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Long term behavior of the positive solutions of the non-autonomous difference equation X n+1= A n x n-1 [divided by] 1+Xn , n = 0,1,2,....

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Long term behavior of the positive solutions of the non-autonomous difference equation:

$$x_{n+1} = \frac{A_n x_{n-1}}{1 + x_n} \ , n = 0,1,2,\ldots \ .$$

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Abstract

It is our goal to investigate the long term behavior of the solutions of the following difference equation:

$$x_{n+1} = \frac{A_n x_{n-1}}{1 + x_n}, \; n = 0,1,2,\ldots,$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers. We will examine how the different periods of the sequence and the relationship of the terms of the sequence affect the long term behavior of the solutions.

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1 Introduction

Our goal in this paper is to investigate the long term behavior of the solutions of the following non-autonomous difference equation:

$$x_{n+1} = \frac{A_n x_{n-1}}{1 + x_n} \ , \ n = 0,1,2,..., \qquad (2)$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers with prime period k+1 and the initial conditions x_0 and x_{-1} are non-negative real numbers.

While examining the long term behavior of the following autonomous difference equation:

$$x_{n+1} = \frac{A x_{n-1}}{1 + x_n} \ , \ n = 0,1,2,..., \qquad (1)$$

where A >0 and the initial conditions x_{-1} and x_0 are non-negative real numbers, the following properties were proved:

- When A <1, every solution of Eq.(1) converges to zero.
- When A =1, every solution of Eq.(1) converges to a period two solution.
- When A > 1, Eq.(1) has unbounded solutions.

2 The Case where $\{A_n\}_{n=0}^{\infty}$ is periodic with prime period 2

In this section, we will assume that $\{A_n\}_{n=0}^{\infty}$ is periodic with prime period two and the initial conditions x_0 and x_{-1} are non-negative real numbers. It is our goal to investigate the long term behavior of the solutions of Eq. (2).

Let

$$M = \max\{A_0, A_1\}.$$

In this section we will prove the following properties of Eq. (2):

- When $M < 1$, every solution of Eq. (2) converges to zero.
- When $M = 1$, every solution of Eq. (2) converges to a period two solution.
- When $M > 1$, Eq.(2) has unbounded solutions.

Lemma 1: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$M = \max\{A_0, A_1\} < 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

Proof: Observe that

$$x_1 = \frac{A_0 x_{-1}}{1 + x_0} \le A_0 x_{-1} \le M x_{-1},$$

$$x_3 = \frac{A_2 x_1}{1 + x_2} = \frac{A_0 x_1}{1 + x_2} \le A_0 x_1 \le M x_1 \le M(M x_{-1}) = M^2 x_{-1},$$

$$x_5 = \frac{A_4 x_3}{1 + x_4} = \frac{A_0 x_3}{1 + x_4} \le A_0 x_3 \le M x_3 \le M(M^2 x_{-1}) = M^3 x_{-1}.$$

It follows by induction that for all $n \ge 0$:

$$0 \le x_{2n+1} \le M^{n+1} x_{-1}.$$

Hence we see that,

$$0 \le \lim_{n\to\infty} x_{2n+1} \le \lim_{n\to\infty} M^{n+1} x_{-1}.$$

Since we know that $M < 1$, then

$$\lim_{n\to\infty} M^{n+1} x_{-1} = 0.$$

Similarly, it follows by induction that:

$$0 \le x_{2n} < M^n x_0 \text{ for all } n \ge 0.$$

So we see that

$$0 \le \lim_{n\to\infty} x_{2n} \le \lim_{n\to\infty} M^n x_0.$$

Thus we get

$$\lim_{n\to\infty} M^n x_0 = 0.$$

Since

$$\lim_{n\to\infty} x_{2n+1} = 0 \quad \text{and} \qquad \lim_{n\to\infty} x_{2n} = 0,$$

it follows that

$$\lim_{n\to\infty} x_n = 0.$$

Lemma 2: Eq. (2) has solutions with prime period two if and only if either:

(2.1) $A_0 = 1$, or

(2.2) $A_1 = 1$.

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). First, suppose that

(2.3) $A_0 = 1, x_0 = 0$ and $x_{-1} > 0$.

The case where $A_1 = 1, x_{-1} = 0$ and $x_0 > 0$ is similar and will be omitted.

It follows via (2.3) that:

$$x_1 = \frac{A_0 x_{-1}}{1 + x_0} = A_0 x_{-1} = x_{-1},$$

$$x_2 = \frac{A_1 x_0}{1 + x_1} = 0 = x_0.$$

Therefore we get solutions with prime period two.

Similarly, prime period two solutions exist when (2.2) occurs.

Now suppose that Eq. (2) has solutions with prime period two. Then it follows that:

$$x_1 = \frac{A_0 x_{-1}}{1 + x_0} = x_{-1},$$
$$x_2 = \frac{A_1 x_0}{1 + x_1} = x_0.$$

So we obtain the following equalities:

$$(2.3)\ A_0 x_{-1} = x_{-1}(1 + x_0) = x_{-1} + x_{-1}x_0,$$

$$(2.4)\ A_1 x_0 = x_0(1 + x_{-1}) = x_0 + x_0 x_{-1}.$$

Then via (2.3) and (2.4) it follows that:

$$(2.5)\ x_{-1}x_0 = A_0 x_{-1} - x_{-1} = x_{-1}(A_0 - 1),$$

$$(2.6)\ x_0 x_{-1} = A_1 x_0 - x_0 = x_0(A_1 - 1).$$

Therefore from (2.5) and (2.6) we get:

$$(2.7)\ \ x_{-1}(A_0 - 1) = x_0(A_1 - 1).$$

Now since we know $x_{-1} \neq x_0$ and $A_0 \neq A_1$, then we see that via (2.7) one of the following conditions must occur:

(i) $\quad x_{-1} = 0$ and $A_1 = 1$, or

(ii) $\quad x_0 = 0$ and $A_0 = 1$.

Hence the result follows.

Lemma 3: Suppose that

$$M = \max\{A_0, A_1\} = 1.$$

Then every positive solution of Eq. (2) converges to a period two solution.

<u>Proof:</u> Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). First we will consider the case where:

$$(3.1)\quad A_0 = 1 \text{ and } A_1 < 1.$$

The case where $A_1 = 1$ and $A_0 < 1$ is similar and will be omitted.

Note that via (3.1) we get:

$$x_1 = \frac{A_0 x_{-1}}{1 + x_0} < A_0 x_{-1} = x_{-1},$$

$$x_3 = \frac{A_2 x_1}{1 + x_2} = \frac{A_0 x_1}{1 + x_2} < A_0 x_1 = x_1,$$

$$x_5 = \frac{A_4 x_3}{1 + x_4} = \frac{A_0 x_3}{1 + x_4} < A_0 x_3 = x_3.$$

It now follows by induction that for all $n \geq 0$:

$$(3.2)\quad 0 < .... < x_{2n+1} < x_{2n-1} < .... < x_1 < x_{-1}.$$

We see that (3.2) is a monotonically decreasing subsequence which is bounded above by x_{-1} and below by zero. Thus, there exists a limit $L \geq 0$ such that:

$$\lim_{n\to\infty} x_{2n+1} = L.$$

Also, observe that via (3.1) we get:

$$x_2 = \frac{A_1 x_0}{1 + x_1} < A_1 x_0,$$

$$x_4 = \frac{A_3 x_2}{1 + x_3} = \frac{A_1 x_2}{1 + x_3} < A_1 x_2 < A_1(A_1 x_0) = A_1{}^2 x_0,$$

$$x_6 = \frac{A_5 x_4}{1+x_5} = \frac{A_1 x_4}{1+x_5} < A_1 x_4 < A_1(A_1{}^2 x_0) = A_1{}^3 x_0 .$$

So it follows by induction that for all $n \geq 0$:

$$0 \leq x_{2n} < A_1{}^n x_0 .$$

Therefore, we get

$$0 \leq \lim_{n\to\infty} x_{2n} < \lim_{n\to\infty} A_1{}^n x_0 .$$

Also from (3.1) it follows that

$$\lim_{n\to\infty} A_1{}^n x_0 = 0 .$$

Thus we see that

$$\lim_{n\to\infty} x_{2n+1} = L \text{ and } \lim_{n\to\infty} x_{2n} = 0 .$$

Hence the result follows.

Lemma 4: Suppose that

$$M = \max\{A_0, A_1\} > 1.$$

Then Eq. (2) has unbounded solutions.

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). First suppose that

$$(4.1) \quad A_1 > 1 \text{ and } x_{-1} = 0 .$$

The case where $A_0 > 1$ and $x_0 = 0$ is similar and will be omitted.

