5-16-2013

Reliability theory: Properties of probability distributions for lifetimes of systems of components

Rachel Santiago

R·I·T

Reliability Theory: Properties of Probability Distributions for Lifetimes of Systems of Components

by

Rachel Santiago

A Thesis Submitted in Partial Fulfillment of the

Requirements for the Degree of Master of Applied Mathematics

School of Mathematics

College of Science

Rochester Institute of Technology

Rochester, NY

May 16, 2013

Committee Approval:

Dr. Tamas Wiandt Graduate Programs Director	May 16, 2013
Dr. James Marengo Advisor	May 16, 2013
Dr. Manuel Lopez Committee Member	May 16, 2013
Dr. Ephraim Agyingi Committee Member	May 16, 2013

Abstract

The purpose of this paper is to make the concepts of reliability theory manageable even for those without an extensive mathematics background. This paper will break down the types of systems, explore the aspects of probability distributions which are important in reliability theory, as well as examine some concepts regarding the differences between system lifetimes and lifetimes of components. The goal of this paper is to allow the reader to gain an understanding of some of the key concepts explored in this theory, to provide examples for the reader to try, and to include proofs that have been broken down for clearer comprehension.

TABLE OF CONTENTS

Chapter One

Introduction

1.1 Systems

Reliability theory is based on the concept of understanding the reliability of systems and their individual components. We are analyzing the probability of the lifetimes for the individual components as well as the system as a whole, i.e. the probability of the system functioning. In order to do this we must first assume that we have a sensible system; this means it functions so long as its components function. We also assume that if the system is non-functioning, turning a component off will not turn the system on. With this idea, it is important to recognize we consider that such a system has a structure function which is increasing and, therefore, considered monotone. We can use the term sensible, or coherent, for this type of system.

In order for a system to function we must consider what kind of component structure is established. We have the following types of structures:

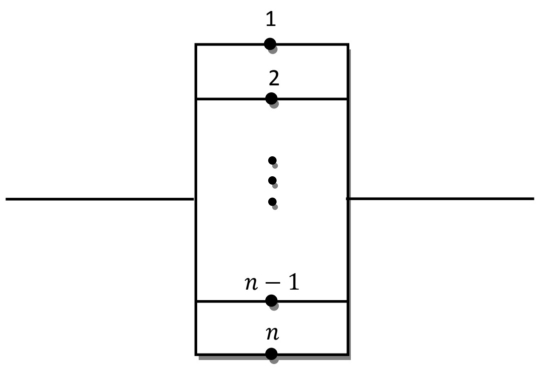
- series system – in order for the system to function, **all** components must be functioning
- parallel system – in order for the system to function, **at least one** of the components must be functioning
- k-out-of-n system – in order for the system to function, **at least k components out of the n** total components must be functioning

It is a fact that any sensible system can be expressed as a parallel system of series subsystems or as a series system of parallel subsystems! Below are pictorial examples of each type of system.

Series System:

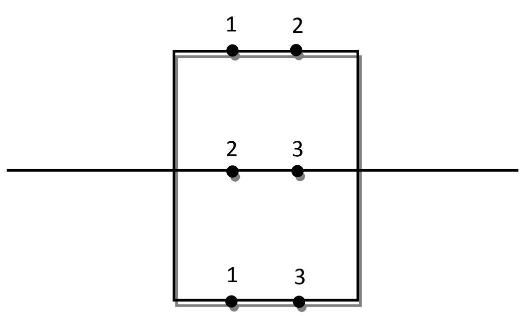
Notice that since ALL components must be functioning, a series system is an n-out-of-n system.

Parallel System:



Notice here that since at least 1 component must function, this is a 1-out-of-n system.

A k-out-of-n system with $k = 2$ and $n = 3$:



Notice that at least 2 out of the 3 components must be functioning in order to have a system that is functioning.

