int_V a^i_{\|i}\, dV \qquad 2.57$$

A generalized divergence theorem can be written as

$$\int\limits_{S} \mathbf{n} * \mathcal{A}dS = \int\limits_{V} \vec{\nabla} * \mathcal{A}dV \qquad\qquad 2.58$$

where $\mathcal{A}$ can represent a scalar, vector or tensor, and $*$ represents a generalized product.

GREEN'S THEOREM: If u and v are two continuous scalar functions of the coordinates in the X frame, then Green's theorem can be written as

$$\int\limits_{V} u\nabla^2 v dV = \int\limits_{S} u\mathbf{n}\cdot\vec{\nabla}v dS - \int\limits_{V} \vec{\nabla}u\cdot\vec{\nabla}v dV \qquad\qquad 2.59$$

or in equivalent symmetric form as

$$\int\limits_{V} \left(u\nabla^2 v - v\nabla^2 u\right)dV = \int\limits_{S} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS \qquad\qquad 2.60$$

or in tensor notation as

$$\int\limits_{V} g^{ij}\left(uv_{i\|j} + v_i u_j\right)dV = \int\limits_{S} \left(uv_i n^i\right)dS \qquad\qquad 2.61$$

STOKES THEOREM: Stokes theorem relates the surface integral of a continuous vector function, over an open surface, to a line integral around the boundary of the surface (Haines [7], p.182), where the direction of integration around the boundary is in the "proper direction". If $\mathbf{a}$ is the vector function (as before) and $\mathbf{n}$ is the unit normal vector to the surface S, and $\mathbf{t}$ is the unit tangent vector to the boundary curve C, then in vector notation Stokes theorem can be written as

$$\int\limits_{S} \mathbf{n}\cdot\text{curl }\mathbf{a}dS = \int\limits_{C} \mathbf{a}\cdot\mathbf{t}ds \qquad\qquad 2.62$$

15

or in tensor notation as

$$\int\limits_{S} \epsilon^{ijk} a_{j\|k} n_i \, dS = \int\limits_{C} a_i \frac{dx^i}{ds} \, ds \qquad 2.63$$

A generalized Stokes Theorem can be written as

$$\int\limits_{S} \left(\overrightarrow{\nabla} \times \mathbf{n}\right) * \mathcal{A} dS = \int\limits_{C} \mathbf{t} * \mathcal{A} ds \qquad 2.64$$

3 HISTORICAL REVIEW

The formulation of three-dimensional elasticity problems including the effect of temperature variation is attributed to Duhamel (1835). G. Green (1840) is credited with deriving what we refer to as Green's strain-energy function. He started with what we refer to as the Principal of Conservation of Energy. He assumed a scalar function which is opposite in sign to the potential energy of the deformed body per unit volume which he expressed in terms of the strain components. The partial derivative of the function with respect to a strain component yields the corresponding stress component. He derived the three-dimensional equations of elasticity (in terms of stresses and strains), containing in the general case 21 constants and in the isotropic case two constants. Lord Kelvin (1855) based the argument for the existence of such a function on the first and second laws of thermodynamics.

Shell theory originated historically as a special case of elastic plates. Sophie Gemaine (1821) provided simplified equations for the vibration of cylindrical shells based on the assumption that the deflections in the plane of the neutral surface were negligble, however the equations contained errors. A. Cauchy and S. D. Poisson (1828-29) achieved a dimensional reduction from the three-dimensional equations by a power-series expansion of the displacement in the direction normal to the middle surface.

Aron (1874) derived general equations for the bending of shells in curvilinear coordinates from the three–dimensional equations of elasticity. Rayleigh (1882) proposed simplifications based on the assumption that the neutral surface was either extensible (bending is unimportant) or nonextensible (bending is important).

G. Kirchhoff (1876) developed a theory for thin plates based on the assumptions that normals to the reference surface remain normal after deformation and that normals do not change length during deformation. A. E. H. Love (1888) applied the theory to shells using the principal of virtual work. The assumptions provided an easy was to

achieve a dimensional reduction but could not be fully reconciled with three-dimensional theory. The difficulty was that when the stress resultants were calculated by integrating the three-dimensional stresses across the thickness the shear resultants did not vanish.

This difficulty prompted some authors to attempt a different approach. Duhem (1893) suggested that physical bodies should be considered as oriented bodies (assemblies of points and directions). E. and F. Cosserat (1907) constructed theories based on this idea. Very little activity followed their initial work. Sudria (1935) noted an error in their theory and gave a different proof of invariance. Ericksen and Truesdell (1958) provided a general theory of oriented bodies in invariant form. Since then, P. M. Naghdi is the most prolific author regarding shell theories developed by assuming the shell can be treated as Cosserat surface.

The two different approaches to developing shell theories appear to have left the research community divided. For example, Niordson [8] writes in his introduction: "It has been demonstrated that there is no hindrance to the construction of such a two-dimensional theory, but the mere fact that a shell is a special case of a three-dimensional body should be a decisive argument against the introduction of any additional hypotheses in the theory. It is also contrary to the strive in science to unify theories."

The above information is contained in: Boley [9], Kraus [10], Love [11], Niordson [8], Soedel [12], and C. Truesdell and R. A. Toupin [13].

4 GEOMETRY OF CONTINUOUS MEDIA

A. Introduction

We begin by considering a general n-dimensional space R, and consider a general coordinate frame X with coordinates x^i and basis $\mathbf{g}_i$. The scalar invariant ds^2, representing the square of the differential distance between two neighboring points is $ds^2 = g_{ij}dx^idx^j$, where g_{ij} is the symmetric covariant metric tensor associated with the covariant base vectors $\mathbf{g}_i$. If the quadratic form ds^2 is positive definite, the space is n-dimensional Riemannian space. If the quadratic form ds^2 is positive definite and if the g_{ij} are constants, the space is referred to as Euclidean. This implies that the Christoffel symbols and the Riemann-Christoffel tensor associated with the X frame vanish identically. We are left with a linear vector space for which all the concepts of linear algebra apply.

We assume, once and for all, that the space of interest S is a three–dimensional Euclidean space in which all events can be ordered in a continuous manner in the time variable t. We assume that the metric of the space is independent of the mass of a body ρ, which may occupy it. We assume that the time variable t and the mass of a body are independent of the motion of any reference system.

We construct, as our frame of reference, a Cartesian frame Y with coordinates y^i, an orthonormal basis $\mathbf{c}_i$, and with the origin at some point O. The location of a point P can be given by a vector $\mathbf{r}$, originating at O and terminating at P, where $\mathbf{r} = y^i\mathbf{c}_i$. If we consider the totality of points P, rather than an individual point, r can be considered a vector field describing the location of all points P in S relative to the Y reference frame. Let $d\mathbf{r}$ represent the vector from P to a neighboring point P'. If the position of a point varies continuously with some variable, say time, the coordinates are functions of the time variable t. The equation $\mathbf{r} = \mathbf{r}(t)$ describes the path (trajectories) of the points P. The velocity vector

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \qquad\qquad 4.A.1$$

and the acceleration vector

$$\mathbf{a} = \frac{d^2\,\mathbf{r}}{dt^2} \qquad\qquad 4.\text{A}.2$$

represent the instantaneous velocity and acceleration of a point as its location varies with time. These can also be thought of as the instantaneous velocity and acceleration fields associated with all points P in S. The velocity vectors are tangent to the curve. We could also parameterize $\mathbf{r}$ relative to some other scalar value, such as curve length s, and perform the same differential operations and have a different physical interpretation of the results.

B. Metric Properties

We could have just as well chosen any number of other coordinate frames as our frame of reference. The vectorial representation of the quantities would be invariant but their analytical form could be different, depending on the frame chosen. We demonstrate this by considering a general curvilinear coordinate frame X with coordinates x^i and basis $\mathbf{g}_i$, and assume that the functional relationship between the coordinates of the Y and the X frames is given by $y^i = y^i(x^1, x^2, x^3, t)$. We assume that the transformation functions are continuous, and the Jacobian of the transformation does not vanish. The scalar invariant ds^2, representing the square of the differential distance between two neighboring points, is

$$ds^2 = g_{ij}\,dx^i\,dx^j \qquad\qquad 4.\text{B}.3$$

where $g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$ and

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial x^i} \qquad\qquad 4.\text{B}.4$$

The base vectors are dependent on the coordinates and are referred to as a local basis. Consider a vector $\mathbf{a}$ given by $\mathbf{a} = a^i \mathbf{g}_i$. The magnitude of vector $\mathbf{a}$ is

$$|\mathbf{a}| = (\mathbf{a}\cdot\mathbf{a})^{\frac{1}{2}} = \left(a^i \mathbf{g}_i \cdot a^j \mathbf{g}_j\right)^{\frac{1}{2}} = g_{ij}\,a^i a^j = a^i a_i \qquad\qquad 4.\text{B}.5$$

If θ is the angle between two vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{g_{ij}a^i b^j}{\sqrt{g_{ij}a^i a^j}\sqrt{g_{ij}b^i b^j}} \qquad\qquad 4.\mathrm{B}.6$$

The scalar V, representing the volume of a parallelepiped with sides parallel to three vectors **a**, **b** and **c** emanating from a corner is (assuming the orientation of **a**, **b** and **c** is such that V is positive) is given by $V = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c}$. The volume of a differential element in the X frame is

$$dV = dx^1\mathbf{g}_1 \cdot dx^2\mathbf{g}_2 \times dx^3\mathbf{g}_3 = \sqrt{g}\,dx^1 dx^2 dx^3 \qquad\qquad 4.\mathrm{B}.7$$

where g is equal to the determinant of the matrix of covariant metric tensor components, $|g_{ij}|$, which is equal to the square of the Jacobian J of the transformation connecting the coordinates in the Y and X frames where

$$J = \left|\frac{\partial y^i}{\partial x^j}\right| \qquad\qquad 4.\mathrm{B}.8$$

We note that the concepts of measurement of length, angle, areas and volumes are all scalar quantities obtained though tensor operations involving the tensor whose components are g_{ij}.

Consider two vectors **a** and **c** whose components are a^i and c^i, respectively. If $\mathbf{a}\cdot\mathbf{c} = 0$ the vectors are orthogonal and if $\mathbf{a}\cdot\mathbf{a} = 1$ the vector **a** is called a unit vector. In tensor notation this would be stated: if $g_{ij}a^i c^j = 0$, the vectors **a** and **c** are orthogonal, and if $g_{ij}a^i a^j = 1$, the vector **a** is called a unit vector. It is common practice in tensor analysis to refer to a vector by its components. Rather than refer to the vector **a**, we refer to the vector a^i, where the appropriate basis is implied.

We will eventually want to reference our results to our physical space. We therefore need to distinguish between tensor components and physical components. For example,

the physical components $a^{(i)}$ of a vector $\mathbf{a}$ are the components of $\mathbf{a}$ in the X frame when referred to a orthonormal basis, or $a^{(i)} = \sqrt{g_{ii}} a^i$, (no sum).

The velocity vector $\mathbf{v}$ is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial x^i} \frac{dx^i}{dt} \qquad \text{4.B.9}$$

or in terms of the base vectors $\mathbf{g}_i$

$$\mathbf{v} = \frac{dx^i}{dt} \mathbf{g}_i = v^i \mathbf{g}_i \qquad \text{4.B.10}$$

The acceleration vector $\mathbf{a}$ is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j} \frac{dx^i}{dt} \frac{dx^j}{dt} \qquad \text{4.B.11}$$

or in terms of the base vectors $\mathbf{g}_i$ and the associated Christoffel symbols

$$\mathbf{a} = \left[\frac{d^2 x^i}{dt^2} + \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \frac{dx^j}{dt} \frac{dx^k}{dt} \right] \mathbf{g}_i = a^i \mathbf{g}_i \qquad \text{4.B.12}$$

The quantity in brackets is referred to as the absolute or intrinsic derivative of the vector v^i with respect to the parameter t and is written as $\frac{\delta v^i}{\delta t}$ or in terms of the covariant derivative

$$\frac{\delta v^i}{\delta t} = v^i_{\|j} \frac{dx^j}{dt} \qquad \text{4.B.13}$$

The acceleration vector can be written as

$$\mathbf{a} = v^i_{\|j} \frac{dx^j}{dt} \mathbf{b}_i = \frac{\delta v^i}{\delta t} \mathbf{b}_i \qquad \text{4.B.14}$$

The length of the path s, traced by $\mathbf{r}$ between two instants of time is given by

$$s = \int_{t_1}^{t_2} \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} \, dt \qquad \text{4.B.15}$$

22

C. Space Curves

If we choose arc length s as our parameter for $\mathbf{r}$, then from differential geometry, a unit vector $\mathbf{t}$, tangent to a curve at P described by $\mathbf{r}$ is given by

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{\partial \mathbf{r}}{\partial x^i}\frac{dx^i}{ds} = \frac{dx^i}{ds}\mathbf{g}_i \qquad\qquad 4.C.16$$

The fact that $\mathbf{t}$ is a unit vector is easily shown by taking the inner product of $\mathbf{t}$ with itself. If we differentiate $\mathbf{t}$ with respect to s, we can define a vector $\mathbf{c}$, referred to as the curvature vector by

$$\mathbf{c} = \frac{d\mathbf{t}}{ds} = \frac{d^2\mathbf{r}}{ds^2} = \kappa\mathbf{n} \qquad\qquad 4.C.17$$

where κ is a scalar, referred to as the curvature of the curve at P, and is the magnitude of $\mathbf{c}$. The vector $\mathbf{n}$ is called the principal normal to the curve at P, and its direction is assumed to be from P towards the center of curvature or towards the center of the osculating circle. The plane containing $\mathbf{t}$ and $\mathbf{n}$ is referred to as the osculating plane. The binormal vector is a unit vector orthogonal to both $\mathbf{t}$ and $\mathbf{n}$ and is given by

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} \qquad\qquad 4.C.18$$

The plane containing $\mathbf{n}$ and $\mathbf{b}$ is sometimes referred to as the normal plane, while the plane containing $\mathbf{t}$ and $\mathbf{b}$ is sometimes referred to as the rectifying plane [14]. The vectors $\mathbf{t}$, $\mathbf{n}$, and $\mathbf{b}$ form a unique, local orthonormal basis at all points P along the curve. By local, we mean that the orientation of the orthonormal triad changes as we move along the curve. The manner in which the basis changes orientation, determines the intrinsic properties of the curve. These properties are given by the Serret-Frenet formulas. In vector notation they are:

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n} \qquad\qquad 4.C.19$$

$$\frac{d\mathbf{b}}{ds} = -\tau\mathbf{n} \qquad\qquad 4.C.20$$

$$\frac{d\mathbf{n}}{ds} = \tau\mathbf{b} - \kappa\mathbf{n} \qquad\qquad 4.C.21$$

where κ is the scalar previously mentioned, and τ is a scalar referred to as the torsion of the curve at P. The second equation is found by differentiating the inner product of **b** and **t** with respect to s and recognizing that $\frac{d\mathbf{b}}{ds}$ is orthogonal to both **b** and **t** and therefore parallel to **n**. The scalar τ is the magnitude of $\frac{d\mathbf{b}}{ds}$. The direction of $\frac{d\mathbf{b}}{ds}$ is assumed to be opposite of **n**, although this is not always the case in all presentations (e.g., Seeley [15]). The third formula is found by differentiating the equation $\mathbf{n} = \mathbf{b} \times \mathbf{t}$ with respect to s and utilizing the first two equations and the relationship between the cross products of **t**, **b**, and **n**. Assuming the components of **t**, **b** and **n** are λ^i, μ^i, and ν^i respectively, the Serret-Frenet formulas can be written in tensor notation as follows:

$$\frac{\delta \lambda^i}{\delta s} = \frac{d\lambda^i}{ds} + \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \lambda^j \frac{dx^k}{ds} = \kappa \mu^i \qquad \text{4.C.22}$$

$$\frac{\delta \mu^i}{\delta s} = \frac{d\mu^i}{ds} + \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \mu^j \frac{dx^k}{ds} = \tau \nu^i - \kappa \lambda^i \qquad \text{4.C.23}$$

$$\frac{\delta \nu^i}{\delta s} = \frac{d\nu^i}{ds} + \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \nu^j \frac{dx^k}{ds} = \kappa \mu^i \qquad \text{4.C.24}$$

The Serret-Frenet equations uniquely (within a rigid body motion) determine a curve C, when the functions κ and τ are given as continuous functions of s along C. If the torsion of a curve is equal to zero, the curve is referred to as a plane curve. Analogous equations are available for surfaces.

D. Surfaces

Recall the vector $\mathbf{r} = \mathbf{r}(y^1, y^2, y^3)$ which describes all points P in Y. When the coordinates are continuous functions of a single parameter, the vector **r** represents a space curve. When the coordinates are continuous functions of two independent parameters, the vector **r** represents a surface.

Let the functional relationship between the coordinates in Y and the two parameters be given by:

$$y^i = y^i(u^1, u^2) \qquad \text{4.D.25}$$

and assume that the functions are continuous and that the Jacobian of the transformation is of rank 2.

The parametric form of the surface is given by

$$\mathbf{r} = \mathbf{r}(u^1, u^2) = y^i(u^1, u^2)\mathbf{c}_i \qquad \text{4.D.26}$$

We could also introduce the transformation to our general X frame at this point, but will not do so. If one of the parameters is held constant, $\mathbf{r}$ describes a curve on the surface. By assigning a series of fixed values to the first parameter and then to the second, a net of curves can be described on the surface. These curves are referred to as coordinate curves. The assumed independence of the two parameters ensures the curves obtained by fixing the first parameter then the second, will intersect at some unique point P on the surface. The value of the parameters at P are referred to the curvilinear or Gaussian coordinates on the surface. If we consider the vector dr from point P on the surface to a neighboring point, then

$$dr = \frac{\partial \mathbf{r}}{\partial u^\alpha} du^\alpha = \mathbf{a}_\alpha du^\alpha \qquad \text{4.D.27}$$

The vector $\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial u^\alpha}$ is tangent to the u^α curve at P. The length of dr is given by

$$ds^2 = dr\cdot dr = \frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} du^\alpha du^\beta = a_{\alpha\beta} du^\alpha du^\beta \qquad \text{4.D.28}$$

The quantity $a_{\alpha\beta} du^\alpha du^\beta$ is called the first fundamental form of the surface and the tensor components $a_{\alpha\beta}$, play the same role on the surface as the metric tensor components g_{ij}, previously described. The plane containing the tangent vectors is called the tangent plane and is tangent to the surface at P. The unit vector $\mathbf{n}$ normal to the tangent plane at P is given by

$$\mathbf{n} = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{|\mathbf{a}_1 \times \mathbf{a}_2|} \qquad \text{4.D.29}$$

The normal vector $\mathbf{n}$ is also normal to all curves on the surface passing through point P and, therefore, is independent of the choice of coordinate curves. The differential

distance along the coordinate curve is $ds_\alpha = \sqrt{a_{\alpha\alpha}}du^\alpha$ (no sum). An element of area is $dA = \sqrt{a}\,du^1 du^2$ where $a = |a_{\alpha\beta}|$. The surface metric tensor determines all intrinsic properties of the surface, but not how the surface appears to an observer in our physical space. A plane, cone, and cylinder are examples of surfaces which can be shown to have the same metric coefficients but obviously appear different to our observer. Surfaces which have the same metric coefficients are referred to as isometric surfaces.

Another quadratic form called the second fundamental form of the surface helps define the true shape of a surface. It is related to the way the unit normal to the surface changes as we move along the surface or to the way the surface deviates from the tangent plane in the neighborhood of P. Its value is approximately one-half the distance between the tangent plane and a point on the surface in the neighborhood of P (Lass [16], p.75; Stoker [17], p.85). Recall that the first fundamental form of the surface was defined by

$$\mathcal{A} \equiv d\mathbf{r} \cdot d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} du^\alpha du^\beta = a_{\alpha\beta} du^\alpha du^\beta \qquad 4.\text{D}.30$$

The second fundamental form of the surface is defined by

$$\mathcal{B} \equiv -d\mathbf{n} \cdot d\mathbf{r} = -\frac{\partial \mathbf{n}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{r}}{\partial u^\beta} du^\alpha du^\beta = b_{\alpha\beta} du^\alpha du^\beta \qquad 4.\text{D}.31$$

The quantities $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are referred to as the first and second fundamental form magnitudes respectively. If the first and second fundamental forms are given as continuous functions of the parameters u^α and are positive definite, and satisfy conditions called the Gauss-Codazzi conditions, the surface is uniquely determined (within a rigid body motion). When $b_{\alpha\beta}$ are zero, the surface is a plane.

A third fundamental form is sometimes introduced. Its form is

$$\mathcal{C} \equiv d\mathbf{n} \cdot d\mathbf{n} = \frac{\partial \mathbf{n}}{\partial u^\alpha} \cdot \frac{\partial \mathbf{n}}{\partial u^\beta} du^\alpha du^\beta = c_{\alpha\beta} du^\alpha du^\beta \qquad 4.\text{D}.32$$

The third fundamental form represents the square of the length of the line element of the spherical image of the surface (Stoker [17], p.98). A spherical image is obtained by translating all of the unit normals to the surface, to the center of a unit sphere. The

surface curve traced by the unit vector **n**, on the surface of the unit sphere, as we move along the actual surface is called the spherical image of the surface.

The three fundamental forms are not independent. This can be shown by using the equations of Gauss and Weingarten (see following section) in the equation for the third fundamental form, with the following result

$$\mathcal{C} = 2H\mathcal{B} - K\mathcal{A} \qquad \text{4.D.33}$$

where H and K are the mean and the total curvature of the surface and are related to the first and second fundamental form magnitudes by

$$H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta} , \qquad K = \frac{b}{a} \qquad \text{4.D.34}$$

where $b = |b_{\alpha\beta}|$ and $a = |a_{\alpha\beta}|$ and the relationship between the contravariant and covariant components of the surface metric tensor is $a^{\alpha\beta} a_{\beta\gamma} = \delta^{\alpha}_{\gamma}$. We write out the tensor representations explicitly for illustration and future use:

$$b = b_{11} b_{22} - (b_{12})^2 \qquad \text{4.D.35}$$

$$a = a_{11} a_{22} - (a_{12})^2 \qquad \text{4.D.36}$$

$$a^{11} = \frac{a_{22}}{a} , \quad a^{12} = a^{21} = -\frac{a_{12}}{a} , \quad a^{22} = \frac{a_{11}}{a} \qquad \text{4.D.37}$$

$$H = \frac{1}{2} \left(\frac{a_{22} b_{11} - 2a_{12} + a_{11} b_{22}}{a_{11} a_{22} - (a_{12})^2} \right) \qquad \text{4.D.38}$$

$$K = \frac{b_{11} b_{22} - (b_{12})^2}{a_{11} a_{22} - (a_{12})^2} \qquad \text{4.D.39}$$

If the parametric directions are the lines of principal curvatures, $a_{12} = b_{12} = 0$, then H and K are related to the principal curvatures K_α by

$$H = \frac{1}{2}(K_1 + K_2) \quad and \quad K = K_1 K_2 \qquad \text{4.D.40}$$

Formulas for the three fundamental forms and the relationship between them will prove useful for deriving the fundamental forms for surfaces which are parallel to a given surface, in terms of the fundamental forms of the given surface.

E. Gauss-Codazzi Conditions

The Gauss-Codazzi equations (conditions) are related to the assumption of the independence of the order of second order, partial derivatives of the tangent and normal vectors (when treated as vectors in three dimension Cartesian space, or $n_{,\alpha\beta} = n_{,\beta\alpha}$ and $a_{\gamma,\alpha\beta} = a_{\gamma,\beta\alpha}$, where the comma in the subscript indicates partial differentiation with respect to the variables following it. Kraus [10] derives the equations, using vector notation, directly from the above assumptions for an orthonormal set of tangent and normal vectors. Sokolnikoff [2] derives the equations for the more general case using general tensor notation (implied base vectors), the more general integrability conditions, and an intermediate coordinate transformation. This presentation follows that of Sokolnikoff, but with an explicit representation of the base vectors and without the intermediate coordinate transformation.

The second fundamental form magnitudes can be written as

$$b_{\alpha\beta} = -\frac{1}{2}\left(\frac{\partial n}{\partial u^\alpha} \cdot \frac{\partial r}{\partial u^\beta} + \frac{\partial n}{\partial u^\beta} \cdot \frac{\partial r}{\partial u^\alpha} \right) = n \cdot \frac{\partial^2 r}{\partial u^\alpha \partial u^\beta} \qquad \text{4.E.41}$$

or by using the comma notation for indicating partial differentiation and the relationship $\frac{\partial r}{\partial u^\alpha} = a_\alpha$, as

$$b_{\alpha\beta} = -\frac{1}{2}\left(n_{,\alpha} \cdot a_\beta + n_{,\beta} \cdot a_\alpha \right) = n \cdot a_{\alpha,\beta} \qquad \text{4.E.42}$$

(We note that assumptions about the direction of the normal are implicit in determining the fact that the quantities above, representing the second fundamental form magnitudes, are

28

assumed to be positive or negative. Kraus [10] and others assume an opposite direction of the normal vector.) The following equation is referred to as the formula(s) of Gauss, and can be shown to yield the same result as 4.E.42, when the inner product of 4.E.43 and $\mathbf{n}$ is taken:

$$\mathbf{n} b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \qquad\qquad 4.E.43$$

Recall that the normal vector $\mathbf{n}$ is a unit vector. Therefore, if we take the partial derivative of the inner product of $\mathbf{n}$ with itself, with respect to one of the surface parameters, we find $\mathbf{n}_{,\alpha} \cdot \mathbf{n} = 0$, which indicates that the two vectors are orthogonal and, therefore, the vector $\mathbf{n}_{,\alpha}$ lies in the tangent plane and can be written in terms of a linear combination of the tangent vectors:

$$\mathbf{n}_{,\alpha} = -a^{\beta\gamma} b_{\beta\alpha} \mathbf{a}_{\gamma} = -b_{\alpha}^{\gamma} \mathbf{a}_{\gamma} \qquad\qquad 4.E.44$$

The above equations are called the Weingarten formulas. Taking the second partial derivative of the above equation

$$\mathbf{n}_{,\alpha\beta} = -(b_{\alpha,\beta}^{\gamma} \mathbf{a}_{\gamma} + b_{\alpha}^{\gamma} \mathbf{a}_{\gamma,\beta}) \qquad\qquad 4.E.45$$

and after making use of

$$\mathbf{a}_{\gamma,\beta} = \left\{ \begin{matrix} \lambda \\ \gamma\beta \end{matrix} \right\} \mathbf{a}_{\lambda} + b_{\gamma\beta} \mathbf{n} \qquad\qquad 4.E.46$$

we find

$$-\mathbf{n}_{,\alpha\beta} = b_{\alpha,\beta}^{\gamma} \mathbf{a}_{\gamma} + b_{\alpha}^{\rho} \left\{ \begin{matrix} \gamma \\ \rho\beta \end{matrix} \right\} \mathbf{a}_{\gamma} + b_{\alpha}^{\gamma} b_{\gamma\beta} \mathbf{n} \qquad\qquad 4.E.47$$

After interchanging α and β we have

$$-\mathbf{n}_{,\beta\alpha} = b_{\beta,\alpha}^{\gamma} \mathbf{a}_{\gamma} + b_{\beta}^{\rho} \left\{ \begin{matrix} \gamma \\ \rho\alpha \end{matrix} \right\} \mathbf{a}_{\gamma} + b_{\beta}^{\gamma} b_{\gamma\alpha} \mathbf{n} \qquad\qquad 4.E.48$$

29

Equating the above two equations and taking the inner product with $\mathbf{a}^\lambda$ and after simplifying we have

$$b^\gamma_{\alpha,\beta} + b^\rho_\alpha \left\{ \begin{matrix} \gamma \\ \rho\beta \end{matrix} \right\} - b^\gamma_{\beta,\alpha} - b^\rho_\beta \left\{ \begin{matrix} \gamma \\ \rho\alpha \end{matrix} \right\} = 0 \qquad 4.E.49$$

Recall that

$$b^\gamma_{\alpha\|\beta} = b^\gamma_{\alpha,\beta} + b^\rho_\alpha \left\{ \begin{matrix} \gamma \\ \rho\beta \end{matrix} \right\} - b^\gamma_\rho \left\{ \begin{matrix} \rho \\ \alpha\beta \end{matrix} \right\} \qquad 4.E.50$$

After substituting the above equation into the previous one with the proper adjustment in the indices we have

$$b^\gamma_{\alpha\|\beta} - b^\gamma_{\beta\|\alpha} = 0 \qquad 4.E.51$$

which are the equations of Codazzi.

Next

$$\mathbf{a}_{\alpha,\beta\gamma} = \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\}_{,\gamma} \mathbf{a}_\lambda + \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\} \mathbf{a}_{\lambda,\gamma} + b_{\alpha\beta,\gamma}\mathbf{n} + b_{\alpha\beta}\mathbf{n}_{,\gamma} = \qquad 4.E.52$$

$$\left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\}_{,\gamma} \mathbf{a}_\lambda + \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \lambda\gamma \end{matrix} \right\} \mathbf{a}_\rho + b_{\lambda\gamma} \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\} \mathbf{n} + b_{\alpha\beta,\gamma}\mathbf{n} + b_{\alpha\beta}\mathbf{n}_{,\gamma} \qquad 4.E.53$$

After interchanging β and γ we have

$$\mathbf{a}_{\alpha,\gamma\beta} = \left\{ \begin{matrix} \lambda \\ \alpha\gamma \end{matrix} \right\}_{,\beta} \mathbf{a}_\lambda + \left\{ \begin{matrix} \lambda \\ \alpha\gamma \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \lambda\beta \end{matrix} \right\} \mathbf{a}_\rho + b_{\lambda\beta} \left\{ \begin{matrix} \lambda \\ \alpha\gamma \end{matrix} \right\} \mathbf{n} + b_{\alpha\gamma,\beta}\mathbf{n} + b_{\alpha\gamma}\mathbf{n}_{,\beta} \qquad 4.E.54$$

Equating the above two equations and taking the inner product with $\mathbf{a}^\delta$ and after simplifying we have

$$\left\{ \begin{matrix} \delta \\ \alpha\beta \end{matrix} \right\}_{,\gamma} - \left\{ \begin{matrix} \delta \\ \alpha\gamma \end{matrix} \right\}_{,\beta} + \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\} \left\{ \begin{matrix} \delta \\ \lambda\gamma \end{matrix} \right\} - \left\{ \begin{matrix} \lambda \\ \alpha\gamma \end{matrix} \right\} \left\{ \begin{matrix} \delta \\ \lambda\beta \end{matrix} \right\} = b_{\alpha\beta}b^\lambda_\gamma - b_{\alpha\gamma}b^\lambda_\beta \qquad 4.E.55$$

The left side of the above equation is recognized as the components of the Riemann-Cristoffel tensor of the surface or

$$R^\delta_{\alpha\gamma\beta} = b_{\alpha\beta}b^\lambda_\gamma - b_{\alpha\gamma}b^\lambda_\beta \qquad 4.E.56$$

30

which is the equation of Gauss.

The two independent equations of Codazzi are

$$b_{\alpha\alpha,\beta} - b_{\alpha\beta,\alpha} = 0, \quad (\alpha \neq \beta), \quad (\text{no sum}) \qquad \text{4.E.57}$$

while the one independent equation of Gauss is

$$b_{11}b_{22} - b_{12}^2 = R_{1212} \qquad \text{4.E.58}$$

The equations of Codazzi in expanded form are

$$\frac{\partial b_{\alpha\alpha}}{\partial u^\beta} - \frac{\partial b_{\alpha b}}{\partial u^\alpha} - b_{\alpha\delta}\left\{\begin{matrix} \delta \\ \alpha\beta \end{matrix}\right\} + b_{\delta\beta}\left\{\begin{matrix} \delta \\ \alpha\alpha \end{matrix}\right\} = 0, \quad (\alpha \neq \beta), \quad (\text{no sum}) \qquad \text{4.E.59}$$

while the equation of Gauss is (sum on β)

$$b_{11}b_{22} - b_{12}^2 = g_{11}\left(\frac{\partial\left\{\begin{matrix}1\\22\end{matrix}\right\}}{\partial u^1} - \frac{\partial\left\{\begin{matrix}1\\21\end{matrix}\right\}}{\partial u^2} + \left\{\begin{matrix}\beta\\22\end{matrix}\right\}\left\{\begin{matrix}1\\\beta 1\end{matrix}\right\} - \left\{\begin{matrix}\beta\\21\end{matrix}\right\}\left\{\begin{matrix}1\\\beta 2\end{matrix}\right\}\right) \qquad \text{4.E.60}$$

Recall that the Gaussian or total curvature of the surface K is given by

$$K = \frac{R_{1212}}{a} \qquad \text{4.E.61}$$

where $a = a_{11}a_{22} - a_{12}^2$, or

$$K = \frac{b_{11}b_{22} - b_{12}^2}{a_{11}a_{22} - a_{12}^2} = \frac{b}{a} \qquad \text{4.E.62}$$

The Gaussian curvature is equal to the product of the principal curvatures. When the parametric curves are lines of the principal curvatures the Codazzi equations take the form

$$\frac{\partial}{\partial u^\beta}\left(\frac{A_\alpha}{R_\alpha}\right) = \frac{1}{R_\beta}\frac{\partial A_\alpha}{\partial u^\beta} \quad (\alpha \neq \beta), \quad (\text{no sum}) \qquad \text{4.E.63}$$

while the Gauss equation is

$$\frac{\partial}{\partial u^1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial u^1}\right) + \frac{\partial}{\partial u^2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial u^2}\right) = -\frac{A_1 A_2}{R_1 R_2} \qquad \text{4.E.64}$$

where $A_\alpha = \sqrt{a_{\alpha\alpha}}$ (no sum), and R_α is the reciprocal of the principal curvature K_α.