Observe that via (4.1), we get:

$$x_1 = \frac{A_0 x_{-1}}{1+x_0} = 0 ,$$

$$x_3 = \frac{A_2 x_1}{1+x_2} = \frac{A_0 x_1}{1+x_2} = 0 ,$$

$$x_5 = \frac{A_4 x_3}{1+x_4} = \frac{A_0 x_3}{1+x_4} = 0,$$

$$\vdots$$

So it follows by induction that $x_{2n-1} = x_{2n+1} = x_{-1} = 0$ for all $n \geq 0$.

Also notice that via (4.1) we get:

$$x_2 = \frac{A_1 x_0}{1+x_1} = A_1 x_0,$$

$$x_4 = \frac{A_3 x_2}{1+x_3} = \frac{A_1 x_2}{1+x_3} = A_1 x_2 = A_1{}^2 x_0,$$

$$x_6 = \frac{A_5 x_4}{1+x_5} = \frac{A_1 x_4}{1+x_5} = A_1 x_4 = A_1{}^3 x_0,$$

$$\vdots$$

It follows by induction that:

$$(4.2)\quad x_{2n} = A_1{}^n x_0 \text{ for all } n \geq 0.$$

Therefore, from (4.1) and (4.2) we see that:

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} A_1{}^n x_0 = \infty .$$

Thus,

$$\lim_{n\to\infty} x_{2n+1} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{2n} = \infty .$$

Hence the result follows.

3 The Case where $\{A_n\}_{n=0}^{\infty}$ is periodic with an even prime period k+1.

In this section, we will assume that $\{A_n\}_{n=0}^{\infty}$ is periodic with an even prime period k+1 and the initial conditions x_0 and x_{-1} are non-negative real numbers. It is our goal to investigate the long term behavior of the solutions of Eq. (2).

Let

$$M = \max\{ A_0, A_1, A_2, ..., A_k \}.$$

In this section, we will prove the following properties of Eq. (2):

- When $M < 1$, every solution of Eq. (2) converges to zero.
- When $M = 1$, we get one of the following situations:
  - Every solution of Eq. (2) converges to a period two solution if either: $A_0 = A_2 = A_4 = ... = A_{k-1} = 1$ or $A_1 = A_3 = A_5 = ... = A_k = 1$
  - Otherwise, every solution of Eq. (2) converges to zero.
- When $M > 1$, Eq. (2) has unbounded solutions.

Lemma 5: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$M = \max\{ A_0, A_1, A_2, ..., A_k \} < 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

<u>Proof:</u>

Proof is similar to the proof of Lemma 1 and will be omitted.

Lemma 6: Eq. (2) has solutions with prime period two if and only if either:

(6.1) $A_0 = A_2 = A_4 = ... = A_{k-1} = 1$, or

(6.2) $A_1 = A_3 = A_5 = ... = A_k = 1$.

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). First, suppose that

$$(6.3)\ \ A_0 = A_2 = A_4 = \ldots = A_{k-1} = 1, x_0 = 0 \text{ and } x_{-1} > 0.$$

The case where (6.2) occurs, $x_{-1} = 0$ and $x_0 > 0$ is similar and will be omitted.

It follows from (6.3) that:

$$x_1 = \frac{A_0 x_{-1}}{1 + x_0} = A_0 x_{-1} = x_{-1},$$

$$x_2 = \frac{A_1 x_0}{1 + x_1} = 0 = x_0,$$

$$x_3 = \frac{A_2 x_1}{1 + x_2} = A_2 x_1 = x_1 = x_{-1},$$

$$x_4 = \frac{A_3 x_2}{1 + x_3} = 0 = x_0,$$

$$\vdots$$

$$x_k = \frac{A_{k-1} x_{k-2}}{1 + x_{k-1}} = A_{k-1} x_{k-2} = x_{-1},$$

$$x_{k+1} = \frac{A_k x_{k-1}}{1 + x_k} = 0 = x_0.$$

By induction, we see that for all $n \geq 0$:

$$x_{2n-1} = x_{2n+1} = x_{-1} \text{ and } x_{2n-2} = x_{2n} = x_0.$$

Therefore, we get solutions with prime period two.

Similarly, prime period two solutions exist when (6.2) occurs.

Also, similarly to Lemma (2), we see that if $x_{-1} = x_1$ and $x_0 = x_2$, then exactly one of the following situations must happen:

(i) (6.1) occurs and $x_0 = 0$, or

(ii) (6.2) occurs and $x_{-1} = 0$.

Lemma 7: Suppose that:

(i) $M = \max\{A_0, A_1, A_2, \ldots., A_k\} = 1$, and
(ii) $A_0 = A_2 = A_4 = \ldots = A_{k-1} = 1$ or $A_1 = A_3 = A_5 = \ldots = A_k = 1$.

Then every positive solution of Eq. (2) converges to a period two solution.

<u>Proof:</u> Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). First, we will consider the case where

$$(7.1)\quad A_0 = A_2 = A_4 = \ldots = A_{k-1} = 1.$$

The case where $A_1 = A_3 = A_5 = \ldots = A_k = 1$ is similar and will be omitted.

Observe that via (7.1) we get:

$$x_1 = \frac{A_0 x_{-1}}{1+x_0} = \frac{x_{-1}}{1+x_0} < x_{-1},$$

$$x_3 = \frac{A_2 x_1}{1+x_2} = \frac{x_1}{1+x_2} < x_1,$$

$$x_5 = \frac{A_4 x_3}{1+x_4} = \frac{x_3}{1+x_4} < x_3.$$

It now follows by induction that for all $n \geq 0$:

$$(7.2)\quad 0 < \ldots. < x_{2n+1} < x_{2n-1} < \ldots. < x_1 < x_{-1}.$$

Notice that (7.2) is a monotonically decreasing subsequence which is bounded above by x_{-1} and below by zero. Thus, there exists a limit $L \geq 0$ such that:

$$(7.3)\quad \lim_{n\to\infty} x_{2n+1} = L.$$

Also note that:

$$x_2 = \frac{A_1 x_0}{1+x_1} < A_1 x_0 \leq x_0,$$

$$x_4 = \frac{A_3 x_2}{1+x_3} < A_3 x_2 \leq x_2,$$

$$x_6 = \frac{A_5 x_4}{1+x_5} < A_5 x_4 \le x_4 .$$

So it follows by induction that for all $n \ge 0$:

$$(7.4)\quad 0 < \ldots . < x_{2n} < x_{2n-2} < \ldots . < x_2 < x_0 .$$

Thus we see that (7.4) is a monotonically decreasing subsequence which is bounded above by x_0 and below by zero. Therefore, there exists a limit $l \ge 0$ such that:

$$(7.5)\quad \lim_{n\to\infty} x_{2n} = l .$$

Hence, the result follows from (7.3) and (7.5).

Lemma 8: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that:

- *(i)* $M = \max\{ A_0, A_1, A_2, \ldots ., A_k \} = 1$.
- *(ii)* There exists an even i, $0 \le i < k$, such that $A_i < 1$, and
- *(iii)* There exists an odd j, $0 < j \le k$, such that $A_j < 1$.

Then

$$\lim_{n\to\infty} x_n = 0 .$$

<u>Proof:</u> First we will consider the following case:

$$(8.1)\quad A_i = 1 \text{ for all } i \in \{0,1,..,k-2\},\ A_{k-1} < 1 \text{ and } A_k < 1 .$$

All other cases are similar and will be omitted.