1.2 Structure Functions and Notation

Before we get into determining whether or not a system functions, we must first describe the state of each component. In reliability theory, we use the vector $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$, our state vector, to indicate which of the components are functioning and which components have failed. If a component is still functioning, we give it a value of 1. If a component has failed, we give it a value of 0. It is defined in this manner:

$$X_i = \begin{cases} 1 \; if \; the \; ith \; component \; is \; functioning \\ 0 \; if \; the \; ith \; component \; has \; failed \end{cases}$$

Once we determine whether or not the components themselves are functioning, we can use this information to determine if the system as a whole is functioning. We will use $\phi(\boldsymbol{X})$ and call this our structure function defined as follows:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 \; if \; the \; system \; is \; functioning \; when \; the \; state \; vector \; is \; \boldsymbol{X} \\ 0 \; if \; the \; system \; has \; failed \; when \; the \; state \; vector \; is \; \boldsymbol{X} \end{cases}$$

Let's examine the structure function for our series, parallel, and k-out-of-n systems. For a series system, it is understood that in order for the system to be functioning as a whole, every single component must be functioning as well. So our structure function has the following mathematical expression:

$$\phi(\boldsymbol{X}) = \min(X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} X_i$$

This structure is sensible in that if any component has failed, it will have a state of 0. If you take the product of the states for all components, if the result is 0, then the system has failed. So long as the product is 1, this ensures that all components are functioning; each component has a state of 1. To be clear, it is necessary, in a series system, for all components to have a state of 1 in order for the system to be functioning. If even one component has a state of 0, then the system will not be functioning.

For a parallel system, we discussed that in order for the system to be functioning at least one of the components must be functioning. The structure function is mathematically expressed as:

$$\phi(\boldsymbol{X}) = \max(X_1, X_2, \dots, X_n) = 1 - \prod_{i=1}^{n}(1 - X_i)$$

The last result is derived by the concept regarding binary variables where we can simply take the product of the cases where the components are not functioning (i.e. where we have the $1 - X_i$) and subtract that quantity from 1 to ensure system functioning. Taking the max value of the states should make sense in that if at least one of the components works, it will have a state of 1 regardless of the states of all other components. We are taking the max value of all of the states in a vector of 1s and 0s. If the max is 1, the system is working. If the max is 0, the system has failed. Essentially, any one component out of the n possible components must have a state of 1. Regardless of the states of all other $n - 1$ components, if at least that one component has a state of 1, then our system is functioning.

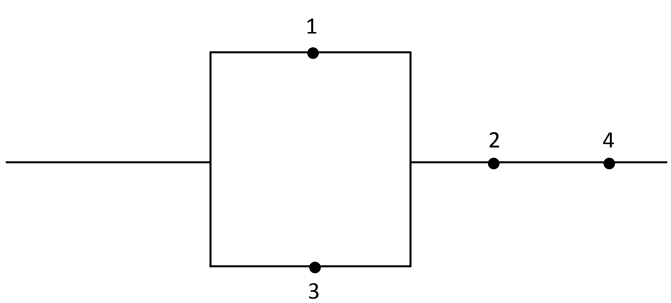
The k-out-of-n system is interesting in that we have to have at least k components working in order for the system to be functioning. The structure function looks like the following:

$$\phi(\boldsymbol{X}) = \begin{cases} 1, if \displaystyle\sum_{i=1}^{n} X_i \geq k \\ 0, if \displaystyle\sum_{i=1}^{n} X_i < k \end{cases}$$

This simply states that the sum of all the components' states must be greater than or equal to the k-component requirement for the system to be working. In other words, at least k components must have a state of 1 in order for the system to function. Additionally, it doesn't matter which k components are functioning so long as that condition is satisfied.

Let us look at a couple examples where we have to determine the structure function for a specific system. Let's say we are given the following diagram:

Diagram1:



What may help at first is to identify which components must be working in order for the system to function. We see that either component 1 can work or component 3 can work, but that components 2 and 4 must work in order for the system to function. Notice that we have a combination here of parallel and series systems. We can write the structure function as follows:

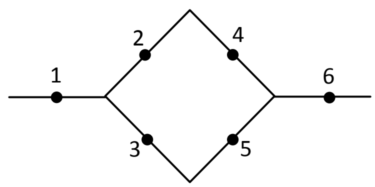
$$\phi(\boldsymbol{X}) = \max(X_1 X_3)\, X_2 X_4$$

$$= (X_1 + X_3 - X_1 X_3) X_2 X_4$$

This concept becomes clearer when we look at the reliability of the system and interpret based on probabilities of functioning for each component.