F. Parallel Surfaces

In Section D we considered the metric properties of a surface described by $r = r(u^1, u^2)$. At every point P on the surface it was possible to construct two independent vectors which were tangent to the surface and a unit vector $\mathbf{n}$ normal to the surface. If we consider the totality of points P' located a distance h along the normal $\mathbf{n}$ from P, the points P' define a surface parallel to the original surface. Let quantities with a bar denote those referencing the parallel surface while those without, reference the original surface. We want to relate quantities on the parallel surface to those on the original surface. The equation of the parallel surface is

$$\bar{r} = \bar{r}(u^1, u^2) = r(u^1, u^2) + h\mathbf{n} \qquad \text{4.F.65}$$

Recall that

$$d\bar{r} = \frac{\partial \bar{r}}{\partial u^\alpha} du^\alpha = \bar{a}_\alpha du^\alpha \qquad \text{4.F.66}$$

$$\bar{n} = \frac{\bar{a}_1 \times \bar{a}_2}{|\bar{a}_1 \times \bar{a}_2|} \qquad \text{4.F.67}$$

Equation 4.F.66 can be written as

$$d\bar{r} = dr + h\, d\mathbf{n} \qquad \text{4.F.68}$$

We compute the first fundamental form for the parallel surface in terms of the quantities associated with the original surface and find

$$\overline{\mathcal{A}} = d\bar{r} \cdot d\bar{r} = dr \cdot dr + 2h\, d\mathbf{n} \cdot dr + d\mathbf{n} \cdot d\mathbf{n} \qquad \text{4.F.69}$$

Recalling the definitions for the three fundamental forms, the above equation reduces to

$$\overline{\mathcal{A}} = \mathcal{A} + 2h\mathcal{B} + \mathcal{C} \qquad \text{4.F.70}$$

But from 4.D.33, which defines the relationship between the three forms, the above equation reduces to

$$\overline{\mathcal{A}} = \mathcal{A}(1 - h^2 K) - 2h\mathcal{B}(1 - hH) \qquad \text{4.F.71}$$

32

The above equation, in terms of the fundamental form magnitudes, is simply

$$\bar{a}_{\alpha\beta} = a_{\alpha\beta}\left(1 - h^2 K\right) - 2hb_{\alpha\beta}(1 - hH) \qquad \text{4.F.72}$$

Similarly, the second fundamental form is

$$\overline{\mathcal{B}} \equiv -d\overline{\mathbf{n}} \cdot d\overline{\mathbf{r}} = -d\overline{\mathbf{n}} \cdot (d\mathbf{r} + h d\mathbf{n}) \qquad \text{4.F.73}$$

Recall that $\overline{\mathbf{n}}$ and $\mathbf{n}$ are unit vectors orthogonal to the tangent planes at P' and P, respectively, and $d\overline{\mathbf{n}}$ and $d\mathbf{n}$ lie in the two respective tangent planes and are therefore orthogonal to the respective normals. If we take the inner product of 4.F.68 with $\mathbf{n}$, we find

$$d\overline{\mathbf{r}} \cdot \mathbf{n} = d\mathbf{r} \cdot \mathbf{n} + h \, d\mathbf{n} \cdot \mathbf{n} = 0 \qquad \text{4.F.74}$$

But $d\overline{\mathbf{r}} \cdot \overline{\mathbf{n}} = 0$, therefore $\mathbf{n} = \overline{\mathbf{n}}$. The equation for the second fundamental can then be written as

$$\overline{\mathcal{B}} = -d\mathbf{n} \cdot d\mathbf{r} - h d\mathbf{n} \cdot d\mathbf{n} \qquad \text{4.F.75}$$

or

$$\overline{\mathcal{B}} = \mathcal{B} - h\mathcal{C} \qquad \text{4.F.76}$$

Using 4.D.33 again

$$\overline{\mathcal{B}} = \mathcal{B}(1 - 2hH) + hK\mathcal{A} \qquad \text{4.F.77}$$

or in terms of the fundamental form magnitudes

$$\bar{b}_{\alpha\beta} = b_{\alpha\beta}(1 - 2hH) + hK a_{\alpha\beta} \qquad \text{4.F.78}$$

5   DEFORMATIONS

We will develop the tensor equations defining the deformation of a region of our physical space. In doing so, we will follow closely the approach and notations used by Sokolnikoff [18] (Chap. 6). Three coordinate frames will be used: a global stationary frame, a local stationary frame (Lagrangian frame) and a local moving frame (Eulerian frame).

We consider a region τ_0 (initial state), of our physical space at a time t_0, and refer to the totality of points contained in the region as P_0. We assume that as time increases, the points move (are displaced) in a continuous manner to a new region of space τ (final state). We are interested in the case when the region τ_0 is deformed as it moves to the region τ.

An orthonormal Cartesian frame Y, with coordinates y^i, and basis c_i, will be used for the global stationary frame. At every point in τ_0, we construct two local general curvilinear coordinate frames X, with convected coordinates x^i. Convected coordinates are coordinates which move with the material points. Any changes in the distance between material points is represented by changes in the base vectors and not in changes in the coordinates. We denote the basis of the local stationary frame as h_i, and the basis of the local frame which moves with the points, as g_i. The points P_0, when located in τ, will be referred to a P. The position vector of P_0 is r_0, and the position vector of P is r. Likewise, the vector from a point to a neighboring point, in a region, is dr_0 and dr, for the initial and final states, respectively. We denote the magnitude of the two vectors as ds_0 and ds, respectively. We refer to the final state as being deformed or strained when $ds \neq ds_0$.

The vectors dr_0 and dr, when expressed in terms of their local basis are:

$$dr_0 = dx^i h_i \quad \text{and} \quad dr = dx^i g_i \qquad\qquad 5.1$$

We denote the covariant metric tensor components associated with τ_0 as h_{ij}, and those associated with τ, as g_{ij}. By calculating the square of the distance element in the initial

state and subtracting it from that in the final state we find,

$$(ds)^2 - (ds_o)^2 = (g_{ij} - h_{ij})dx^i dx^j \qquad 5.2$$

We let

$$E_{ij} = \frac{1}{2}(g_{ij} - h_{ij}) \qquad 5.3$$

We note that the E_{ij} represent the components of a symmetric tensor, due to the manner in which it was formed. We let $\mathbf{E}$ denote the tensor when referred to the base vectors in the initial state and $\mathbf{E}^*$ denote the tensor when referred to the base vectors in the final state. Operations on the components of $\mathbf{E}$ and $\mathbf{E}^*$, involving the metric tensor, require use of the metric associated with their respective basis, and the results distinguished as such. The tensor $\mathbf{E}$ is referred to as the Lagrangian strain tensor, and $\mathbf{E}^*$ as the Eulerian strain tensor. If we express both tensors in terms of their covariant components then $\mathbf{E} = E_{ij}\mathbf{h}^i\mathbf{h}^j$ and $\mathbf{E}^* = E_{ij}^*\mathbf{g}^i\mathbf{g}^j$.

First, from the viewpoint of an observer in the initial state, let e_i represent the change in length per unit length (elongation) of the base vectors in the initial state, then

$$e_i = \frac{|\mathbf{g}_i| - |\mathbf{h}_i|}{|\mathbf{h}_i|} \text{ (no sum)} \qquad 5.4$$

The above equation, when expressed in terms of the metric tensor components, results in

$$e_i = \frac{\sqrt{g_{ii}} - \sqrt{h_{ii}}}{\sqrt{h_{ii}}} = \sqrt{1 + \frac{g_{ii} - h_{ii}}{h_{ii}}} - 1 = \sqrt{1 + \frac{2E_{ii}}{h_{ii}}} - 1 \text{ (no sum)} \qquad 5.5$$

This can also be written as

$$\sqrt{g_{ii}} = (1 + e_i)\sqrt{h_{ii}} \text{ (no sum)} \qquad 5.6$$

If we rewrite the fraction under the radical as (no sum)

$$\left(\frac{g_{ii} - h_{ii}}{h_{ii}}\right) = \frac{|\mathbf{g}_i|^2 - |\mathbf{h}_i|^2}{|\mathbf{h}_i|^2} = \frac{(|\mathbf{g}_i| + |\mathbf{h}_i|)}{|\mathbf{h}_i|} \frac{(|\mathbf{g}_i| - |\mathbf{h}_i|)}{|\mathbf{h}_i|} = \frac{(|\mathbf{g}_i| + |\mathbf{h}_i|)}{|\mathbf{h}_i|} e_i \qquad 5.7$$

we note that when the elongation of the base vectors is small compared to unity, the quantity to the left of the equality sign in the equation above is likewise. Therefore, the quantity under the radical can be expanded in the form of a Maclaurin series as

$$\sqrt{1 + \frac{g_{ii} - h_{ii}}{h_{ii}}} = 1 + \frac{1}{2}\left(\frac{g_{ii} - h_{ii}}{h_{ii}}\right) - \cdots, \quad \text{(no sum)} \qquad 5.8$$

when only the first two terms are retained. The elongations of the base vectors in the initial state, is then given by

$$e_i = \frac{1}{2}\left(\frac{g_{ii} - h_{ii}}{h_{ii}}\right) \quad \text{(no sum)} \qquad 5.9$$

Utilizing the definition of the strain tensor components in terms of the metric tensor components with the above, we find

$$e_i = \frac{E_{ii}}{h_{ii}} \quad \text{(no sum)} \qquad 5.10$$

The strain tensor components E_{ii} (no sum) are related to the elongations in the direction of the base vectors and are referred to as the normal components. When the base vectors in the initial state are unit vectors the above simplifies to

$$e_i = E_{ii} \quad \text{(no sum)} \qquad 5.11$$

From the viewpoint of an observer in the final state

$$e_i = \frac{|\mathbf{h}_i| - |\mathbf{g}_i|}{|\mathbf{g}_i|} \quad \text{(no sum)} \qquad 5.12$$

or

$$e_i = \frac{\sqrt{h_{ii}} - \sqrt{g_{ii}}}{\sqrt{g_{ii}}} = 1 - \sqrt{1 - \frac{g_{ii} - h_{ii}}{g_{ii}}} = 1 - \sqrt{1 - \frac{2E_{ii}^*}{g_{ii}}} \quad \text{(no sum)} \qquad 5.13$$

When assumptions, similar to those made in the initial state, are made, we find

$$e_i = \frac{E_{ii}^*}{g_{ii}} \quad \text{(no sum)} \qquad 5.14$$

A. Shearing Components

Let θ_{ij} represent the angle between the base vectors $\mathbf{g}_i$ and $\mathbf{g}_j$ in the final state while $\theta_{ij}{}^o$, represent the angle between the base vectors $\mathbf{h}_i$ and $\mathbf{h}_j$ in the initial state. The strain tensor components, in terms of the base vectors and the angles between them, can be written as

$$E_{ij} = \frac{1}{2}\left(|\mathbf{g}_i||\mathbf{g}_j|\cos\theta_{ij} - |\mathbf{h}_i||\mathbf{h}_j|\cos\theta_{ij}{}^o\right) \qquad \text{5.A.15}$$

or in terms of the metric tensor components and angles as

$$E_{ij} = \frac{1}{2}\left(\sqrt{g_{ii}}\sqrt{g_{jj}}\cos\theta_{ij} - \sqrt{h_{ii}}\sqrt{h_{jj}}\cos\theta_{ij}{}^o\right) \ \text{(no sum)} \qquad \text{5.A.16}$$

If we assume that the base vectors of the initial state are orthogonal and let $\theta_{ij} = \frac{\pi}{2} - \alpha_{ij}$, the above equation reduces to

$$E_{ij} = \frac{1}{2}\left(\sqrt{g_{ii}}\sqrt{g_{jj}}\sin\alpha_{ij}\right) \ \text{(no sum)} \qquad \text{5.A.17}$$

or when expressed in terms of the elongations and metric components of the initial state

$$E_{ij} = \frac{1}{2}\left((1+e_i)(1+e_i)\sqrt{h_{ii}}\sqrt{h_{jj}}\sin\alpha_{ij}\right) \qquad \text{5.A.18}$$

The strain tensor components E_{ij} $(i \neq j)$ are referred to as the shearing components and are related to the change in the angle between elements which are orthogonal in the initial state. If the elongations are small compared to unity and α_{ij} are small, then

$$E_{ij} = \frac{1}{2}\sqrt{h_{ii}}\sqrt{h_{jj}}\alpha_{ij} \ \text{(no sum)} \qquad \text{5.A.19}$$

From the viewpoint of an observer in the final state

$$E_{ij}^* = \frac{1}{2}\left((1-e_i)(1-e_i)\sqrt{g_{ii}}\sqrt{g_{jj}}\sin\beta_{ij}\right) \ \text{(no sum)} \qquad \text{5.A.20}$$

where the e_i are the elongations along the base vectors in the final state, and $\beta = \theta_{ij}{}^o - \frac{\pi}{2}$.

B. Strain Invariants

We are interested in determining invariant forms related to the strain tensor components in the initial and final state. We will use the viewpoint of an observer in the final state first. Recall that a unit vector in the direction of dr in the final state, has the following form

$$\frac{d\mathbf{r}}{ds} = \frac{dx^i}{ds}\mathbf{g}_i = \lambda^i \mathbf{g}_i \qquad \text{5.B.21}$$

The λ^i are components of a unit vector in the final state which determines the direction of dr. We seek a quadratic form involving the strain tensor components and the unit vector components. We proceed as follows:

$$\frac{d\mathbf{r}}{ds} \cdot \frac{d\mathbf{r}}{ds} = \frac{(ds)^2}{(ds)^2} = g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 1 \qquad \text{5.B.22}$$

$$\frac{d\mathbf{r_o}}{ds} = \mathbf{h}_i\frac{dx^i}{ds} \qquad \text{5.B.23}$$

$$\frac{d\mathbf{r_o}}{ds} \cdot \frac{d\mathbf{r_o}}{ds} = \frac{(ds_o)^2}{(ds)^2} = h_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \qquad \text{5.B.24}$$

$$\frac{(ds)^2}{(ds)^2} - \frac{(ds_o)^2}{(ds)^2} = (g_{ij} - h_{ij})\frac{dx^i}{ds}\frac{dx^j}{ds} \qquad \text{5.B.25}$$

The equation above may be rewritten as

$$\frac{(ds)^2 - (ds_o)^2}{2(ds)^2} = E_{ij}^*\lambda^i\lambda^j \qquad \text{5.B.26}$$

We want to determine the directions for which the quadratic form $Q(\lambda) = E_{ij}^*\lambda^i\lambda^j$ has extreme values. To do this we maximize the form subject to the constraint

$$\phi(\lambda) = g_{ij}\lambda^i\lambda^j - 1 = 0 \qquad \text{5.B.27}$$

Using the Lagrange multiplier method, where ϵ is the multiplier, we find

$$\frac{\partial Q}{\partial \lambda^i} - \epsilon\frac{\partial \phi}{\partial \lambda^i} = 0 \qquad \text{5.B.28}$$

which may be simplified to

$$\left(E^*_{ij} - \epsilon g_{ij}\right)\lambda^j = 0 \qquad\qquad \text{5.B.29}$$

or upon use of the contravariant metric components

$$\left(E^{*k}_j - \epsilon \delta^k_j\right)\lambda^j = 0 \qquad\qquad \text{5.B.30}$$

This system has a nontrivial solution if and only if

$$\left|E^{*k}_j - \epsilon \delta^k_j\right| = 0 \qquad\qquad \text{5.B.31}$$

Recall from linear algebra that any real symmetric form can be reduced by means of a similarity transformation to a diagonal form. The eigenvalues of the diagonal form are identical to those of the original form, and all other similar forms, and are, therefore, invariant. The eigenvalues of the diagonal form are simply the diagonal elements and their associated eigenvectors are orthogonal. Let ϵ_i be the diagonal elements (eigenvalues) of the diagonalized form, then

$$\left|g^{ik}E^*_{ij} - \epsilon \delta^k_j\right| = -\epsilon^3 + \vartheta_1 \epsilon^2 - \vartheta_2 \epsilon + \vartheta_3 = 0 \qquad\qquad \text{5.B.32}$$

where the coefficients are

$$\vartheta_1 = \epsilon_1 + \epsilon_2 + \epsilon_3 \qquad\qquad \text{5.B.33}$$

$$\vartheta_2 = \epsilon_1 \epsilon_2 + \epsilon_2 \epsilon_3 + \epsilon_1 \epsilon_3 \qquad\qquad \text{5.B.34}$$

$$\vartheta_3 = \epsilon_1 \epsilon_2 \epsilon_3 \qquad\qquad \text{5.B.35}$$

The eigenvalues are invariant and therefore the ϑ_i are likewise. Recall that the eigenvectors $\lambda^i_1, \lambda^i_2, \lambda^i_3$ are determined by successive substitution of the eigenvalues $\epsilon_1, \epsilon_2, \epsilon_3$, into

$\left(E_j^{*k} - \epsilon\delta_j^k\right)\lambda^j = 0$. If we let the components of the eigenvectors be the coordinates of an orthonormal Cartesian frame in the final state and denote the coordinates by z^i then

$$Q(z) = \epsilon_1\left(z^1\right)^2 + \epsilon_2\left(z^2\right)^2 + \epsilon_3\left(z^3\right)^2 \qquad \text{5.B.36}$$

The quantities $\epsilon_1, \epsilon_2, \epsilon_3$ are referred to as the principal strains. The principal directions are those orthogonal directions for which the tensor components representing the shearing strain vanish. The only way the shearing strain components can vanish is when the change in the angle between elements which are orthogonal in the initial state is identically zero. We can, therefore, conclude that the principal directions are those orthogonal directions in the initial state which remain orthogonal in the final state.

The equation $E_{ij}^*(x)\lambda^i\lambda^j = $ const. defines a quadratic surface at every point P in the final state. The principal directions are coincident with the major axes.

The invariants, in the final state, when expressed in terms of the E_j^{*i} , are as follows:

$$\vartheta_1 \equiv E_i^{*i} \qquad \text{5.B.37}$$

$$\vartheta_2 \equiv \frac{1}{2!}\delta_{kl}^{ij}E_i^{*k}E_j^{*l} \qquad \text{5.B.38}$$

$$\vartheta_3 \equiv \frac{1}{3!}\delta_{lmn}^{ijk}E_i^{*l}E_j^{*m}E_k^{*n} \qquad \text{5.B.39}$$

where $\delta_{j_1\cdots j_k}^{i_1\cdots i_k}$ is the generalized Kronecker delta.

Since the principal strains are invariant we can, without loss of generality, reference them to an orthonormal Cartesian frame and the equation for the elongation in the principal directions reduces to

$$e_i = \frac{ds_o^i - ds^i}{ds^i} = 1 - \sqrt{1 - 2\varepsilon_i} \qquad \text{5.B.40}$$

40

From the viewpoint of an observer in the initial state similar invariant quantities can be developed and referenced to an orthonormal Cartesian frame. The equation for the elongation in the principal directions, in the initial state, is then

$$e_i{}^o = \frac{ds^i - ds_o^i}{ds_o^i} = \sqrt{1 + 2\epsilon_i{}^o} - 1 \qquad\qquad 5.\text{B}.41$$

By solving each of the two equations above for $\frac{ds_o^i}{ds^i}$ and equating the results we can find the relationship between the principal strains in the initial and final states, or

$$\epsilon_i{}^o = \frac{\epsilon_i}{1 - 2\epsilon_i} \qquad\qquad 5.\text{B}.42$$

$$\epsilon_i = \frac{\epsilon_i{}^o}{1 + 2\epsilon_i{}^o} \qquad\qquad 5.\text{B}.43$$

Further, the above equations allow us to relate, similarly, $\vartheta_i{}^o$ and ϑ_i. We note when the principal strains are "small", the principal strains and their associated invariants in the initial state are identical to those in the final state.

C. Volume Elements

We will now relate the change in the differential volume elements in the initial and final states to the strain invariants previously described. Let the element in the initial state be denoted by dV_o while the element in the final state, by dV. From the previous definition of a volume element we find, for the initial state,

$$dV_o = \sqrt{h}\, dx^1 dx^2 dx^3 \qquad\qquad 5.\text{C}.44$$

and for the final state

$$dV = \sqrt{g}\, dx^1 dx^2 dx^3 \qquad\qquad 5.\text{C}.45$$

or after combining

$$\frac{dV_o}{dV} = \sqrt{\frac{h}{g}} \qquad\qquad 5.\text{C}.46$$

41

The metric tensor components associated with the initial state can be considered the components of a tensor defined in the final state and can, therefore, be contracted by operations involving the metric tensor of the final state, or $g^{ik}h_{ij} = h_j^k$ and $g_{ik}h_j^k = h_{ij}$. From the theory of determinants we know that the determinant of a product is equal to the product of the determinants, or

$$\left| g_{ik} h_j^k \right| = \left| g_{ik} \right| \left| h_j^k \right| = \left| h_{ij} \right| \qquad \text{5.C.47}$$

which reduces to

$$\left| h_j^k \right| = \frac{h}{g} \qquad \text{5.C.48}$$

Recall the definition of the strain tensor components, which after rearranging yields

$$h_{ij} = g_{ij} - 2E_{ij}^* \qquad \text{5.C.49}$$

or after use of the contravariant components of the metric tensor associated with the final state

$$h_j^i = \delta_j^i - 2E_j^{*i} \qquad \text{5.C.50}$$

Therefore,

$$\frac{dV_o}{dV} = \sqrt{\left| \delta_j^i - 2E_j^{*i} \right|} \qquad \text{5.C.51}$$

After expanding the determinant and substituting the strain invariants, we find

$$\left| \delta_j^i - 2E_j^{*i} \right| = 1 - 2\vartheta_1 + 4\vartheta_2 - 8\vartheta_3 \qquad \text{5.C.52}$$

If we disregard the terms involving the products of the strains, consistent with the linear theory, we have

$$\frac{dV_o}{dV} = \sqrt{1 - 2\vartheta_1} \qquad \text{5.C.53}$$

But if the strains are "small", after a Maclaurin series expansion and disregarding higher order terms, we find

$$\frac{dV_{\mathrm{o}}}{dV} = 1 - \vartheta_1 \qquad\qquad 5.C.54$$

which may be rewritten as

$$\frac{dV - dV_{\mathrm{o}}}{dV} = \vartheta_1 \qquad\qquad 5.C.55$$

The equation above represents the change in volume per unit volume for an observer in the final state, and the quantity on the right hand side is referred to as the dilatation. For an observer in the initial state, with the same assumptions as above, we find

$$\frac{dV - dV_{\mathrm{o}}}{dV_{\mathrm{o}}} = \vartheta_1{}^{\mathrm{o}} \qquad\qquad 5.C.56$$

6 STRAIN-DISPACEMENT

We want to relate the components of the strain tensor to the components of a displacement vector. We define a displacement vector $\mathbf{d}$, which has the form $\mathbf{d} = u^i \mathbf{h}_i = w^i \mathbf{g}_i$, when written in terms of its components in the initial and final states, respectively. We consider first our frame of reference to be the initial state. The vector $\mathbf{d}$, when written in terms of the position vectors, is given by $\mathbf{d} = \mathbf{r} - \mathbf{r}_o$. Taking the covariant derivative of $\mathbf{d}$ with respect to x^i and recalling the definition of the base vectors (equation 4.B.4), we find

$$\mathbf{g}_i = \mathbf{h}_i + \frac{\partial \mathbf{d}}{\partial x^i} \qquad\qquad 6.1$$

Computing the inner product of the base vectors in the final state and using the equation above, we find

$$\mathbf{g}_i \cdot \mathbf{g}_j = \mathbf{h}_i \cdot \mathbf{h}_j + \mathbf{h}_i \cdot \frac{\partial \mathbf{d}}{\partial x^j} + \mathbf{h}_j \cdot \frac{\partial \mathbf{d}}{\partial x^i} + \frac{\partial \mathbf{d}}{\partial x^j} \cdot \frac{\partial \mathbf{d}}{\partial x^i} \qquad\qquad 6.2$$

Rearranging the above equation and recalling the definition of the strain tensor we find

$$2E_{ij} = \mathbf{h}_i \cdot \frac{\partial \mathbf{d}}{\partial x^j} + \mathbf{h}_j \cdot \frac{\partial \mathbf{d}}{\partial x^i} + \frac{\partial \mathbf{d}}{\partial x^j} \cdot \frac{\partial \mathbf{d}}{\partial x^i} \qquad\qquad 6.3$$

The relationship between the partial derivative of the displacement vector with respect to x^i and the covariant derivative of its components is

$$\frac{\partial \mathbf{d}}{\partial x^i} = u^k_{\|i} \mathbf{h}_k \qquad\qquad 6.4$$

Using this equation in 6.3 we find

$$2E_{ij} = u^l_{\|j} h_{il} + u^k_{\|i} h_{kj} + u^k_{\|i} u^l_{\|j} h_{lk} \qquad\qquad 6.5$$

where the covariant derivative involves use of the metric tensor components of the initial state. If we used the final state as our frame of reference, vector $\mathbf{d}$ would have the form $\mathbf{d} = \mathbf{r}_o - \mathbf{r}$. The relationship between the base vectors and displacement vectors are

$$\mathbf{h}_i = \mathbf{g}_i + \frac{\partial \mathbf{d}}{\partial x^i} \qquad\qquad 6.6$$

44

Computing the inner product of the base vectors in the initial state and using the equation above and after carrying out operations similar to those previously performed, in terms of the final state we find

$$2E_{ij}^* = w_{\|j}^l g_{il} + w_{\|i}^k g_{kj} - w_{\|i}^k w_{\|j}^l g_{lk} \qquad 6.7$$

where the covariant derivative involves use of the metric tensor components of the final state.

The last term in both equations involves products of the derivatives of the displacements, products of the displacements, and products of the derivatives of the displacements and the displacements. When these product terms are disregarded we have a set of linear equations and the theory using them is referred to as a linear theory of strain. The equations associated with the linear theory are

$$2E_{ij} = u_{\|j}^l h_{il} + u_{\|i}^k h_{kj} \qquad 6.8$$

$$2E_{ij}^* = w_{\|j}^l g_{il} + w_{\|i}^k g_{kj} \qquad 6.9$$

If the displacements are small enough so that the strain tensor components can be regarded as infinitesimal, the two equations above are identical and the associated theory is called the infinitesimal theory.

If the functions describing the displacements are given in terms of the initial state or the final state the strain components can be calculated. If the functions describing the strain components are given, we usually seek to integrate the equations in order to determine the displacement fields.

45

7 COMPATIBILITY CONDITIONS

In order to integrate the field equations relating the strain and displacement tensor components, the strain tensor components must satisfy a set of conditions referred to as the compatibility or integrability conditions. These conditions are obtained using the equation relating the strain tensor components to the metric tensor components in the initial and final states, $g_{ij} = h_{ij} + 2E_{ij}$, and the equations for the Riemann-Christoffel tensors formed from the metric tensor components of the initial state and the metric tensor components of the final state. Recall that the Riemann-Christoffel tensor consists of various combinations of terms involving the appropriate metric tensor components.

The Riemann-Christoffel tensors formed from the metric tensor components of the initial state and the metric tensor components of the final state, must both vanish identically due to the previous definition of our physical space. We first form the Riemann-Christoffel tensor for the final state in terms of the g_{ij} and set it equal to zero. We replace the terms containing the g_{ij} with terms containing h_{ij} and E_{ij} by making use of $h_{ij} = g_{ij} - 2E_{ij}$. We next rearrange some of the terms containing h_{ij} into the form of the Riemann-Christoffel tensor of the initial state and eliminate them due to the vanishing of the tensor components. We are left with an equation of the form

$$\epsilon_{ijkl} + g^{rs}\left(\epsilon_{jks}\epsilon_{ilr} - \epsilon_{jls}\epsilon_{ikr}\right) = 0 \qquad 7.1$$

where

$$\epsilon_{ijkl} \equiv E_{jl\|ik} + E_{ik\|jl} - E_{ij\|kl} - E_{kl\|ij} \qquad 7.2$$

and

$$\epsilon_{ijk} \equiv E_{ik\|j} + E_{kj\|i} - E_{ij\|k} \qquad 7.3$$

which are referred to as the Christoffel deviators. The equations can be linearized by disregarding the terms in parentheses which are the product terms. Upon doing so we obtain

$$E_{jl\|ik} + E_{ik\|jl} - E_{ij\|kl} - E_{kl\|ij} = 0 \qquad 7.4$$

These equations reduce to those of Saint Venant, when the general coordinate frames are orthonormal Cartesian and the strains are infinitesimal.

8 KINEMATICS

In Chapters 5 and 6 we introduced the concept of strain and the strain-displacement relationships in general tensor notation in terms of convected coordinates. In this chapter we revisit the subject in direct notation without the use of convected coordinates. In addition, other stain measures and strain rated are introduced.

Assuming that "a body can be mapped smoothly onto a domain," Truesdell [19] describes four equivalent methods of describing its motion. These are referred to as the material, the referential, the spatial and the relative descriptions. The material description uses the actual particles and the time as the independent variables while the referential description uses the coordinates of the particles relative to some fixed, arbitrary frame of reference and the time as the independent variables. The spatial description uses the coordinates of a point in space and time as the independent variables. The relative description uses a variable referential description. The referential and the relative descriptions are the ones most commonly used in the mechanics of solids. When the referential description is taken at time t=0 it is called Lagrangian and when it is taken at some variable time t it is referred to as the relative description [20].

Let $\mathbf{R}$ and $\mathbf{r}$ denote vector fields describing the position of material points of a body in an initial and current configuration, respectively. Assume that both vector fields originate from the origin of a single orthonormal Cartesian frame Y, with coordinates Y^i. Assume a functional transformation between the orthonormal frame and two general curvilinear frames X and x with coordinates X^i and x^i, in the initial and current configuration, respectively, where

$$Y^i = Y^i(X^1, X^2, X^3) = Y^i(x^1, x^2, x^3) \qquad 8.1$$

Let $d\mathbf{R}$ and $d\mathbf{r}$ denote vector fields describing distances to neighboring points in the respective configurations with magnitudes dS and ds respectively. The relationships between $\mathbf{R}$ and $\mathbf{r}$ is given by $\mathbf{r} = \psi(\mathbf{R}, t)$ and $\mathbf{R} = \Psi(\mathbf{r}, t)$, where the vector functions

are smooth and continuous and where r= $\psi(\mathbf{R}, t_o= 0) = \mathbf{R}$. Let $\mathbf{F}$ and its inverse $\mathbf{F}^{-1}$ denote the two-point tensors representing the material and spatial deformation gradients, respectively, where

$$dr = \mathbf{F} \, d\mathbf{R} = d\mathbf{R} \cdot \mathbf{F}^T = \left(r\overleftarrow{\nabla}_X \right) \cdot d\mathbf{R} = d\mathbf{R} \cdot \left(\overrightarrow{\nabla}_X r \right) \tag{8.2}$$

$$d\mathbf{R} = \mathbf{F}^{-1} \cdot dr = dr \cdot \left(\mathbf{F}^{-1} \right)^T = \left(\mathbf{R}\overleftarrow{\nabla}_x \right) \cdot dr = dr \cdot \left(\overrightarrow{\nabla}_x \mathbf{R} \right) \tag{8.3}$$

and

$$\det \mathbf{F} = J > 0 \tag{8.4}$$

Let $d\mathbf{R}$ and dr have the following representation in terms of general curvilinear frames in the reference and current configuration, respectively

$$d\mathbf{R} = dX^I \mathbf{G}_I = dX_I \mathbf{G}^I \tag{8.5}$$

$$dr = dx^i \mathbf{g}_i = dx_i \mathbf{g}^i \tag{8.6}$$

where the symmetric metric tensors associated with the reference and current configuration, respectively, are given by

$$G^{IJ} \mathbf{G}_I \mathbf{G}_J = G_{IJ} \mathbf{G}^I \mathbf{G}^J = \delta_I^J \mathbf{G}^I \mathbf{G}_J \tag{8.7}$$

$$g^{ij} \mathbf{g}_i \mathbf{g}_j = g_{ij} \mathbf{g}^i \mathbf{g}^j = \delta_i^j \mathbf{g}^i \mathbf{g}_j \tag{8.8}$$

with the respective components given by

$$\mathbf{G}_I \cdot \mathbf{G}^J = \delta_I^J \tag{8.9}$$

$$\mathbf{G}_I \cdot \mathbf{G}_J = G_{IJ} \tag{8.10}$$

$$\mathbf{G}^I \cdot \mathbf{G}^J = G^{IJ} \tag{8.11}$$

49

and

$$g_j \cdot g^i = \delta_j^{\cdot i} \qquad\qquad 8.12$$

$$g_i \cdot g_j = g_{ij} \qquad\qquad 8.13$$

$$g^i \cdot g^j = g^{ij} \qquad\qquad 8.14$$

Let **u** represent the displacement vector field such that $r = R + u$. The deformation gradients can be expressed in terms of the displacement gradients by

$$\mathbf{F} = 1 + \mathbf{u}\overleftarrow{\nabla}_X \qquad\qquad 8.15$$

and

$$\mathbf{F}^T = 1 + \overrightarrow{\nabla}_X \mathbf{u} = 1 + \left(\mathbf{u}\overleftarrow{\nabla}_X\right)^T \qquad\qquad 8.16$$

when **u** is expressed in terms of the coordinates, X^i, of the reference configuration and

$$\mathbf{F}^{-1} = 1 - \mathbf{u}\overleftarrow{\nabla}_x \qquad\qquad 8.17$$

$$\left(\mathbf{F}^{-1}\right)^T = 1 - \overrightarrow{\nabla}_x \mathbf{u} = 1 - \left(\mathbf{u}\overleftarrow{\nabla}_x\right)^T \qquad\qquad 8.18$$

when **u** is expressed in terms of the coordinates, x^i, of the current configuration. In terms of the curvilinear bases the deformation gradients and their transposes have the following forms

$$\mathbf{F} = \frac{\partial x^j}{\partial X^I} g_j G^I \qquad\qquad 8.19$$

$$\mathbf{F}^T = \frac{\partial x^j}{\partial X^I} G^I g_j \qquad\qquad 8.20$$

$$\mathbf{F}^{-1} = \frac{\partial X^J}{\partial x^i} G_J g^i \qquad\qquad 8.21$$

50

$$\left(\mathbf{F}^{-1}\right)^{T} = \frac{\partial X^{J}}{\partial x^{i}} \mathbf{g}^{i} \mathbf{G}_{J} \qquad 8.22$$

If the curvilinear coordinates are convected (i.e., $x^{i} = X^{i}$) then the component representation of the deformation gradients assume the following simple forms:

$$\mathbf{F} = \mathbf{g}_{i} \mathbf{G}^{i} \qquad 8.23$$

$$\mathbf{F}^{T} = \mathbf{G}^{i} \mathbf{g}_{i} \qquad 8.24$$

$$\mathbf{F}^{-1} = \mathbf{G}_{i} \mathbf{g}^{i} \qquad 8.25$$

$$\left(\mathbf{F}^{-1}\right)^{T} = \mathbf{g}^{i} \mathbf{G}_{i} \qquad 8.26$$

Let $\mathbf{C}$ and $\mathbf{B}$ denote the right and the left Cauchy-Green deformation tensors, respectively, where

$$(ds)^{2} = d\mathbf{r} \cdot d\mathbf{r} = d\mathbf{R} \cdot \left(\mathbf{F} \cdot \mathbf{F}^{T}\right) \cdot d\mathbf{R} = d\mathbf{R} \cdot \mathbf{C} \cdot d\mathbf{R} \qquad 8.27$$

$$(dS)^{2} = d\mathbf{R} \cdot d\mathbf{R} = d\mathbf{r} \cdot \left[\left(\mathbf{F}^{-1}\right)^{T} \cdot \mathbf{F}^{-1}\right] \cdot d\mathbf{r} = d\mathbf{R} \cdot \mathbf{B}^{-1} \cdot d\mathbf{R} \qquad 8.28$$

Let $\mathbf{E}$ and $\mathbf{E}^{*}$ denote the Lagrangian and Eulerian strain tensors respectively, where

$$(ds)^{2} - (dS)^{2} = 2 \, d\mathbf{R} \cdot \mathbf{E} \cdot d\mathbf{R} \qquad 8.29$$

$$(ds)^{2} - (dS)^{2} = 2 \, d\mathbf{r} \cdot \mathbf{E}^{*} \cdot d\mathbf{r} \qquad 8.30$$

The relationship between the strain tensors, the deformation gradients, the deformation tensors and the displacement gradients is given by

$$\mathbf{E} = \frac{1}{2} \left[\mathbf{F}^{T} \cdot \mathbf{F} - \mathbf{1}\right] = \frac{1}{2} \left[\mathbf{C} - \mathbf{1}\right] \qquad 8.31$$

$$E = \frac{1}{2}\left(u\overleftarrow{\nabla}_X + u\overleftarrow{\nabla}_X^T + u\overleftarrow{\nabla}_X \cdot u\overleftarrow{\nabla}_X^T\right) \qquad 8.32$$

$$E^* = \frac{1}{2}\left[1 - \left(F^{-1}\right)^T \cdot F^{-1}\right] = \frac{1}{2}\left[1 - B^{-1}\right] \qquad 8.33$$

$$E^* = \frac{1}{2}\left(u\overleftarrow{\nabla}_x + u\overleftarrow{\nabla}_x^T - u\overleftarrow{\nabla}_x \cdot u\overleftarrow{\nabla}_x^T\right) \qquad 8.34$$

By use of the polar decomposition theorem (see for example Billington [5] or Malvern [6]) the deformation gradient can be written as

$$F = R \cdot U = V \cdot R \qquad 8.35$$

where R now represents the second order rotation tensor and U, and V are the right, and the left stretch tensors, respectively, where $R \cdot R^T = 1$ The deformation tensors can be expressed in terms of the stretch tensors by

$$C = U^2 = F^T \cdot F \qquad 8.36$$

$$B = V^2 = F \cdot F^T \qquad 8.37$$

and the relationship between the two stretch tensors and the two deformation tensors is given by

$$V = R \cdot U \cdot R^T \qquad B = R \cdot C \cdot R^T \qquad 8.38$$

By making use of the above three equations in equations 8.31 and 8.33, the strain tensors can represented in various equivalent forms. For example the Green-Lagrange strain tensor can be given by

$$E = \frac{1}{2}\left[F^T \cdot F - 1\right] = \frac{1}{2}[C - 1] = \qquad 8.39$$
$$\frac{1}{2}[U^2 - 1] = \frac{1}{2}\left[R^T \cdot V^2 \cdot R - 1\right] = \frac{1}{2}\left[R^T \cdot B \cdot R - 1\right]$$

The above equation is useful for determining relationships between the principal scalar invariants associated with the various tensors.