It follows from (8.1) that:

$$x_1 = \frac{A_0 x_{-1}}{1+x_0} = \frac{x_{-1}}{1+x_0} < x_{-1},$$
$$x_3 = \frac{A_2 x_1}{1+x_2} = \frac{x_1}{1+x_2} < x_1,$$
$$x_5 = \frac{A_4 x_3}{1+x_4} = \frac{x_3}{1+x_4} < x_3,$$

$$\vdots$$

$$x_{k-2} = \frac{A_{k-3}x_{k-4}}{1+x_{k-3}} = \frac{x_{k-4}}{1+x_{k-3}} < x_{k-4}.$$

Hence we see that:

$$(8.2)\ \ 0 < \ldots < x_{2k+1} < x_{2k-1} < \ldots < x_k < x_{k-2} < \ldots < x_5 < x_3 < x_1 < x_{-1}.$$

Now via (8.1) and (8.2), notice that:

$$x_k = \frac{A_{k-1}x_{k-2}}{1+x_{k-1}} < A_{k-1}x_{k-2} < A_{k-1}x_{-1},$$

$$x_{k+2} = \frac{A_{k+1}x_k}{1+x_{k+1}} < A_{k+1}x_k = A_0x_k = x_k,$$

$$\vdots$$

$$x_{2k-1} = \frac{A_{2k-2}x_{2k-3}}{1+x_{2k-2}} < A_{2k-2}x_{2k-3} = A_{k-2}x_{2k-3} = x_{2k-3},$$

$$x_{2k+1} = \frac{A_{2k}x_{2k-1}}{1+x_{2k}} < A_{2k}x_{2k-1} = A_{k-1}x_{2k-1} < A_{k-1}x_k < A_{k-1}{}^2x_{-1},$$

$$x_{2k+3} = \frac{A_{2k+2}x_{2k+1}}{1+x_{2k+2}} < A_{2k+2}x_{2k+1} = A_0x_{2k+1} = x_{2k+1},$$

$$\vdots$$

So by induction we observe the following property:

$$(8.3)\ \ x_{(k+1)(t+1)-1} < A_{k-1}{}^{t+1}x_{-1} \text{ for all } t \geq 0.$$

Thus from (8.1) and (8.3) we get:

$$(8.4)\ \ 0 \leq \lim_{t\to\infty} x_{(k+1)(t+1)-1} < \lim_{t\to\infty} A_{k-1}{}^{t+1}x_{-1} = 0.$$

Also note that,

$$x_2 = \frac{A_1 x_0}{1 + x_1} = \frac{x_0}{1 + x_1} < x_0,$$
$$x_4 = \frac{A_3 x_2}{1 + x_3} = \frac{x_2}{1 + x_3} < x_2,$$
$$x_6 = \frac{A_5 x_4}{1 + x_5} = \frac{x_4}{1 + x_5} < x_4,$$
$$\vdots$$
$$x_{k-1} = \frac{A_{k-2} x_{k-3}}{1 + x_{k-2}} = \frac{x_{k-3}}{1 + x_{k-2}} < x_{k-3}.$$

It follows that:

$$(8.5)\ \ 0 < \ldots < x_{2k+2} < x_{2k} < \ldots < x_{k+1} < x_{k-1} < \ldots < x_6 < x_4 < x_2 < x_0.$$

Hence via (8.1) and (8.5), we see that:

$$x_{k+1} = \frac{A_k x_{k-1}}{1 + x_k} < A_k x_{k-1} < A_k x_0,$$
$$x_{k+3} = \frac{A_{k+2} x_{k+1}}{1 + x_{k+2}} = \frac{A_1 x_{k+1}}{1 + x_{k+2}} = \frac{x_{k+1}}{1 + x_{k+2}} < x_{k+1},$$
$$\vdots$$
$$x_{2k} = \frac{A_{2k-1} x_{2k-2}}{1 + x_{2k-1}} = \frac{A_{k-2} x_{2k-2}}{1 + x_{2k-1}} = \frac{x_{2k-2}}{1 + x_{2k-1}} < x_{2k-2},$$
$$x_{2k+2} = \frac{A_{2k+1} x_{2k}}{1 + x_{2k+1}} < A_{2k+1} x_{2k} = A_k x_{2k} < A_k x_{k-1} < A_k{}^2 x_0,$$
$$x_{2k+4} = \frac{A_{2k+3} x_{2k+2}}{1 + x_{2k+3}} < A_{2k+3} x_{2k+2} = A_1 x_{2k+2} = x_{2k+2}.$$

So by induction we observe the following property:

$$(8.6)\ \ x_{(k+1)(t+1)} < A_k{}^{t+1} x_0 \text{ for all } t \geq 0.$$

Note that (8.5) is a bounded monotonically decreasing sequence that has a limit L, such that $0 \leq L < x_0$. Also notice that all the terms of (8.6) are a subsequence of (8.5).
Now from (8.6) and (8.1) we get:

$$(8.7)\ \ 0 \le \lim_{t\to\infty} x_{(k+1)(t+1)} < \lim_{t\to\infty} A_k{}^{t+1} x_0 = 0.$$

Therefore, since (8.6) is a subsequence of a monotonically decreasing sequence (8.5) bounded below by zero, we see that via (8.5) and (8.7):

$$(8.8)\ \ \lim_{n\to\infty} x_{2n} = 0\,.$$

Similarly, via (8.2) and (8.4) we see that

$$(8.9)\ \ \lim_{n\to\infty} x_{2n+1} = 0\,.$$

Hence, the result follows from (8.8) and (8.9).

Lemma 9: Suppose that

M =max{ $A_0, A_1, A_2, \ldots., A_k$ } >1.

Then Eq.(2) has unbounded solutions.

<u>Proof:</u> Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$(9.1)\ \ \min\{A_1, A_3, A_5, \ldots., A_k\} > 1 \text{ and } x_{-1} = 0\,.$$

The case where $\min\{A_0, A_2, A_4, \ldots., A_{k-1}\} > 1$ and $x_0 = 0$ is similar and will be omitted.

Observe that via (9.1), we get:

$$x_1 = \frac{A_0 x_{-1}}{1+x_0} = 0\,,$$
$$x_3 = \frac{A_2 x_1}{1+x_2} = 0\,,$$
$$x_5 = \frac{A_4 x_3}{1+x_4} = 0\,,$$
$$\vdots$$

So it follows by induction that $x_{2n-1} = x_{2n+1} = x_{-1} = 0$ for all $n \ge 0$.

In addition, note that via (9.1) we get:

$$x_2 = \frac{A_1 x_0}{1 + x_1} = A_1 x_0 \geq \min\{A_1, A_3, A_5, ..., A_k\} x_0,$$

$$x_4 = \frac{A_3 x_2}{1 + x_3} = A_3 x_2 \geq \left(\min\{A_1, A_3, A_5, ..., A_k\}\right)^2 x_0,$$

$$x_6 = \frac{A_5 x_4}{1 + x_5} = A_5 x_4 \geq \left(\min\{A_1, A_3, A_5, ..., A_k\}\right)^3 x_0.$$

It follows by induction that:

$$(9.2)\quad x_{2n} \geq \left(\min\{A_1, A_3, A_5, ..., A_k\}\right)^n x_0 \text{ for all } n \geq 0.$$

Also from (9.1) we see that:

$$(9.3)\quad \lim_{n\to\infty}\left[\left(\min\{A_1, A_3, A_5, ..., A_k\}\right)^n x_0\right] = \infty\ .$$

Now notice that from (9.2) and (9.3) we get:

$$\lim_{n\to\infty} x_{2n} = \infty\ ,$$

from which the result follows.

4 The Case where $\{A_n\}_{n=0}^{\infty}$ is periodic with prime period 3

In this section, we will assume that $\{A_n\}_{n=0}^{\infty}$ is periodic with prime period three and the initial conditions x_0 and x_{-1} are non-negative real numbers. It is our goal to investigate the long term behavior of the solutions of Eq. (2).

Let

$$M = \max\{ A_0, A_1, A_2 \}.$$

In this section, we will prove the following properties of Eq. (2):

- When $M \leq 1$, every solution of Eq. (2) converges to zero.
- When $M > 1$, we get one of the following three situations:
  - If $A_0 A_1 A_2 < 1$, then every solution of Eq. (2) converges to zero.
  - If $A_0 A_1 A_2 = 1$, then every solution of Eq. (2) converges to a period six solution.
  - If $A_0 A_1 A_2 > 1$, then Eq. (2) has unbounded solutions.

Lemma 10: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$M = \max\{ A_0, A_1, A_2 \} < 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

Proof:

Proof is similar to the proof of Lemma 1 and will be omitted.

Lemma 11: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$M = \max\{ A_0, A_1, A_2 \} = 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

Proof: We will consider the case

$$(11.1)\quad A_0 = 1,\ A_1 = 1 \text{ and } A_2 < 1.$$

All other cases are similar and will be omitted.

It follows by computation and via (11.1) that:

$$x_1 = \frac{A_0 x_{-1}}{1+x_0} = \frac{x_{-1}}{1+x_0} < x_{-1},$$
$$x_2 = \frac{A_1 x_0}{1+x_1} = \frac{x_0}{1+x_1} < x_0,$$
$$x_3 = \frac{A_2 x_1}{1+x_2} < A_2 x_1 < x_1,$$
$$x_4 = \frac{A_3 x_2}{1+x_3} = \frac{A_0 x_2}{1+x_3} = \frac{x_2}{1+x_3} < x_2,$$
$$x_5 = \frac{A_4 x_3}{1+x_4} = \frac{A_1 x_3}{1+x_4} = \frac{x_3}{1+x_4} < x_3,$$
$$x_6 = \frac{A_5 x_4}{1+x_5} = \frac{A_2 x_4}{1+x_5} < A_2 x_4 < x_4,$$
$$x_7 = \frac{A_6 x_5}{1+x_6} = \frac{A_0 x_5}{1+x_6} = \frac{x_5}{1+x_6} < x_5,$$
$$x_8 = \frac{A_7 x_6}{1+x_7} = \frac{A_1 x_6}{1+x_7} = \frac{x_6}{1+x_7} < x_6,$$
$$x_9 = \frac{A_8 x_7}{1+x_8} = \frac{A_2 x_7}{1+x_8} < A_2 x_7 < x_7.$$

So by induction, we observe the following properties:

$$(11.2)\quad 0 < \ldots < x_{2n+2} < x_{2n} < \ldots < x_6 < x_4 < x_2 < x_0 \text{ for all } n \geq 0,$$

$$(11.3)\quad x_{3(2t+2)} < A_2^{t+1} x_0 \text{ for all } t \geq 0,$$

$$(11.4)\quad 0 < \ldots < x_{2n+1} < x_{2n-1} < \ldots < x_5 < x_3 < x_1 < x_{-1} \text{ for all } n \geq 0,$$

$$(11.5)\quad x_{3(2t+1)} < A_2^{t+1} x_{-1} \text{ for all } t \geq 0.$$

Note that (11.2) is a bounded monotonically decreasing sequence that has a limit L, such that $0 \leq L < x_0$. Also note that the all the terms of (11.3) are a subsequence of (11.2). In addition, from (11.3) and (11.1) notice that:

$$(11.6)\ \ 0 \le \lim_{t\to\infty} x_{3(2t+2)} < \lim_{t\to\infty} A_2^{\,t+1} x_0 = 0.$$

Therefore, since (11.3) is a subsequence of a monotonically decreasing sequence (11.2) bounded below by zero, we see that via (11.2) and (11.6):

$$\lim_{n\to\infty} x_{2n} = 0.$$

Similarly, we observe that:

$$\lim_{n\to\infty} x_{2n+1} = 0.$$

Hence, the result follows that:

$$\lim_{n\to\infty} x_n = 0.$$

The following theorem is a consequence of Lemmas 10 and 11.

Theorem: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$M = \max\{ A_0, A_1, A_2 \} \le 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

Now we will assume that $\max\{ A_0, A_1, A_2 \} > 1$.

Lemma 12: Let $\{x_n\}_{n=-1}^{\infty}$ be a positive solution of Eq. (2). Suppose that

$$(12.1)\ \ A_0 A_1 A_2 < 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

Proof: Observe that via (12.1) we get:

$$x_1 = \frac{A_0 x_{-1}}{1 + x_0} < A_0 x_{-1},$$
$$x_2 = \frac{A_1 x_0}{1 + x_1} < A_1 x_0,$$

$$x_3 = \frac{A_2 x_1}{1+x_2} < A_2 x_1,$$

$$x_4 = \frac{A_3 x_2}{1+x_3} = \frac{A_0 x_2}{1+x_3} < A_0 x_2,$$

$$x_5 = \frac{A_4 x_3}{1+x_4} = \frac{A_1 x_3}{1+x_4} < A_1 x_3,$$

$$x_6 = \frac{A_5 x_4}{1+x_5} = \frac{A_2 x_4}{1+x_5} < A_2 x_4 < A_2 (A_0 x_2) < A_0 A_1 A_2 x_0,$$

$$x_7 = \frac{A_6 x_5}{1+x_6} = \frac{A_0 x_5}{1+x_6} < A_0 x_5 < A_0 (A_1 x_3) < A_0 A_1 A_2 x_1,$$

$$x_8 = \frac{A_7 x_6}{1+x_7} = \frac{A_1 x_6}{1+x_7} < A_1 x_6 < A_1 (A_2 x_4) < A_0 A_1 A_2 x_2,$$

$$x_9 = \frac{A_8 x_7}{1+x_8} = \frac{A_2 x_7}{1+x_8} < A_2 x_7 < A_0 A_1 A_2 x_3,$$

$$x_{10} = \frac{A_9 x_8}{1+x_9} = \frac{A_0 x_8}{1+x_9} < A_0 x_8 < A_0 A_1 A_2 x_4,$$

$$x_{11} = \frac{A_{10} x_9}{1+x_{10}} = \frac{A_1 x_9}{1+x_{10}} < A_1 x_9 < A_1 A_2 x_7 < A_0 A_1 A_2 x_5,$$

$$x_{12} = \frac{A_{11} x_{10}}{1+x_{11}} = \frac{A_2 x_{10}}{1+x_{11}} < A_2 x_{10} < A_0 A_2 x_8 < A_0 A_1 A_2 x_6,$$

$$x_{13} = \frac{A_{12} x_{11}}{1+x_{12}} = \frac{A_0 x_{11}}{1+x_{12}} < A_0 x_{11} < A_0 A_1 A_2 x_7 < (A_0 A_1 A_2)^2 x_1,$$

$$x_{14} = \frac{A_{13} x_{12}}{1+x_{13}} = \frac{A_1 x_{12}}{1+x_{13}} < A_1 x_{12} < A_0 A_1 A_2 x_8 < (A_0 A_1 A_2)^2 x_2,$$

$$x_{15} = \frac{A_{14} x_{13}}{1+x_{14}} = \frac{A_2 x_{13}}{1+x_{14}} < A_2 x_{13} < A_0 A_2 x_{11} < A_0 A_1 A_2 x_9 < (A_0 A_1 A_2)^2 x_3,$$

$$x_{16} = \frac{A_{15} x_{14}}{1+x_{15}} = \frac{A_0 x_{14}}{1+x_{15}} < A_0 x_{14} < A_0 A_1 x_{12} < A_0 A_1 A_2 x_{10} < (A_0 A_1 A_2)^2 x_4,$$

$$x_{17} = \frac{A_{16} x_{15}}{1+x_{16}} = \frac{A_1 x_{15}}{1+x_{16}} < A_1 x_{15} < A_0 A_1 A_2 x_{11} < (A_0 A_1 A_2)^2 x_5,$$

$$x_{18} = \frac{A_{17} x_{16}}{1+x_{17}} = \frac{A_2 x_{16}}{1+x_{17}} < A_2 x_{16} < A_0 A_1 A_2 x_{12} < (A_0 A_1 A_2)^3 x_0.$$

It follows by induction that for all $n \geq 0$:

$$(12.2)\ \ 0 \leq x_{6n+6} < (A_0 A_1 A_2)^{n+1} x_0,$$

$$(12.3)\ \ 0 \leq x_{6n+7} < (A_0 A_1 A_2)^{n+1} x_1,$$

$$(12.4)\ \ 0 \le x_{6n+8} < (A_0 A_1 A_2)^{n+1} x_2 ,$$

$$(12.5)\ \ 0 \le x_{6n+9} < (A_0 A_1 A_2)^{n+1} x_3 ,$$

$$(12.6)\ \ 0 \le x_{6n+10} < (A_0 A_1 A_2)^{n+1} x_4 ,$$

$$(12.7)\ \ 0 \le x_{6n+11} < (A_0 A_1 A_2)^{n+1} x_5 .$$

Notice that via (12.1) and (12.2), it follows that:

$$0 \le \lim_{n\to\infty} x_{6n+6} < \lim_{n\to\infty} (A_0 A_1 A_2)^{n+1} x_0 = 0 .$$

Therefore, we see that:

$$\lim_{n\to\infty} x_{6n+6} = 0 .$$

Similarly, via (12.1), (12.2), (12.3), (12.4), (12.5), (12.6), and (12.7) we get:

$$(12.8)\ \ \lim_{n\to\infty} x_{6n+6+j} = 0 \text{ for all } j \in \{0,1,2,3,4,5\} .$$

Hence the result follows from (12.8).

Lemma 13: Eq. (2) has solutions with prime period six if:

$$(13.1)\ \ A_0 A_1 A_2 = 1.$$

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that $x_{-1} = 0$ and $x_0 > 0$.

The case where $x_0 = 0$ and $x_{-1} > 0$ is similar and will be omitted.