The fully coupled equations of thermoelasticity contain terms which include a measure of the rate strain and therefore we need relationships for the Lagrangian and Eulerian strain rates. Recall that $dr = \mathbf{F} \cdot d\mathbf{R}$ and $d\mathbf{R} = \mathbf{F}^{-1} \cdot dr$. If we differentiate the first relationship with respect to time and substitute the second relationship into the result we find

$$\frac{d}{dt}dr = dv = \frac{d\mathbf{F}}{dt} \cdot d\mathbf{R} = \frac{d\mathbf{F}}{dt} \cdot \mathbf{F}^{-1} \cdot dr \qquad 8.40$$

where we made use of the fact that

$$\frac{d}{dt}d\mathbf{R} = \mathbf{0} \qquad 8.41$$

Let

$$\mathbf{L} = \frac{d\mathbf{F}}{dt} \cdot \mathbf{F}^{-1} \qquad 8.42$$

where $\mathbf{L}$ is the spatial velocity gradient tensor [5]. The above equation can be rearranged as

$$\frac{d\mathbf{F}}{dt} = \mathbf{L} \cdot \mathbf{F} \qquad 8.43$$

and

$$\frac{d\mathbf{F}^T}{dt} = \mathbf{F}^T \cdot \mathbf{L}^T \qquad 8.44$$

The velocity gradient tensor can be written as the sum of a symmetric and skew-symmetric tensor (Cauchy–Stokes decomposition [5]) or, $\mathbf{L} = \mathbf{D} + \mathbf{W}$ where $\mathbf{D}$ and $\mathbf{W}$ represent the symmetric rate of deformation (stretching) tensor and skew-symmetric spin (vorticity) tensor respectively, and

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) \qquad 8.45$$

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right) \qquad 8.46$$

The appropriateness of the description of $\mathbf{D}$ as the rate of deformation tensor can be shown by differentiating $(ds)^2 = d\mathbf{r} \cdot d\mathbf{r}$ with respect to time or

$$\frac{d}{dt}(ds)^2 = 2d\mathbf{r} \cdot \frac{d}{dt}d\mathbf{r} = 2d\mathbf{r} \cdot d\mathbf{v} \qquad 8.47$$

and after making use of equations 8.40, 8.45, and 8.46 we find

$$\frac{d}{dt}(ds)^2 = 2d\mathbf{r} \cdot \mathbf{L} \cdot d\mathbf{r} = 2d\mathbf{r} \cdot \mathbf{D} \cdot d\mathbf{r} + 2d\mathbf{r} \cdot \mathbf{W} \cdot d\mathbf{r} \qquad 8.48$$

Recall that $\mathbf{W}$ is a skew-symmetric tensor which implies that the last term in the above equation is identically zero or

$$\frac{d}{dt}(ds)^2 = 2d\mathbf{r} \cdot \mathbf{D} \cdot d\mathbf{r} \qquad 8.49$$

The above equation relates the rate of change of the square of the differential length element in the current configuration to the tensor $\mathbf{D}$.

Recall that the Green-Lagrange strain tensor can be expressed in terms of the material deformation gradient tensor. Rate forms of the strain tensor will involve rate forms of the material deformation gradient tensor. If we differentiate the Green-Lagrange tensor with respect to time we find

$$\frac{d\mathbf{E}}{dt} = \frac{1}{2}\left(\frac{d\mathbf{F}^T}{dt} \cdot \mathbf{F} + \mathbf{F}^T \cdot \frac{d\mathbf{F}}{dt}\right) \qquad 8.50$$

or after use of equations 8.43 and 8.44

$$\frac{d\mathbf{E}}{dt} = \frac{1}{2}(\mathbf{F}^T \cdot \mathbf{L}^T \cdot \mathbf{F} + \mathbf{F}^T \cdot \mathbf{L} \cdot \mathbf{F}) = \frac{1}{2}(\mathbf{F}^T \cdot (\mathbf{L}^T + \mathbf{L}) \cdot \mathbf{F}) \qquad 8.51$$

The Lagrangian strain rate is

$$\frac{d\mathbf{E}}{dt} = \mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F} \qquad 8.52$$

where use was made of equation 8.45. The Eulerian strain rate is

$$\frac{d\mathbf{E}^*}{dt} = \mathbf{D} - \left(\mathbf{E}^* \cdot \mathbf{L} + \mathbf{L}^T \cdot \mathbf{E}^*\right) \qquad 8.53$$

If we let dv and dV represent differential volume elements in the final and initial configurations, respectively, and let $d\mathbf{r}$, $d\mathbf{r}'$ and $d\mathbf{r}''$ represent the lengths of a parallelepiped in the final state then

$$dv = (d\mathbf{r} \times d\mathbf{r}') \cdot d\mathbf{r}'' \qquad\qquad 8.54$$

and

$$dV = (d\mathbf{R} \times d\mathbf{R}') \cdot d\mathbf{R}'' \qquad\qquad 8.55$$

After substituting $d\mathbf{r} = \mathbf{F} \cdot d\mathbf{R}$ into equation 8.54 we have

$$dv = (\mathbf{F}{\cdot}d\mathbf{R} \times \mathbf{F}{\cdot}d\mathbf{R}') \cdot (\mathbf{F}{\cdot}d\mathbf{r}'') \qquad\qquad 8.56$$

Utilizing the relationship [5]

$$(\mathbf{F}{\cdot}d\mathbf{R} \times \mathbf{F}{\cdot}d\mathbf{R}') = \mathbf{F}^{\mp} \cdot (d\mathbf{R} \times d\mathbf{R}') \qquad\qquad 8.57$$

and (see equation 2.5)

$$\mathbf{F}^{\mp} = \det \mathbf{F}(\mathbf{F}^{-1})^{T} \qquad\qquad 8.58$$

and the definition of the transpose of a tensor, we find

$$dv = J dV \qquad\qquad 8.59$$

where recall that $\det \mathbf{F} = J$.

The relationship between the differential areas in the final and initial configurations is found by a similar process and is given by

$$\mathbf{n} da = J \mathbf{N} \cdot \mathbf{F}^{-1} dA \qquad\qquad 8.60$$

or alternately as

$$\mathbf{da} = J \mathbf{dA} \cdot \mathbf{F}^{-1} \qquad\qquad 8.61$$

where $\mathbf{da} = \mathbf{n} da$ represents the area vector in the final configuration and $\mathbf{dA} = \mathbf{N} dA$ represents the area vector in the initial configuration and $\mathbf{n}$ and $\mathbf{N}$ are unit normals to the area elements in the final and initial configurations, respectively.

9 CONSERVATION OF MASS

Let ρ_0 and ρ represent the mass per unit volume in the initial and final states, respectively. The mass in the final state is equal to mass in the initial state or

$$\int_v \rho\, dv = \int_V \rho_0\, dV \qquad\qquad 9.1$$

The relationship between the differential volume elements was given by equation 8.59 or

$$dv = J\, dV \qquad\qquad 9.2$$

When the equation above is substituted into 9.1 we find

$$\int_V \rho J\, dV = \int_V \rho_0\, dV \qquad\qquad 9.3$$

or

$$\rho_0 = J\rho \qquad\qquad 9.4$$

which is a form of the Law of Conservation of Mass in the Lagrangian frame.

If we had selected a Eulerian frame (i.e., we focus on a region of space rather than on the material points) the equation for the Law of Conservation of Mass would have a different form. Assuming that mass is not created or destroyed in the region, the rate of change of mass in the region is equal to the rate of mass flow into the region.

Let the total mass in the region M, be given by

$$M = \int_v \rho\, dv \qquad\qquad 9.5$$

The rate of change of mass in the region is given by

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} = \int_v \frac{\partial \rho}{\partial t}\, dv \qquad\qquad 9.6$$

56

where the total derivative reduces to a partial derivative due to the fact that the volume remain constant. The rate of mass flow into the region through a surface S with an outward normal $\mathbf{n}$ and differential area da is

$$-\int_S \rho\mathbf{v}\cdot\mathbf{n}\,da = -\int_v \overrightarrow{\nabla_x}\cdot(\rho\mathbf{v})\,dv \qquad 9.7$$

where $\mathbf{v}$ is the velocity vector. Equating the above two equations we find

$$\frac{\partial\rho}{\partial t} + \overrightarrow{\nabla_x}\cdot(\rho\mathbf{v}) = 0 \qquad 9.8$$

which is the equation for the Law of Conservation of Mass in the Eulerian frame.

10 STRESS and EQUILIBRIUM

Let the Cauchy stress tensor be represented by $\mathbf{T}$ and the first and the second Piola-Kirchhoff stress tensors by $\mathbf{T}^0$ and $\widetilde{\mathbf{T}}$, respectively. Some authors use $\mathbf{T_R}$ for the first Piola-Kirchhoff stress tensor, where

$$\mathbf{T_R} = \left(\mathbf{T}^0\right)^T \qquad\qquad 10.1$$

If the Cauchy stress tensor is defined by

$$\mathbf{t} = \frac{d\mathbf{P}}{da} = \mathbf{n}\cdot\mathbf{T} = \mathbf{T}^T\cdot\mathbf{n} \qquad\qquad 10.2$$

or rewritten as

$$d\mathbf{P} = \mathbf{t}\,da = \mathbf{n}\cdot\mathbf{T}\,da \qquad\qquad 10.3$$

where $\mathbf{n}$ is a unit vector normal to the differential surface area element da, in the current configuration and $d\mathbf{P}$ is the actual force transmitted across the surface S and $\mathbf{t}$ is the Cauchy stress vector. In this description when the tensor is written in terms of its components the second index indicates the direction of the component.

If in the current configuration, $\mathbf{b}$ represents the body forces per unit mass and ρ the mass per unit volume and $\mathbf{v}$ the velocity, then the integral form of the equation for conservation of linear momentum can be written as

$$\int_S \mathbf{t}\,da + \int_V \rho\mathbf{b}\,dv = \frac{d}{dt}\int_V \rho\mathbf{v}\,dv \qquad\qquad 10.4$$

or

$$\int_S \mathbf{n}\cdot\mathbf{T}\,da + \int_V \rho\mathbf{b}\,dv = \frac{d}{dt}\int_V \rho\mathbf{v}\,dv \qquad\qquad 10.5$$

By making use of the divergence theorem to transform the surface integral to a volume integral we find or

$$\int_V \left(\overrightarrow{\nabla_x}\cdot\mathbf{T} + \rho\mathbf{b} - \rho\frac{d\mathbf{v}}{dt}\right)dv = 0 \qquad\qquad 10.6$$

or

$$\vec{\nabla}_x \cdot \mathbf{T} + \rho \mathbf{b} = \rho \frac{d\mathbf{v}}{dt} \qquad 10.7$$

which is Cauchy's first law of motion.

The integral form of the equation for conservation of angular momentum is given by

$$\int_S (\mathbf{r} \times \mathbf{t})\,da + \int_V (\mathbf{r} \times \rho \mathbf{b})\,dv = \frac{d}{dt}\int_V (\mathbf{r} \times \rho \mathbf{v})\,dv \qquad 10.8$$

The term to the right side of the equal sign in the above equation can be written as

$$\frac{d}{dt}\int_V (\mathbf{r} \times \rho \mathbf{v})\,dv = \int_V (\mathbf{v} \times \rho \mathbf{v})\,dv + \int_V (\mathbf{r} \times \rho \frac{d\mathbf{v}}{dt})\,dv = \int_V (\mathbf{r} \times \rho \frac{d\mathbf{v}}{dt})\,dv \qquad 10.9$$

Substituting the above into equation 10.8 and after simplifying we find

$$\int_S (\mathbf{r} \times \mathbf{t})\,da = \int_V (\rho \mathbf{r} \times (\frac{d\mathbf{v}}{dt} - \mathbf{b}))\,dv \qquad 10.10$$

or

$$\int_S (\mathbf{r} \times \mathbf{T}^{\mathbf{T}} \cdot \mathbf{n})\,da = \int_V (\rho \mathbf{r} \times (\frac{d\mathbf{v}}{dt} - \mathbf{b}))\,dv \qquad 10.11$$

After using the divergence theorem in the following form (Billington, p.70 [5])

$$\int_S \mathbf{r} \times (\mathbf{T}^{\mathbf{T}} \cdot \mathbf{n})\,da = \int_V (\mathbf{r} \times (\vec{\nabla}_x \cdot \mathbf{T}) + \mathbf{T}^{\mathbf{T}} - \mathbf{T})\,dv \qquad 10.12$$

in the above equation and simplifying we find

$$\mathbf{T}^{\mathbf{T}} = \mathbf{T} \qquad 10.13$$

which implies the symmetry of the Cauchy stress tensor, which is Cauchy's second law of motion [6]. In the linearized theory of thermoelasticity the symmetry of the Cauchy stress tensor is a consequence of the symmetry of the strain and elasticity tensors [21] in the constitutive relationships.

The actual force in the current configuration can be related to a force $\widetilde{d\mathbf{P}}$ in the reference configuration by

$$\widetilde{d\mathbf{P}} = \mathbf{F}^{-1} \cdot d\mathbf{P} = d\mathbf{P} \cdot (\mathbf{F}^{-1})^T \qquad 10.14$$

which is similar to

$$dR = F^{-1} \cdot dr = dr \cdot \left(F^{-1} \right)^T \qquad 10.15$$

The Cauchy stress tensor associates the actual force per unit area in the current configuration with the base vectors in the current configuration. The first Piola-Kirchhoff stress tensor (sometimes referred to as the engineering stress tensor [22]) associates the actual force in the current configuration per unit area in the reference configuration with the base vectors of the current configuration or

$$d\mathbf{P} = \mathbf{t}_o dA = (\mathbf{N} \cdot \mathbf{T}^o) dA \qquad 10.16$$

The second Piola-Kirchhoff stress tensor associates the force $\widetilde{d\mathbf{P}}$ per unit area in the reference configuration with the base vectors of the reference configuration and is given by

$$\widetilde{d\mathbf{P}} = \widetilde{\mathbf{t}} dA = \left(\mathbf{N} \cdot \widetilde{\mathbf{T}} \right) dA \qquad 10.17$$

The relationship between the first Piola-Kirchhoff stress tensor and the Cauchy stress tensor is found by equating equations 10.3 and 10.16 and making use of equation 8.60. The relationship between the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor is found by equating equations 10.14 and 10.17 and making use of 10.3 and 8.60. The relationship between the three tensors is as follows:

$$\mathbf{T}^0 = J \, \mathbf{F}^{-1} \cdot \mathbf{T} = \widetilde{\mathbf{T}} \cdot \mathbf{F}^T \qquad 10.18$$

$$\widetilde{\mathbf{T}} = J \, \mathbf{F}^{-1} \cdot \mathbf{T} \cdot \left(\mathbf{F}^{-1} \right)^T = \mathbf{T}^0 \cdot (\mathbf{F}^{-1})^T \qquad 10.19$$

$$\mathbf{T} = J^{-1} \, \mathbf{F} \cdot \widetilde{\mathbf{T}} \cdot \mathbf{F}^T = J^{-1} \, \mathbf{F} \cdot \mathbf{T}^o \qquad 10.20$$

The three stress tensors can be expressed in terms of their components and base vectors as

$$\mathbf{T} = T^{ij} \mathbf{g}_i \mathbf{g}_j \qquad 10.21$$

$$\mathbf{T}^o = {}_oT^{Ij}\mathbf{G}_I\mathbf{g}_j \qquad\qquad 10.22$$

$$\widetilde{\mathbf{T}} = \widetilde{T}^{IJ}\mathbf{G}_I\mathbf{G}_J \qquad\qquad 10.23$$

where

$$_oT^{Ij} = J\frac{\partial X^I}{\partial x^i}T^{ij} \qquad\qquad 10.24$$

$$\widetilde{T}^{IJ} = J\frac{\partial X^I}{\partial x^i}\frac{\partial X^J}{\partial x^j}T^{ij} \qquad\qquad 10.25$$

In [23] Mason describes two additional stress measures. The first, which he refers to as the Lagrangian stress tensor, relates the actual force in the current configuration per unit area in the reference configuration with the base vectors of the reference configuration. If $\mathbf{T}^L$ denotes the Lagrangian stress tensor then

$$\mathbf{T}^L = {}_LT^{IJ}\mathbf{G}_I\mathbf{G}_J = \mathbf{T}^0 \cdot \left(\mathbf{Z}^{-1}\right)^T \qquad\qquad 10.26$$

where $\mathbf{Z}$ represents the two point shifter tensor which transforms the components of a vector in one coordinate system to the components of the same vector in another coordinate system such that

$$\mathbf{p} = \mathbf{Z} \cdot \mathbf{P} \qquad\qquad 10.27$$

where $\mathbf{P}$ and $\mathbf{p}$ are the same vector in the reference and current configuration respectively. In terms of the curvilinear systems we have

$$\mathbf{Z} = g^i_{.L}\mathbf{g}_i\mathbf{G}^L \qquad\qquad 10.28$$

and

$$\mathbf{Z}^{-1} = g^L_{.i}\mathbf{G}_L\mathbf{g}^i \qquad\qquad 10.29$$

61

The components of the shifter tensor are

$$g^k_{\cdot K} = \mathbf{g}^k \cdot \mathbf{G}_K = \mathbf{G}_K \cdot \mathbf{g}^k = g^{\cdot k}_K \qquad 10.30$$

$$g^K_{\cdot k} = \mathbf{G}^K \cdot \mathbf{g}_k = \mathbf{g}_k \cdot \mathbf{G}^K = g^{\cdot K}_k \qquad 10.31$$

The components of the Lagrangian stress tensor are

$$_L T^{IJ} = g^{\cdot J}_j \; _o T^{Ij} = J\frac{\partial X^I}{\partial x^i} g^{\cdot J}_j \, T^{ij} \qquad 10.32$$

The second is a tensor $\mathbf{T}^\pi$ which is the Cauchy stress tensor expressed in terms of the base vectors in the reference configuration or

$$\mathbf{T}^\pi = {}_\pi T^{iJ}\mathbf{g}_i\mathbf{G}_J = \mathbf{T} \cdot \left(\mathbf{Z}^{-1}\right)^T \qquad 10.33$$

where

$$_\pi T^{iJ} = T^{ik}g^{\cdot J}_k \qquad 10.34$$

In [24] the Kirchhoff stress tensor $\mathbf{T}^k$ is defined by

$$\mathbf{T}^k = J\,\mathbf{T} \qquad 10.35$$

If the material behavior includes stress rates then material derivatives of one of the stress tensors is required. For example if $\mathbf{T}$ is an objective tensor (see equation 12.6) [5] then

$$\mathbf{T}^* = \mathbf{Q} \cdot \mathbf{T} \cdot \mathbf{Q}^T \qquad 10.36$$

must hold. The material time derivative of $\mathbf{T}$, $\dot{\mathbf{T}}$, is not objective but the co-rotational stress rate $\mathbf{T}^r$ and the convected stress rate $\mathbf{T}^c$ are objective tensors [6,5], where

$$\mathbf{T}^r = \dot{\mathbf{T}} - \mathbf{W} \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{W} \qquad 10.37$$

and

$$\mathbf{T}^c = \dot{\mathbf{T}} + \mathbf{L}^T \cdot \mathbf{T} + \mathbf{T} \cdot \mathbf{L} \qquad 10.38$$

The equations of motion in the current state are

$$\vec{\nabla}_x \cdot \mathbf{T} + \rho\mathbf{b} = \rho\frac{d^2\mathbf{r}}{dt^2} \qquad 10.39$$

while the equations of motion in the initial state are

$$\vec{\nabla}_X \cdot \mathbf{T}^\circ + \rho_o\mathbf{b}_o = \rho_o\frac{d^2\mathbf{r}}{dt^2} \qquad 10.40$$

or

$$\vec{\nabla}_X \cdot \left[\tilde{\mathbf{T}} \cdot \mathbf{F}^T\right] + \rho_o\mathbf{b}_o = \rho_o\frac{d^2\mathbf{r}}{dt^2} \qquad 10.41$$

where $\mathbf{b}_o$ is the body force per unit volume in the initial state.

11   ENERGY EQUATION

For the nonpolar case in the current configuration the local form of the first law of thermodynamics (energy balance) is

$$\rho \frac{de}{dt} = \mathbf{T} \cdot \cdot \mathbf{L} + \rho r - \vec{\nabla}_x \cdot \mathbf{q} \qquad 11.1$$

with the classical form of the Fourier law of heat conduction

$$\mathbf{q} = -\mathbf{K} \cdot \mathbf{g} \qquad 11.2$$

where the spatial temperature gradient $\mathbf{g}$ is

$$\mathbf{g} = \vec{\nabla}_x \theta \qquad 11.3$$

and r is the internal heat supply per unit mass, e the internal energy per unit mass, $\mathbf{q}$ the outward directed heat flux vector, $\mathbf{K}$ the conductivity tensor, θ the absolute temperature, assumed greater than zero, and

$$\mathbf{T} \cdot \cdot \mathbf{L} = \mathrm{tr}(\mathbf{T} \cdot \mathbf{L}) \qquad 11.4$$

is the stress power (power per unit volume) in the current configuration. Assuming that the Cauchy stress tensor is symmetric, the stress power in the reference configuration is

$$\tilde{\mathbf{T}} \cdot \cdot \frac{d\mathbf{E}}{dt} = \mathbf{T}^\circ \cdot \cdot \frac{d\mathbf{F}}{dt} \qquad 11.5$$

The following conjugate pairs of stress and strain variables are listed in [20] and attributed to [25].

$$\{\mathbf{T}, \mathbf{L}\}, \left\{ \tilde{\mathbf{T}}, \frac{d\mathbf{E}}{dt} \right\}, \left\{ \frac{\rho_0}{\rho} \mathbf{F}^T \cdot \mathbf{T} \cdot \mathbf{F}, \frac{d\mathbf{E}^*}{dt} \right\}, \left\{ \mathbf{T}^\circ \cdot \mathbf{R}, \frac{d\mathbf{U}}{dt} \right\}, \left\{ \mathbf{T}^\circ, \frac{d\mathbf{F}}{dt} \right\} \qquad 11.6$$

The heat flux vector and the temperature gradient in the reference configuration are $\mathbf{q}_0$ and $\mathbf{g}_0$ and are defined by [26]

$$\mathbf{q}_0 = J \mathbf{F}^{-1} \cdot \mathbf{q} \qquad 11.7$$

$$\mathbf{g_o} = \mathbf{F}^T \cdot \mathbf{g} \qquad\qquad 11.8$$

The local form of the Clausius-Duhem inequality which is one form of the second law of thermodynamics (entropy inequality) is

$$\frac{d\eta}{dt} \geq \frac{r}{\theta} - \frac{1}{\rho}\overrightarrow{\nabla}_x \cdot \frac{\mathbf{q}}{\theta} \qquad\qquad 11.9$$

where η is the entropy per unit mass. It is worth noting that this mathematical statement of second law is not without criticism. Coleman and Noll [27] used the above form for obtaining restrictions on the thermo-mechanical behavior (constitutive relationships) of elastic materials. Green and Laws [28], Green and Naghdi [29], and Day [30] raise objections to use of the Clausius-Duhem inequality citing particular examples where its use allows heat to flow from a colder region to a warmer one, which is in direct violation of the essence of the second law.

The Helmholtz free energy per unit mass ψ can be expressed in terms of the entropy, internal energy and absolute temperature by

$$\psi = e - \eta\theta \qquad\qquad 11.10$$

Differentiating the above equation with respect to time t and rearranging we find

$$\frac{de}{dt} = -\frac{d\psi}{dt} + \frac{d\eta}{dt}\theta + \frac{d\theta}{dt}\eta \qquad\qquad 11.11$$

Solving equation 11.1 for r and substituting the result into equation 11.2 and after using the above equation and simplifying we find

$$\rho\frac{d\psi}{dt} + \rho\frac{d\theta}{dt}\eta - \mathbf{T}\cdot\cdot\mathbf{L} + \frac{1}{\theta}\mathbf{q}\cdot\mathbf{g} \leq 0 \qquad\qquad 11.12$$

which is the local dissipation inequality [21] in terms of the Helmholtz free energy per unit mass, the absolute temperature, the entropy per unit mass, the Cauchy stress tensor, the temperature gradient vector and the heat conduction vector for the current configuration.

12 CONSTITUTIVE EQUATIONS

According to Truesdell [19] "a constitutive equation is a relation between forces and motions. In [13] Truesdell and Toupin list a number of general mathematical principles to be used as an aid in formulating constitutive equations.

They are as follows:

1. Consistency - The equations must be consistent with the general balance laws.

2. Coordinate invariance - The equations must be valid in all inertial coordinate systems.

3. Isotropy and aeolotropy - If the materials exhibit no preferred directions of response (isotropy) or symmetry with respect to certain preferred directions (aeolotropy) then these properties should be mathematically precise.

4. Just setting - When the equations are combined with the general balance laws, a unique solution, which is continuous in the variables, should result for appropriate initial and boundary conditions.

5. Dimensional invariance - All dimensionally independent material constants should be included in each constitutive equation (but need not be listed). This is accomplished by using Buckingham's π theorem. According to Shames [31], the total number of independent dimensionless groups required to describe a phenomenon which involves n variables is n–r where r is the rank of the dimensional matrix.

6. Material indifference - The response of the material should be independent of the observer.

7. Equipresence - A variable present as a variable in one constitutive equation should appear in all of the constitutive equations.

The most frequently used is the principle of material frame indifference which is briefly described as follows.

66

If f, **a**, and **A** are arbitrary scalar, vector, and tensor fields, respectively, referred to an *orthonormal Cartesian frame* and are subjected to the observer transformation

$$\mathbf{r}^* = \mathbf{c}(t) + \mathbf{Q}(t) \cdot \mathbf{r} \qquad\qquad 12.1$$

$$t^* = t - a \qquad\qquad 12.2$$

where a is a scalar, **c** is a vector and **Q** is a proper rotation tensor where

$$\mathbf{Q} \cdot \mathbf{Q}^{\mathrm{T}} = \mathbf{1} \qquad\qquad 12.3$$

and if

$$f(\mathbf{r}^*,t^*) = f(\mathbf{r},t) \qquad\qquad 12.4$$

$$\mathbf{a}(\mathbf{r}^*, t^*) = \mathbf{Q}(t) \cdot \mathbf{a}(\mathbf{r}, t) \qquad\qquad 12.5$$

$$\mathbf{A}(\mathbf{r}^*, t^*) = \mathbf{Q}(t) \cdot \mathbf{A}(\mathbf{r}, t) \cdot \mathbf{Q}^{T}(t) \qquad\qquad 12.6$$

then f, **a**, and **A** are material indifferent or objective fields. The deformation gradient tensor **F** transforms as a vector when a=0, or

$$\mathbf{F}(\mathbf{r}^*, t) = \mathbf{Q}(t) \cdot \mathbf{F}(\mathbf{r}, t) \qquad\qquad 12.7$$

As an example Carlson in [21], presents an elastic material defined by the following constitutive equations:

$$\psi = \widehat{\psi}(\mathbf{F}, \theta, \mathbf{g}, \mathbf{r}) \qquad\qquad 12.8$$

$$\mathbf{T} = \widehat{\mathbf{T}}(\mathbf{F}, \theta, \mathbf{g}, \mathbf{r}) \qquad\qquad 12.9$$

$$\eta = \widehat{\eta}(\mathbf{F}, \theta, \mathbf{g}, \mathbf{r}) \qquad\qquad 12.10$$

$$\mathbf{q} = \widehat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{g}, \mathbf{r}) \qquad\qquad 12.11$$

In general the constitutive equations could also contain first and higher order spatial and temporal derivatives of **F**, θ, **g**, and **r** or other quantities and their derivatives depending on the nature of the material we are concerned with. For example Ghoneim and Dalo [32] (see also Ghoneim [33]) introduce an elastic heat flow vector and a set of internal state variables into the constitutive equations to develop a set of coupled thermoviscoplasticity equations which include second sound effects (thermo-mechanical disturbances propagate with finite speed).

Assuming a nonpolar thermoelastic medium (no assigned traction couples or body couples) the Clausius-Duhem inequality can be put in the following form

$$\left(\mathbf{T} - \rho \frac{\partial \hat{\psi}}{\partial \mathbf{F}} \mathbf{F}^T \right) : \mathbf{L} - \rho \left(\eta + \frac{\partial \hat{\psi}}{\partial \theta} \right) \dot{\theta} - \rho \frac{\partial \hat{\psi}}{\partial \mathbf{g}} \cdot \dot{\mathbf{g}} - \frac{1}{\theta} \mathbf{g} \cdot \mathbf{q} \geq 0 \qquad 12.12$$

where a superimposed dot indicates the material derivative. From the equation above, for an arbitrary admissible thermodynamic process the following relationships must hold, where dependence on **r** is implied:

$$\psi = \hat{\psi}(\mathbf{F}, \theta), \quad \mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}, \theta), \quad \eta = \hat{\eta}(\mathbf{F}, \theta) \qquad 12.13$$

$$\mathbf{T} = \rho \frac{\partial \hat{\psi}}{\partial \mathbf{F}} \mathbf{F}^T, \quad \eta = -\frac{\partial \hat{\psi}}{\partial \theta} \qquad 12.14$$

$$\hat{\mathbf{q}} \cdot \mathbf{g} \leq 0 \qquad 12.15$$

also

$$\frac{\partial^2 \psi}{\partial \theta \partial \mathbf{F}} = \frac{\partial \hat{\mathbf{T}}}{\partial \theta} = -\frac{\partial \hat{\eta}}{\partial \mathbf{F}} \qquad 12.16$$

which is referred to as the Maxwell relation. The energy equation can now be put in the form

$$\theta \dot{\eta} = \rho r - \vec{\nabla}_x \cdot \mathbf{q} \qquad 12.17$$

68

Due to the Helmholtz free energy equation 11.10, the internal energy takes the form

$$e = \hat{e}(\mathbf{F}, \theta)$$ 12.18

Now, by partially differentiating the Helmholtz free energy equation with respect to θ

$$\frac{\partial \hat{\psi}}{\partial \theta} = \left(\frac{\partial \hat{e}}{\partial \theta} - \theta \frac{\partial \hat{\eta}}{\partial \theta} \right) - \eta = \left(\frac{\partial \hat{e}}{\partial \theta} - \theta \frac{\partial \hat{\eta}}{\partial \theta} \right) + \frac{\partial \hat{\psi}}{\partial \theta}$$ 12.19

we find

$$\frac{\partial \hat{e}}{\partial \theta} = \theta \frac{\partial \hat{\eta}}{\partial \theta}$$ 12.20

Let c_E given by

$$c_E = c_E(\mathbf{F}, \theta) = \frac{\partial \hat{e}}{\partial \theta} = \theta \frac{\partial \hat{\eta}}{\partial \theta}$$ 12.21

represent the specific heat at constant strain.