Observe that via (13.1) we get:

$$x_1 = \frac{A_0 x_{-1}}{1 + x_0} = 0 ,$$

$$x_2 = \frac{A_1 x_0}{1 + x_1} = A_1 x_0 ,$$

$$x_3 = \frac{A_2 x_1}{1 + x_2} = 0 ,$$

$$x_4 = \frac{A_3 x_2}{1+x_3} = A_0 x_2,$$

$$x_5 = \frac{A_4 x_3}{1+x_4} = 0 = x_{-1},$$

$$x_6 = \frac{A_5 x_4}{1+x_4} = A_2 x_4 = A_2(A_0 x_2) = A_0 A_1 A_2 x_0 = x_0,$$

$$x_7 = \frac{A_6 x_5}{1+x_6} = 0 = x_1,$$

$$x_8 = \frac{A_7 x_6}{1+x_7} = A_1 x_6 = A_0 A_1 A_2 x_2 = x_2,$$

$$x_9 = \frac{A_8 x_7}{1+x_8} = 0 = x_3,$$

$$x_{10} = \frac{A_9 x_8}{1+x_9} = A_0 x_8 = A_0(A_1 x_6) = A_0 A_1 A_2 x_4 = x_4,$$

$$x_{11} = \frac{A_{10} x_9}{1+x_{10}} = 0 = x_5.$$

Therefore, it follows by induction that:

$$\begin{aligned}
x_{6n} &= x_{6n+6} \quad \text{for all } n \geq 0, \\
x_{6n+1} &= x_{6n+7} \quad \text{for all } n \geq 0, \\
x_{6n+2} &= x_{6n+8} \quad \text{for all } n \geq 0, \\
x_{6n+3} &= x_{6n+9} \quad \text{for all } n \geq 0, \\
x_{6n+4} &= x_{6n+10} \quad \text{for all } n \geq 0, \\
x_{6n+5} &= x_{6n+11} \quad \text{for all } n \geq 0.
\end{aligned}$$

Thus the result follows.

Lemma 14: Suppose that

$$(14.1) \quad A_0 A_1 A_2 = 1.$$

Then every positive solution of Eq. (2) converges to a period six solution.

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Observe that via (14.1) we get:

$$x_1 = \frac{A_0 x_{-1}}{1+x_0} < A_0 x_{-1},$$

$$x_2 = \frac{A_1 x_0}{1+x_1} < A_1 x_0,$$

$$x_3 = \frac{A_2 x_1}{1+x_2} < A_2 x_1,$$

$$x_4 = \frac{A_3 x_2}{1+x_3} = \frac{A_0 x_2}{1+x_3} < A_0 x_2,$$

$$x_5 = \frac{A_4 x_3}{1+x_4} = \frac{A_1 x_3}{1+x_4} < A_1 x_3,$$

$$x_6 = \frac{A_5 x_4}{1+x_5} = \frac{A_2 x_4}{1+x_5} < A_2 x_4 < A_2(A_0 x_2) < A_0 A_1 A_2 x_0 = x_0,$$

$$x_7 = \frac{A_6 x_5}{1+x_6} = \frac{A_0 x_5}{1+x_6} < A_0 x_5 < A_0(A_1 x_3) < A_0 A_1 A_2 x_1 = x_1,$$

$$x_8 = \frac{A_7 x_6}{1+x_7} = \frac{A_1 x_6}{1+x_7} < A_1 x_6 < A_1(A_2 x_4) < A_0 A_1 A_2 x_2 = x_2,$$

$$x_9 = \frac{A_8 x_7}{1+x_8} = \frac{A_2 x_7}{1+x_8} < A_2 x_7 < A_0 A_1 A_2 x_3 = x_3,$$

$$x_{10} = \frac{A_9 x_8}{1+x_9} = \frac{A_0 x_8}{1+x_9} < A_0 x_8 < A_0 A_1 A_2 x_4 = x_4,$$

$$x_{11} = \frac{A_{10} x_9}{1+x_{10}} = \frac{A_1 x_9}{1+x_{10}} < A_1 x_9 < A_1(A_2 x_7) < A_0 A_1 A_2 x_5 = x_5,$$

$$x_{12} = \frac{A_{11} x_{10}}{1+x_{11}} = \frac{A_2 x_{10}}{1+x_{11}} < A_2 x_{10} < A_0(A_2 x_8) < A_0 A_1 A_2 x_6 = x_6,$$

$$x_{13} = \frac{A_{12} x_{11}}{1+x_{12}} = \frac{A_0 x_{11}}{1+x_{12}} < A_0 x_{11} < A_0(A_1 A_2 x_7) = x_7 < x_1,$$

$$x_{14} = \frac{A_{13} x_{12}}{1+x_{13}} = \frac{A_1 x_{12}}{1+x_{13}} < A_1 x_{12} < A_0 A_1 A_2 x_8 = x_8 < x_2,$$

$$x_{15} = \frac{A_{14} x_{13}}{1+x_{14}} = \frac{A_2 x_{13}}{1+x_{14}} < A_2 x_{13} < A_0 A_2 x_{11} < A_0 A_1 A_2 x_9 = x_9 < x_3,$$

$$x_{16} = \frac{A_{15} x_{14}}{1+x_{15}} = \frac{A_0 x_{14}}{1+x_{15}} < A_0 x_{14} < A_0 A_1 x_{12} < A_0 A_1 A_2 x_{10} = x_{10} < x_4,$$

$$x_{17} = \frac{A_{16} x_{15}}{1+x_{16}} = \frac{A_1 x_{15}}{1+x_{16}} < A_1 x_{15} < A_0 A_1 A_2 x_{11} = x_{11} < x_5,$$

$$x_{18} = \frac{A_{17} x_{16}}{1+x_{17}} = \frac{A_2 x_{16}}{1+x_{17}} < A_2 x_{16} < A_0 A_1 A_2 x_{12} = x_{12} < x_6 < x_0,$$

$$\vdots$$

Thus, we observe the following properties by induction for all $n \geq 0$:

$$(14.2)\ 0 < \ldots < x_{6n+6} < x_{6n} < \ldots < x_{12} < x_6 < x_0,$$

$$(14.3)\ \ 0 < \ldots . < x_{6n+7} < x_{6n+1} < \ldots . < x_{13} < x_7 < x_1,$$

$$(14.4)\ \ 0 < \ldots . < x_{6n+8} < x_{6n+2} < \ldots . < x_{14} < x_8 < x_2,$$

$$(14.5)\ \ 0 < \ldots . < x_{6n+9} < x_{6n+3} < \ldots . < x_{15} < x_9 < x_3,$$

$$(14.6)\ \ 0 < \ldots . < x_{6n+10} < x_{6n+4} < \ldots . < x_{16} < x_{10} < x_4,$$

$$(14.7)\ \ 0 < \ldots . < x_{6n+11} < x_{6n+5} < \ldots . < x_{17} < x_{11} < x_5.$$

Now notice that (14.2) is a monotonically decreasing subsequence which is bounded above by x_0 and below by zero. Thus there exists a limit $L \geq 0$ such that:

$$\lim_{n\to\infty} x_{6n} = L.$$

Hence, the result follows via (14.2), (14.3), (14.4), (14.5), (14.6) and (14.7).

Lemma 15: Suppose that

$$(15.1)\ \ A_0 A_1 A_2 > 1.$$

Then Eq. (2) has unbounded solutions.

<u>Proof:</u> Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$x_{-1} = 0 \text{ and } x_0 > 0\ .$$

The case where $x_0 = 0$ and $x_{-1} > 0$ is similar and will be omitted.

Observe that via (15.1) we get:

$$x_1 = \frac{A_0 x_{-1}}{1 + x_0} = 0,$$
$$x_2 = \frac{A_1 x_0}{1 + x_1} = A_1 x_0,$$
$$x_3 = \frac{A_2 x_1}{1 + x_2} = 0,$$
$$x_4 = \frac{A_3 x_2}{1 + x_3} = A_0 x_2,$$

$$x_5 = \frac{A_4 x_3}{1+x_4} = 0,$$

$$x_6 = \frac{A_5 x_4}{1+x_4} = A_2 x_4 = A_2(A_0 x_2) = A_0 A_1 A_2 x_0,$$

$$x_7 = \frac{A_6 x_5}{1+x_6} = 0,$$

$$x_8 = \frac{A_7 x_6}{1+x_7} = A_1 x_6 = A_0 A_1 A_2 x_2,$$

$$x_9 = \frac{A_8 x_7}{1+x_8} = 0,$$

$$x_{10} = \frac{A_9 x_8}{1+x_9} = A_0 x_8 = A_0(A_1 x_6) = A_0 A_1 A_2 x_4,$$

$$x_{11} = \frac{A_{10} x_9}{1+x_{10}} = 0,$$

$$x_{12} = \frac{A_{11} x_{10}}{1+x_{11}} = A_2 x_{10} = A_2(A_0 A_1 x_6) = (A_0 A_1 A_2)^2 x_0,$$

$$x_{13} = \frac{A_{12} x_{11}}{1+x_{12}} = 0,$$

$$x_{14} = \frac{A_{13} x_{12}}{1+x_{13}} = A_1 x_{12} = A_1(A_2 x_{10}) = A_1(A_2 A_0 x_8) = (A_0 A_1 A_2)^2 x_2,$$

$$x_{15} = \frac{A_{14} x_{13}}{1+x_{14}} = 0,$$

$$x_{16} = \frac{A_{15} x_{14}}{1+x_{15}} = A_0 x_{14} = A_0(A_1 A_2 x_{10}) = (A_0 A_1 A_2)^2 x_4,$$

$$\vdots$$

So it follows by induction that:

$$(15.2)\ \ x_{2n+1} = 0 \text{ for all } n \geq 0,$$

$$(15.3)\ \ x_{6n} = (A_0 A_1 A_2)^n x_0 \text{ for all } n \geq 1,$$

$$(15.4)\ \ x_{6n+2} = (A_0 A_1 A_2)^n x_2 \text{ for all } n \geq 1,$$

$$(15.5)\ \ x_{6n+4} = (A_0 A_1 A_2)^n x_4 \text{ for all } n \geq 1.$$

Therefore, we see that via (15.1), (15.2), (15.3), (15.4), and (15.5) we get:

$$\lim_{n\to\infty} x_{6n} = \lim_{n\to\infty} (A_0 A_1 A_2)^n x_0 = \infty ,$$

$$\lim_{n\to\infty} x_{6n+2} = \lim_{n\to\infty} (A_0 A_1 A_2)^n x_2 = \infty ,$$

$$\lim_{n\to\infty} x_{6n+4} = \lim_{n\to\infty} (A_0 A_1 A_2)^n x_4 = \infty .$$

Hence the result follows.

5 The Case $\{A_n\}_{n=0}^{\infty}$ is periodic with an odd prime period k+1.

In this section, we will assume that $\{A_n\}_{n=0}^{\infty}$ is periodic with an odd prime period k+1 and the initial conditions x_0 and x_{-1} are non-negative real numbers. It is our goal to investigate the long term behavior of the solutions of Eq. (2).

Let

$$M = \max\{ A_0, A_1, A_2, \ldots., A_k \}.$$

In this section, we will prove the following properties of Eq. (2):

- When $M \le 1$, every solution of Eq. (2) converges to zero.
- When $M > 1$, then we get the following three situations:
  - If $\prod_i A_i < 1$, then every solution of Eq. (2) converges to zero.
  - If $\prod_i A_i = 1$, then every solution of Eq. (2) converges to a period 2(k+1) solution.
  - If $\prod_i A_i > 1$, then Eq. (2) has unbounded solutions.

Lemma 16: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$M = \max\{ A_0, A_1, A_2, \ldots., A_k \} < 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

<u>Proof:</u>

Proof is similar to the proof of Lemma 1 and will be omitted.

Lemma 17: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$M = \max\{ A_0, A_1, A_2, \ldots., A_k \} = 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

Proof: Consider the case

$$(17.1)\quad A_i = 1 \text{ for all } i \in \{0,1,..,k-1\} \text{ and } A_k < 1.$$

All other cases are similar and will be omitted.

It follows by computation and via (17.1) that:

$$x_1 = \frac{A_0 x_{-1}}{1+x_0} = \frac{x_{-1}}{1+x_0} < x_{-1},$$

$$x_2 = \frac{A_1 x_0}{1+x_1} = \frac{x_0}{1+x_1} < x_0,$$

$$x_3 = \frac{A_2 x_1}{1+x_2} = \frac{x_1}{1+x_2} < x_1,$$

$$\vdots$$

$$x_k = \frac{A_{k-1} x_{k-2}}{1+x_{k-1}} = \frac{x_{k-2}}{1+x_{k-1}} < x_{k-2},$$

$$x_{k+1} = \frac{A_k x_{k-1}}{1+x_k} < A_k x_{k-1},$$

$$x_{k+2} = \frac{A_{k+1} x_k}{1+x_{k+1}} < A_{k+1} x_k = A_0 x_k = x_k,$$

$$x_{k+3} = \frac{A_{k+2} x_{k+1}}{1+x_{k+2}} < A_{k+2} x_{k+1} = A_1 x_{k+1} = x_{k+1},$$

$$\vdots$$

$$x_{2k+1} = \frac{A_{2k} x_{2k-1}}{1+x_{2k}} < A_{2k} x_{2k-1} = A_{k-1} x_{2k-1} = x_{2k-1},$$

$$x_{2k+2} = \frac{A_{2k+1} x_{2k}}{1+x_{2k+1}} < A_{2k+1} x_{2k} = A_k x_{2k},$$

$$x_{2k+3} = \frac{A_{2k+2} x_{2k+1}}{1+x_{2k+2}} < A_{2k+2} x_{2k+1} = A_0 x_{2k+1} = x_{2k+1},$$

$$\vdots$$

$$x_{3k+2} = \frac{A_{3k+1}x_{3k}}{1+x_{3k+1}} < A_{3k+1}x_{3k} = A_{k-1}x_{3k} = x_{3k},$$

$$x_{3k+3} = \frac{A_{3k+2}x_{3k+1}}{1+x_{3k+21}} < A_{3k+2}x_{3k+1} = A_k x_{3k+1},$$

$$x_{3k+4} = \frac{A_{3k+3}x_{3k+2}}{1+x_{3k+3}} < A_{3k+3}x_{3k+2} = A_0 x_{3k+2} = x_{3k+2},$$

$$\vdots$$

$$x_{4k+3} = \frac{A_{4k+2}x_{4k+1}}{1+x_{4k+2}} < A_{4k+2}x_{4k+1} = A_{k-1}x_{4k+1} = x_{4k+1},$$

$$x_{4k+4} = \frac{A_{4k+3}x_{4k+2}}{1+x_{4k+31}} < A_{4k+3}x_{4k+2} = A_k x_{4k+2},$$

$$x_{4k+5} = \frac{A_{4k+4}x_{4k+3}}{1+x_{4k+4}} < A_{4k+4}x_{4k+3} = A_0 x_{4k+3} = x_{4k+3},$$

$$\vdots$$

By induction, we observe the following properties:

$$(17.2)\quad 0 < \ldots < x_{2n+2} < x_{2n} < \ldots < x_6 < x_4 < x_2 < x_0 \text{ for all } n \geq 0,$$

$$(17.3)\quad x_{(k+1)(2t+2)} < A_k{}^{t+1}x_0 \text{ for all } t \geq 0,$$

$$(17.4)\quad 0 < \ldots < x_{2n+3} < x_{2n+1} < \ldots < x_5 < x_3 < x_1 < x_{-1} \text{ for all } n \geq 0,$$

$$(17.5)\quad x_{(k+1)(2t+1)} < A_k{}^{t+1}x_{-1} \text{ for all } t \geq 0.$$

Notice that (17.2) is a bounded monotonically decreasing sequence that has a limit L, such that $0 \leq L < x_0$. Also note that all the terms of (17.3) are a subsequence of (17.2). In addition, via (17.3) and (17.1) we see that:

$$(17.6)\quad 0 \leq \lim_{t\to\infty} x_{(k+1)(2t+2)} < \lim_{t\to\infty} A_k{}^{t+1}x_0 = 0.$$

Therefore, we see that (17.3) is a subsequence of a monotonically decreasing sequence (17.2) which is bounded below by zero. Thus, it follows via (17.2) and (17.6) that:

$$\lim_{n\to\infty} x_{2n} = 0.$$

Similarly, via (17.1), (17.4) and (17.5) we get:

$$0 \leq \lim_{t\to\infty} x_{(k+1)(2t+1)} < \lim_{t\to\infty} A_k{}^{t+1} x_{-1} = 0,$$

from which we see that:

$$\lim_{n\to\infty} x_{2n+1} = 0.$$

Hence the result follows.

The following theorem is a consequence of Lemmas 16 and 17.

Theorem: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$M = \max\{ A_0, A_1, A_2, \ldots, A_k \} \leq 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

Now we will assume that $\max\{ A_0, A_1, A_2, \ldots, A_k \} > 1$.