The principle of material frame indifference requires

$$\psi = \hat{\psi}(\mathbf{F}, \theta) = \hat{\psi}(\mathbf{Q} \cdot \mathbf{F}, \theta)$$ 12.22

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}, \theta) = \mathbf{Q}^T \cdot \hat{\mathbf{T}}(\mathbf{Q} \cdot \mathbf{F}, \theta) \cdot \mathbf{Q}$$ 12.23

$$\eta = \hat{\eta}(\mathbf{F}, \theta) = \hat{\eta}(\mathbf{Q} \cdot \mathbf{F}, \theta)$$ 12.24

$$\mathbf{q} = \hat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{g}) = \mathbf{Q}^T \cdot \hat{\mathbf{q}}(\mathbf{Q} \cdot \mathbf{F}, \theta, \mathbf{Q} \cdot \mathbf{g})$$ 12.25

By letting $\mathbf{Q} = \mathbf{R}^T$ and recalling that from the polar decomposition theorem $\mathbf{F} = \mathbf{R} \cdot \mathbf{U}$, the equations can be put in the following form:

$$\psi = \hat{\psi}(\mathbf{F}, \theta) = \hat{\psi}(\mathbf{U}, \theta)$$ 12.26

$$\mathbf{T} = \hat{\mathbf{T}}(\mathbf{F}, \theta) = \mathbf{F} \cdot \mathbf{U}^{-1} \cdot \hat{\mathbf{T}}(\mathbf{U}, \theta) \cdot \mathbf{U}^{-1} \cdot \mathbf{F}^T$$ 12.27

69

$$\eta = \hat{\eta}(\mathbf{F}, \theta) = \hat{\eta}(\mathbf{U}, \theta) \tag{12.28}$$

$$\mathbf{q} = \hat{\mathbf{q}}(\mathbf{F}, \theta, \mathbf{g}) = \mathbf{F} \cdot \mathbf{U}^{-1} \cdot \hat{\mathbf{q}}\left(\mathbf{U}, \theta, \mathbf{U}^{-1} \cdot \mathbf{F}^T \cdot \mathbf{g}\right) \tag{12.29}$$

Recalling that

$$\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1}) = \frac{1}{2}\left(\mathbf{F}^T \cdot \mathbf{F} - \mathbf{1}\right) \tag{12.30}$$

and

$$\frac{\rho_o}{\rho} = J = \det\mathbf{F} = \mathrm{III}_{\mathbf{F}} \tag{12.31}$$

or

$$\mathrm{III}_{\mathbf{F}} = (\mathrm{III}_{\mathbf{U}})^{\frac{1}{2}} = (1 + 2\mathrm{I}_{\mathbf{E}} + 4\mathrm{II}_{\mathbf{E}} + 8\mathrm{III}_{\mathbf{E}})^{\frac{1}{2}} \tag{12.32}$$

the equations can be written as

$$\psi = \tilde{\psi}(\mathbf{E}, \theta) \tag{12.33}$$

$$\mathbf{T} = \mathbf{F} \cdot \tilde{\mathbf{T}}(\mathbf{E}, \theta) \cdot \mathbf{F}^T \tag{12.34}$$

$$\eta = \tilde{\eta}(\mathbf{E}, \theta) \tag{12.35}$$

$$\mathbf{q} = J^{-1}\mathbf{F} \cdot \tilde{\mathbf{q}}_o(\mathbf{E}, \theta, \mathbf{g}_o) \tag{12.36}$$

Taking the material derivative of 12.33 we find

$$\dot{\psi} = \frac{\partial \tilde{\psi}}{\partial \mathbf{E}} : \frac{d\mathbf{E}}{dt} + \frac{\partial \tilde{\psi}}{\partial \theta} \frac{d\theta}{dt} \tag{12.37}$$

But

$$\frac{\partial \tilde{\psi}}{\partial \mathbf{E}} : \frac{d\mathbf{E}}{dt} = \frac{\partial \tilde{\psi}}{\partial \mathbf{E}} : \left(\mathbf{F}^T \cdot \mathbf{D} \cdot \mathbf{F}\right) = \mathbf{F} \cdot \frac{\partial \tilde{\psi}}{\partial \mathbf{E}} \cdot \mathbf{F}^T : \mathbf{D} \tag{12.38}$$

70

where we recall that

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right)$$

12.39

Due to the symmetry of both $\frac{\partial \tilde{\psi}}{\partial \mathbf{E}}$ and $\mathbf{D}$, 12.37 can be written as

$$\dot{\psi} = \mathbf{F} \cdot \frac{\partial \tilde{\psi}}{\partial \mathbf{E}} \cdot \mathbf{F}^T : \mathbf{L} + \frac{\partial \tilde{\psi}}{\partial \theta}\frac{d\theta}{dt}$$

12.40

Now taking the material derivative of $\psi = \hat{\psi}(\mathbf{F},\theta)$ we find

$$\dot{\psi} = \frac{\partial \hat{\psi}}{\partial \mathbf{F}} \cdot \mathbf{F}^T : \mathbf{L} + \frac{\partial \hat{\psi}}{\partial \theta}\frac{d\theta}{dt}$$

12.41

Equating equations 12.40 and 12.41 we have

$$\frac{\partial \hat{\psi}}{\partial \mathbf{F}} = \mathbf{F} \cdot \frac{\partial \tilde{\psi}}{\partial \mathbf{E}} \qquad \frac{\partial \hat{\psi}}{\partial \theta} = \frac{\partial \tilde{\psi}}{\partial \theta}$$

12.42

Recalling equation 12.14 and utilizing the first of the above equations we have

$$\mathbf{T} = \rho\frac{\partial \hat{\psi}}{\partial \mathbf{F}} \cdot \mathbf{F}^T = \rho\mathbf{F} \cdot \frac{\partial \tilde{\psi}}{\partial \mathbf{E}} \cdot \mathbf{F}^T$$

12.43

which shows that

$$\tilde{\mathbf{T}}(\mathbf{E}, \theta) = \rho\frac{\partial \tilde{\psi}}{\partial \mathbf{E}}$$

12.44

Recalling the relationship between the Cauchy and Piola-Kirchhoff stress tensors we find

$$\mathbf{T} = \rho\mathbf{F}\frac{\partial \tilde{\psi}}{\partial \mathbf{E}}\mathbf{F}^T$$

12.45

$$\tilde{\mathbf{T}} = \rho_\mathrm{o}\frac{\partial \tilde{\psi}}{\partial \mathbf{E}}$$

12.46

$$\mathbf{T_R} = (\mathbf{T}^\mathrm{o})^T = \rho_\mathrm{o}\mathbf{F} \cdot \frac{\partial \tilde{\psi}}{\partial \mathbf{E}}$$

12.47

13 THERMOELASTICITY

The equations for the nonlinear theory of thermoelasticity expressed in the reference configuration consist of:

The balance equation for linear momentum

$$\vec{\nabla}_X \cdot \mathbf{T}^o + \rho_o \mathbf{b}_o = \rho_o \frac{d^2 \mathbf{x}}{dt^2} \qquad \text{13.1}$$

or

$$\vec{\nabla}_X \cdot \left[\tilde{\mathbf{T}} \cdot \mathbf{F}^T \right] + \rho_o \mathbf{b}_o = \rho_o \frac{d^2 \mathbf{x}}{dt^2} \qquad \text{13.2}$$

and the balance equation for thermal and mechanical energy

$$\rho_o \theta \dot{\eta} = \rho_o r - \vec{\nabla}_X \cdot \mathbf{q}_o \qquad \text{13.3}$$

The constitutive equations

$$\psi = \tilde{\psi}(\mathbf{E}, \theta) \qquad \text{13.4}$$

$$\eta = \tilde{\eta}(\mathbf{E}, \theta) \qquad \text{13.5}$$

$$\mathbf{T_R} = (\mathbf{T}^o)^T = J\,\mathbf{F} \cdot \tilde{\mathbf{T}}(\mathbf{E}, \theta) \qquad \text{13.6}$$

or

$$\tilde{\mathbf{T}} = J\,\tilde{\mathbf{T}}(\mathbf{E}, \theta) \qquad \text{13.7}$$

$$\mathbf{q}_o = \tilde{\mathbf{q}}_o(\mathbf{E}, \theta, \mathbf{g}_o) \qquad \text{13.8}$$

with

$$\mathbf{F} = \mathbf{1} + \mathbf{u}\overleftarrow{\nabla}_X \qquad \text{13.9}$$

72

$$E = \frac{1}{2}\left(F^T \cdot F - 1\right)$$ 13.10

$$g_o = \vec{\nabla}_X \theta$$ 13.11

$$J = \frac{\rho_o}{\rho}$$ 13.12

and subject to the thermodynamic restrictions

$$\tilde{\eta}(E, \theta) = -\frac{\partial \tilde{\psi}(E, \theta)}{\partial \theta}$$ 13.13

$$\tilde{T}(E, \theta) = \rho \frac{\partial \tilde{\psi}(E, \theta)}{\partial E}$$ 13.14

$$\tilde{q}_o(E, \theta, g_o) \cdot g_o \leq 0$$ 13.15

The conclusion that the heat flux vector vanishes with the spatial gradient of the temperature (equation 13.15) is demonstrated by Carlson in [21] by use of a Taylor series expansion and Chadwick and Seet in [26] by use of the mean-value theorem. Following Chadwick and Seet [26] and applying the mean-value theorem to $\tilde{q}_o(E, \theta, g_o)$ between $\tilde{q}_o(E, \theta, 0)$ and $\tilde{q}_o(E, \theta, g_o)$ and assuming $\delta > |g_o| > 0$, we find

$$\frac{\partial \tilde{q}_o(E, \theta, \lambda g_o)}{\partial g_o} \cdot g_o = \tilde{q}_o(E, \theta, g_o) - \tilde{q}_o(E, \theta, 0)$$ 13.16

with $1 > \lambda > 0$. If we let

$$K = -\frac{\partial \tilde{q}_o(E, \theta, \lambda g_o)}{\partial g_o}$$ 13.17

and

$$g_o = \varepsilon a$$ 13.18

where $|\varepsilon| < \delta$ and $\mathbf{a}$ is an arbitrary unit vector, solve for $\tilde{\mathbf{q}}_o(\mathbf{E}, \theta, \mathbf{g}_o)$ and substitute into 13.15 we find

$$\varepsilon \mathbf{a} \cdot \tilde{\mathbf{q}}_o(\mathbf{E}, \theta, 0) - \varepsilon^2 \mathbf{a} \cdot \{\mathbf{K}(\mathbf{E}, \theta, \varepsilon \lambda \mathbf{g}_o) \cdot \mathbf{a}\} \leq 0 \qquad 13.19$$

Dividing by ε and allowing $\varepsilon \to 0$ we find

$$\tilde{\mathbf{q}}_o(\mathbf{E}, \theta, 0) = 0 \qquad 13.20$$

which implies that the heat flux vector is identically zero when the spatial gradient of the temperature vanishes. By making use of the above relationship equation 13.19 can be rewritten as

$$\varepsilon^2 \mathbf{a} \cdot \{\mathbf{K}(\mathbf{E}, \theta, \varepsilon \lambda \mathbf{g}_o) \cdot \mathbf{a}\} \geq 0 \qquad 13.21$$

After dividing by ε^2 and allowing $\varepsilon \to 0$ we find that $\mathbf{K}(\mathbf{E}, \theta, 0)$ is a positive semi-definite second order tensor.

The linearized versions of the equations of thermoelasticity can be obtained by expanding the various scalar, vector and tensor functions in a Taylor series (see also Hughes and Pister [34]) about $\mathbf{E} = 0$, $\mathbf{q}_o = 0$ and $\theta = \theta_o$ and assuming that the following quantities are less than or equal to δ

$$\left|\mathbf{u}\overleftarrow{\nabla}_X\right|, \quad \left|\dot{\mathbf{u}}\overleftarrow{\nabla}_X\right|, \quad |\theta - \theta_o|, \quad \left|\dot{\theta}\right|, \quad |\mathbf{g}_o| \qquad 13.22$$

where $\delta << 1$. Recall that

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{u}\overleftarrow{\nabla}_X + \mathbf{u}\overleftarrow{\nabla}_X^T + \mathbf{u}\overleftarrow{\nabla}_X \cdot \mathbf{u}\overleftarrow{\nabla}_X^T\right) \qquad 13.23$$

which can be rewritten as

$$\mathbf{E} = \frac{1}{2}\mathbf{e} + \frac{1}{2}\left(\mathbf{u}\overleftarrow{\nabla}_X \cdot \mathbf{u}\overleftarrow{\nabla}_X^T\right) = \frac{1}{2}\mathbf{e} + O(\delta^2) \qquad 13.24$$

where

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{u}\overleftarrow{\nabla}_X + \mathbf{u}\overleftarrow{\nabla}_X^T\right) = O(\delta) \qquad 13.25$$

74

Also

$$\mathbf{F} = \mathbf{1} + O(\delta) \qquad\qquad 13.26$$

and

$$J = 1 + O(\delta) \qquad\qquad 13.27$$

For example

$$\tilde{\mathbf{T}}(\mathbf{E}, \theta) = \tilde{\mathbf{T}}(\mathbf{0}, \theta_{\mathrm{o}}) + \left(\frac{\partial \tilde{\mathbf{T}}(\mathbf{E}, \theta)}{\partial \mathbf{E}} \Big|_{\substack{\mathbf{E} = \mathbf{0} \\ \theta = \theta_{\mathrm{o}}}} \right) [\mathbf{E}] + \left(\frac{\partial \tilde{\mathbf{T}}(\mathbf{E}, \theta)}{\partial \theta} \Big|_{\substack{\mathbf{E} = \mathbf{0} \\ \theta = \theta_{\mathrm{o}}}} \right) (\theta - \theta_{\mathrm{o}}) + O(\delta^2)$$

$$13.28$$

By assuming that the residual stress is equal to zero, i.e.,

$$\tilde{\mathbf{T}}(\mathbf{0}, \theta_{\mathrm{o}}) = \mathbf{0} \qquad\qquad 13.29$$

and letting

$$\mathbf{C} = \frac{\partial \tilde{\mathbf{T}}(\mathbf{E}, \theta)}{\partial \mathbf{E}} \Big|_{\substack{\mathbf{E} = \mathbf{0} \\ \theta = \theta_{\mathrm{o}}}} \qquad\qquad 13.30$$

$$\mathbf{M} = \frac{\partial \tilde{\mathbf{T}}(\mathbf{E}, \theta)}{\partial \theta} \Big|_{\substack{\mathbf{E} = \mathbf{0} \\ \theta = \theta_{\mathrm{o}}}} \qquad\qquad 13.31$$

and substituting

$$\mathbf{E} = \mathbf{e} + O(\delta) \qquad\qquad 13.32$$

equation 13.28 can be written as

$$\tilde{\mathbf{T}}(\mathbf{E}, \theta) = \mathbf{C}[\mathbf{e}] + \mathbf{M}(\theta - \theta_{\mathrm{o}}) + O(\delta) \qquad\qquad 13.33$$

The equations of the linear theory of thermoelasticity consist of the following:

- Strain − displacement relationship

$$\mathbf{e} = \frac{1}{2} \left(\mathbf{u}\overleftarrow{\nabla}_{\mathbf{X}} + \mathbf{u}\overleftarrow{\nabla}_{\mathbf{X}}^T \right) \qquad\qquad 13.34$$

75

- Conservation of linear momentum

$$\overrightarrow{\nabla}_X \cdot \tilde{\mathbf{T}} \cdot \mathbf{F}^T + \rho_o \mathbf{b}_o = \rho_o \frac{\partial^2 \mathbf{u}}{\partial t^2} \qquad 13.35$$

- Stress – strain – temperature relationship

$$\tilde{\mathbf{T}} = C[\mathbf{e}] + \mathbf{M}(\theta - \theta_o) \qquad 13.36$$

- Conservation of energy

$$-\overrightarrow{\nabla}_X \cdot \mathbf{q}_o + \theta_o \mathbf{M} : \dot{\mathbf{e}} + \rho_o r = \rho_o c \dot{\theta} \qquad 13.37$$

- Heat conduction relationship

$$\mathbf{q}_o = -\mathbf{K} \cdot \overrightarrow{\nabla}_X \theta \qquad 13.38$$

In 13.36 C is the fourth-order elasticity tensor and in 13.36 and 13.37 $\mathbf{M}$ is the second-order stress-temperature tensor. Assuming that the inverse of C exists then

$$\mathbf{e} = C^{-1}\left[\tilde{\mathbf{T}}\right] + C^{-1}[\mathbf{M}](\theta - \theta_o) \qquad 13.39$$

If we let

$$\mathbf{A} = C^{-1}[\mathbf{M}] \qquad 13.40$$

then

$$\mathbf{e} = C^{-1}\left[\tilde{\mathbf{T}}\right] + \mathbf{A}(\theta - \theta_o) \qquad 13.41$$

where C^{-1} is the compliance tensor and $\mathbf{A}$ is the thermal expansion tensor.

When the material of the body is isotropic the equations reduce to

1. Strain – displacement relationship

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{u}\overset{\leftarrow}{\nabla}_X + \mathbf{u}\overset{\leftarrow}{\nabla}_X^T\right) \qquad\qquad 13.42$$

2. Conservation of linear momentum

$$\overset{\rightarrow}{\nabla}_X \cdot \tilde{\mathbf{T}} \cdot \mathbf{F}^T + \rho_o \mathbf{b}_o = \rho_o \frac{\partial^2 \mathbf{u}}{\partial t^2} \qquad\qquad 13.43$$

3. Stress – strain – temperature relationship

$$\tilde{\mathbf{T}} = 2\mu\mathbf{e} + \lambda(\mathrm{tr}\mathbf{e})\mathbf{1} + m(\theta - \theta_o)\mathbf{1} \qquad\qquad 13.44$$

4. Conservation of energy

$$-\overset{\rightarrow}{\nabla}_X \cdot \mathbf{q}_o + m\theta_o(\mathrm{tr}\dot{\mathbf{e}}) + \rho_o r = \rho_o c\dot{\theta} \qquad\qquad 13.45$$

5. Heat conduction relationship

$$\mathbf{q}_o = -k\overset{\rightarrow}{\nabla}_X \theta \qquad\qquad 13.46$$

where use is made of the following for isotropic tensors. Given a general fourth-order tensor, $\mathbf{C}$, where

$$\mathbf{C} = C^{ijkl}\mathbf{g}_i\mathbf{g}_j\mathbf{g}_k\mathbf{g}_l \qquad\qquad 13.47$$

If $\mathbf{C}$ is isotropic then [1]

$$C^{ijkl} = \lambda_1\delta^{ij}\delta^{kl} + \lambda_2\delta^{ik}\delta^{jl} + \lambda_3\delta^{il}\delta^{jk} \qquad\qquad 13.48$$

77

where $\lambda_1, \lambda_2, \lambda_3$ are scalars. If, in addition, C is symmetric then

$$C^{jikl} = \lambda_1 \delta^{ji}\delta^{kl} + \lambda_2 \delta^{jk}\delta^{il} + \lambda_3 \delta^{jl}\delta^{ik} \qquad 13.49$$

Subtracting the above two equations, one from another, and rearranging we have

$$(\lambda_2 - \lambda_3)(\delta^{ik}\delta^{jl} - \delta^{jk}\delta^{il}) = 0 \qquad 13.50$$

which implies

$$\lambda_2 = \lambda_3 \qquad 13.51$$

If we let

$$\lambda = \lambda_1 \qquad 13.52$$

and

$$\mu = \lambda_2 = \lambda_3 \qquad 13.53$$

we have

$$C^{ijkl} = \lambda \delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \qquad 13.54$$

Given a general second-order tensor **A**, where

$$\mathbf{A} = A^{ij}\mathbf{g}_i\mathbf{g}_j \qquad 13.55$$

if the tensor is isotropic then [5]

$$A^{ij} = \phi \delta^{ij} \qquad 13.56$$

where ϕ is a scalar. Let

$$\mathbf{T} = \mathbf{C}[e] \qquad 13.57$$

where **T** and **e** are second-order tensors and **C** is a fourth-order symmetric isotropic tensor, then

$$T^{ij} = \lambda \delta^{ij}\delta^{kl}e_{lk} + \mu \delta^{ik}\delta^{jl}e_{lk} + \mu \delta^{il}\delta^{jk}e_{lk} \qquad 13.58$$

78

which can be rewritten as

$$T^{ij} = \lambda \delta^{ij} \delta^{kl} e_{lk} + \mu \delta^{ik} \delta^{jl} e_{lk} + \mu \delta^{ik} \delta^{jl} e_{kl} \qquad 13.59$$

If **e** is symmetric then

$$T^{ij} = \lambda \delta^{ij} \delta^{kl} e_{lk} + 2\mu \delta^{ik} \delta^{jl} e_{lk} \qquad 13.60$$

where λ and μ are the Lamé constants. In direct notation the above equation has the form

$$\mathbf{T} = \mathbf{C}[\mathbf{e}] = \lambda(\text{tr}\mathbf{e})\mathbf{1} + 2\mu\mathbf{e} \qquad 13.61$$

The second order isotropic tensors can be written as

$$\mathbf{M} = m\mathbf{1} \qquad 13.62$$

$$\mathbf{K} = k\mathbf{1} \qquad 13.63$$

$$\mathbf{A} = \alpha\mathbf{1} \qquad 13.64$$

where m, k, and α are scalars representing the stress-temperature modulus, the thermal conductivity and the coefficient of thermal expansion, respectively. Taking the trace of equation 13.44 and rearranging we find

$$\text{tr}\mathbf{e} = \frac{1}{2\mu + 3\lambda}\text{tr}\tilde{\mathbf{T}} - \frac{3m}{2\mu + 3\lambda}(\theta - \theta_0) \qquad 13.65$$

Substituting the above into 13.44 and rearranging yields

$$\mathbf{e} = \frac{1}{2\mu}\tilde{\mathbf{T}} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\left(\text{tr}\tilde{\mathbf{T}}\right)\mathbf{1} - \frac{m}{(2\mu + 3\lambda)}(\theta - \theta_0)\mathbf{1} \qquad 13.66$$

or

$$\mathbf{e} = \frac{1}{2\mu}\tilde{\mathbf{T}} - \frac{\lambda}{2\mu(2\mu + 3\lambda)}\left(\text{tr}\tilde{\mathbf{T}}\right)\mathbf{1} + \alpha(\theta - \theta_0)\mathbf{1} \qquad 13.67$$

where the coefficient of thermal expansion α is given by

$$\alpha = -\frac{m}{(2\mu + 3\lambda)} \qquad 13.68$$

and

$$m = -\alpha(2\mu + 3\lambda) \qquad 13.69$$

The relationship between the Lamé constants and Poisson's ratio and Young's modulus are given (Sokolnikoff [18]) as:

$$\mu = \frac{E}{2(1+\nu)} \quad , \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad 13.70$$

The converse relationships (Gurtin [35]) are

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \quad , \quad \nu = \frac{\lambda}{2(\mu + \lambda)} \qquad 13.71$$

The initial value – boundary value problem of linear thermoelasticity consists of the following:

- Strain – displacement relationship

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{u}\overleftarrow{\nabla}_X + \mathbf{u}\overleftarrow{\nabla}_X^T\right) \qquad 13.72$$

- Conservation of linear momentum

$$\overrightarrow{\nabla}_X \cdot \tilde{\mathbf{T}} \cdot \mathbf{F}^T + \rho_o \mathbf{b}_o = \rho_o \frac{\partial^2 \mathbf{u}}{\partial t^2} \qquad 13.73$$

- Stress – strain – temperature relationship

$$\tilde{\mathbf{T}} = 2\mu\mathbf{e} + \lambda(\mathrm{tr}\mathbf{e})\mathbf{1} + m(\theta - \theta_o)\mathbf{1} \qquad 13.74$$

- Conservation of energy

$$-\vec{\nabla}_X \cdot \mathbf{q}_o + m\theta_o(\mathrm{tr}\dot{\mathbf{e}}) + \rho_o r = \rho_o c_E \dot{\theta} \qquad 13.75$$

- Heat conduction relationship

$$\mathbf{q}_o = -k\vec{\nabla}_X \theta \qquad 13.76$$

and the following initial conditions for the body for t=0 :

- Displacement $\mathbf{u} = \mathbf{u}_o$

- Velocity $\dot{\mathbf{u}} = \dot{\mathbf{u}}_o$

- Temperature difference $\vartheta = \theta - \theta_o = \vartheta_o$

and the following boundary conditions on any boundary:

- Displacement $\mathbf{u} = \hat{\mathbf{u}}$

  or

- Traction $\mathbf{t} = \mathbf{T} \cdot \mathbf{n} = \hat{\mathbf{t}}$

  and

- Temperature difference $\vartheta = \theta - \theta_o = \hat{\vartheta}$

  or

- Heat flux $q = \mathbf{q} \cdot \mathbf{n} = \hat{q}$

The momentum and energy equations are fully coupled through the strain rate tensor which implies that changes in the strain produce changes in the temperature field and an increase in entropy (thermoelastic dissipation) or changes in temperature produce changes in strain (thermally induced vibrations). The equations imply that thermal and elastic disturbances propagate at infinite speeds which is contrary to physical observations [36].

81

There are two generally accepted theories which resolve the discrepancy [37]. The classical form of the Fourier law of heat conduction

$$\mathbf{q} = -\mathbf{K} \cdot \mathbf{g} \qquad\qquad 13.77$$

can be written in a generalized form [38] as

$$\mathbf{q} + \tau \frac{\partial \mathbf{q}}{\partial t} = -\mathbf{K} \cdot \mathbf{g} \qquad\qquad 13.78$$

where τ is the relaxation time. H. W. Lord and Y. Shulman [39] are credited with incorporating this modification into a thermoelasticity theory. The second theory attributed to A. E. Green and K. A. Lindsay [40] involves two relaxation times and is based on an alternate representation of the entropy inequality (second law of thermodynamics). Ignaczak [37] describes the classical theory (C), the Lord–Shulman (L-S) theory, and the Green-Lindsay (G-S) theory succinctly with the following equations which he refers to as the equations of generalized dynamic thermoelasticity (GDT):

- Strain – displacement relationship

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{u}\overleftarrow{\nabla}_X + \mathbf{u}\overleftarrow{\nabla}_X^{\,T} \right) \qquad\qquad 13.79$$

- Conservation of linear momentum

$$\overrightarrow{\nabla}_X \cdot \tilde{\mathbf{T}} \cdot \mathbf{F}^T + \rho_o \mathbf{b}_o = \rho_o \frac{\partial^2 \mathbf{u}}{\partial t^2} \qquad\qquad 13.80$$

- Stress – strain – temperature relationship

$$\tilde{\mathbf{T}} = 2\mu \mathbf{e} + \lambda(\mathrm{tr}\,\mathbf{e})\mathbf{1} + m(\theta - \theta_o)\mathbf{1} + m\tau_1 \frac{\partial \theta}{\partial t}\mathbf{1} \qquad\qquad 13.81$$

- Conservation of energy

$$-\overrightarrow{\nabla}_X \cdot \mathbf{q}_o + m\theta_o\left(\mathrm{tr}\frac{\partial \mathbf{e}}{\partial t} \right) + \rho_o r = \rho_o c_E \frac{\partial \theta}{\partial t} + \rho_o c_E \tau_o \frac{\partial^2 \theta}{\partial t^2} \qquad\qquad 13.82$$

- Heat conduction relationship

$$q_o + \tau \frac{\partial q_o}{\partial t} = -k \vec{\nabla}_X \theta \qquad\qquad 13.83$$

where τ, τ_o, τ_1, are relaxation times. The equations reduce to the classical theory (C) if $\tau = \tau_o = \tau_1 = 0$. The equations reduce to the Lord–Shulman (L-S) theory if $\tau > 0$ and $\tau_o = \tau_1 = 0$. The equations reduce to the Green-Lindsay (G-S) theory if $\tau = 0$ and $\tau_1 \geq \tau_o > 0$. Kranyš in [41] introduces a general form of hyperbolic operators for converting various parabolic equations to hyperbolic form.

The stress/strain/displacement field and temperature field have to solved for simultaneously. The equations uncouple when changes in temperature due to mechanically induced straining can be ignored. Therefore, we can solve for the temperature field independently and then solve for the stress/strain/displacement field. When in addition the time rate of change of load application is gradual enough that the inertia term can be disregarded, the resulting equations are referred to as the quasi-static problem.

In the previous formulation is was assumed that the displacements were to be treated as the unknown variables. When the strains are treated as the unknown variables the compatibility equation

$$\vec{\nabla}_X \times e \times \overleftarrow{\nabla}_X = 0 \qquad\qquad 13.84$$

needs to be added to the set of equations. When equation 13.44 is substituted into the above equation the compatibility conditions can be expressed in terms of the stresses.

14 SHELL THEORIES

A. Introduction

A thin shell is defined by Kraus [10] as a body bounded by two closely spaced curved surfaces called faces and having three identifying features: a reference surface, a thickness, and edges. The edges are usually assumed to be perpendicular to the reference surface however in [42] Libai and Simmonds point out that Koiter has attempted to treat beveled (non-perpendicular) edges in some unpublished work. Shells without edges are called closed shells.

The goal in developing a shell theory is to be able to exploit the fact that the distance between the faces, the thickness, is small compared to the other dimensions of the shell. Shells are usually classified as thick or thin based on the magnitude of the ratio of the thickness to some characteristic length on the reference surface. Some shells of constant thickness can be classified as both. In [43] Rubin and Florence point out that although a conical shell may be considered thin at its base, it has to be considered thick near the tip. Thin shell theories generally disregard transverse shear and changes in thickness. When the shell is considered thin enough that bending can be disregarded, the shell is called a membrane shell (e.g., balloons, bubbles). Shells are usually classified as deep or shallow based the ratio of the characteristic length to the minimum principal radius of curvature of the reference surface. Shallow shell theories typically disregard various terms involving the curvature.

The approaches to developing shell theories can be classified as either direct or indirect. The indirect approach consists of the reduction of the three-dimensional equations to a set of two dimensional equations which are valid for the reference surface of the shell. In the three-dimensional formulations we are dealing with various thermo-mechanical quantities which are expressed per unit volume or per unit area. In the

two-dimensional formulations we seek to represent the quantities per unit area and per unit length. The reduction is generally accomplished by one of two general methods.

The first method is based on asymptotic approximations. The three-dimensional equations are expressed as various series expansions in terms of some parameter which is a function of the shell thickness. By appealing to Saint-Venant's principle, (see Horgan [44]) the three-dimensional stress distribution is replaced by a statically equivalent set of resultant forces and moments which are obtained by integrating the stresses across the thickness. The resultants are expressed per unit length of elements on and intersecting the reference surface. The reactions at the edges of the shell (open shells) are treated similarly. This is the most generally used approach and the one which will be presented here. The second method involves a priori error estimates of the various quantities associated with the three-dimensional equations (see for example John [45,46]).

The direct approach is concerned with regarding the shell as a Cosserat surface. The shell is regarded a priori as a two dimensional body, a surface, with a field of non-tangential vectors attached to it. The vectors are called directors and can be used to represent the thickness of the shell. All of the thermo-mechanical quantities are expressed initially, per unit area of the surface and per unit length of curves on and intersecting the surface. Although this approach simplifies the formulation of the kinematic and kinetic relationships it complicates the formulation of constitutive relationships.

Isothermal elastic shells via Cosserat surface theory are treated by Langhaar in [47] and by Zhilin in [48]. Elastic shells, including thermal effects, via Cosserat surface theory are treated by Green, Naghdi, et al. in [49–53]. In [54] Rubin considers a uniqueness proof for generalized boundary conditions which allows for mechanical contact and thermal radiation.

In addition to the different ways of developing shell theories briefly described above, the equations can be presented in either vector or component form and they may be formulated either in terms of the strain measures (intrinsic formulation) or in terms of

the displacements. Recall that when the strains are treated as the unknown variables, rather then the displacements, the general set of equations must be supplemented with the compatibility equations. Axelrad and Emmerling discuss the advantages of the vector form of the intrinsic relationships in [55].

The general shell theories developed are usually described as either nonlinear or linear. The nonlinear theories can be subdivided into those involving geometric non-linearities and those involving material nonlinearities. The material nonlinearities are a result of nonlinear constitutive relationships (stress-strain relationships). Geometric nonlinearities result from retaining product terms involving the displacements and their gradients in the strain measures.

Recall that the strain measures in general involve changes in the lengths of curves and changes in the angles between intersecting curves. The change in the angle between two intersecting curves can be described as a rotation. In the case of shells we are concerned with rotations of elements on the reference surface and the rotation of the normal to the reference surface. When the rotations are finite they do not transform as vectors [56], and care must taken to ensure the formulation is invariant (see for example [57] and [58]). Pietraszkiewicz in [59] suggests subdividing finite rotations into categories where the two rotations are described as large/small, large/moderate, and large/large where the second term describes the inplane rotations. The definition of small, moderate, and large are provided in [59]. The large/small description implies the shell is undergoing large displacements with small strains. Axelrad and Emmerling in [60] describe this class of shell as a flexible shell. They further argue that within this class, the deformation typically varies more strongly with one of the surface coordinates than with the other. They term displacements of this type as realizable large displacements. In [61] Nolte et al. provide a comparison of large rotation shell theories.

The equations are formulated in either a Lagrangian, relative or Eulerian reference frame or some variation thereof. In [62] Lévesque and Bertrand discuss the disadvantages

of using a quasi-Eulerian for problems involving fluid-structure interactions.

We will follow the Lagrangian formulation (see for example [59,63]), and in addition we follow the formulation used by Başar in [64] and Başar and Krätzig in [65]. The formulation used is for shells undergoing finite deformations. They introduce an independent rotation vector to describe the rotation of the normal to the reference surface. Other authors (eg., Libai and Simmonds [42], Axelrad and Emmerling [55], Taber [66]) use a formulation involving an orthonormal frame which rigidly rotates and translates with a material point.

B. Coordinates

In addition to being concerned with descriptions of the metric properties of the initial and final configuration and the connection between them, we have to be able to define metric properties at points off the reference surface in terms of metric properties at points on the reference surface.

We begin by considering a vector $\mathbf{r}$, originating from the origin of an orthonormal Cartesian frame, which describes a general surface, where $\mathbf{r}$ has the form $\mathbf{r} = \mathbf{r}(x^1, x^2)$, where x^1 and x^2 are independent variables. We construct a unit vector $\mathbf{n}$ normal to the surface at the terminus of $\mathbf{r}$ and directed out from the surface (opposite to the direction of the normal to the lines of principal curvature). The position vector $\mathbf{p}$ to any point P in the shell, in terms of the vector $\mathbf{r}$ to the reference surface and the unit vector $\mathbf{n}$, normal to the reference surface is given by

$$\mathbf{p} = \mathbf{r} + x^3 \mathbf{n} \qquad \text{14.B.1}$$

where x^3 is the distance from the reference surface along the normal. The space described by this vector is referred to as the normal space of the shell (Lukasiewicz [67]). The vector from P to a neighboring point is then

$$d\mathbf{p} = d\mathbf{r} + dx^3 \mathbf{n} + x^3 d\mathbf{n} \qquad \text{14.B.2}$$

The scalar invariant representing the square of the distance from point P to a neighboring point is then

$$ds^2 = d\mathbf{p} \cdot d\mathbf{p} = d\mathbf{r} \cdot d\mathbf{r} + 2x^3 d\mathbf{n} \cdot d\mathbf{r} + \left(x^3\right)^2 d\mathbf{n} \cdot d\mathbf{n} + \left(dx^3\right)^2 \qquad \text{14.B.3}$$

where the following relationships have been used

$$d\mathbf{n} \cdot \mathbf{n} = 0 \qquad \text{14.B.4}$$

$$d\mathbf{r} \cdot \mathbf{n} = 0 \qquad \text{14.B.5}$$

$$\mathbf{n} \cdot \mathbf{n} = 1 \qquad \text{14.B.6}$$

Recalling the definition of the first, second and third fundamental forms of the reference surface and substituting those into equation 14.B.3 we find

$$ds^2 = \mathcal{A} - 2x^3 \mathcal{B} + \left(x^3\right)^2 \mathcal{C} + \left(dx^3\right)^2 \qquad \text{14.B.7}$$

The first three terms are recognized as the first fundamental form of a surface parallel to the reference surface and offset a distance x^3 from it. By making use of the equation for the relationship between the three fundamental forms

$$\mathcal{C} = 2H\mathcal{B} - K\mathcal{A} \qquad \text{14.B.8}$$

the above equation can be reduced to

$$ds^2 = \mathcal{A}(1 - \left(x^3\right)^2 K) - 2x^3 \mathcal{B}\left(1 - x^3 H\right) + \left(dx^3\right)^2 \qquad \text{14.B.9}$$

where K and H are the Gaussian and mean curvature of the reference surface, respectively. Recall that

$$ds^2 = d\mathbf{p} \cdot d\mathbf{p} = g_{ij} dx^i dx^j = g_{\alpha\beta} dx^\alpha dx^\beta + dx^3 dx^3 \qquad \text{14.B.10}$$

where g_{ij} are covariant components of the metric tensor in the normal space of the shell. Equating the above two equations we find the relationship between the metric tensor

components in the normal space of the shell in terms of those on the reference surface

or

$$g_{\alpha\beta} = a_{\alpha\beta}\left(1 - \left(x^3\right)^2 K\right) - 2x^3 b_{\alpha\beta}\left(1 - x^3 H\right) \qquad 14.\text{B}.11$$

$$g_{33} = 1$$

The relationship for the second fundamental form magnitudes (components of the curvature tensor) is

$$\bar{b}_{\alpha\beta} = b_{\alpha\beta} - u^3 c_{\alpha\beta} \qquad 14.\text{B}.12$$

or

$$\bar{b}_{\alpha\beta} = \left(1 - 2x^3 H\right) b_{\alpha\beta} + x^3 K a_{\alpha\beta} \qquad 14.\text{B}.13$$

where the overbar indicates the components are for the normal space of the shell.

15 SHELL EQUATIONS

Although references have been included where appropriate, the following authors provide an extensive treatment of the shell theory: Lukasiewicz [67], Libai and Simmonds [42], Naghdi [68,69], Niordson [8].

Let x^i be a set of convected curvilinear coordinates which describe a collection of material points in a body, where relative to a right-handed orthonormal Cartesian frame with material coordinates z^i, the following transformations hold.

$$x^i = x^i\left(z^1, z^2, z^3\right) \qquad\qquad 15.1$$

$$z^i = z^i\left(x^1, x^2, x^3\right) \qquad\qquad 15.2$$

In what follows Latin indices have values of 1,2,3 while Greek indices have values of 1,2. Let the boundary surfaces of the body be specified by

$$x^3 = \alpha(x^\alpha) , \qquad x^3 = \beta(x^\alpha) , \qquad \alpha < 0 < \beta \qquad\qquad 15.3$$

The surface defined by

$$x^3 = 0 \qquad\qquad 15.4$$

will be referred to as the reference surface and is located between the two bounding surfaces. The above equations loosely describe a shell of variable thickness. In this section a comma is used to denote partial differentiation relative to the convected coordinates. For example,

$$\frac{\partial \mathbf{r}}{\partial x^\alpha} = \mathbf{r}_{,\alpha} \qquad\qquad 15.5$$

An overbar is used to indicate quantities referred to the current configuration while the same quantities referred to the reference configuration are indicated without the overbar.

The quantities referred to the current configuration are in general time dependant, while the quantities referred to the reference configuration are not.

Let $\mathbf{r}$ and $\mathbf{p}$ be position vectors to a point on the reference surface and in the shell space, respectively. Let the metric tensors in the shell space be denoted by

$$g^{ij}\mathbf{g}_i\mathbf{g}_j = g_{ij}\mathbf{g}^i\mathbf{g}^j = \delta_i^j\mathbf{g}^i\mathbf{g}_j \qquad 15.6$$

while the metric tensors on the surface are

$$a^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta = a_{\alpha\beta}\mathbf{a}^\alpha\mathbf{a}^\beta = \delta_\beta^\alpha\mathbf{a}_\alpha\mathbf{a}^\beta \qquad 15.7$$

where

$$\mathbf{p},_i = \mathbf{g}_i \qquad 15.8$$

$$\mathbf{r},_\alpha = \mathbf{a}_\alpha \qquad 15.9$$

Let $\mathbf{a}_3$ represent the unit normal to the same point on the reference surface defined by

$$\mathbf{a}_3 = \mathbf{a}^3 = \frac{1}{2}\varepsilon^{\alpha\beta}(\mathbf{a}_\alpha \times \mathbf{a}_\beta) \qquad 15.10$$

where the permutation tensor of the surface is

$$\varepsilon^{\alpha\beta} = \frac{e^{\alpha\beta}}{\sqrt{a}} \qquad 15.11$$

Let the curvature tensor of the surface be given by

$$\mathbf{B} = -\mathbf{a}_{3,\alpha}\mathbf{a}^\alpha \qquad 15.12$$

with components

$$b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{a}_{3,\beta} \qquad 15.13$$

Let $\mathbf{U}$ and $\mathbf{u}$ represent the displacement fields for the shell space and the reference surface such that

$$\bar{\mathbf{p}} = \mathbf{p} + \mathbf{u} \qquad 15.14$$

$$\bar{\mathbf{r}} = \mathbf{r} + \mathbf{U} \qquad 15.15$$

Let the position vectors to points in the normal space of the shell in the reference and current configurations be given by

$$\mathbf{p}\left(x^\alpha, x^3\right) = \mathbf{r}(x^\alpha) + \sum_{n=1}^\infty \left(x^3\right)^n \mathbf{d}_n\left(x^\alpha\right) \qquad 15.16$$

$$\overline{\mathbf{p}}\left(x^\alpha, x^3, t\right) = \overline{\mathbf{r}}(x^\alpha, t) + \sum_{n=1}^\infty \left(x^3\right)^n \overline{\mathbf{d}}_n\left(x^\alpha, t\right) \qquad 15.17$$

where the terms involving $\overline{\mathbf{d}}_n$ and $\mathbf{d}_n$ represent vectors attached to the reference surface and are referred to as directors. In terms of the displacement vectors

$$\mathbf{u}\left(x^\alpha, x^3, t\right) = \mathbf{U}(x^\alpha, t) + \sum_{n=1}^\infty \left(x^3\right)^n \mathbf{w}_n(x^\alpha, t) \qquad 15.18$$

where

$$\mathbf{w}_n = \overline{\mathbf{d}}_n - \mathbf{d}_n \qquad 15.19$$

The series term in (15.18) allows the displacements in the shell space to be approximated to any desired degree. Other quantities such as temperature can be similarly expressed. The base vectors in the shell space are given by

$$\mathbf{g}_i = \mathbf{p}_{,i} = \mathbf{r}_{,i} + \sum_{n=1}^\infty \left(x^3\right)^n_{,i} \mathbf{d}_n + \sum_{n=1}^\infty \left(x^3\right)^n \mathbf{d}_{n,i} \qquad 15.20$$

If we assume a shell of constant thickness h and assume the reference surface to be located midway between the two bounding surfaces and disregard terms in the equation above for n>2 and assume the director in the reference configuration is normal to the surface or

$$\left(x^3\right)^n_{,i} = 0 \quad , \quad i \neq 3 \qquad 15.21$$

$$-\frac{h}{2} \leq x^3 \leq +\frac{h}{2} \qquad 15.22$$

$$\mathbf{d}_1 = \mathbf{a}_3 \qquad 15.23$$

$$\overline{\mathbf{d}}_1 = \mathbf{d} \qquad 15.24$$

$$\overline{\mathbf{d}}_n = \mathbf{d}_n = 0 \quad n \geq 2 \qquad 15.25$$

$$\mathbf{w}_1 = \mathbf{w} \qquad 15.26$$

equations (15.16), (15.17), and (15.18) can be rewritten simply as

$$\mathbf{p}\left(x^\alpha, x^3\right) = \mathbf{r}(x^\alpha) + x^3 \mathbf{a}_3 \ (x^\alpha) \qquad\qquad 15.27$$

$$\overline{\mathbf{p}}\left(x^\alpha, x^3, t\right) = \overline{\mathbf{r}}(x^\alpha, t) + x^3 \mathbf{d}(x^\alpha, t) \qquad\qquad 15.28$$

$$\mathbf{u}\left(x^\alpha, x^3, t\right) = \mathbf{U}(x^\alpha, t) + x^3 \mathbf{w}(x^\alpha, t) \qquad\qquad 15.29$$

where

$$\overline{\mathbf{p}} - \mathbf{p} = \mathbf{u} = \mathbf{U} + x^3 \mathbf{w} \qquad\qquad 15.30$$

and

$$\mathbf{w} = \mathbf{d} - \mathbf{a}^3 \qquad\qquad 15.31$$

The base vectors for the normal space in the reference configuration are

$$\mathbf{g}_\alpha = \mathbf{p}_{,\alpha} = \mathbf{r}_{,\alpha} + x^3 \mathbf{a}_{3,\alpha} = \mathbf{a}_\alpha + x^3 \mathbf{a}_{3,\alpha} \qquad\qquad 15.32$$

$$\mathbf{g}_3 = \mathbf{a}_3 \qquad\qquad 15.33$$

while those in the current configuration are

$$\overline{\mathbf{g}}_\alpha = \overline{\mathbf{P}}_{,\alpha} = \overline{\mathbf{r}}_{,\alpha} + x^3 \mathbf{d}_{,\alpha} = \overline{\mathbf{a}}_\alpha + x^3 \mathbf{d}_{,\alpha} \qquad\qquad 15.34$$

$$\overline{\mathbf{g}}_3 = \mathbf{d} \qquad\qquad 15.35$$

Let $\mathbf{Z}$ represent the shifter tensor (see Wagner [70]) which relates quantities in the shell space to those on the reference surface, be given by

$$\mathbf{Z} = z_j^i \mathbf{a}_i \mathbf{a}^j \ = \mathbf{a}_\alpha \mathbf{a}^\alpha + \mathbf{a}_3 \mathbf{a}^3 - x^3 \ \mathbf{B} \qquad\qquad 15.36$$

The components of the shifter tensor can be written as

$$z_\beta^\alpha = \delta_\beta^\alpha - x^3 b_\beta^\alpha \qquad\qquad 15.37$$

$$z_3^3 = 1 \qquad\qquad 15.38$$

The determinant of the shifter tensor components can be expressed in terms of the determinants of the two metric tensor components or in terms of the mean and total curvature as

$$|z_j^i| = z = \frac{\sqrt{g}}{\sqrt{a}} = 1 - 2x^3 H + \left(x^3\right)^2 K \qquad 15.39$$

where

$$H = \frac{1}{2}a^{\alpha\beta}b_{\alpha\beta} \quad , \quad K = \frac{b}{a} \qquad 15.40$$

The components of the inverse of the shifter tensor are [68]

$$\left(z_\beta^\alpha\right)^{-1} = \frac{1}{z}\left[\delta_\beta^\alpha + x^3\left(b_\beta^\alpha - 2H\delta_\beta^\alpha\right)\right] \qquad 15.41$$

$$\left(z_3^3\right)^{-1} = 1 \qquad 15.42$$

The relationship between the covariant base vectors can be written as

$$\mathbf{g}_i = \mathbf{Z} \cdot \mathbf{a}_i = \mathbf{a}_i \cdot \mathbf{Z}^T \qquad 15.43$$

while the relationship between the contravariant base is

$$\mathbf{g}^i = \mathbf{Z}^{-1} \cdot \mathbf{a}^i = \mathbf{a}^i \cdot \left(\mathbf{Z}^{-1}\right)^T \qquad 15.44$$

The relationship between the covariant and contravariant components of the metric tensor of the shell space and those on the reference surface, in terms of the components of the shifter tensor is

$$g_{ij} = z_i^k z_j^l a_{kl} \qquad 15.45$$

and

$$g^{ij} = (z_k^i)^{-1}(z_l^j)^{-1} a^{kl} \qquad 15.46$$

The displacement vectors expressed in terms of the base vectors in the reference configuration are

$$\mathbf{U} = U^\alpha \mathbf{a}_\alpha + U^3 \mathbf{a}_3 \qquad 15.47$$

$$\mathbf{w} = w^\alpha \mathbf{a}_\alpha + w^3 \mathbf{a}_3 \qquad 15.48$$

From

$$\overline{p} = p{+}u = p{+}U + x^3 w \qquad 15.49$$

we can obtain

$$\overline{P}_{,i} = p_{,i}{+}u_{,i} = p_{,i}{+}U_{,i} + \delta_i^3 w + x^3 w_{,i} \qquad 15.50$$

or

$$\overline{g}_i = g_i{+}u_{,i} = g_i{+}U_{,i} + \delta_i^3 w + x^3 w_{,i} \qquad 15.51$$

Taking the dot product of the above equation with itself we obtain

$$\overline{g}_{ij} - g_{ij} = g_i{\cdot}u_{,j} + g_j \cdot u_{,i} + u_{,i} \cdot u_{,j} \qquad 15.52$$

Recall (see equation A.C.57) that

$$u_{,i} = u_{\|i}^k g_k = u_{k\|i} g^k \qquad 15.53$$

where the double vertical bar denotes covariant differentiation with respect to the metric of the normal space. The components of the Green-Lagrange strain tensor for the normal space are

$$E_{ij} = \frac{1}{2}\left(\overline{g}_{ij} - g_{ij}\right) \qquad 15.54$$

Making use of the above three equations we have

$$E_{ij} = \frac{1}{2}\left(u_{\|j}^k g_{ik} + u_{\|i}^k g_{kj} + u_{\|i}^k u_{\|j}^l g_{kl}\right) \qquad 15.55$$

which reduces to

$$E_{ij} = \frac{1}{2}\left(u_{i\|j} + u_{j\|i} + u_{\|i}^k u_{k\|j}\right) \qquad 15.56$$

The strains can be separated into strains in the plane of the reference surface and those not in the plane or

$$E_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha\|\beta} + u_{\beta\|\alpha} + u_{\|\alpha}^\gamma u_{\gamma\|\beta} + u_{\|\alpha}^3 u_{3\|\beta}\right) \qquad 15.57$$

$$E_{\alpha 3} = \frac{1}{2}\left(u_{\alpha\|3} + u_{3\|\alpha} + u_{\|\alpha}^{\gamma} u_{\gamma\|3} + u_{\|\alpha}^{3} u_{3\|3}\right)$$ 15.58

$$E_{33} = \frac{1}{2}\left(2u_{3\|3} + u_{\|3}^{\gamma} u_{\gamma\|3} + u_{\|3}^{3} u_{3\|3}\right)$$ 15.59

where equation 15.57 represents the inplane strains, equation 15.58 represents the transverse shear strain, equation 15.59 represents the transverse normal strain. We can derive the relationship between the covariant derivatives in the normal space and those on the surface by making use of

$$\mathbf{u}_{,i} = u_{\|i}^{k}\mathbf{g}_{k} = u_{,i}^{\alpha}\mathbf{a}_{\alpha} + u^{\alpha}\mathbf{a}_{\alpha,i} + u_{,i}^{3}\mathbf{a}_{3} + u^{3}\mathbf{a}_{3,i}$$ 15.60

and

$$\mathbf{g}_{k} = z_{k}^{i}\mathbf{a}_{i} \quad , \quad \mathbf{g}^{m} = (z_{j}^{m})^{-1}\mathbf{a}^{j}$$ 15.61

The covariant derivatives in the normal space can be expressed as

$$u_{\|i}^{k} = (z_{j}^{k})^{-1}(u_{,i}^{\alpha}\delta_{\alpha}^{j} + u^{\alpha}\mathbf{a}^{j}\cdot\mathbf{a}_{\alpha,i} + u_{,i}^{3}\delta_{3}^{j} + u^{3}\mathbf{a}^{j}\cdot\mathbf{a}_{3,i})$$ 15.62

or

$$u^{k}\|_{i} = (z_{\alpha}^{k})^{-1}(u_{,i}^{\alpha} + u^{\beta}\mathbf{a}^{\alpha}\cdot\mathbf{a}_{\beta,i} + u^{3}\mathbf{a}^{\alpha}\cdot\mathbf{a}_{3,i}) +$$ 15.63
$$(z_{3}^{k})^{-1}(u_{,i}^{3} + u^{\beta}\mathbf{a}^{3}\cdot\mathbf{a}_{\beta,i})$$

By making use of the Weingarten equations

$$\mathbf{a}^{\alpha}\cdot\mathbf{a}_{3,\beta} = -b_{\beta}^{\alpha}$$ 15.64

$$\mathbf{a}^{3}\cdot\mathbf{a}_{\alpha,\beta} = b_{\alpha\beta}$$ 15.65

and the following relationships

$$\mathbf{a}^{3}\cdot\mathbf{a}_{3,\beta} = 0$$ 15.66

$$\mathbf{a}_{3,3} = 0$$ 15.67

$$\mathbf{a}_{\alpha,3} = 0$$ 15.68

$$z_{\alpha}^{3} = z_{3}^{\alpha} = 0$$ 15.69

the covariant derivatives can be written as

$$u^\gamma\|_\beta = (z_\alpha^\gamma)^{-1}(u_{|\beta}^\alpha - u^3 b_\beta^\alpha)$$ 15.70

$$u^\gamma\|_3 = (z_\alpha^\gamma)^{-1}u_{,3}^\alpha$$ 15.71

$$u^3\|_\beta = u_{,\beta}^3 + u^\alpha b_{\alpha\beta}$$ 15.72

$$u^3\|_3 = u_{,3}^3$$ 15.73

where the single vertical bar denotes covariant differentiation with respect to the metric of the reference surface. Recalling that

$$g_{ik} = z_i^j z_k^l a_{jl}$$ 15.74

$$u_{i\|j} = u_{\|j}^k g_{ik} = u_{\|j}^k z_i^j z_k^l a_{jl}$$ 15.75

$$z_k^i(z_j^k)^{-1} = \delta_j^i$$ 15.76

we also can find the relationship for the covariant derivatives of the contravariant components or

$$u_{\gamma\|\beta} = z_\gamma^\alpha(u_{\alpha|\beta} - u^3 b_{\alpha\beta})$$ 15.77

$$u_{\gamma\|3} = z_\gamma^\alpha u_{\alpha,3}$$ 15.78

$$u_{3\|\beta} = u_{\|\beta}^3$$ 15.79

$$u_{3\|3} = u^3_{\|3} \qquad\qquad 15.80$$

When we substitute into the equations for the strain components (equations 15.57, 15.58, 15.59) we have for the normal space of the shell

$$E_{\alpha\beta} = \frac{1}{2}(z^\lambda_\alpha(u_{\lambda|\beta} - u^3 b_{\lambda\beta}) + z^\lambda_\beta(u_{\lambda|\alpha} - u^3 b_{\lambda\alpha}) + \qquad\qquad 15.81$$
$$(u^\rho_{|\alpha} - u^3 b^\rho_\alpha)(u_{\rho|\beta} - u^3 b_{\rho\beta}) +$$
$$(u^3_{,\alpha} + u^\lambda b_{\lambda\alpha})(u^3_{,\beta} + u^\rho b_{\rho\beta})$$

$$E_{\alpha3} = \frac{1}{2}(z^\lambda_\alpha u_{\lambda,3} + u^3_{,\alpha} + u^\lambda b_{\lambda\alpha} + (u^\rho_{|\alpha} - u^3 b^\rho_\alpha)u_{\rho,3} + (u^3_{,\alpha} + u^\lambda b_{\lambda\alpha})u^3_{,3}) \qquad\qquad 15.82$$

$$E_{33} = \frac{1}{2}(2u^3_{,3} + u^\alpha_{,3} u_{\alpha,3} + (u^3_{,3})^2) \qquad\qquad 15.83$$

Recalling that

$$\mathbf{u} = \mathbf{U} + x^3 \mathbf{w} \qquad\qquad 15.84$$

$$\mathbf{u}_{,\alpha} = \mathbf{U}_{,\alpha} + x^3 \mathbf{w}_{,\alpha} \qquad\qquad 15.85$$

$$\mathbf{u}_{,3} = \mathbf{w} \qquad\qquad 15.86$$

we can find that

$$u^\gamma\|_\beta = (z^\gamma_\alpha)^{-1}((U^\alpha_{|\beta} - U^3 b^\alpha_\beta) + x^3(w^\alpha_{|\beta} - w^3 b^\alpha_\beta)) \qquad\qquad 15.87$$

$$u^\gamma\|_3 = (z^\gamma_\alpha)^{-1} w^\alpha \qquad\qquad 15.88$$

$$u^3\|_\beta = (U^3_{,\beta} + U^\alpha b_{\alpha\beta}) + x^3(w^3_{,\beta} + w^\alpha b_{\alpha\beta}) \qquad\qquad 15.89$$

$$u^3\|_3 = w^3 \qquad\qquad 15.90$$

$$u_{\gamma\|\beta} = z_\gamma^\alpha((U_{\alpha|\beta} - U^3 b_{\alpha\beta}) + x^3(w_{\alpha|\beta} - w^3 b_{\alpha\beta})) \qquad\qquad 15.91$$

$$u_{\gamma\|3} = z_\gamma^\alpha w_\alpha \qquad\qquad 15.92$$

When the above results are substituted in equations 15.81, 15.82, and 15.83 we find

$$E_{\alpha\beta} = \frac{1}{2}(z_\alpha^\lambda(U_{\lambda|\beta} - U^3 b_{\lambda\beta} + x^3(w_{\lambda|\beta} - w^3 b_{\lambda\beta})) + \qquad\qquad 15.93$$
$$z_\beta^\lambda(U_{\lambda|\alpha} - U^3 b_{\lambda\alpha} + x^3(w_{\lambda|\alpha} - w^3 b_{\lambda\alpha})) +$$
$$(U_{|\alpha}^\rho - U^3 b_\alpha^\rho + x^3(w_{|\alpha}^\rho - w^3 b_\alpha^\rho))(U_{\rho|\beta} - U^3 b_{\rho\beta} + x^3(w_{\rho|\beta} - w^3 b_{\rho\beta})) +$$
$$(U_{,\alpha}^3 + U^\lambda b_{\lambda\alpha} + x^3(w_{,\alpha}^3 + w^\lambda b_{\lambda\alpha}))(U_{,\beta}^3 + U^\rho b_{\rho\beta} + x^3(w_{,\beta}^3 + w^\rho b_{\rho\beta}))$$

$$E_{\alpha 3} = \frac{1}{2}(z_\alpha^\lambda w_\lambda + U^\lambda b_{\lambda\alpha} + x^3(w_{,\alpha}^3 + w^\lambda b_{\lambda\alpha}) + \qquad\qquad 15.94$$
$$((U_{|\alpha}^\rho - U^3 b_\alpha^\rho) + x^3(w_{|\alpha}^\rho - w^3 b_\alpha^\rho))w_\rho +$$
$$((U_{,\alpha}^3 + U^\lambda b_{\lambda\alpha}) + x^3(w_{,\alpha}^3 + w^\lambda b_{\lambda\alpha}))w^3)$$

$$E_{33} = \frac{1}{2}(w^3(2 + w^3) + w^\alpha w_\alpha) \qquad\qquad 15.95$$

In order to simplify the results when the strain tensor components are presented the following (or variations of) substitutions are often made:

$$\mathbf{U}_{,\alpha} = \phi_{\cdot\alpha}^\lambda \mathbf{a}_\lambda + \phi_{3\alpha} \mathbf{a}_3 = \phi_{\lambda\alpha} \mathbf{a}^\lambda + \phi_{3\alpha} \mathbf{a}_3 \qquad\qquad 15.96$$

$$\mathbf{w}_{,\alpha} = \psi_{\cdot\alpha}^\lambda \mathbf{a}_\lambda + \psi_{3\alpha} \mathbf{a}_3 = \psi_{\lambda\alpha} \mathbf{a}^\lambda + \psi_{3\alpha} \mathbf{a}_3 \qquad\qquad 15.97$$

where

$$\phi_{\alpha\beta} = U_{\alpha|\beta} - U_3 b_{\alpha\beta} \quad , \quad \psi_{\alpha\beta} = w_{\alpha|\beta} - w_3 b_{\alpha\beta} \qquad 15.98$$

$$\phi^\alpha_{.\beta} = U^\alpha_{|\beta} - U_3 b^\alpha_\beta \quad , \quad \psi^\alpha_{.\beta} = w^\alpha_{|\beta} - w_3 b^\alpha_\beta \qquad 15.99$$

$$\phi_{3\beta} = U_{3,\beta} + U_\lambda b^\lambda_\beta \quad , \quad \psi_{3\beta} = w_{3,\beta} + w_\lambda b^\lambda_\beta \qquad 15.100$$

When we substitute the above equations into the equations for the covariant derivatives in the normal space of the shell we find

$$u^\gamma\|_\beta = (z^\gamma_\alpha)^{-1}(\phi^\alpha_{.\beta} + x^3\psi^\alpha_{.\beta}) \qquad 15.101$$

$$u^\gamma\|_3 = (z^\gamma_\alpha)^{-1}w^\alpha \qquad 15.102$$

$$u^3\|_\beta = \phi_{3\beta} + x^3\psi_{3\beta} \qquad 15.103$$

$$u_{\gamma\|\beta} = z^\alpha_\gamma(\phi_{\alpha\beta} + x^3\psi_{\alpha\beta}) \qquad 15.104$$

$$u_{\gamma\|3} = z^\alpha_\gamma w_\alpha \qquad 15.105$$

$$u_{3\|\beta} = u^3_{\|\beta} \qquad 15.106$$

$$u_{3\|3} = u^3_{\|3} \qquad 15.107$$

$$u^3\|_3 = w^3 \qquad 15.108$$

When the equations above are substituted into 15.57, 15.58, and 15.59 we have

$$E_{\alpha\beta} = \frac{1}{2}(z^\lambda_\alpha(\phi_{\lambda\beta} + x^3\psi_{\lambda\beta}) + z^\lambda_\beta(\phi_{\lambda\alpha} + x^3\psi_{\lambda\alpha}) + \qquad 15.109$$

$$(\phi^\rho_{.\alpha} + x^3\psi^\rho_{.\alpha})(\phi_{\rho\beta} + x^3\psi_{\rho\beta}) + (\phi_{3\alpha} + x^3\psi_{3\alpha})(\phi_{3\beta} + x^3\psi_{3\beta})$$

$$E_{\alpha 3} = \frac{1}{2}(z_\alpha^\lambda(U_{\lambda,3} + x^3 w_{\lambda,3} + w_\lambda) + \phi_{3\alpha} + x^3 \psi_{3\alpha} +$$

$$(\phi_{.\alpha}^\rho + x^3 \psi_{.\alpha}^\rho)w_\rho + (\phi_{3\alpha} + x^3 \psi_{3\alpha})w^3)$$

15.110

$$E_{33} = \frac{1}{2}(w^3(2 + w^3) + w^\alpha w_\alpha)$$

15.111

or after substitution for the shifter tensor

$$E_{\alpha\beta} = \frac{1}{2}((\phi_{\alpha\beta} + \phi_{\beta\alpha} + \phi_{.\alpha}^\rho \phi_{\rho\beta} + \phi_{3\alpha}\phi_{3\beta})$$

15.112

$$+x^3(\psi_{\alpha\beta} + \psi_{\beta\alpha} + \phi_{.\alpha}^\rho \psi_{\rho\beta} + \psi_{.\alpha}^\rho \phi_{\rho\beta} - b_\alpha^\lambda \phi_{\lambda\beta} - b_\beta^\lambda \phi_{\lambda\alpha} + \psi_{3\beta}\phi_{3\alpha} + \psi_{3\alpha}\phi_{3\beta})$$

$$(x^3)^2(\psi_{3\alpha}\psi_{3\beta} + \psi_{.\alpha}^\rho \psi_{\rho\beta} - b_\alpha^\lambda \psi_{\lambda\beta} - b_\beta^\lambda \psi_{\lambda\alpha}))$$

$$E_{\alpha 3} = \frac{1}{2}(w_\alpha + \phi_{3\alpha} + \phi_{3\alpha}w^3 + \phi_{.\alpha}^\rho w_\rho +$$

15.113

$$x^3(-b_\alpha^\lambda w_\lambda + \psi_{3\alpha} + \psi_{.\alpha}^\rho w_\rho + \psi_{3\alpha}w^3))$$

$$E_{33} = \frac{1}{2}(w^3(2 + w^3) + w^\alpha w_\alpha)$$

15.114

which are the full three-dimensional, nonlinear strain displacement relationships.

In order to proceed further, assumptions regarding the behavior of the shell are usually made. In equation 15.112 the last term, which is quadratic in x^3 is usually disregarded. Equation 15.112 is assumed to be of the form

$$E_{\alpha\beta} = \alpha_{\alpha\beta} + x^3 \kappa_{\alpha\beta}$$

15.115

where

$$\alpha_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta})$$

15.116

are membrane strains and

$$\kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta})$$

15.117

101

are the bending strains. The base vectors in the current configuration can be expressed in terms of the base vectors of the reference configuration by

$$\bar{\mathbf{a}}_\alpha = \mathbf{a}_\rho(\delta_\alpha^\rho + \phi_{.\alpha}^\rho) + \phi_{3\alpha}\mathbf{a}_3 \qquad 15.118$$

The unit normal to the deformed surface is given by

$$\bar{\mathbf{a}}^3 = \frac{1}{2}\bar{\varepsilon}^{\alpha\beta}(\bar{\mathbf{a}}_\alpha \times \bar{\mathbf{a}}_\beta) \qquad 15.119$$

or

$$\bar{\mathbf{a}}^3 = \sqrt{\frac{a}{\bar{a}}}\left[(-\phi_{3\gamma} + \delta_{\lambda\gamma}^{\alpha\beta}\phi_{.\beta}^\lambda\phi_{3\alpha})\mathbf{a}^\gamma + (1 + \phi_{.\alpha}^\alpha + \frac{1}{2}\delta_{\rho\gamma}^{\alpha\beta}\phi_{.\alpha}^\rho\phi_{.\beta}^\gamma)\mathbf{a}^3\right] \qquad 15.120$$

From which

$$\sqrt{\frac{\bar{a}}{a}} = (1 + 2\alpha_\alpha^\alpha + 2(\alpha_\alpha^\alpha\alpha_\beta^\beta - \alpha_\beta^\alpha\alpha_\alpha^\beta))^{\frac{1}{2}} \qquad 15.121$$

where

$$\alpha_\alpha^\beta = \frac{1}{2}(\phi_{\alpha\rho} + \phi_{\rho\alpha} + \phi_{\lambda\alpha}\phi_{.\rho}^\lambda + \phi_{3\alpha}\phi_{3\rho})a^{\rho\beta} \qquad 15.122$$

and

$$\alpha_\alpha^\alpha = \phi_\alpha^\alpha + \frac{1}{2}(\phi_{\lambda\alpha}\phi_{.\rho}^\lambda + \phi_{3\alpha}\phi_{3\rho})a^{\rho\alpha} \qquad 15.123$$

The components of the curvature tensor in the deformed configuration are given by

$$\bar{b}_{\alpha\beta} = -\bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_{,\beta}^3 \qquad 15.124$$

By using equation 15.118 the membrane strains are given by

$$\alpha_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha\beta} + \phi_{\beta\alpha} + \phi_{\lambda\alpha}\phi_\beta^\lambda + \phi_{3\alpha}\phi_{3\beta}) \qquad 15.125$$

In [64] it is assumed that the shear deformation is constant throughout the shell or

$$\mathbf{d} \cdot \mathbf{d} = \text{constant} \qquad 15.126$$

from which

$$(\mathbf{d} \cdot \mathbf{d})_{,\alpha} = 0 \qquad 15.127$$

102

or

$$\mathbf{d} \cdot \mathbf{d}_{,\alpha} = 0 \tag{15.128}$$

Recalling that

$$\mathbf{d} = \mathbf{w} + \mathbf{a}^3 = w_\alpha \mathbf{a}^\alpha + (w_3+1)\mathbf{a}^3 = w^\alpha \mathbf{a}_\alpha + (w_3+1)\mathbf{a}^3 \tag{15.129}$$

$$\mathbf{d}_{,\alpha} = \mathbf{w}_{,\alpha} + \mathbf{a}^3_{,\alpha} = \psi^\lambda_{.\alpha} \mathbf{a}_\lambda + \psi_{3\alpha} \mathbf{a}_3 + \mathbf{a}^3_{,\alpha} \tag{15.130}$$

$$\mathbf{d} \cdot \mathbf{d}_{,\alpha} = \psi_{3\alpha} + w_\beta \psi^\beta_{.\alpha} + w_3 \psi_{3\alpha} - w_\beta b^\beta_\alpha = 0 \tag{15.131}$$

which shows that the second term in equation 15.113 vanishes. When changes in the thickness of the shell are neglected or

$$\mathbf{d} \cdot \mathbf{d} = 1 \tag{15.132}$$

We find

$$w^3(2 + w^3) + w^\alpha w_\alpha = 0 \tag{15.133}$$

or the transverse normal strain (equation 15.114) is identically zero. In addition this assumption allows for calculation of the normal component of $\mathbf{w}$ in terms of the tangential components (Başar and Krätzig [65], Başar [64]) or

$$w_3 = -1 \pm \sqrt{(1 - w_\alpha w^\alpha)} \tag{15.134}$$

and the introduction of a rotation vector Ω which relates $\mathbf{w}$ to the rotation of the normal on the undeformed reference surface or

$$\Omega = \mathbf{a}_3 \times \mathbf{w} \tag{15.135}$$

The magnitude of Ω in terms of the angle ω between $\mathbf{a}_3$ and $\mathbf{d}$ is simply $|\Omega| = |\sin\omega|$.