Lemma 18: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$(18.1) \prod_{i=0}^{k} A_i < 1.$$

Then

$$\lim_{n\to\infty} x_n = 0.$$

Proof: Observe that via (18.1):

$$x_{2(k+1)} = \frac{A_{2(k+1)-1} x_{2(k+1)-2}}{1+ x_{2(k+1)-1}} = \frac{A_{2k+1} x_{2k}}{1+ x_{2k+1}} = \frac{A_k x_{2k}}{1+ x_{2k+1}} < A_k x_{2k} = A_k \left[\frac{A_{2k-1} x_{2k-2}}{1+ x_{2k-1}} \right]$$

$$< A_k A_{2k-1} x_{2k-2} = A_k A_{k-2} x_{2k-2} = A_k A_{k-2} \left[\frac{A_{2k-3} x_{2k-4}}{1+ x_{2k-3}} \right] < A_k A_{k-2} A_{2k-3} x_{2k-4}$$

$$A_k A_{k-2} A_{k-4} x_{2k-4} < \ldots < A_k A_{k-2} A_{k-4} \ldots A_0 x_{2k-k} = A_k A_{k-2} A_{k-4} \ldots A_0 x_k =$$

$$A_k A_{k-2} A_{k-4} \ldots. A_0 \left[\frac{A_{k-1} x_{k-2}}{1+ x_{k-1}} \right] < \left(A_k A_{k-2} A_{k-4} \ldots. A_0 \right) A_{k-1} x_{k-2} =$$

$$\left(A_k A_{k-2} A_{k-4} \ldots. A_0 \right) A_{k-1} \left[\frac{A_{k-3} x_{k-4}}{1+ x_{k-3}} \right] < \left(A_k A_{k-2} A_{k-4} \ldots. A_0 \right) A_{k-1} A_{k-3} x_{k-4} < \ldots\ldots$$

$$< \left(A_k A_{k-2} A_{k-4} \ldots. A_0 \right) \left(A_{k-1} A_{k-3} A_{k-5} \ldots. A_1 \right) x_0 = \left[\prod_{i=0}^{k} A_i \right] x_0 < x_0 ,$$

$$x_{4(k+1)} = \frac{A_{4(k+1)-1} x_{4(k+1)-2}}{1+ x_{4(k+1)-1}} = \frac{A_{4k+3} x_{4k+2}}{1+ x_{4k+3}} = \frac{A_k x_{4k+2}}{1+ x_{4k+3}} < A_k x_{4k+2} = A_k \left[\frac{A_{4k+1} x_{4k}}{1+ x_{4k+1}} \right]$$

$$< A_k A_{4k+1} x_{4k} = A_k A_{k-2} x_{4k} = A_k A_{k-2} \left[\frac{A_{4k-1} x_{4k-2}}{1+ x_{4k-1}} \right] < A_k A_{k-2} A_{4k-1} x_{4k-2} =$$

$$A_k A_{k-2} A_{k-4} x_{4k-2} < \ldots. < A_k A_{k-2} A_{k-4} \ldots. A_0 x_{4k-(k-2)} = A_k A_{k-2} A_{k-4} \ldots. A_0 x_{3k+2} =$$

$$A_k A_{k-2} A_{k-4} \ldots. A_0 \left[\frac{A_{3k+1} x_{3k}}{1+ x_{3k+1}} \right] < A_k A_{k-2} A_{k-4} \ldots. A_0 A_{3k+1} x_{3k} = \left(A_k A_{k-2} A_{k-4} \ldots. A_0 \right) A_{k-1} x_{3k} =$$

$$\left(A_k A_{k-2} A_{k-4} \ldots. A_0 \right) A_{k-1} \left[\frac{A_{3k-1} x_{3k-2}}{1+ x_{3k-1}} \right] < \left(A_k A_{k-2} A_{k-4} \ldots. A_0 \right) A_{k-1} A_{3k-1} x_{3k-2} =$$

$$\left(A_k A_{k-2} A_{k-4} \ldots. A_0 \right) A_{k-1} A_{k-3} x_{3k-2} < \ldots. < \left(A_k A_{k-2} A_{k-4} \ldots. A_0 \right) \left(A_{k-1} A_{k-3} A_{k-5} \ldots. A_1 \right) x_{2(k+1)} =$$

$$= \left[\prod_{i=0}^{k} A_i \right] x_{2(k+1)} < \left[\prod_{i=0}^{k} A_i \right]^2 x_0 < x_0 .$$

So it follows by induction that for all $n \geq 0$:

$$(18.2) \quad 0 \leq x_{2(k+1)n+2(k+1)} < \left[\prod_{i=0}^{k} A_i \right]^{n+1} x_0 .$$

Therefore, via (18.1) and (18.2) we get:

$$\lim_{n\to\infty} x_{2(k+1)n+2(k+1)} \le \lim_{n\to\infty}\left[\prod_{i=0}^{k} A_i\right]^{n+1} x_0 = 0.$$

Similarly, we see that:

$$(18.3)\ \lim_{n\to\infty} x_{2(k+1)n+2(k+1)+j} = 0 \text{ for all } j \in \{0,1,2,\ldots,2(k+1)-1\}.$$

Hence the result follows from (18.3).

Lemma 19: Eq. (2) has solutions with period 2(k+1) if:

$$(19.1)\ \prod_{i=0}^{k} A_i = 1.$$

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that $x_{-1} = 0$ and $x_0 > 0$.

The case where $x_0 = 0$ and $x_{-1} > 0$ is similar and will be omitted.

Observe that:

$$x_1 = \frac{A_0 x_{-1}}{1 + x_0} = 0,$$

$$x_3 = \frac{A_2 x_1}{1 + x_2} = 0,$$

$$x_5 = \frac{A_4 x_3}{1 + x_4} = 0,$$

$$\vdots$$

So it follows by induction that $x_{2n-1} = x_{2n+1} = x_{-1} = 0$ for all $n \ge 0$.
Note that via (19.1) we get:

$$x_{2(k+1)} = \frac{A_{2(k+1)-1}x_{2(k+1)-2}}{1 + x_{2(k+1)-1}} = A_{2k+1}x_{2k} = A_k x_{2k} = A_k\left[\frac{A_{2k-1}x_{2k-2}}{1 + x_{2k-1}}\right]$$

$$= A_k A_{2k-1} x_{2k-2} = A_k A_{k-2} x_{2k-2} = A_k A_{k-2}\left[\frac{A_{2k-3}x_{2k-4}}{1 + x_{2k-3}}\right] = A_k A_{k-2} A_{2k-3} x_{2k-4}$$

$$A_k A_{k-2} A_{k-4} x_{2k-4} = \ldots\ldots = A_k A_{k-2} A_{k-4} \ldots. A_0 x_{2k-k} = A_k A_{k-2} A_{k-4} \ldots. A_0 x_k =$$

$$A_k A_{k-2} A_{k-4} \dots . A_0 \left[\frac{A_{k-1} x_{k-2}}{1+ x_{k-1}} \right] = \left(A_k A_{k-2} A_{k-4} \dots . A_0 \right) A_{k-1} x_{k-2} =$$

$$\left(A_k A_{k-2} A_{k-4} \dots . A_0 \right) A_{k-1} \left[\frac{A_{k-3} x_{k-4}}{1+ x_{k-3}} \right] = \left(A_k A_{k-2} A_{k-4} \dots . A_0 \right) A_{k-1} A_{k-3} x_{k-4} = \dots \dots$$

$$= \left(A_k A_{k-2} A_{k-4} \dots . A_0 \right) \left(A_{k-1} A_{k-3} A_{k-5} \dots . A_1 \right) x_0 = \left[\prod_{i=0}^{k} A_i \right] x_0 = x_0 .$$

It follows by induction that for all $n \geq 0$:

$$x_{2(k+1)n} = x_{2(k+1)n+2(k+1)} .$$

Similarly, we see that:

$$\begin{aligned} x_{2(k+1)n+1} &= x_{2(k+1)n+2(k+1)+1} , \\ x_{2(k+1)n+2} &= x_{2(k+1)n+2(k+1)+2} , \\ x_{2(k+1)n+3} &= x_{2(k+1)n+2(k+1)+3} , \\ &\vdots \\ x_{2(k+1)n+[2(k+1)-1]} &= x_{2(k+1)n+2(k+1)+[2(k+1)-1]} . \end{aligned}$$

Hence the result follows.

Lemma 20: Suppose that

$$(20.1) \quad \prod_{i=0}^{k} A_i = 1.$$

Then every solution of Eq. (2) converges to a period 2(k+1) solution.