The Kirchhoff-Love assumptions consist of:

- Normals to the reference surface do not change length after deformation or

$$\mathbf{d} \cdot \mathbf{d} = 1 \qquad\qquad 15.136$$

- Normals to the reference surface remain normal after deformation or

$$\bar{\mathbf{a}}^3 = \mathbf{d} = \mathbf{a}^3 + \mathbf{w} \qquad\qquad 15.137$$

If we assume that

$$\bar{\mathbf{a}}^3 = \mathbf{d} = \mathbf{a}^3 + \mathbf{w} \qquad\qquad 15.138$$

and recall that

$$\bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}^3_{,\beta} = -\bar{b}_{\alpha\beta} \qquad\qquad 15.139$$

we find

$$-(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) = \psi_{\beta\alpha} - \phi_{\rho\beta} b^\rho_\alpha + \phi^\rho_\beta \psi_{\rho\alpha} + \psi_{3\alpha}\phi_{3\beta} \qquad\qquad 15.140$$

and due to the symmetry of the curvature tensors

$$-(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) = \frac{1}{2}(\psi_{\beta\alpha} - \phi_{\rho\beta} b^\rho_\alpha + \phi^\rho_\beta \psi_{\rho\alpha} + \psi_{3\alpha}\phi_{3\beta} + \qquad\qquad 15.141$$
$$\psi_{\alpha\beta} - \phi_{\rho\alpha} b^\rho_\beta + \phi^\rho_\alpha \psi_{\rho\beta} + \psi_{3\beta}\phi_{3\alpha})$$

If we assume small but finite strains such that products of the strains can be disregarded we find

$$\sqrt{\frac{\bar{a}}{a}} \approx (1 + 2\alpha^\alpha_\alpha)^{\frac{1}{2}} = 1 + \alpha^\alpha_\alpha \qquad\qquad 15.142$$

$$\bar{\mathbf{a}}^3 = \sqrt{\frac{a}{\bar{a}}}\,(n_\gamma \mathbf{a}^\gamma + n_3 \mathbf{a}^3) \qquad\qquad 15.143$$

If we retain the Kirchhoff-Love assumptions we find

$$E_{\alpha\beta} = \alpha_{\alpha\beta} + x^3 \kappa_{\alpha\beta} \qquad \text{15.144}$$

$$e_{\alpha 3} = 0 \qquad \text{15.145}$$

$$e_{33} = 0 \qquad \text{15.146}$$

where

$$\alpha_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha\beta} + \phi_{\beta\alpha} + \phi_{\lambda\alpha}\phi_{\beta}^{\lambda} + \phi_{3\alpha}\phi_{3\beta}) \qquad \text{15.147}$$

$$\kappa_{\alpha\beta} = \frac{1}{2}(\psi_{\beta\alpha} + \psi_{\alpha\beta} - \phi_{\rho\beta}b_{\alpha}^{\rho} - \phi_{\rho\alpha}b_{\beta}^{\rho} + \qquad \text{15.148}$$

$$\phi_{\beta}^{\rho}\psi_{\rho\alpha} + \psi_{3\alpha}\phi_{3\beta} + \phi_{\alpha}^{\rho}\psi_{\rho\beta} + \psi_{3\beta}\phi_{3\alpha})$$

When we reintroduce the previous substitutions in equations 15.147 and 15.148 we find the strain measures in terms of the displacements or

$$\alpha_{\alpha\beta} = \frac{1}{2}(U_{\alpha|\beta} + U_{\beta|\alpha} - 2U^3 b_{\alpha\beta} \qquad \text{15.149}$$

$$(U_{|\alpha}^{\rho} - U^3 b_{\alpha}^{\rho})(U_{\rho|\beta} - U^3 b_{\rho\beta}) + (U_{,\alpha}^3 + U^{\lambda}b_{\lambda\alpha})(U_{,\beta}^3 + U^{\rho}b_{\rho\beta}))$$

$$\kappa_{\alpha\beta} = \frac{1}{2}(w_{\alpha|\beta} + w_{\beta|\alpha} - 2w^3 b_{\alpha\beta} - \qquad \text{15.150}$$

$$(U_{\rho|\beta} - U^3 b_{\rho\beta})b_{\alpha}^{\rho} - (U_{\rho|\alpha} - U^3 b_{\rho\alpha})b_{\beta}^{\rho} +$$

$$(U_{|\beta}^{\rho} - U^3 b_{\beta}^{\rho})(w_{\rho|\alpha} - w^3 b_{\rho\alpha}) + (U_{|\alpha}^{\rho} - U^3 b_{\alpha}^{\rho})(w_{\rho|\beta} - w^3 b_{\rho\beta}) +$$

$$(U_{,\alpha}^3 + U^{\lambda}b_{\lambda\alpha})(w_{,\beta}^3 + w^{\rho}b_{\rho\beta}) + (U_{,\beta}^3 + U^{\lambda}b_{\lambda\beta})(w_{,\alpha}^3 + w^{\rho}b_{\rho\alpha}))$$

When equations 15.147 and 15.148 are linearized by dropping terms involving products of the displacement gradients we find

$$\alpha_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha\beta} + \phi_{\beta\alpha}) \qquad \text{15.151}$$

$$\kappa_{\alpha\beta} = \frac{1}{2}(\psi_{\beta\alpha} + \psi_{\alpha\beta} - \phi_{\rho\beta}b_\alpha^\rho - \phi_{\rho\alpha}b_\beta^\rho) \qquad \text{15.152}$$

$$\sqrt{\frac{\overline{a}}{a}} = 1 + \phi_\alpha^\alpha \qquad \text{15.153}$$

$$\sqrt{\frac{a}{\overline{a}}} = 1 - \phi_\alpha^\alpha \qquad \text{15.154}$$

and

$$\overline{\mathbf{a}}^3 \approx -\phi_{3\gamma}\mathbf{a}^\gamma + \mathbf{a}^3 = w_\gamma\mathbf{a}^\gamma + (1 + w_3)\mathbf{a}^3 \qquad \text{15.155}$$

from which

$$w_\gamma = -\phi_{3\gamma} \qquad \text{15.156}$$

$$w_3 = 0 \qquad \text{15.157}$$

The two linearized strain tensors (equations 15.151 and 15.152) are symmetric and when the displacements are substituted in, can be written as

$$\alpha_{\alpha\beta} = \frac{1}{2}(U_{\alpha|\beta} + U_{\beta|\alpha} - 2U^3 b_{\alpha\beta}) \qquad \text{15.158}$$

$$\kappa_{\alpha\beta} = \frac{1}{2}(w_{\alpha|\beta} + w_{\beta|\alpha} - (U_{\rho|\beta} - U^3 b_{\rho\beta})b_\alpha^\rho - (U_{\rho|\alpha} - U^3 b_{\rho\alpha})b_\beta^\rho) \qquad \text{15.159}$$

$$\kappa_{\alpha\beta} = -\frac{1}{2}(\phi_{3\alpha|\beta} + \phi_{3\beta|\alpha} + (U_{\rho|\beta} - U^3 b_{\rho\beta})b_\alpha^\rho + (U_{\rho|\alpha} - U^3 b_{\rho\alpha})b_\beta^\rho) \qquad \text{15.160}$$

where

$$\phi_{3\alpha|\beta} = U_{3|\alpha\beta} + U_\lambda b_{\alpha|\beta}^\lambda + b_\alpha^\lambda U_{\lambda|\beta} \qquad \text{15.161}$$

$$U_{3|\alpha\beta} = U_{3,\alpha\beta} - \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\} U_{3,\lambda} \qquad \text{15.162}$$

or after substituting the above and making use of the compatibility equations

$$\kappa_{\alpha\beta} = -(U_{3|\alpha\beta} + U_\lambda b^\lambda_{\alpha|\beta} + b^\lambda_\alpha U_{\lambda|\beta} + U_{\rho|\beta} b^\rho_\alpha - U^3 b_{\rho\beta} b^\rho_\alpha)$$

15.163

For convected coordinates the deformation gradient is given by

$$\mathbf{F} = \overline{\mathbf{g}}_i \mathbf{g}^i$$

15.164

The Green-Lagrangian strain tensor in the shell space is denoted by $\mathbf{E}_{\mathfrak{B}}$, and is defined by

$$\mathbf{E}_{\mathfrak{B}} = \frac{1}{2}(\overline{g}_{ij} - g_{ij})\mathbf{g}^i\mathbf{g}^j$$

15.165

The Green-Lagrangian strain tensor on the reference surface is denoted by $\mathbf{E}$, and is defined by

$$\mathbf{E} = \left(\mathbf{Z}^{-1}\right)^T \cdot \mathbf{E}_{\mathfrak{B}} \cdot \mathbf{Z}^{-1}$$

15.166

Likewise if $\tilde{\mathbf{T}}_{\mathfrak{B}}$ and $\tilde{\mathbf{T}} = \tilde{T}^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta$ represent the second Piola-Kirchhoff stress tensor in the shell space and on the reference surface, respectively, then

$$\tilde{\mathbf{T}} = \mathbf{Z}^T \cdot \tilde{\mathbf{T}}_{\mathfrak{B}} \cdot \mathbf{Z}$$

15.167

The work conjugate stress resultant and stress couple tensors are [70] given by

$$\mathbf{N} = \int_{-h/2}^{h/2} \tilde{T}^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta \det\mathbf{Z}\, dx^3$$

15.168

$$\mathbf{M} = \int_{-h/2}^{h/2} x^3 \tilde{T}^{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta \det\mathbf{Z}\, dx^3$$

15.169

$$\mathbf{Q} = \int_{-h/2}^{h/2} \tilde{T}^{\alpha3}\mathbf{a}_\alpha\mathbf{a}_3 \det\mathbf{Z}\, dx^3$$

15.170

where the relationship between the first and second Piola-Kirchhoff and Cauchy stress tensors is given by

$$\widetilde{\mathbf{T}}_{\mathfrak{B}} = J\,\mathbf{F}^{-1}\cdot\mathbf{T}_{\mathfrak{B}}\cdot\left(\mathbf{F}^{-1}\right)^T = \mathbf{T}^0_{\mathfrak{B}}\left(\mathbf{F}^{-1}\right)^T$$

15.171

$$\mathbf{T}_{\mathfrak{B}} = J^{-1}\,\mathbf{F}\cdot\tilde{\mathbf{T}}_{\mathfrak{B}}\cdot\mathbf{F}^{T} = J^{-1}\,\mathbf{F}\cdot\mathbf{T}_{\mathfrak{B}}^{\mathrm{o}} \tag{15.172}$$

$$\mathbf{T}_{\mathfrak{B}}^{0} = J\,\mathbf{F}^{-1}\cdot\mathbf{T}_{\mathfrak{B}} = \tilde{\mathbf{T}}_{\mathfrak{B}}\cdot\mathbf{F}^{T} \tag{15.173}$$

where

$$J = \frac{d\bar{v}}{dv} = \frac{(\bar{\mathbf{g}}_{\alpha}\times\bar{\mathbf{g}}_{\beta})\cdot\bar{\mathbf{g}}_{3}}{(\mathbf{g}_{\alpha}\times\mathbf{g}_{\beta})\cdot\mathbf{g}_{3}} \tag{15.174}$$

The tensor $\mathbf{N}$ is the stress resultant tensor, $\mathbf{M}$ the moment tensor and $\mathbf{Q}$ the shear stress vector and are work conjugate to the strain tensors with components $\alpha_{\alpha\beta}$, $\kappa_{\alpha\beta}$, and $E_{\alpha3}$. The general equations presented become reasonably complex when expressed in component form. In [55] Axelrad and Emmerling discuss the advantage of using an intrinsic form of the equations. (i.e., using the strains as unknowns rather than the classical approach of using the displacements as unknowns.)

The equations of motion and constitutive relationships can be derived by appropriate substitutions into the following:

- Conservation of linear momentum

$$\vec{\nabla}_{X}\cdot\tilde{\mathbf{T}}_{\mathfrak{B}}\cdot\mathbf{F}^{T} + \rho_{o}\mathbf{b}_{o} = \rho_{o}\frac{\partial^{2}\mathbf{u}}{\partial t^{2}} \tag{15.175}$$

- Stress – strain – temperature relationship

$$\tilde{\mathbf{T}}_{\mathfrak{B}} = 2\mu\mathbf{E}_{\mathfrak{B}} + \lambda(\mathrm{tr}\mathbf{E}_{\mathfrak{B}})\mathbf{1} + m(\theta - \theta_{o})\mathbf{1} \tag{15.176}$$

- Conservation of energy

$$-\vec{\nabla}_{X}\cdot\mathbf{q}_{o} + m\theta_{o}\left(\mathrm{tr}\dot{\mathbf{E}}_{\mathfrak{B}}\right) + \rho_{o}r = \rho_{o}c_{E}\dot{\theta} \tag{15.177}$$

- Heat conduction relationship

$$q_o = -k\overrightarrow{\nabla}_X \theta \qquad\qquad 15.178$$

For example the stress – strain – temperature relationship in direct notation for the shell space is

$$\tilde{\mathbf{T}}_{\mathfrak{B}} = 2\mu \mathbf{E}_{\mathfrak{B}} + \lambda(\mathrm{tr}\mathbf{E}_{\mathfrak{B}})\mathbf{1} + m(\theta - \theta_o)\mathbf{1} \qquad\qquad 15.179$$

while in component notation (after dropping the subscript $\mathfrak{B}$ for clarity)

$$\tilde{T}_j{}^i = \lambda E_n^n \delta_j^i + 2\mu E_j^i + m(\theta - \theta_o)\delta_j^i \qquad\qquad 15.180$$

or

$$\tilde{T}^{ik} = (\lambda E_n^n \delta_j^i + 2\mu E_j^i + m(\theta - \theta_o)\delta_j^i)g^{jk} \qquad\qquad 15.181$$

$$\tilde{T}^{ik} = (\lambda E_{nk}g^{kn}\delta_j^i + 2\mu E_{jn}g^{in} + m(\theta - \theta_o)\delta_j^i)g^{jk} \qquad\qquad 15.182$$

Use of the relationship

$$g^{ij} = (z_k^i)^{-1}(z_l^j)^{-1}a^{kl} \qquad\qquad 15.183$$

in equation 15.182 shifts the stress tensor from the normal space to the reference surface.

16 COMPATIBILITY EQUATIONS

The compatibility equations connect the two measures of strain previously described. They can be derived from the Gauss equation

$$\overline{R}_{\delta\gamma\beta\alpha} = \overline{b}_{\alpha\gamma}\overline{b}_{\beta\delta} - \overline{b}_{\beta\gamma}\overline{b}_{\alpha\delta} \qquad\qquad 16.1$$

where the left side of the above equation when expressed in terms of the Christoffel symbols is

$$\overline{R}_{\delta\gamma\beta\alpha} = \overline{[\gamma\alpha,\delta]}_{,\beta} - \overline{[\gamma\beta,\delta]}_{,\alpha} + \overline{\left\{\begin{matrix}\lambda\\\gamma\beta\end{matrix}\right\}}\overline{[\delta\alpha,\lambda]} - \overline{\left\{\begin{matrix}\lambda\\\gamma\alpha\end{matrix}\right\}}\overline{[\delta\beta,\lambda]} \qquad 16.2$$

and the Codazzi equations

$$\overline{b}_{\alpha\beta|\gamma} - \overline{b}_{\alpha\gamma|\beta} = 0 \qquad\qquad 16.3$$

where the Riemann-Christoffel tensor and covariant differentiation are associated with the Christoffel symbols in the deformed configuration. The Gauss equation produces one distinct equation

$$\overline{R}_{\beta\alpha\beta\alpha} = \overline{[\alpha\alpha,\beta]}_{,\beta} - \overline{[\alpha\beta,\beta]}_{,\alpha} + \overline{\left\{\begin{matrix}\lambda\\\alpha\beta\end{matrix}\right\}}\overline{[\alpha\alpha,\lambda]} - \overline{\left\{\begin{matrix}\lambda\\\alpha\alpha\end{matrix}\right\}}\overline{[\beta\beta,\lambda]} \qquad 16.4$$

$$\overline{R}_{\beta\alpha\beta\alpha} = \overline{b}_{\alpha\alpha}\overline{b}_{\beta\beta} - \overline{b}_{\beta\alpha}\overline{b}_{\alpha\beta} \qquad\qquad 16.5$$

The Codazzi equations produce two distinct equations

$$\overline{b}_{\alpha\alpha|\beta} - \overline{b}_{\alpha\beta|\alpha} = 0 \quad \text{no sum} \quad \alpha \neq \beta \qquad\qquad 16.6$$

Recall that the relationship between the strain and the metrics of the deformed and undeformed reference surface is given by

$$\alpha_{\alpha\beta} = \frac{1}{2}(\overline{a}_{\alpha\beta} - a_{\alpha\beta}) \qquad\qquad 16.7$$

and

$$\kappa_{\alpha\beta} = \bar{b}_{\alpha\beta} - b_{\alpha\beta}$$ 16.8

from which the covariant components of the metric tensor of the deformed reference surface in terms of the strain measure and metric tensor of the undeformed reference surface are

$$\bar{a}_{\alpha\beta} = 2\alpha_{\alpha\beta} + a_{\alpha\beta}$$ 16.9

$$\bar{b}_{\alpha\beta} = \kappa_{\alpha\beta} + b_{\alpha\beta}$$ 16.10

The contravariant components [67] are given by

$$\bar{a}^{\rho\psi} = \bar{\varepsilon}^{\rho\alpha}\bar{\varepsilon}^{\psi\beta}\bar{a}_{\alpha\beta} = \frac{a}{\bar{a}}(2\alpha_{\alpha\beta}\varepsilon^{\rho\alpha}\varepsilon^{\psi\beta} + \alpha^{\rho\psi})$$ 16.11

where use of the following has been made

$$\bar{\varepsilon}^{\alpha\beta} = \frac{e^{\alpha\beta}}{\sqrt{\bar{a}}} \quad , \quad \varepsilon^{\alpha\beta} = \frac{e^{\alpha\beta}}{\sqrt{a}}$$ 16.12

The relationship between the Christoffel symbols for the deformed and undeformed reference surface can be found by considering the difference between Christoffel symbols, which is referred to by Niordson [8] as the Christoffel deviator. First we express the Christoffel symbols for the deformed reference surface in terms of its' associated metric tensor.

$$\overline{\left\{{\gamma \atop \alpha\beta}\right\}} = \frac{1}{2}\bar{a}^{\gamma\lambda}(\bar{a}_{\lambda\alpha,\beta} + \bar{a}_{\lambda\beta,\alpha} - \bar{a}_{\alpha\beta,\lambda})$$ 16.13

Next, subtract the Christoffel symbols of the undeformed reference surface and multiply by the covariant components of the metric tensor of deformed reference surface to eliminate the contravariant components from the right side of the equation

$$\bar{a}_{\gamma\delta}\left(\overline{\left\{{\gamma \atop \alpha\beta}\right\}} - \left\{{\gamma \atop \alpha\beta}\right\}\right) = \frac{1}{2}(\bar{a}_{\delta\alpha,\beta} + \bar{a}_{\delta\beta,\alpha} - \bar{a}_{\alpha\beta,\delta}) - \bar{a}_{\gamma\delta}\left\{{\gamma \atop \alpha\beta}\right\}$$ 16.14

Next, substitute 16.9 into the above equation and recall the formula for covariant differentiation of the covariant components of a second order tensor

$$\alpha_{\alpha\beta|\gamma} = \alpha_{\alpha\beta,\gamma} - \left\{ \begin{array}{c} \lambda \\ \alpha\gamma \end{array} \right\} \alpha_{\lambda\beta} - \left\{ \begin{array}{c} \lambda \\ \beta\gamma \end{array} \right\} \alpha_{\alpha\lambda} \qquad 16.15$$

and

$$\overline{a}_{\gamma\delta}\left(\overline{\left\{ \begin{array}{c} \gamma \\ \alpha\beta \end{array} \right\}} - \left\{ \begin{array}{c} \gamma \\ \alpha\beta \end{array} \right\} \right) = \frac{1}{2}(a_{\delta\alpha|\beta} + a_{\delta\beta|\alpha} - a_{\alpha\beta|\delta}) + (\alpha_{\delta\alpha|\beta} + \alpha_{\delta\beta|\alpha} - \alpha_{\alpha\beta|\delta}) \qquad 16.16$$

Recalling that the covariant derivative of the metric tensor is zero the above equation can be written simply as

$$\overline{\left\{ \begin{array}{c} \gamma \\ \alpha\beta \end{array} \right\}} = \left\{ \begin{array}{c} \gamma \\ \alpha\beta \end{array} \right\} + \overline{a}^{\gamma\delta}(\alpha_{\delta\alpha|\beta} + \alpha_{\delta\beta|\alpha} - \alpha_{\alpha\beta|\delta}) \qquad 16.17$$

$$\overline{[\alpha\beta,\lambda]} = [\alpha\beta,\lambda] + (\alpha_{\lambda\alpha|\beta} + \alpha_{\lambda\beta|\alpha} - \alpha_{\alpha\beta|\lambda}) \qquad 16.18$$

$$\overline{R}_{\beta\alpha\beta\alpha} = \overline{[\alpha\alpha,\beta]}_{,\beta} - \overline{[\alpha\beta,\beta]}_{,\alpha} + \overline{\left\{ \begin{array}{c} \lambda \\ \alpha\beta \end{array} \right\}} \overline{[\alpha\alpha,\lambda]} - \overline{\left\{ \begin{array}{c} \lambda \\ \alpha\alpha \end{array} \right\}} \overline{[\beta\beta,\lambda]} \qquad 16.19$$

$$\overline{R}_{\beta\alpha\beta\alpha} = \overline{b}_{\alpha\alpha}\overline{b}_{\beta\beta} - \overline{b}_{\beta\alpha}\overline{b}_{\alpha\beta} \qquad 16.20$$

The Codazzi equations produce two distinct equations

$$\overline{b}_{\alpha\alpha|\beta} - \overline{b}_{\alpha\beta|\alpha} = 0 \quad \text{no sum} \quad \alpha \neq \beta \qquad 16.21$$

The Gauss and Codazzi equations can be written [67] in more condensed form as

$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}\left(\overline{[\alpha,\beta\mu]}_{,\lambda} + \overline{\left\{ \begin{array}{c} \kappa \\ \alpha\mu \end{array} \right\}} \overline{[\kappa,\beta\lambda]} + \overline{b}_{\alpha\mu}\overline{b}_{\beta\lambda} \right) \qquad 16.22$$

$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}\overline{b}_{\beta\lambda|\mu} = 0 \qquad 16.23$$

respectively. After making the appropriate substitutions we find

$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}(\kappa_{\beta\lambda|\mu} + \overline{a}^{\kappa\rho}(b_{\kappa\lambda} - \kappa_{\kappa\lambda})(\alpha_{\rho\beta|\mu} + \alpha_{\rho\mu|\beta} - \alpha_{\alpha\mu|\rho})) = 0 \qquad 16.24$$

112

$$K\alpha_\lambda^\lambda + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}(\alpha_{\alpha\mu|\beta\lambda} - b_{\alpha\mu}\kappa_{\beta\lambda} + \frac{1}{2}\kappa_{\alpha\mu}\kappa_{\beta\lambda} + \qquad\qquad 16.25$$

$$\overline{a}^{\kappa\rho}(\alpha_{\kappa\alpha|\mu} + \alpha_{\kappa\mu|\alpha} - \alpha_{\alpha\mu|\kappa})(\alpha_{\rho\beta|\mu} + \alpha_{\rho\mu|\beta} - \alpha_{\alpha\mu|\rho})) = 0 \qquad 16.26$$

where

$$\overline{a}^{\kappa\rho} = \frac{a}{\overline{a}}(2\alpha_{\pi\omega}\varepsilon^{\kappa\pi}\varepsilon^{\rho\omega} + \alpha^{\kappa\rho}) \qquad\qquad 16.27$$

The linear equations are obtained by dropping terms involving products of the strains and their derivatives. When this is done we find

$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}(\kappa_{\beta\lambda|\mu} + \frac{a}{\overline{a}}a^{\kappa\rho}b_{\kappa\lambda}(\alpha_{\rho\beta|\mu} + \alpha_{\rho\mu|\beta} - \alpha_{\alpha\mu|\rho})) = 0 \qquad 16.28$$

$$K\alpha_\lambda^\lambda + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}(\alpha_{\alpha\mu|\beta\lambda} - b_{\alpha\mu}\kappa_{\beta\lambda}) = 0 \qquad\qquad 16.29$$

17 SPHERICAL SHELLS

Next to shells of cylindrical shape, shells of spherical shape are probably the most often analyzed in journal articles. The reasons are possibly as follows:

1. Shells of spherical shape are in very common usage as parts of structures, manufacturing equipment, and many recreational items. They are constructed in a spherical shape in order to exploit particular features of the geometry (e.g., the property of a spherical shape to maximize the volume contained within the bounding surface. The ability to react to uniform internal or external pressures loads without bending, under some circumstances. In other cases the symmetry associated with a closed spherical shell has some value. The sports of basketball, soccer, volleyball, tennis, etc. would be radically different without a spherical shell.

2. Shells of nearly spherical shape frequently occur in nature.

3. The mathematics associated with spherical coordinates is well known.

4. Components which are not truly spherical are often treated as such for purposes of simplifying the analysis. For example in [71] Hodges et al. treat a parabolic mirror constructed of two layers of CFRP (carbon fiber reinforced plastic) with a aluminum honeycomb sandwich as a spherical shell. In the field of biomechanics, Takamizawa and Matsudaa ([72]) use a thick-walled spherical model to analyze the left ventricle of the heart.

5. Since the spherical shell represents the simplest shell of nonzero Gaussian curvature, it is often used to investigate more complicated displacements, material behaviors, loads, boundary conditions etc.

Spherical shells find applications in the civil, mechanical, nuclear, aerospace, and ordnance engineering fields among others. Although the military, nuclear, and aerospace applications are usually considered the most extreme from a thermal loading standpoint, this obviously depends on the properties from which a component is constructed and its

intended use. From an engineering standpoint, thermal stresses need to be considered simply because they are always present to some degree and they are either undesirable or desirable. The nuclear or aerospace engineer's task would be greatly simplified if he had structural materials with a coefficient of thermal expansion equal to zero, whereas an engineer working on instruments to detect thermal displacements would have his task greatly complicated with the same materials. Bimetallic shallow spherical shells are sometimes used as an elastic element for thermal sensitivity in precision instruments [73].

The need for consideration of thermal effects is evident by considering some of the temperature extremes objects can be subjected. Objects in orbit around the Earth are subjected to temperature ranges of from 45°F to –325°F depending on whether they are in sunlight or shadow [71]. For radiation problems the Sun can be considered as a blackbody radiator with an effective temperature of approximately 10,000°F [74]. The components used in pressure vessels in the nuclear energy industry can be subjected to equilibrium temperatures approaching 1600°F [75].

In addition the use of high powered lasers can produce large heat fluxes. The phenomena of thermal ratcheting and creep and thermal shock and stability are important design considerations. The extreme design parameters may require use of ceramics, plastics and reinforced composites which maybe more sensitive to thermal effects than metals.

18 SPHERICAL SHELL EQUATIONS

The transformation between an orthonormal Cartesian coordinate system and a spherical coordinate system is given

$$z^1 = r \sin \phi \cos \theta \qquad 18.1$$

$$z^2 = r \sin \phi \sin \theta \qquad 18.2$$

$$z^3 = r \cos \phi \qquad 18.3$$

where r is the length of a vector from the origin to an arbitrary point, ϕ is a measure of the azimuthal angle from the z^3 axis and θ is a measure of the longitudinal angle from the z^1 axis. The transformation holds for

$$0 < \phi < \pi \qquad 18.4$$

$$0 \le \theta \le 2\pi \qquad 18.5$$

$$0 < r < \infty \qquad 18.6$$

We can utilize the results from the previous sections by letting

$$\phi = x^1 \quad , \quad \theta = x^2 \qquad 18.7$$

and let the distance in the direction perpendicular to the tangent plane and away from the center of curvature be represented by x^3. The matrix of covariant and contravariant metric tensor components are

$$[g_{ij}] = \begin{bmatrix} r^2 & 0 & 0 \\ 0 & (r \sin \phi)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 18.8$$

and

$$[g^{ij}] = \begin{bmatrix} r^{-2} & 0 & 0 \\ 0 & (r \sin \phi)^{-2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 18.9$$

respectively. The scalar invariant representing the square of distance between neighboring points is

$$(ds)^2 = (r d\phi)^2 + \left(r \sin \phi d\theta^2\right)^2 + (dr)^2 \qquad \text{18.10}$$

The nonzero Christoffel symbols of the second kind are

$$\begin{aligned}
&\left\{ \begin{smallmatrix} r \\ \phi\phi \end{smallmatrix} \right\} = -r \quad \left\{ \begin{smallmatrix} r \\ \theta\theta \end{smallmatrix} \right\} = -r(\sin \phi)^2 \quad \left\{ \begin{smallmatrix} \phi \\ \theta\theta \end{smallmatrix} \right\} = -\sin \phi \cos \phi \\
&\left\{ \begin{smallmatrix} \phi \\ r\phi \end{smallmatrix} \right\} = r^{-1} \quad \left\{ \begin{smallmatrix} \theta \\ r\theta \end{smallmatrix} \right\} = r^{-1} \qquad\qquad \left\{ \begin{smallmatrix} \theta \\ \phi\theta \end{smallmatrix} \right\} = \cot \phi
\end{aligned} \qquad \text{18.11}$$

When the above relationships are used with the three-dimensional equations we can find the component form of the equations describing the thermoelastic behavior of a solid in spherical coordinates.

If we let the length of the radius vector remain constant, we describe the surface of a sphere. We are interested in describing various quantities on the surface of the sphere and outside and inside the surface (i.e., the normal space of the shell) for both the reference and current configuration. The metric tensor components for the surface are

$$\left[a_{\alpha\beta} \right] = \begin{bmatrix} r^2 & 0 \\ 0 & (r \sin \phi)^2 \end{bmatrix} \qquad \text{18.12}$$

$$\left[a^{\alpha\beta} \right] = \begin{bmatrix} r^{-2} & 0 \\ 0 & (r \sin \phi)^{-2} \end{bmatrix} \qquad \text{18.13}$$

$$\left[b^{\alpha}_{\beta} \right] = - \begin{bmatrix} r^{-1} & 0 \\ 0 & r^{-1} \end{bmatrix} \qquad \text{18.14}$$

$$\left[b_{\alpha\beta} \right] = - \begin{bmatrix} r & 0 \\ 0 & r(\sin \phi)^2 \end{bmatrix} \qquad \text{18.15}$$

and the nonzero Christoffel symbols of the second kind for the surface are

$$\left\{ \begin{smallmatrix} \phi \\ \theta\theta \end{smallmatrix} \right\} = -\sin \phi \cos \phi \quad , \quad \left\{ \begin{smallmatrix} \theta \\ \phi\theta \end{smallmatrix} \right\} = \cot \phi \qquad \text{18.16}$$

When the above relationships are used with the two-dimension equations we can find the equations describing the thermoelastic behavior of a spherical shell.

As an example of the process involved we develop the linearized strain measures in terms of the physical components of a spherical shell. The covariant derivatives on the surface are given by

$$U_{\phi|\phi} = U_{\phi,\phi} \qquad\qquad 18.17$$

$$U_{\phi|\theta} = U_{\phi,\theta} - U_\theta \cot \phi$$

$$U_{\theta|\theta} = U_{\theta,\theta} + U_\phi \sin \phi \cos \phi$$

$$U_{\theta|\phi} = U_{\theta,\phi} - U_\theta \cot \phi$$

and

$$U^\phi_{|\phi} = U^\phi_{,\phi} \qquad\qquad 18.18$$

$$U^\phi_{|\theta} = U^\phi_{,\theta} - U^\theta \sin \phi \cos \phi$$

$$U^\theta_{|\theta} = U^\theta_{,\theta} + U^\phi \cot \phi$$

$$U^\theta_{|\phi} = U^\theta_{,\phi} + U^\phi \cot \phi$$

Recall that the functions involved with the linearized strain measures were given by

$$\phi_{\alpha\beta} = U_{\alpha|\beta} - U_3 b_{\alpha\beta} \qquad\qquad 18.19$$

$$\phi^\alpha_{\cdot\beta} = U^\alpha_{|\beta} - U_3 b^\alpha_\beta \qquad\qquad 18.20$$

$$\phi_{3\beta} = U_{3,\beta} + U_\alpha b^\alpha_\beta \qquad\qquad 18.21$$

and the covariant derivative of $\phi_{3\alpha}$ by

$$\phi_{3\alpha|\beta} = U_{3|\alpha\beta} + U_\lambda b^\lambda_{\alpha|\beta} + b^\lambda_\alpha U_{\lambda|\beta} \qquad\qquad 18.22$$

From equation 18.14 we find

$$b^\lambda_{\alpha|\beta} = 0 \qquad\qquad 18.23$$

Recall that

$$U_{3|\alpha\beta} = U_{3,\alpha\beta} - \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\} U_{3,\lambda} \qquad\qquad 18.24$$

or

$$\phi_{3\alpha|\beta} = U_{3,\alpha\beta} - \left\{ \begin{matrix} \lambda \\ \alpha\beta \end{matrix} \right\} U_{3,\lambda} + b_\alpha^\lambda U_{\lambda|\beta} \qquad\qquad 18.25$$

The functions associated with the strain measures are

$$\phi_{\phi\phi} = U_{\phi,\phi} + rU_3 \qquad\qquad 18.26$$

$$\phi_{\phi\theta} = U_{\phi,\theta} - U_\theta \cot\phi$$

$$\phi_{\theta\theta} = U_{\theta,\theta} + U_\phi \sin\phi \cos\phi + U_3 r \sin^2\phi$$

$$\phi_{\theta\phi} = U_{\theta,\phi} - U_\theta \cot\phi$$

$$\phi_{.\phi}^{\phi} = U_{,\phi}^{\phi} + \frac{1}{r}U_3 \qquad\qquad 18.27$$

$$\phi_{.\theta}^{\phi} = U_{,\theta}^{\phi} - U^\theta \sin\phi \cos\phi$$

$$\phi_{.\theta}^{\theta} = U_{,\theta}^{\theta} + U^\phi \cot\phi + \frac{1}{r}U_3$$

$$\phi_{.\phi}^{\theta} = U_{,\phi}^{\theta} + U^\phi \cot\phi$$

$$\phi_{3\phi} = U_{3,\phi} + \frac{1}{r}U_\phi \qquad\qquad 18.28$$

$$\phi_{3\theta} = U_{3,\theta} + \frac{1}{r}U_\theta$$

$$\phi_{3\phi|\phi} = U_{3,\phi\phi} - \frac{1}{r}U_{\phi,\phi} \qquad\qquad 18.29$$

$$\phi_{3\phi|\theta} = U_{3,\phi\theta} - U_{3,\phi}\cot\phi - \frac{1}{r}U_{\phi,\theta} + \frac{1}{r}U_\theta \cot\phi$$

$$\phi_{3\theta|\theta} = U_{3,\theta\theta} + U_{3,\phi}\sin\phi\cos\phi - \frac{1}{r}U_{\theta,\theta} - \frac{1}{r}U_\phi \sin\phi\cos\phi$$

$$\phi_{3\theta|\phi} = U_{3,\theta\phi} - U_{3,\theta}\cot\phi - \frac{1}{r}U_{\theta,\phi} + \frac{1}{r}U_\theta \cot\phi$$

Recall that the linearized membrane strains were given by

$$\alpha_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha\beta} + \phi_{\beta\alpha})$$ 18.30

or in terms of the surface coordinates

$$\alpha_{\phi\phi} = U_{\phi,\phi} + rU_3$$ 18.31

$$\alpha_{\theta\theta} = U_{\theta,\theta} + U_\phi \sin\phi \cos\phi + U_3 r \sin^2\phi$$ 18.32

$$\alpha_{\phi\theta} = \frac{1}{2}(U_{\phi,\theta} + U_{\theta,\phi} - 2U_\theta \cot\phi)$$ 18.33

The curvature strains were given by

$$\kappa_{\alpha\beta} = \frac{1}{2}(\psi_{\beta\alpha} + \psi_{\alpha\beta} - \phi_{\rho\beta}b_\alpha^\rho - \phi_{\rho\alpha}b_\beta^\rho)$$ 18.34

$$\kappa_{\alpha\beta} = -\frac{1}{2}(\phi_{3\alpha|\beta} + \phi_{3\beta|\alpha} + \phi_{\rho\beta}b_\alpha^\rho + \phi_{\rho\alpha}b_\beta^\rho)$$ 18.35

$$\kappa_{\alpha\beta} = -\frac{1}{2}(\phi_{3\alpha|\beta} + \phi_{3\beta|\alpha} - \frac{2}{r}\alpha_{\alpha\beta})$$ 18.36

or in terms of the surface coordinates

$$\kappa_{\phi\phi} = -(U_{3,\phi\phi} - \frac{2}{r}U_{\phi,\phi} + U_3)$$ 18.37

$$\kappa_{\phi\theta} = -\frac{1}{2}(U_{3,\phi\theta} - U_{3,\phi}\cot\phi - \frac{2}{r}U_{\phi,\theta} + \frac{4}{r}U_\theta\cot\phi + U_{3,\theta\phi} - U_{3,\theta}\cot\phi - \frac{2}{r}U_{\theta,\phi})$$ 18.38

$$\kappa_{\theta\theta} = -(U_{3,\theta\theta} + U_{3,\phi}\sin\phi\cos\phi - \frac{2}{r}U_{\theta,\theta} - \frac{2}{r}U_\phi\sin\phi\cos\phi - \frac{1}{r}U_3\sin^2\phi)$$ 18.39

The physical components of the displacement vector are given by

$$U_\phi = U_{\langle\phi\rangle} r \qquad\qquad 18.40$$

$$U_\theta = U_{\langle\theta\rangle} r \sin\phi \qquad\qquad 18.41$$

$$U^\phi = U_{\langle\phi\rangle}\frac{1}{r} \qquad\qquad 18.42$$

$$U^\theta = U_{\langle\theta\rangle}\frac{1}{r \sin\phi} \qquad\qquad 18.43$$

The relationship between the partial derivatives is given by

$$U_{\phi,\phi} = rU_{\langle\phi\rangle,\phi} \qquad\qquad 18.44$$

$$U_{\phi,\theta} = rU_{\langle\phi\rangle,\theta} \qquad\qquad 18.45$$

$$U_{\theta,\theta} = U_{\langle\theta\rangle,\theta} r \sin\phi \qquad\qquad 18.46$$

$$U_{\theta,\phi} = U_{\langle\theta\rangle,\phi} r \sin\phi + U_{\langle\theta\rangle} r \cos\phi \qquad\qquad 18.47$$

The physical components of a second order tensor are given by

$$e_{\langle\phi\phi\rangle} = r^{-2} e_{\phi\phi} \qquad\qquad 18.48$$

$$e_{\langle\theta\theta\rangle} = (r \sin\phi)^{-2} e_{\theta\theta} \qquad\qquad 18.49$$

$$e_{\langle\theta\phi\rangle} = r^{-2}(\sin\phi)^{-1} e_{\theta\phi} \qquad\qquad 18.50$$

The physical components of the strain-displacement relationships have the familiar form

$$\alpha_{\langle\phi\phi\rangle} = \frac{1}{r}(U_{\langle\phi\rangle,\phi} + U_3) \qquad\qquad 18.51$$

$$\alpha_{\langle\theta\theta\rangle} = \frac{1}{r}((\sin\phi)^{-1}U_{\langle\theta\rangle,\theta} + U_{\langle\phi\rangle}\cot\phi + U_3) \qquad\qquad 18.52$$

$$\alpha_{\langle\phi\theta\rangle} = \frac{1}{2r}((\sin\phi)^{-1}U_{\langle\phi\rangle,\theta} + U_{\langle\theta\rangle,\phi} - U_{\langle\theta\rangle}\cot\phi) \qquad\qquad 18.53$$

$$\kappa_{\langle\phi\phi\rangle} = -r^{-2}(U_{3,\phi\phi} - 2U_{\langle\phi\rangle,\phi} + U_3) \qquad\qquad 18.54$$

$$\kappa_{\langle\phi\theta\rangle} = -\frac{1}{2r^2\sin\phi}(U_{3,\phi\theta} - U_{3,\phi}\cot\phi - 2U_{\langle\phi\rangle,\theta} + 2U_{\langle\theta\rangle}\cos\phi + \qquad\qquad 18.55$$
$$U_{3,\theta\phi} - U_{3,\theta}\cot\phi - 2U_{\langle\theta\rangle,\phi}\sin\phi)$$

$$\kappa_{\langle\theta\theta\rangle} = -\frac{1}{r^2 \sin^2\phi}(U_{3,\theta\theta} + U_{3,\phi}\sin\phi\cos\phi - 2U_{\langle\theta\rangle,\theta}\sin\phi -$$

$$2U_{\langle\phi\rangle}\sin\phi\cos\phi - \frac{1}{r}U_3\sin^2\phi)$$

18.56

19 REVIEW OF THE LITERATURE

In preparation for this work a reasonably comprehensive review of the literature associated with the general subject of thermal stresses in shells was done [76]. Additional information was collected regarding the general subjects of thermoelasticity and shell theory.

Some of the names which appeared frequently in connection with the general subject of thermoelasticity are: V. K. Agarwal, M. Anwar, Z. Bem, B. A. Boley, D. E. Carlson, P. Chadwick, B. D. Coleman, W. A. Day, R. Dhaliwal, H. Ghoneim, A. E. Green, R. B. Hetnarski, D. Iesan, J. Ignaczak, K. E. Lindsay, H. W. Lord, I. Müller, P. M. Naghdi, W. Nowacki, J. L. Nowinski, Y. H. Pao, H. Parkus, Y. Schulman, L. T. C. Seet, H. Sherief, Y. Takeuti, J. H. Weiner.

Some of the names which appeared frequently in connection with the general subject of shells are: S. A. Ambartsumian, E. L. Axelrad, Y. Başar, B. Budiansky, W. Z. Chien, F. A. Emmerling, W. Flügge, A. E. Green, A.L. Gol'denveizer, W. T. Koiter, W. B. Krätzig, H. Kraus, S. A. Lukasiewicz, A. I. Lur'e, K. Marguerre, P. M. Naghdi, F. I. Niordson, R. P. Nordgen, V. Novozhilov, W. Pietraszkiewicz, E. Reissner, J. L. Sanders Jr., J. L. Synge, G. Wempner, W. Zerna.

There are a large variety of problems covered under the general category of thermal stresses in spherical shells. When spherical shells are used as containment vessels for fluids, an opening in the shell structure is required. These openings (discontinuities) can result in stress concentrations. The analysis of spherical shells with discontinuities is given in [77–80].

When a shell is described as thick, the effect of transverse shear and/or changes in the thickness of the shell have to be considered. Doxsee [81] points out that when laminated composites are heated they expand more in the direction normal to the plane of the laminate than in the plane of the laminate. These materials are often used in shell

structures and use of a thin shell theory to predict their behavior would prove inadequate. Thermal stresses in thick-walled spherical shells are considered in [43,82–95] while spherical shells made specifically of composite materials are addressed in [71,75,96–99]. The behavior of spherical shells composed of thermoplastic materials is addressed in [86,87,92,93,97,100–105]. The fully coupled equations of thermoelasticity allow for the possibility of thermally induced vibrations which are treated in [106,107]. If the loads are time dependant then the problem is transient in nature. Transient problems are considered in [85,88,105,108–111].

The subject of creep in spherical shells is considered in [82,112].

If the thermal loading occurs abruptly the phenomena of thermal shock must be considered. Problems involving thermal shock are given in [100,113–116].

Thermal ratcheting, the phenomenon of net strain accumulation due to plastic straining under cyclic thermal loading, an important factor in the design of nuclear reactor pressure vessels, is considered in [117]. The subject of stability of spherical shells is considered in [23,73,101,102,112,118–126].

If the thermal-mechanical loads are axisymmetric and/or the shell can be considered shallow the equations assume a much simpler form. Either or both of these assumptions is often made when studying nonlinear material behavior in spherical shells. Spherical shells subjected to axisymmetric loads are addressed in [85,114,118,123,127–133]. Problems involving shallow spherical shells are given in [23,73,75,77–79,99,101,102,112] and [121,124,127,132,134–140].

20 CONCLUSIONS

The intent of this work was to: (1) Discuss the basic elements of three-dimensional continuum mechanics and develop the basic equations which describe the behavior of a linear thermoelastic solid. (2) Describe the process of reduction of the three-dimensional formulation to the two-dimensional equations of a general shell theory. (3) Show how the two-dimensional general shell equations reduce to those describing the behavior of a spherical shell. The equations were, in most cases, presented in both direct and component form. The use of direct notation provides a succinct way of describing the kinematic, kinetic, and constitutive relationships and the associated boundary and initial conditions independent of any particular coordinate system.

However, from an engineering viewpoint, we eventually need a solution to the equations. The need for a solution requires that the equations be expressed in component form. In addition, we want numbers which represent the magnitudes of the variables of interest. For example, given some thermo-mechanical loadings and a set of material properties, we might want to know the magnitudes of the stresses and displacements in the body of interest. These numbers have to have units associated with them, which relate to our three-dimensional locally Cartesian world. In other words we need the solution variables expressed in terms of physical components.

The shell equations are by their vary nature approximations to the three-dimensional equations. The assumptions made during equation formulation determine the range of applicability to various problems. In many cases the assumptions are not unambiguous unless the equations are written out in component form. In component form the terms which are disregarded or some how altered can be described explicitly. The descent from expressing the equations in direct notation to expressing them in component form in terms of general curvilinear coordinates is, in my opinion, the most logical. The equations maintain their invariant quality and are still reasonably compact. The introduction of

the appropriate representation of the metric tensors allows the general equations to be applied to a specific geometry. The descent to Cartesian tensors is easily accomplished. In addition, the majority of the current literature on shell theory uses a combination of direct and general tensor component notation, in a variety of different forms. There is not a standard mathematical representation for the numerous variables and the various mathematical operations involved in problem formulation. There is even variation from paper to paper for the same author.

The goal in this work was to be consistent in terminology and notation. While this, when expressed in words, seems a rather simple and straightforward task, in practice is reasonably difficult. The intent here, is not to make excuses for the inadequacies of this work, but alert those who may attempt to follow a similar path to the obstacles they will have to deal with. One of the major limitations is the inability of various publishing "type" software to easily produce the variety of symbols required. For example, the software used to produce this paper was incapable of producing bold lowercase Greek symbols and various accented symbols.

The effort to deal with the subjects covered in this paper was at the expense of not discussing a number of equally important, related topics. These topics when covered in some reasonable depth could constitute papers by themselves. Some examples are the subjects of curvature measures, constitutive relationships, boundary conditions, variational methods, uniqueness theorems, finite element formulations, static-geoemetric analogies, error estimation, non-dimensional formulations, and solution methods for various classes of shell problems.

Appendix A

A. Metric Tensor

The metric tensor components of a general curvilinear system are related to various combinations of the inner products of the base vectors and their reciprocals. The covariant components are given by

$$\mathbf{g}_i \cdot \mathbf{g}_j = g_{ij} \qquad \text{A.A.1}$$

The contravariant components are given by

$$\mathbf{g}^i \cdot \mathbf{g}^j = g^{ij} \qquad \text{A.A.2}$$

while the mixed components are simply the components of the Kronecker delta

$$\mathbf{g}_i \cdot \mathbf{g}^j = g_i{}^j = \delta_i^j \qquad \text{A.A.3}$$

due to the definition of a reciprocal basis. The tensor is symmetric due to the symmetry of the inner product. The metric tensor is a unit tensor for the space and can be written as

$$\mathbf{1} = \mathbf{g}_i \mathbf{g}^i = g^{ij} \mathbf{g}_i \mathbf{g}_j = g_{ij} \mathbf{g}^i \mathbf{g}^j \qquad \text{A.A.4}$$

The relationship between the base vectors and the metric tensor components and a coordinate transformation can be demonstrated as follows. Let a vector $\mathbf{r}$ when referred to an orthonormal coordinate system with coordinates y^i and base vectors $\mathbf{b}_i$ be given by

$$\mathbf{r} = y^i \mathbf{b}_i \qquad \text{A.A.5}$$

where by definition of an orthonormal coordinate system

$$y^i = y_i \qquad \text{A.A.6}$$

$$\mathbf{b}_i = \mathbf{b}^i$$

$$\mathbf{b}_i \cdot \mathbf{b}_j = \delta_{ij}$$

Then

$$\mathbf{dr} = \frac{\partial \mathbf{r}}{\partial y^i} dy^i = dy^i \mathbf{b}_i \qquad \text{A.A.7}$$

Let the relationship between the coordinates in this system with those in a general curvilinear system with coordinates x^i be given by

$$y^i = y^i(x^1, x^2, \ldots, x^n) \qquad \text{A.A.8}$$

$$x^i = x^i(y^1, y^2, \ldots, y^n) \qquad \text{A.A.9}$$

We find

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j \qquad \text{A.A.10}$$

Substituting the above equation into A.A.7 we find

$$\mathbf{dr} = \frac{\partial \mathbf{r}}{\partial y^i} dy^i = dy^i \mathbf{b}_i = \left(\frac{\partial y^i}{\partial x^j} \mathbf{b}_i \right) dx^j \qquad \text{A.A.11}$$

If the vector $\mathbf{r}$ is given as a function of the coordinates in the general curvilinear system then $d\mathbf{r}$ is given by

$$\mathbf{dr} = \frac{\partial \mathbf{r}}{\partial x^i} dx^i = dx^i \mathbf{g}_i \qquad \text{A.A.12}$$

From equations A.A.11 and A.A.12 we find

$$\mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial x^j} = \frac{\partial y^i}{\partial x^j} \mathbf{b}_i \qquad \text{A.A.13}$$

Substituting the above into A.A.1 we find

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial \mathbf{r}}{\partial x^j} = \frac{\partial y^k}{\partial x^i} \frac{\partial y^k}{\partial x^j} \qquad \text{A.A.14}$$

The contravariant components are given by

$$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j = \frac{\partial \mathbf{r}}{\partial y^i} \cdot \frac{\partial \mathbf{r}}{\partial y^j} = \frac{\partial x^k}{\partial y^i} \frac{\partial x^k}{\partial y^j} \qquad \text{A.A.15}$$

128

The definition of the metric tensor components allows us to define the relationship between the covariant, contravariant, and mixed components of other tensors. For example, let a vector **v** be defined by

$$\mathbf{v} = v^i \mathbf{g}_i = v_i \mathbf{g}^i \qquad\qquad \text{A.A.16}$$

Taking the inner product of the above with the covariant base vector and then the contravariant base vector we find

$$v_i = v^j g_{ij} \qquad\qquad \text{A.A.17}$$

$$v^i = v_j g^{ij} \qquad\qquad \text{A.A.18}$$

Let a tensor **T** be defined by

$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j = T^i_j\mathbf{g}_i\mathbf{g}^j = T_{ij}\mathbf{g}^i\mathbf{g}^j \qquad\qquad \text{A.A.19}$$

we find

$$T_{ij} = T^k_{\cdot j} g_{ik} = T^{kl} g_{ik} g_{jl} \qquad\qquad \text{A.A.20}$$

$$T^{ij} = T^{\cdot j}_k g^{ik} = T_{kl} g^{ik} g^{jl} \qquad\qquad \text{A.A.21}$$

Taking the partial derivative of A.A.13 we find

$$\mathbf{g}_{i,j} = \mathbf{g}_{j,i} \qquad\qquad \text{A.A.22}$$

B. Christoffel Symbols

The Christoffel symbols are related to the inner product of the base vectors with their partial derivatives with respect to the coordinates. The Christoffel symbol of the first kind is given by

$$[i, jk] = \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_j}{\partial x^k} = \mathbf{g}_i \cdot \mathbf{g}_{j,k} \qquad\qquad \text{A.B.23}$$

Recall that

$$g_{i,j} = g_{j,i} \qquad \text{A.B.24}$$

so that

$$[i, jk] = [i, kj] \qquad \text{A.B.25}$$

The Christoffel symbols can be expressed in terms of various combinations of partial derivatives of the metric tensor. Recall that

$$\mathbf{g}_i \cdot \mathbf{g}_j = g_{ij} \qquad \text{A.B.26}$$

If we take the partial derivative of the above equation we find

$$\mathbf{g}_j \cdot \mathbf{g}_{i,k} + \mathbf{g}_i \cdot \mathbf{g}_{j,k} = g_{ij,k} \qquad \text{A.B.27}$$

After permuting the indices in the above equation we find

$$\mathbf{g}_j \cdot \mathbf{g}_{k,i} + \mathbf{g}_k \cdot \mathbf{g}_{j,i} = g_{kj,i} \qquad \text{A.B.28}$$

$$\mathbf{g}_k \cdot \mathbf{g}_{i,j} + \mathbf{g}_i \cdot \mathbf{g}_{k,j} = g_{ik,j} \qquad \text{A.B.29}$$

Making use of A.B.24 in the above three equations they can be rewritten as

$$\mathbf{g}_j \cdot \mathbf{g}_{i,k} + \mathbf{g}_i \cdot \mathbf{g}_{j,k} = g_{ij,k} \qquad \text{A.B.30}$$

$$\mathbf{g}_j \cdot \mathbf{g}_{i,k} + \mathbf{g}_k \cdot \mathbf{g}_{j,i} = g_{kj,i} \qquad \text{A.B.31}$$

$$\mathbf{g}_k \cdot \mathbf{g}_{j,i} + \mathbf{g}_i \cdot \mathbf{g}_{j,k} = g_{ik,j} \qquad \text{A.B.32}$$

Adding A.B.30 and A.B.32 we find

$$2\mathbf{g}_i \cdot \mathbf{g}_{j,k} = g_{ij,k} + g_{ik,j} - (\mathbf{g}_j \cdot \mathbf{g}_{i,k} + \mathbf{g}_k \cdot \mathbf{g}_{i,j}) \qquad \text{A.B.33}$$

and after substitution of A.B.31 and rearranging we find

$$\mathbf{g}_i \cdot \mathbf{g}_{k,j} = \frac{1}{2}(g_{ij,k} + g_{ik,j} - g_{kj,i}) \qquad \text{A.B.34}$$

or after making use of A.B.23 and A.B.25 we find the desired result

$$[i, jk] = \frac{1}{2}(g_{ij,k} + g_{ik,j} - g_{kj,i}) \qquad \text{A.B.35}$$

The Christoffel symbols of the second kind are given by

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \mathbf{g}^i \cdot \mathbf{g}_{j,k} \qquad \text{A.B.36}$$

The relationship between the Christoffel symbols of the first and second kind are

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = g^{il}[l, jk] \qquad \text{A.B.37}$$

The remaining combinations of the inner products of the base vectors with their partial derivatives can be derived by making use of the relationship between the inner product of the base vectors with their reciprocals or

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j \qquad \text{A.B.38}$$

Taking the partial derivative of the above equation we find

$$\mathbf{g}^j \cdot \mathbf{g}_{i,k} + \mathbf{g}_i \cdot \mathbf{g}^j_{,k} = 0 \qquad \text{A.B.39}$$

or after substitution of the Christoffel symbols and rearranging

$$\mathbf{g}_i \cdot \mathbf{g}^j_{,k} = -\left\{ \begin{matrix} j \\ ik \end{matrix} \right\} \qquad \text{A.B.40}$$

If we multiply the above equation by the covariant components of the metric tensor we have

$$\mathbf{g}^r \cdot \mathbf{g}^j_{,k} = -g^{ir}\left\{ \begin{matrix} j \\ ik \end{matrix} \right\} \qquad \text{A.B.41}$$

In summary we have in terms of the base vectors

$$\mathbf{g}_i \cdot \mathbf{g}_{j,k} = [i, jk] \qquad \text{A.B.42}$$

131

$$\mathbf{g}^i \cdot \mathbf{g}_{j,k} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\} \qquad\qquad \text{A.B.43}$$

$$\mathbf{g}_i \cdot \mathbf{g}^j_{,k} = -\left\{ \begin{array}{c} j \\ ik \end{array} \right\} \qquad\qquad \text{A.B.44}$$

$$\mathbf{g}^r \cdot \mathbf{g}^j_{,k} = -g^{ir}\left\{ \begin{array}{c} j \\ ik \end{array} \right\} \qquad\qquad \text{A.B.45}$$

or in terms of the metric tensor

$$[i, jk] = \frac{1}{2}(g_{ij,k} + g_{ik,j} - g_{kj,i}) \qquad\qquad \text{A.B.46}$$

$$\left\{ \begin{array}{c} i \\ jk \end{array} \right\} = \frac{1}{2}g^{il}(g_{lj,k} + g_{lk,j} - g_{kj,l}) \qquad\qquad \text{A.B.47}$$

where

$$\left\{ \begin{array}{c} i \\ jk \end{array} \right\} = g^{il}[l, jk] \qquad\qquad \text{A.B.48}$$

$$\left\{ \begin{array}{c} r \\ ir \end{array} \right\} = (\ln \sqrt{g})_{,i} \qquad\qquad \text{A.B.49}$$

The Christoffel symbols of the first and second kind are often [67,68,1] written as Γ_{ijk} and Γ^i_{jk} respectively.

The partial derivatives of the base vectors are given by

$$\mathbf{g}_{j,k} = [i, jk]\mathbf{g}^i \qquad\qquad \text{A.B.50}$$

$$\mathbf{g}_{j,k} = \left\{ \begin{array}{c} i \\ jk \end{array} \right\}\mathbf{g}_i \qquad\qquad \text{A.B.51}$$

$$\mathbf{g}^j_{,k} = -\left\{ \begin{array}{c} j \\ ik \end{array} \right\}\mathbf{g}^i \qquad\qquad \text{A.B.52}$$

$$g^j_{,k} = -g^{ir} \left\{ \begin{matrix} j \\ ik \end{matrix} \right\} g_r \qquad\qquad \text{A.B.53}$$

C. Covariant Derivative

Let the representation of a vector in a general curvilinear coordinate system, in terms of its contravariant components be given by

$$\mathbf{u} = u^i \mathbf{g}_i \qquad\qquad \text{A.C.54}$$

If we take the partial derivative of the vector with respect to the coordinates of the general curvilinear system we find

$$\frac{\partial \mathbf{u}}{\partial x^j} = \frac{\partial u^i}{\partial x^j} \mathbf{g}_i + u^i \frac{\partial \mathbf{g}^i}{\partial x^j} \qquad\qquad \text{A.C.55}$$

or if we use a comma to denote partial differentiation then

$$\mathbf{u}_{,j} = u^i_{,j} \mathbf{g}_i + u^i \mathbf{g}_{i,j} \qquad\qquad \text{A.C.56}$$

If we assume that the partial derivative of the vector can be expressed as

$$\mathbf{u}_{,j} = u^i_{\|j} \mathbf{g}_i \qquad\qquad \text{A.C.57}$$

then we can equate the above two equations, and after taking the inner product of the result with the contravariant base vectors we find

$$u^i_{\|k} = u^i_{,k} + u^j (\mathbf{g}^i \cdot \mathbf{g}_{j,k}) \qquad\qquad \text{A.C.58}$$

or in terms of the Christoffel symbols of the second kind

$$u^i_{\|k} = u^i_{,k} + u^j \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} \qquad\qquad \text{A.C.59}$$

If the vector is expressed in terms of its covariant components then

$$u_{i\|k} = u_{i,k} - u_j \left\{ \begin{matrix} j \\ ik \end{matrix} \right\} \qquad\qquad \text{A.C.60}$$

133

The quantities $u^i_{\|j}$ and $u_{i\|j}$ are referred to as the mixed and covariant components of the covariant derivative of the vector **u**. Covariant differentiation raises the rank of the tensor by one and is also a tensor and can therefore be differentiated covariantly again. The covariant derivative of higher order tensors can be found in a similar manner. For example, if the representation of a second order tensor in the general curvilinear coordinate system is given by

$$\mathbf{T} = T^{ij}\mathbf{g}_i\mathbf{g}_j \qquad\qquad \text{A.C.61}$$

then the partial derivative of the tensor is

$$\mathbf{T},_k = T^{ij}_{,k}\mathbf{g}_i\mathbf{g}_j + T^{ij}\mathbf{g}_{i,k}\mathbf{g}_j + T^{ij}\mathbf{g}_i\mathbf{g}_{j,k} \qquad\qquad \text{A.C.62}$$

If we assume

$$\mathbf{T},_k = T^{ij}_{\|k}\mathbf{g}_i\mathbf{g}_j \qquad\qquad \text{A.C.63}$$

then after equating the above two equations and taking the inner product of the result with the contravariant base vectors (twice) and using the Christoffel symbols of the second kind we find

$$T^{ij}_{\|k} = T^{ij}_{,k} + T^{mj}\left\{ \begin{array}{c} i \\ mk \end{array} \right\} + T^{im}\left\{ \begin{array}{c} j \\ mk \end{array} \right\} \qquad\qquad \text{A.C.64}$$

Formulas for the covariant derivative of the mixed and covariant components of second order tensors are given below

$$T^i_{j\|k} = T^i_{j,k} - T^i_m\left\{ \begin{array}{c} m \\ jk \end{array} \right\} + T^m_j\left\{ \begin{array}{c} i \\ mk \end{array} \right\} \qquad\qquad \text{A.C.65}$$

$$T_{ij\|k} = T_{ij,k} - T_{im}\left\{ \begin{array}{c} m \\ jk \end{array} \right\} - T_{mj}\left\{ \begin{array}{c} m \\ ik \end{array} \right\} \qquad\qquad \text{A.C.66}$$

The second covariant derivative of a tensor can be found by following the above forms. For example, the second covariant derivative of the covariant components of the vector **u**, which is a third order tensor, is given by

$$u_{i\|jk} = (u_{i\|j}),_k - u_{i\|m}\left\{ \begin{array}{c} m \\ jk \end{array} \right\} - u_{m\|j}\left\{ \begin{array}{c} m \\ ik \end{array} \right\} \qquad\qquad \text{A.C.67}$$

134

The formulas for covariant differentiation of sums and products of tensors can be shown to be the same as for ordinary differentiation. The metric tensor and the Kronecker deltas behave as constants during covariant differentiation.

D. Intrinsic Derivative

Given a vector $\mathbf{u}$ in a general curvilinear coordinate system which is a function of the coordinates x^i, where

$$x^i = x^i(t), \qquad t_1 \le t \le t_2 \qquad\qquad \text{A.D.68}$$

and t is a scalar parameter, then

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial x^i}\frac{dx^i}{dt} = u^j_{\|i}\frac{dx^i}{dt}\mathbf{g}_j \qquad\qquad \text{A.D.69}$$

or after substituting for the covariant derivative

$$\frac{d\mathbf{u}}{dt} = u^j_{\|i}\frac{dx^i}{dt}\mathbf{g}_j = \left(\frac{\partial u^j}{\partial x^i}\frac{dx^i}{dt} + u^k\left\{\begin{matrix} j \\ ki \end{matrix}\right\}\frac{dx^i}{dt}\right)\mathbf{g}_j \qquad \text{A.D.70}$$

which reduces to

$$\frac{d\mathbf{u}}{dt} = \left(\frac{\partial u^j}{\partial t} + u^k\left\{\begin{matrix} j \\ ki \end{matrix}\right\}\frac{dx^i}{dt}\right)\mathbf{g}_j = \frac{\delta u^j}{\delta t}\mathbf{g}_j \qquad \text{A.D.71}$$

where

$$\frac{\delta u^j}{\delta t} = \frac{\partial u^j}{\partial t} + u^k\left\{\begin{matrix} j \\ ki \end{matrix}\right\}\frac{dx^i}{dt} \qquad\qquad \text{A.D.72}$$

is referred to as the intrinsic derivative of the contravariant components of the vector $\mathbf{u}$. Formulas for intrinsic differentiation follow from the results presented here and in the previous section.

The formulas for intrinsic differentiation of sums and products of tensors can be shown to be the same as for ordinary differentiation. The metric tensor and the Kronecker deltas behave as constants during intrinsic differentiation.

E. Riemann-Christoffel Tensor

The Riemann-Christoffel tensor is a tensor of rank 4, consisting of various combinations of partial derivatives of the Christoffel symbols. It sometimes is referred to as the curvature tensor or the acceleration tensor. The following definitions and properties are from [2]. The mixed Riemann-Christoffel tensor or the Riemann-Christoffel symbol of the second kind written in terms of determinants is :

$$R^i_{jkl} = \begin{vmatrix} \frac{\partial}{\partial x^k} & \frac{\partial}{\partial x^l} \\ \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} & \left\{ \begin{matrix} i \\ jl \end{matrix} \right\} \end{vmatrix} + \begin{vmatrix} \left\{ \begin{matrix} i \\ mk \end{matrix} \right\} & \left\{ \begin{matrix} i \\ ml \end{matrix} \right\} \\ \left\{ \begin{matrix} m \\ jk \end{matrix} \right\} & \left\{ \begin{matrix} m \\ jl \end{matrix} \right\} \end{vmatrix} \qquad \text{A.E.73}$$

or when written out as

$$R^i_{jkl} = \left\{ \begin{matrix} i \\ jl \end{matrix} \right\}_{,k} - \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}_{,l} + \left\{ \begin{matrix} i \\ mk \end{matrix} \right\} \left\{ \begin{matrix} m \\ jl \end{matrix} \right\} - \left\{ \begin{matrix} i \\ ml \end{matrix} \right\} \left\{ \begin{matrix} m \\ jk \end{matrix} \right\} \qquad \text{A.E.74}$$

Similarly, the covariant Riemann-Christoffel tensor or the Riemann-Christoffel symbol of the first kind is

$$R_{ijkl} = \begin{vmatrix} \frac{\partial}{\partial x^k} & \frac{\partial}{\partial x^l} \\ [jk,i] & [jl,i] \end{vmatrix} + \begin{vmatrix} \left\{ \begin{matrix} m \\ jk \end{matrix} \right\} & \left\{ \begin{matrix} m \\ jl \end{matrix} \right\} \\ [ik,m] & [il,m] \end{vmatrix} \qquad \text{A.E.75}$$

or

$$R_{ijkl} = [jl,i]_{,k} - [jk,i]_{,l} + \left\{ \begin{matrix} m \\ jk \end{matrix} \right\} [il,m] - \left\{ \begin{matrix} m \\ jl \end{matrix} \right\} [ik,m] \qquad \text{A.E.76}$$

The relationship between the two tensors is:

$$R_{ijkl} = g_{i\alpha} R^{\alpha}_{jkl} \qquad \text{A.E.77}$$

$$R^i_{jkl} = g^{i\alpha} R_{\alpha jkl} \qquad \text{A.E.78}$$

The Riemann-Christoffel tensor has the following properties:

1. The tensor is skew-symmetric with respect to the first two and last two indices or

$$R_{jikl} = -R_{ijkl} \qquad \text{A.E.79}$$

$$R_{ijlk} = -R_{ijkl} \qquad \text{A.E.80}$$

136

2. The tensor is symmetric with respect to groups of the first two and last two indices or

$$R_{klij} = R_{ijlk} \qquad \text{A.E.81}$$

3.

$$R_{ijlk} + R_{iklj} + R_{iljk} = 0 \qquad \text{A.E.82}$$

$$R^i_{jlk} + R^i_{klj} + R^i_{ljk} = 0 \qquad \text{A.E.83}$$

4. If three or more of the indices are equal, then

$$R_{ijkl} = 0 \qquad \text{A.E.84}$$

Due to the above properties, the number of distinct nonzero components, N, is given by

$$N = \frac{n^2}{12}(n^2 - 1) \qquad \text{A.E.85}$$

where n is the dimension of the space. If $R^i_{jkl} = 0$ then the order of covariant differentiation in immaterial.