<u>Proof:</u> Let $\left\{ x_n \right\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Observe that via (20.1):

$$x_{2(k+1)} = \frac{A_{2(k+1)-1} x_{2(k+1)-2}}{1+ x_{2(k+1)-1}} = \frac{A_{2k+1} x_{2k}}{1+ x_{2k+1}} = \frac{A_k x_{2k}}{1+ x_{2k+1}} < A_k x_{2k} = A_k \left[\frac{A_{2k-1} x_{2k-2}}{1+ x_{2k-1}} \right]$$

$$< A_k A_{2k-1} x_{2k-2} = A_k A_{k-2} x_{2k-2} = A_k A_{k-2} \left[\frac{A_{2k-3} x_{2k-4}}{1+ x_{2k-3}} \right] < A_k A_{k-2} A_{2k-3} x_{2k-4} =$$

$$A_k A_{k-2} A_{k-4} x_{2k-4} < \ldots\ldots < A_k A_{k-2} A_{k-4} \ldots . A_0 x_{2k-k} = A_k A_{k-2} A_{k-4} \ldots . A_0 x_k =$$

$$A_k A_{k-2} A_{k-4} \ldots . A_0 \left[\frac{A_{k-1} x_{k-2}}{1 + x_{k-1}} \right] < \left(A_k A_{k-2} A_{k-4} \ldots . A_0 \right) A_{k-1} x_{k-2} =$$

$$\left(A_k A_{k-2} A_{k-4} \ldots . A_0 \right) A_{k-1} \left[\frac{A_{k-3} x_{k-4}}{1 + x_{k-3}} \right] < \left(A_k A_{k-2} A_{k-4} \ldots . A_0 \right) A_{k-1} A_{k-3} x_{k-4} < \ldots\ldots$$

$$< \left(A_k A_{k-2} A_{k-4} \ldots . A_0 \right) \left(A_{k-1} A_{k-3} A_{k-5} \ldots . A_1 \right) x_0 = \left[\prod_{i=0}^{k} A_i \right] x_0 = x_0 .$$

It follows by induction for all $n \geq 0$:

$$(20.2)\ \ 0 < x_{2(k+1)n+2(k+1)} < x_{2(k+1)n} .$$

Similarly, note that for all $n \geq 0$:

$$\begin{gathered} 0 < x_{2(k+1)n+2(k+1)+1} < x_{2(k+1)n+1} , \\ 0 < x_{2(k+1)n+2(k+1)+2} < x_{2(k+1)n+2} , \\ 0 < x_{2(k+1)n+2(k+1)+3} < x_{2(k+1)n+3} , \\ \vdots \\ \vdots \\ \vdots \\ 0 < x_{2(k+1)n+2(k+1)+[2(k+1)-1]} < x_{2(k+1)n+[2(k+1)-1]} . \end{gathered}$$

Observe that (20.2) is a monotonically decreasing subsequence bounded above by x_0 and below by zero. Thus, there exists a limit $L \geq 0$ such that:

$$\lim_{n \to \infty} x_{2(k+1)n} = L .$$

Similarly, we see that the remaining 2(k+1)-1 decreasing subsequences will also have finite limits, from which the result follows.

Lemma 21: Suppose that

$$(21.1)\ \ \prod_{i=0}^{k} A_i > 1.$$

Then Eq. (2) has unbounded solutions.

Proof: Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (2). Suppose that

$$x_{-1} = 0 \text{ and } x_0 > 0.$$

The case where $x_0 = 0$ and $x_{-1} > 0$ is similar and will be omitted.

Observe that:

$$x_1 = \frac{A_0 x_{-1}}{1+x_0} = 0,$$
$$x_3 = \frac{A_2 x_1}{1+x_2} = 0,$$
$$x_5 = \frac{A_4 x_3}{1+x_4} = 0,$$
$$\vdots$$

So it follows by induction that $x_{2n-1} = x_{2n+1} = x_{-1} = 0$ for all $n \ge 0$.

Now notice that via (21.1):

$$x_{2(k+1)} = \frac{A_{2(k+1)-1}x_{2(k+1)-2}}{1+x_{2(k+1)-1}} = A_{2k+1}x_{2k} = A_k x_{2k} = A_k\left[\frac{A_{2k-1}x_{2k-2}}{1+x_{2k-1}}\right]$$

$$= A_k A_{2k-1}x_{2k-2} = A_k A_{k-2}x_{2k-2} = A_k A_{k-2}\left[\frac{A_{2k-3}x_{2k-4}}{1+x_{2k-3}}\right] = A_k A_{k-2}A_{2k-3}x_{2k-4}$$

$$A_k A_{k-2}A_{k-4}x_{2k-4} = \ldots\ldots = A_k A_{k-2}A_{k-4}\ldots.A_0 x_{2k-k} = A_k A_{k-2}A_{k-4}\ldots.A_0 x_k =$$

$$A_k A_{k-2}A_{k-4}\ldots.A_0\left[\frac{A_{k-1}x_{k-2}}{1+x_{k-1}}\right] = \left(A_k A_{k-2}A_{k-4}\ldots.A_0\right)A_{k-1}x_{k-2} =$$

$$\left(A_k A_{k-2}A_{k-4}\ldots.A_0\right)A_{k-1}\left[\frac{A_{k-3}x_{k-4}}{1+x_{k-3}}\right] = \left(A_k A_{k-2}A_{k-4}\ldots.A_0\right)A_{k-1}A_{k-3}x_{k-4} = \ldots\ldots$$

$$= \left(A_k A_{k-2}A_{k-4}\ldots.A_0\right)\left(A_{k-1}A_{k-3}A_{k-5}\ldots.A_1\right)x_0 = \left[\prod_{i=0}^{k} A_i\right]x_0 > x_0,$$

$$x_{4(k+1)} = \frac{A_{4(k+1)-1}x_{4(k+1)-2}}{1+x_{4(k+1)-1}} = A_{4k+3}x_{4k+2} = A_k x_{4k+2} = A_k\left[\frac{A_{4k+1}x_{4k}}{1+x_{4k+1}}\right]$$

$$= A_k A_{4k+1}x_{4k} = A_k A_{k-2}x_{4k} = A_k A_{k-2}\left[\frac{A_{4k-1}x_{4k-2}}{1+x_{4k-1}}\right] = A_k A_{k-2}A_{4k-1}x_{4k-2}$$

$$A_k A_{k-2}A_{k-4}x_{4k-2} = \ldots\ldots = A_k A_{k-2}A_{k-4}\ldots.A_0 x_{4k-(k-2)} = A_k A_{k-2}A_{k-4}\ldots.A_0 x_{3k+2} =$$

$$A_k A_{k-2}A_{k-4}\ldots.A_0\left[\frac{A_{3k+1}x_{3k}}{1+x_{3k+1}}\right] = \left(A_k A_{k-2}A_{k-4}\ldots.A_0\right)A_{k-1}x_{3k} =$$

$$\left(A_k A_{k-2}A_{k-4}\ldots.A_0\right)A_{k-1}\left[\frac{A_{3k-1}x_{3k-2}}{1+x_{3k-1}}\right] = \left(A_k A_{k-2}A_{k-4}\ldots.A_0\right)A_{k-1}A_{k-3}x_{3k-2} = \ldots\ldots$$

$$= \left(A_k A_{k-2}A_{k-4}\ldots.A_0\right)\left(A_{k-1}A_{k-3}A_{k-5}\ldots.A_1\right)x_{2(k+1)} = \left[\prod_{i=0}^{k} A_i\right]x_{2(k+1)} = \left[\prod_{i=0}^{k} A_i\right]^2 x_0 > x_0 .$$

So it follows by induction that for all $n \geq 1$:

$$(21.2)\quad x_{2(k+1)n} = \left[\prod_{i=0}^{k} A_i\right]^n x_0 .$$

Therefore via (21.1) and (21.2) we see that:

$$\lim_{n\to\infty} x_{2(k+1)n} = \lim_{n\to\infty}\left[\prod_{i=0}^{k} A_i\right]^n x_0 = \infty \ .$$

Similarly note that:

$$\begin{gathered}\lim_{n\to\infty} x_{2(k+1)n+2} = \infty \ ,\\ \lim_{n\to\infty} x_{2(k+1)n+4} = \infty \ ,\\ \vdots \\ \vdots \\ \vdots \\ \lim_{n\to\infty} x_{2(k+1)n+2(k+1)-2} = \infty \ .\end{gathered}$$

Hence the result follows.

6 Conclusions and Future Work

In conclusion, we discovered that the behavior of the solutions of Eq. (2) is similar to the behavior of the solutions of Eq.(1); depending on either the maximum value or on the product of the terms of the periodic sequence $\{A_n\}_{n=0}^{\infty}$.

First, we can consider the behavior of the solutions of Eq. (2) when $\{A_n\}_{n=0}^{\infty}$ is an infinite sequence of positive real numbers that has a non-negative limit L. From computer observations, we conjecture the following:

- When $L < 1$, every solution of Eq.(2) converges to zero.
- When $L = 1$, every solution of Eq.(2) converges to a period two solution.
- When $L > 1$, Eq.(2) has unbounded solutions.

Furthermore, it is of paramount interest to study the long term behavior of the solutions of the following difference equation:

$$x_{n+1} = \frac{A_n x_{n-1}}{B_n + x_n}, \quad n = 0,1,2,\ldots,$$

where $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ are periodic sequences of positive real numbers.

Moreover, future work on this problem is to study the long term behavior of the solutions of the following difference equation:

$$x_{n+1} = \frac{A_n x_{n-k}}{1 + x_{n-l}}, \quad n = 0,1,2,\ldots,$$

where we have added delays k and l respectively; $k = 1,2,3,\ldots.$ and $l = 0,1,2,\ldots.$

7 References

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