In [16,2] a number of other associated tensors are developed. The Ricci tensor is given by

$$R_{ij} = R^r_{ijr} = \left\{ {r \atop ir} \right\}_{,j} - \left\{ {r \atop ij} \right\}_{,r} + \left\{ {r \atop mj} \right\} \left\{ {m \atop ir} \right\} - \left\{ {r \atop mr} \right\} \left\{ {m \atop ij} \right\} \qquad \text{A.E.86}$$

which can be shown to be symmetric by use of

$$\left\{ {r \atop ir} \right\} = (\ln \sqrt{g})_{,i} \qquad \text{A.E.87}$$

which implies that the number of distinct components, N, is given by

$$N = \frac{n}{2}(n + 1) \qquad \text{A.E.88}$$

where n is the dimension of the space. For a four dimensional space, with the Ricci tensor set equal to zero, one obtains ten partial differential equations which were used by Einstein in his general theory of relativity [2].

From the Bianchi identity which is

$$R^i_{jkl\|m} + R^i_{jlm\|k} + R^i_{jmk\|l} = 0 \qquad \text{A.E.89}$$

which can be written as

$$R_{ijkl\|m} + R_{ijlm\|k} + R_{ijmk\|l} = 0 \qquad \text{A.E.90}$$

which after using property 1 we find

$$R_{ijkl\|m} - R_{ijml\|k} - R_{jimk\|l} = 0 \qquad \text{A.E.91}$$

Next multiply by $g^{il} g^{jk}$ and find

$$g^{jk} R^l_{.jkl\|m} - g^{jk} R^l_{.jml\|k} - g^{il} R^k_{.imk\|l} = 0 \qquad \text{A.E.92}$$

which after substituting for Ricci's tensor we have

$$g^{jk} R_{jk\|m} - g^{jk} R_{jm\|k} - g^{il} R_{im\|l} = 0 \qquad \text{A.E.93}$$

and simplifying

$$R^k_{k\|m} - 2R^k_{m\|k} = 0 \qquad \text{A.E.94}$$

or alternatively as

$$(R^k_m - \frac{1}{2}\delta^k_m R^i_i)_{\|k} = 0 \qquad \text{A.E.95}$$

where the quantity in parentheses is known as Einstein's tensor [2], and denoted by

$$G^k_m = R^k_m - \frac{1}{2}\delta^k_m R^i_i \qquad \text{A.E.96}$$

The quantity R^i_i in called the scalar curvature [16].

138

References

[1] B. A. Dubrovin, A. T. Fomenko, and S. P. Novikov, *Modern Geometry-Methods and Application, Part 1*. New York: SpringerVerlag, 1984.

[2] I. S. Sokolnikoff, *Tensor Analysis, Theory and Applications to Geometry and Mechanics of Continua*. New York: John Wiley and Sons, Inc., 2nd ed., 1964.

[3] K. Washizu, *Variational Methods in Elasticity and Plasticity*. New York: Pergamon Press, 1982.

[4] W. Nowacki, *Thermoelasticty*. Warszawa: Polish Scientific Publishers, 1962.

[5] E. W. Billington, *Introduction to the Mechanics and Physics of Solids*. Bristol, England: Adam Hilger Ltd., 1986.

[6] L. E. Malvem, *Introduction to the Mechanics of a Continuous Medium*. New Jersey: Prentice-Hall,Inc., 1969.

[7] C. W. Haines, *Analysis for Engineers*. New York: West Publishing Co., 1974.

[8] F. I. Niordson, *Shell Theory*. New York: North-Holland, 1985.

[9] B. A. Boley and J. H. Weiner, *Theory of Thermal Stresses*. New York: John Wiley and Sons, Inc., 1960.

[10] H. Kraus, *Thin Elastic Shells*. New York: John Wiley & Sons,Inc,, 1967.

[11] A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*. New York: Dover Publications, 1944.

[12] W. Soedel, *Vibrations of Shells and Plates*. New York: Marcel Dekker, Inc., 1981.

[13] C. Truesdell and R. A. Toupin, "The Classical Field Theories," in *Encyclopedia of Physics Vol. III/1 : Principles of Classical Mechanics and Field Theory* (S. Fluegge, ed.), pp. 226–793, Berlin: Springer-Verlag, 1960.

[14] E. Kreyszig, *Advanced Engineering Mathematics*. New York: John Wiley and Sons, Inc., 1979.

[15] R. T. Seeley, *Calculus of Serveral Variables*. Glenview, Illinois: Scott, Foresman and Co., 1970.

[16] H. Lass, *Vector and Tensor Analysis*. New York: McGraw-Hill Book Co., Inc., 1950.

[17] J. J. Stoker, *Differential Geometry*. New York: John Wiley and Sons, Inc., 1969.

[18] I. S. Sokolnikoff, *Mathematical Theory of Elasticty*. New York: McGraw-Hill Book Co., Inc., 1956.

[19] C. Truesdell, *The Elements of Continuum Mechanics*. New York: Springer-Verlag, 1966.

[20] M. S. Gadala, M. A. Dokainish, and G. A. Oravas, "Formulation Methods of Geometric and Material Nonlinearity Problems," *International Journal for Numerical Methods in Engineering*, vol. 20, pp. 887–914, 1984.

[21] D. E. Carlson, "Linear Thermoelasticity," in *Encyclopedia of Physics (S. Fluegge, chief ed.): Mechanics of Solids II* (C. T. Truesdell, ed.), pp. 297–345, Berlin: Springer-Verlag, 1972.

[22] M. F. Beatty, "Topics in Finite Elasticity: Hyperelasticity of Rubber, Elastomers, and Biological Tissues-With Examples," *Applied Mechanics Reviews*, vol. 40, no. 12, pp. 1699–1734, 1987.

[23] J. Mason, "Variational, Incremental and Energy Methods in Solid Mechanics and Shell Theory," in *Studies in Applied Mechanics*, vol. 4, Amsterdam: Elsevier Applied Science, 1980.

[24] A. F. Saleeb, T. Y. Chang, W. Graf, and S. Yingyeunyoung, "A Hybrid/Mixed Model for Non-Linear Shell Analysis and Its Applications to Large-Rotation Problems," *International Journal for Numerical Methods in Engineering*, vol. 29, pp. 407–446, 1990.

[25] R. Hill, "On Constitutive Inequalities for Simple Materials-I," *J. Mech. Phys. Solids*, vol. 16, pp. 229–242, 1968.

[26] P. Chadwick and L. T. Seet, "Second-Order Thermoelasticity Theory," in *Trends in Elasticity and Thermoelasticty*, pp. 29–57, Groningen, The Netherlands: Wolters-Noordoff Publishing, 1971.

[27] B. D. Coleman and W. Noll, "The Thermodynamics of Elastic Materials with Heat Conduction and Viscosity," *Archive for Rational Mechanics and Analysis*, vol. 13, pp. 167–178, 1963.

[28] A. E. Green and N. Laws, "On the Entropy Inequality," *Arch. Rational Mech. Anal.*, vol. 41, pp. 319–332, 1971.

[29] A. E. Green and P. M. Naghdi, "On Thermodynamics and the Nature of the Second Law," *Proceedings of the Royal Society of London Series A-Mathematical and Physical Sciences*, vol. 357, pp. 253–270, 1977.

[30] W. A. Day, "An Objection to Using Entropy as a Primitive Concept in Continuum THermodynamics," *Acta Mechanica*, vol. 27, pp. 251–255, 1977.

[31] I. H. Shames, *Mechanics of Fluids*. New York: McGraw-Hill, 1962.

[32] H. Ghoneim and D. N. Dalo, "Thermoplasticity with Second Sound Effects," *Journal of Thermal Stresses*, vol. 10, pp. 357–366, 1987.

[33] H. Ghoneim, "Analysis and Application of a Coupled Thermoviscoplasticity Theory," *Journal of Applied Mechanics*, vol. 57, pp. 828–835, 1990.

[34] T. J. R. Hughes and K. S. Pister, "Consistenr Linearization in Mechanics of Solids and Structures," *Computers and Structures*, vol. 8, pp. 391–397, 1978.

[35] M. E. Gurtin, "The Linear Theory of Elasticity," in *Enclyclopedia of Physics (S. Fluegge, chief ed.): Mechanics of Solids II* (C. T. Truesdell, ed.), pp. 1–295, Berlin: Springer-Verlag, 1972.

[36] H. H. Sherief and M. N. Anwar, "Problem in Generalized Thermoelasticity," *Journal of Thermal Stresses*, vol. 9, pp. 165–181, 1986.

[37] J. Ignaczak, "Linear Dynamic Thermoelasticity - A Survey," *Shock and Vibration Digest*, vol. 13, no. 9, pp. 3–8, 1981.

[38] J. Ignaczak, "Uniqueness in Generalized Thermoelasticity," *Journal of Thermal Stresses*, vol. 2, pp. 171–175, 1979.

[39] H. W. Lord and Y. Shulman, "A Generalized Dynamical Theory of Thermoelasticty," *Journal of Mechanics and Physics of Solids*, vol. 15, pp. 229–309, 1967.

[40] A. E. Green and K. A. Lindsay, "Thermoelasticity," *Elasticity*, vol. 2, pp. 1–7, 1972.

[41] M. Kranys, "Causal Theories of Evolution and Wave Propagation in Mathematical Physics," *Applied Mechanics Reviews*, vol. 42, no. 11, pp. 305–322, 1989.

[42] A. Libai and J. G. Simmonds, "Nonlinear Elastic Shell Theory," in *Advances in Applied Mechanics*, vol. 23, pp. 271–371, New York: Academic Press, 1983.

[43] M. B. Rubin and A. L. Florence, "Thermomechanical Response of Shells with Emphasis on a Conical Shell: Final rept. 1 Dec. 83–30 Nov. 84," Tech. Rep. AFOSR-TR-85-1239, Air Force Office of Scientific Research, Bolling AFB, D.C., 1984.

[44] C. O. Horgan, "Recent Developments Concerning Saint-Venant's Principle: An Update," *Applied Mechanics Reviews*, vol. 42, no. 11, pp. 295–303, 1989.

[45] F. John, "Refined Interior Shell Equations," in *Theory of Thin Shells* (F. I. Niordson, ed.), (Berlin), pp. 1–14, IUTAM,2nd Symposium,Copenhagen,1967, Springer-Verlag, 1969.

[46] F. John, "Estimates for the Derivatives of the Stresses in a Thin Shell and Interior Shell Equations," *Communications on Pure and Applied Mathematics*, vol. XVII, pp. 235–267, 1965.

[47] H. L. Langhaar, "Elastic Surfaces and THeories of Shells," *Acta Mechanica*, vol. 19, pp. 109–128, 1974.

[48] P. A. Zhilin, "Mechanics of Deformable Directed Surfaces," *International Journal of Solids and Structures*, vol. 12, pp. 635–648, 1976.

[49] A. E. Green, P. M. Naghdi, and W. L. Wainwright, "A General Theory of a Cosserat Surface," *Archive for Rational Mechanics and Analysis*, vol. 20, pp. 297–308, 1965.

[50] A. E. Green and P. M. Naghdi, "On Thermal Effects in the Theory of Shells," *Proceedings of the Royal Society of London Series A-Mathematical and Physical Sciences*, vol. 365, pp. 161–190, 1979.

[51] A. E. Green and P. M. Naghdi, "Shells in the light of Generalized Continua," in *Theory of Thin Shells* (F. I. Niordson, ed.), (Berlin), pp. 39–58, IUTAM,2nd Symposium,Copenhagen,1967, Springer-Verlag, 1969.

[52] P. M. Naghdi, "The Theory of Shells and Plates," in *Enclyclopedia of Physics (S. Fluegge, chief ed.): Mechanics of Solids II* (C. T. Truesdell, ed.), pp. 425–640, Berlin: Springer-Verlag, 1972.

[53] M. M. Carroll and P. M. Naghdi, "The Influence of the Reference Geometry on the Response of Elastic Shells," Tech. Rep. AM-72-2, California Univ., Berkeley. Div. of Applied Mechanics, 1972.

[54] M. B. Rubin, "A Uniqueness Theorem for Thermoelastic Shells with Generalized Boundary Conditions," *Quarterly of Applied Mathematics*, vol. XLIV, no. 3, pp. 431–440, 1986.

[55] E. L. Axelrad and F. A. Emmerling, "Vector and Tensor form of Intrinsic Shell-Theory Relations," *International Journal of Non-Linear Mechanics*, vol. 23, no. 1, pp. 9–23, 1988.

[56] F. Gruttmann, E. Stein, and P. Wriggers, "Theory and Numerics of Thin Elastic Shells with Finite Rotations," *Ingenieur-Archiv*, vol. 59, pp. 54–67, 1989.

[57] H. Parisch, "An Investigation of a Finite Rotation Four Node Assumed Strain Shell Element," *International Journal for Numerical Methods in Engineering*, vol. 31, pp. 127–150, 1991.

[58] H. Cheng and K. C. Gupta, "An Historical Note an Finite Rotations," *Journal of Applied Mechanics*, vol. 56, pp. 139–145, 1989.

[59] W. Pietraszkiewicz, "Lagrangian Description and Incremental Formulation in the Non-Linear Theory of Thin Shells," *International Journal of Non-Linear Mechanics*, vol. 19, no. 2, pp. 115–140, 1983.

[60] E. L. Axelrad and F. A. Emmerling, "Intrinsic Shell Theory Relevant for Realizable Large Displacements," *International Journal of Non-Linear Mechanics*, vol. 22, no. 2, pp. 139–150, 1987.

[61] L. P. Nolte, J. Makowski, and H. Stumpf, "On the Derivation and Comparative Analysis of Large Rotation Shell Theories," *Ingenieur-Archiv*, vol. 56, pp. 145–160, 1986.

[62] D. Levesque and L. Bertrand, "A Finite Element Formulation for the Coupled Thermoviscoelastic Response of Solid-Fluid Systems," *International Journal for Numerical Methods in Engineering*, vol. 29, pp. 953–968, 1990.

[63] W. Pietraszkiewicz and M. L. Szwabowicz, "Entirely Lagrangian Nonlinear Theory of Thin Shells," *Acta Mechanica*, vol. 33, pp. 273–288, 1981.

[64] Y. Basar, "A Consistent Theory of Geometrically Non-linear Shells with An Independent Rotation Vector," *International Journal of Solids and Structures*, vol. 23, no. 10, pp. 1401–1415, 1987.

[65] Y. Basar and W. B. Kratzig, "A Consistent Shell Theory for Finite Deformations," *Acta Mechanica*, vol. 76, pp. 73–87, 1989.

[66] L. Taber, "On a Theory for Large Elastic Deformation of Shells of Revolution Including Torsion and Thick-Shell Effects," *International Journal of Solids and Structures*, vol. 24, no. 9, pp. 973–985, 1988.

[67] S. A. Lukasiewicz, "Thermal Stresses in Shells," in *Thermal Stresses III* (R. B. Hetnarski, ed.), pp. 355–555, New York: Elsevier Applied Science, 1989.

[68] P. M. Naghdi, "Foundations of Elastic Shell Theory," in *Progress in Solid Mechanics*, vol. 4, pp. vii–90, Amsterdam: North-Holland Publishing Co., 1963.

[69] L. Brillouin, *Tensors in Mechanics and Elasticity*. New York: Academic Press, 1964.

[70] W. Wagner, "A Finite Element Model For Non-Linear Shells of Revolution with Finite Rotations," *International Journal for Numerical Methods in Engineering*, vol. 29, pp. 1455–1471, 1990.

[71] J. Hodges, M. I. Darby, and B. Yates, "Thermal Expansion of CFRP/Aluminium/CFRP Composite Spherical Shells," *High Temperatures-High Pressures*, vol. 17, no. 3, pp. 325–331, 1985.

[72] K. Takamizawa and T. Matsuda, "Kinematics for Bodies Undergoing Residual Stress and its Applications to the Left Ventricle," *Journal of Applied Mechanics*, vol. 57, pp. 321–329, 1990.

[73] R. H. Liu, "Non-Linear Thermal Stability of Bimetallic Shallow Shells of Revolution," *International Journal of Non-Linear Mechanics*, vol. 18, no. 5, pp. 409–429, 1983.

[74] L. C. Thomas, *Fundamentals of Heat Transfer*. New Jersey: Prentice-Hall,Inc., 1980.

[75] P. P. Raju, "On Shallow Shells of Transversely Isotropic Materials," *Journal of Pressure Vessel Technology, Transactions of the ASME, Series J*, vol. 97, pp. 185–191, 1975.

[76] F. W. Keene and R. B. Hetnarski, "Bibliography on Thermal Stresses in Shells," *Journal of Thermal Stresses*, vol. 13, no. 4, pp. 339–542, 1990.

[77] P. Prakash and K. P. Rao, "Thermal Stresses in a Spherical Shell with a Circular Elastic Inclusion," in *Int. Conf. on Struct. Mech. in React. Technol., 3rd, Trans., London, Engl., Sept. 1–5, 1975, vol. 5, part L2/1*, Brussels, Belg.: Comm. of the Eur. Communities, 1975.

[78] P. Prakash and K. P. Rao, *Circular Inclusions, Radial Nozzles and Reinforced Openings in Spherical Shells*. Ph.d thesis, Indian Inst. of Science, Bangalore. Dept. of Aeronautical Engineering, 1975.

[79] K. P. Rao, "Stress Concentrations Around Discontinuities in Thin Cylindrical and Spherical Shells," in *Theory of shells; Proceedings of the Third Symposium, Tiflis, Georgian SSR, Aug. 22–28, 1978*, pp. 473–494, Amsterdam,: North-Holland, 1980.

[80] J. Jayaraman and K. P. Rao, "Thermal Stresses in a Spherical Shell with a Conical Nozzle," *Nuclear Engineering and Design*, vol. 48, no. 2–3, pp. 367–375, 1978.

[81] L. E. J. Doxsee, "A Higher-Order Theory of Hygrothermal Behavior of Laminated Composite Shells," *International Journal of Solids and Structures*, vol. 25, no. 4, pp. 339–355, 1989.

[82] V. Kumar and S. Mukherjee, "Creep Analysis of Metallic Structures in the Presence of Thermal Gradients Using Newer Constitutive Relations," *Journal of Pressure Vessel Technology, Transactions of the ASME, Series J*, vol. 99, pp. 272–280, 1977.

[83] M. B. Tate, "Thermal Stresses in Ogival Radomes with Temperature-Dependent Material Properties," Tech. Rep. APL-TG-1073, Johns Hopkins Univ., Silver Spring Md., Applied Physics Lab., 1969.

[84] J. Venkataramana, "Thermal Stresses in Heterogeneous Thick Isotropic Shells," *Nuclear Engineering and Design*, vol. 30, no. 3, pp. 369–378, 1974.

[85] G. Cinelli, "Thermoelastic Vibrations of Thick Elastic Anisotropic Shells," in *Developments in Mechanics, Proc. of 11th Midwestern Mech. Conf., Aug. 18–20, 1969, Iowa State Univ., Ames*, vol. 5, pp. 451–468, Midwestern Mech. Conf. Publ., 1969.

[86] T. K. Hellen, N. G. Galluzzo, and A. P. Kfouri, "Elastic-Plastic Analysis of a Thick Spherical Shell under Thermal Loading-A Comparison of Three Numerical Procedures," *International Journal of Mechanical Sciences*, vol. 19, no. 4, pp. 209–221, 1977.

[87] H. A. Nied, *Coupled Thermoplasticity with Applications to Thick-Walled Spheres*. Ph.d thesis, Pennsylvania Univ., Philadelphia, 1969.

[88] M. B. Rubin, "Heat Conduction in Plates and Shells with Emphasis on a Conical Shell," *International Journal of Solids and Structures*, vol. 22, pp. 527–551, 1986.

[89] A. Szekeres, "Strength Analysis of Thick-Walled Spheres in Spheroid Symmetrical Thermal State," *Periodica Polytechnica, Mechanical Engineering*, vol. 17, no. 1, pp. 91–84, 1973.

[90] F. P. Avril, R. F. Parisse, and S. V. Nardo, "Thermal Stresses in an Ablating Thick-Walled Sphere," Tech. Rep. PIBAL-930, Polytechnic Inst. of Brooklyn, N. Y. Dept. of Aerospace Engineering and Applied Mechanics., 1966.

[91] N. M. Belyaev, V. I. Zavelion, and A. A. Ryadno, "Analysis of Temperature Fields of Bodies in the Shape of Shells," *Journal of Engineering Physics (English trans. of Inzhenerno-Fizicheskii Zhurnal)*, vol. 43, no. 4, pp. 1144–1148, 1983.

[92] S. R. Bodner and Y. Partom, "A Large Deformation Elastic-Viscoplastic Analysis of a Thick-Walled Spherical Shell," in *ASME Paper 72-APM-K (for meeting 1972)*, Israel Inst. of Technology Haifa, 1972.

[93] D. Durban, "Thermoplastic Behavior of a Thick-Walled Sphere," *AIAA Journal*, vol. 19, pp. 826–828, 1981.

[94] D. Durban, "Thermo-Elastic/Plastic Behaviour of a Strain-Hardening Thick-Walled Sphere," *International Journal of Solids and Structures*, vol. 19, pp. 643–652, 1983.

[95] Y. M. Kolyano and I. N. Makhorkin, "Thermal Stresses in a Spherical Thick-Walled Shell with a Filler," *Strength of Materials (English trans. of Problemy Prochnosti)*, vol. 8, pp. 1337–1340, 1977.

[96] M. A. Gorgui, D. M. Darwish, and O. L. Moustafa, "Thermal Stresses in Cylindrical and Spherical Composite Bodies," in *Computational and Asymptotic Methods for Boundary and Interior Layers; Proceedings of the Second BAIL Conference, Dublin, Ireland, June 16–18, 1982*, pp. 250–256, Dublin: Boole Press, 1982.

[97] T. A. Hahn, *Thermal Expansion Behavior and Analysis for Aluminum-Silicon Eutectic Alloys and the Composite Material Systems: Aluminum-Silicon-Carbide and Aluminum-Tungsten*. Ph.d thesis, Univ. of Maryland, 1986.

[98] A. F. Pojur, B. Yates, and B. T. Kelly, "Thermal Expansion at Elevated Temperatures. III-A Hemispherical Laminar Composite of Pyrolytic Graphite, Silicon Carbide and its Constituents Between 300 and 800 K," *Journal of Physics, Part D-Applied Physics*, vol. 5, pp. 1321–1329, 1972.

[99] H. E. Williams, "Axisymmetric Thermal Stresses in Sandwich Shells of Revolution with Application to Shallow, Spherical Shells," Tech. Rep. NASA-CR-76428, Jet Propulsion Lab., California Inst. of Tech., Pasadena, 1966.

[100] B. Raniecki, "Spherical Thermoplastic Stress-Waves," *Proceedings of Vibration Problems*, vol. 6, no. 4, pp. 379–399, 1965.

[101] T. J. Chung, J. T. Oden, R. L. Eidson, J. F. Jenkins, and A. E. Masters, "Dynamic Elastoplastic Response of Geometrically Nonlinear Arbitrary Shells of Revolution under Impulsive Mechanical and Thermal Loadings," *The Shock and Vibration Bull., Part 7*, vol. 41, pp. 81–88, 1971.

[102] R. Riff and G. J. Simitses, "The Dynamic Aspects of Thermo-Elasto-Viscoplastic Snap-Through and Creep Buckling Phenomena," Tech. Rep. NASA-CR-181411, Georgia Inst. of Tech., Atlanta, 1987.

[103] R. Delpak and W. M. Hague, "An Experimental and Theoretical Investigation of the Frequencies and Mode Shapes of Axisymmetric Shell Models," *Journal of Sound and Vibration*, vol. 72, pp. 235–249, 1980.

[104] E. I. Grigolyuk and A. V. Borodin, "Thermoelastoplastic Problem for a Hollow Sphere Exposed to Radiation," *Strength of Materials (English trans. of Problemy Prochnosti, Dec. 1975, pp. 72–77)*, vol. 7, no. 12, pp. 1504–1510, 1976.

[105] H. Ishikawa, "Transient Thermoelastoplastic Stress Analysis for a Hollow Sphere Using the Incremental Theory of Plasticity," *International Journal of Solids and Structures*, vol. 13, no. 7, pp. 645–655, 1977.

[106] T. F. Chen, *Thermoelastic Vibrations of Spherical Shells*. Ph.d thesis, Univ. of Wisconsin-Milwaukee, 1974.

[107] H. Kraus, "Thermally Induced Vibrations of Thin Non-Shallow Spherical Shells," *American Inst. of Aeronautics and Astronautics, Annual Meeting, 2nd, San Francisco, Calif., Jul. 26–29, 1965, Paper 65-409*, 1965.

[108] B. N. Cassenti and S. V. Nardo, "Temperatures and Thermal Stresses in Thin-Walled Spherical Segments," Tech. Rep. PIBAL-69–20, Polytechnic Inst. of Brooklyn N. Y., Dept. of Aerospace Engineering and Applied Mechanics, 1969.

[109] H. Kraus, "Dynamic Thermal and Pressure Stresses in Thin Spherical Shell: Interim Report," Tech. Rep. PWA-2329, Pratt and Whitney Aircraft East Hartford Conn, 1964.

[110] E. C. Rodabaugh, "Exploratory Study of the Feasibility of Developing Stress Indices for Thermal Gradients in Pressure Vessel Nozzles, Phase Report 117–11," Tech. Rep. GRAI7524; NSA3206, Battelle Columbus Labs., Ohio, 1975.

[111] W. E. Warren, "Transient Axisymmetric Thermoelastic Problem for the Hollow Sphere," *AIAA Journal*, vol. 1, pp. 2569–2576, 1963.

[112] A. M. Vinogradov and P. G. Glockner, "Buckling of Spherical Viscoelastic Shells," *Journal of the Structural Division, Proceedings of the ASCE*, vol. 106, no. 1, pp. 59–67, 1980.

[113] T. A. Zaker, *Dynamic Thermal Shock in Spherical Shells*. Ph.d thesis, IIT Research Inst., Chicago, Ill., 1967.

[114] T. Hata, "Thermal Shock in a Hollow Sphere Caused by Rapid Uniform Heating," *Journal of Applied Mechanics*, vol. 58, pp. 64–69, 1991.

[115] T. A. Zaker, "Dynamic Thermal Shock in Hollow Spheres," *Quarterly of Applied Mathematics*, vol. 26, pp. 503–520, 1969.

[116] M. Laudanna and F. Quori, "Approximate Study of the Thermal Shock," Tech. Rep. AFOSR-70–1118TR, Air Force Office of Scientific Research, Bolling AFB, D.C., 1969.

[117] D. P. Updike, "Thermal Ratcheting under Biaxial Stress States," *Nuclear Engineering and Design*, vol. 33, no. 3, pp. 387–397, 1975.

[118] D. Shilkrut, "Solutions of Some Stability Problems in the Theory of Geometrically Non-Linear Shells," *Israel Journal of Technology*, vol. 18, no. 1–2, pp. 76–83, 1980.

[119] M. Sunakawa, "Elastic Stability of Spherical Shells Subjected to External Pressure," *Tokyo, U., Inst. of Space and Aeronautical Science*, vol. 33, no. 423, pp. 125–148, 1968.

[120] W. Thielemann, "Structural Stability of Conical and Spherical Thin Shells: Bibliography 4," Tech. Rep. STAR6508, Advisory Group for Aeronautical Research and Development, Paris, 1961.

[121] H. D. Liu, X. W. Liu, T. T. Loo, D. Y. Nie, and J. X. Shi, "Thermal Snap-Through Buckling of a Bimetallic Shallow Spherical Shell," *Acta Mechanica Sinica*, vol. 9, pp. 1–13, 1966.

[122] M. Sunakawa, "Axisymmetric Buckling of Shallow Spherical Shells under External Pressure," in *Japan National Congress for Applied Mechanics, 16th, U. of Tokyo, Tokyo, Japan, Oct. 19, 20, 1966, Proceedings*, pp. 158–163, Tokyo: Central Scientific Publishers, 1968.

[123] V. I. Weingarten and Y. S. Wang, "Stability of Shells Attached to an Elastic Core," *Journal of the Engineering Mechanics Division, Proceedings of the ASCE*, vol. 102, no. 5, pp. 839–849, 1976.

[124] B. D. Aggarwala and E. Saibel, "Thermal Stability of Bimetallic Shallow Spherical Shells," *International Journal of Non-Linear Mechanics*, vol. 5, pp. 49–62, 1970.

[125] S. J. Han, "Instability Localized at the Inner Surface of an Imploding Spherical Shell," in *American Physical Society Spring Meeting, Balitmore, Md., 18 Apr., 1983*, Los Alamos National Lab., N.M., LA-UR-83-1087; CONF-830463-1, 1983.

[126] V. A. Krysko, G. M. Guba, and V. G. Fomin, "Dynamics of Hollow Spherical Shell under the Action of Thermal Shock," *Soviet Applied Mechanics (English trans. of Prikladnaya Mekhanika)*, vol. 22, no. 5, pp. 409–414, 1986.

[127] V. D. Afinogenov, "Thermal Stresses in a Shallow Spherical Shell under Axisymmetrical Heating," in *English trans. of Vestn. Mosk. Univ., Ser. 1, Mat. Mekh. no. 3, 1968, pp. 114–118*, Boston Spa, (England): National Lending Library for Science and Technology, NLL-RTS-5888, 1970.

[128] S. S. Issa, "Over Determinate Solution for Shallow Spherical Shells," *Mechanics Research Communications*, vol. 14, no. 3, pp. 181–186, 1987.

[129] Z. F. Baczynski and J. Ignaczak, "Thermoelastic Stress Analysis of Reactor Secondary Containment," in *Trans. of the Int. Conf. on Struct. Mech. in React. Technol., 4th, vol. J, (a) Loading Cond. and Struct. Anal. of React. Containment, San Francisco, Calif., Aug. 15–19, 1977*, Luxemb.: Comm. of the Eur. Communities, 1977.

[130] S. H. Chu, M. A. Goldberg, and S. V. Nardo, "Temperature and Thermal Stress Problems of Thin-walled Spheres and Cones with Aerodynamic Heating, Research Report," Tech. Rep. PIBAL-68-21, Polytechnic Inst. of Brooklyn, New York, N.Y. Dept. of Aerospace Engineering and Applied Mechanics, 1968.

[131] R. C. Davis, "Direct Solution for Thermal Stresses in a Nose Cap under an Arbitrary Axisymmetric Temperature Distribution," in *Structures, Structural Dynamics and Materials Conference, 29th, Williamsburg, Va., Apr. 18–20, 1988, Technical Papers. Part 1*, pp. 289–295, Washington, D.C.: American Inst. of Aeronautics and Astronautics, 1988.

[132] A. M. Novikova and I. A. Motovilovets, "Thermal Stress of a Shallow Spherical Shell," in *English trans. of Akad. Nauk Ukr. SSR, Inst. Mekhan. Tepl. Napryazh. v Elementakh Konstrukt. (USSR), no. 9, 1970, pp.23–34*, National Aeronautical Lab., Bangalore (India) NAL-TR-113, 1972.

[133] V. M. Porksheyan, "Steady-State Thermomechanical Vibrations of a Spherical Thermoviscoelastic Dome under Cyclical Loading," *Mechanics of Solids (English trans. of Izvestiya Akademii Nauk SSSR, Mekhanika Tverdogo Tela)*, vol. 21, no. 2, pp. 144–151, 1986.

[134] E. T. Cranch and O. H. Griffith, "Discontinuity Thermal Stresses in Shallow Spherical Shells," in *ASME, First Int. Conf. on Pressure Vessel Technology, Delft, Netherlands, Sept. 29-Oct. 2, 1969, Part 1*, pp. 435–449, American Society of Mechanical Engineers, 1969.

[135] A. K. Galinsh, "Refined Equations of Motion for Orthotropic Thermoelastic Shells," in *Edited trans. of Issledovaniya po Teorii Plastin i Obolochek (USSR) no., 8 pp. 365–392, 1972*, Foreign Technology Div. Wright-Patterson AFB, Ohio, FTD-HT-23-366-74, 1974.

[136] N. Kamiya and Y. Sawaki, "Boundary Element Method Applied to Finite Deflections of Shallow Elastic Shells," in *Methodes Numeriques de l'Ingenieur, Comptes Rendus du Troisieme Congres International, Paris, Fr., 1983, Mar. 14–16*, vol. 2, pp. 739–748, Paris: Pluralis, 1983.

[137] A. W. Leissa and A. S. Kadi, "Analysis of Shallow Shells by the Method of Point Matching: Technical Rept. No. 7 (Final), 1 Dec. 66–31 Mar. 69," Tech. Rep. AFFDL-TR-69-71, Ohio State Univ. Research Foundation Columbus, 1969.

[138] P. K. Mehta, "Nonsymmetric Thermal Bowing of Curved Circular Mirrors," in *Structural Mechanics of Optical Systems II; Proceedings of the Meeting, Los Angeles, Calif., Jan. 13–15, 1987*, pp. 187–205, Bellingham, Wa.: Society of Photo-Optical Instrumentation Engineers, 1987.

[139] A. K. Noor, "On Making Large Nonlinear Problems Small," *Computer Methods in Applied Mechanics and Engineering*, vol. 34, pp. 955–985, 1982.

[140] D. N. Paliwal and R. N. Rai, "Shallow Spherical Shell on Pasternak Foundation Subjected to Elevated Temperature," *Thin-Walled Structures*, vol. 5, no. 5, pp. 343–349, 1987.