Let us now try to write the structure function corresponding to the following diagram:

Diagram 2:



We see from this diagram that component 1 has to work no matter what, then either 2 and 4 work or 3 and 5 work, but that 6 must work no matter what as well. Therefore, the structure function can then be written as:

$$\phi(\boldsymbol{X}) = X_1 \mathrm{max}(X_2X_4, X_3X_5)X_6$$

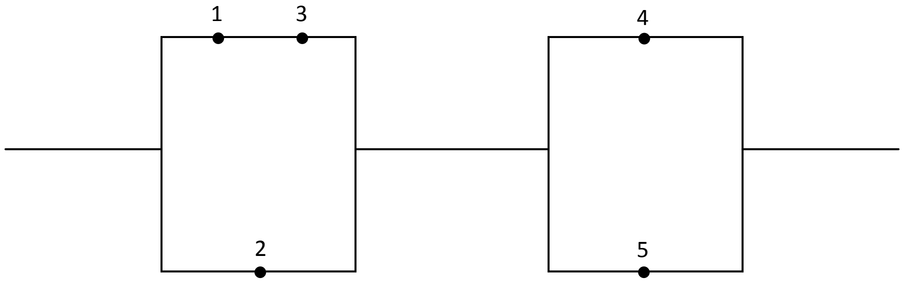
$$= X_1X_6(X_2X_4 + X_3X_5 - X_2X_4X_3X_5)$$

The above function can be verified by looking at the inclusion-exclusion concepts in probability related topics.

1.3 Minimal Path Sets and Minimal Cut Sets

In reliability theory, we examine the ideas of minimal path sets and minimal cut sets. First, let's discuss what it means to be a minimal path set. Examine the following diagram:

Diagram 3:



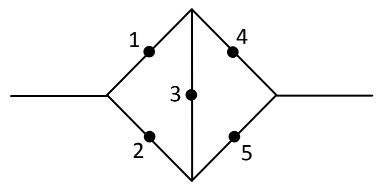
When we say minimal path set we mean a non-redundant set of components so that when these are on the system is functioning. This should imply naturally that if we were to turn off one of the components on any of these direct paths, then the system would turn off. The minimal path sets are essentially defined the exact same way we defined our structure functions for each of our practice examples. Looking at diagram 3, we see that there are four minimal path sets: {1,3,4}, {2,4}, {1,3,5}, {2,5}. When we find our structure function for a diagram like this, we would define it as:

$$\phi(\boldsymbol{X}) = \max(X_1X_3, X_2)\max(X_4, X_5)$$

$$= (X_1X_3 + X_2 - X_1X_3X_2)(X_4 + X_5 - X_4X_5)$$

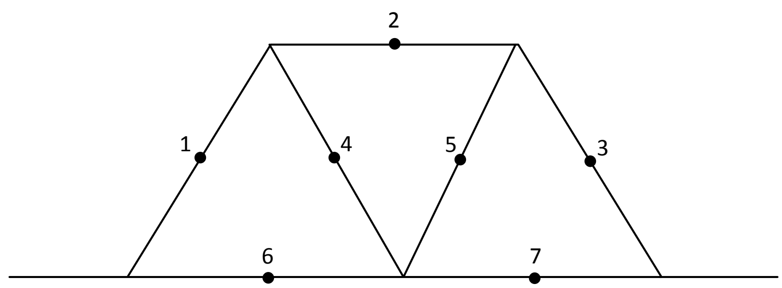
Minimal cut sets are the opposite of minimal path sets. What we are looking for here is the answer to the question, "Which components, if turned off, will leave the entire system non-functioning?" Obviously, if we turn off all of the components in the sensible system then it will not function. By using the term minimal, we want to find a non-redundant set of components which, if turned off, would leave the system non-functioning. Let's examine the following bridge diagram:

Diagram 4:

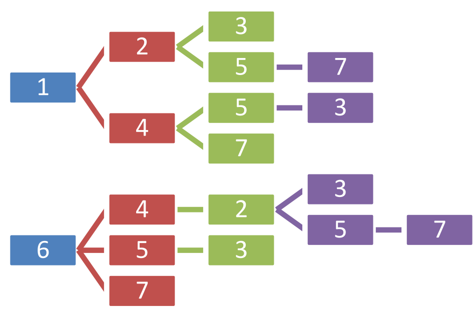


We can find what the minimal path sets are by taking any of the paths that will lead to the system functioning. These are {1,4}, {1,3,5}, {2,3,4}, {2,5}. Now let's examine the minimal cut paths. If we turn off component 1, we won't be able to use either minimal path set containing component 1, but we can still access the other two paths. However, if we shut off both components 1 and 2, we have successfully shut off the system. We can also easily notice that if we shut off components 4 and 5 we also successfully shut off the system. These are not the only minimal cut sets! One might think shutting component 3 off would ensure the system is off, but there are ways to get around component 3. With this being said, the last two minimal cut sets are {1,3,5} and {2,3,4}. What might help to understand this is to start by shutting off a component, say component 1, and then following the path containing component 2. At this point, notice that to block you from continuing, you have to shut off both components 3 and 5 in order to prevent you from moving forward in turning the system on. Let's try an example where we have to find both the minimal path sets and the minimal cut sets. Take a look at the following diagram and write down what you believe to be the minimal path and minimal cut sets:

Diagram 5:



We can identify the minimal path sets as {1,2,3}, {1,4,7}, {1,2,5,7}, {1,4,5,3}, {6,7}, {6,4,2,5,7}, {6,4,2,3}, and {6,5,3}. Using a tree diagram to illustrate the minimal path sets, we have:



Now we can identify the minimal cut sets as well. First we notice that if we cut off components 1 and 6 we can’t have a functioning system. Similarly, we can cut off components 3 and 7 and we will never have a functioning system. Everything in between gets to be a bit more interesting. The minimal cut sets are {1,6}, {3,7}, {2,4,6}, {2,5,7}, {1,4,5,7}, {3,4,5,6}. To verify these are accurate minimal cut sets, start by assuming all elements were on except those elements in a proposed minimal cut set. If you were to turn on **any one component** in any set, the system would then function.

Chapter Two

Reliability

2.1 Notation

Now that we have covered how to write appropriate structure functions of coherent systems, it is time to examine the reliability of systems of independent components. Let us define X_i, the state of the i^{th} component, as a random variable with the following properties:

$$P\{X_i = 1\} = p_i = 1 - P\{X_i = 0\}$$

This is saying that our component has a probability, p_i, of functioning and that is the same as 1 minus the probability of that component failing. That probability, p_i, is known as the reliability of the ith component. We can take this a step further and define the reliability of the system using the letter, r, as follows:

$$r = P\{\phi(\boldsymbol{X}) = 1\}, where\ \boldsymbol{X} = (X_1, X_2, \dots, X_n)$$

Since the components are independent, we can express the reliability of the system as a function of the reliabilities of its components:

$$r = r(\boldsymbol{p}), where\ \boldsymbol{p} = (p_1, p_2, \dots, p_n)$$

Since we established the fact that our components are independent, we can now just use the fact that $r(\boldsymbol{p})$ is our reliability function! Going back to our initial types of systems, we can establish the reliability functions for our series, parallel, and k-out-of-n systems.

Series System:

$$r(\boldsymbol{p}) = P\{\phi(\boldsymbol{X}) = 1\}$$

$$= P\{X_i = 1 \; for \; all \; i = 1, 2, \dots, n\}$$

$$= \prod_{i=1}^{n} p_i$$

Parallel System:

$$r(\boldsymbol{p}) = P\{\phi(\boldsymbol{X}) = 1\}$$

$$= P\{X_i = 1 \; for \; some \; i = 1, 2, \dots, n\}$$

$$= 1 - P\{X_i = 0 \; for \; all \; i = 1, 2, \dots, n\}$$

$$= 1 - \prod_{i=1}^{n} (1 - p_i)$$

This formula may seem like a strange way to determine the reliability of a parallel structure unless you keep in mind that in order to ensure a system with a parallel structure functions, we can simply find the probability that it won't function and subtract that from 1. This takes care of all the possible minimal path sets in a parallel structure system without having to go through each and every single one of them! In other words, it can be easier to obtain the probabilities of the minimal cut sets versus the minimal path sets and simply subtract that probability from 1.

k-out-of-n system (with equal probabilities):

We will first state that we can say each component has the same probability of functioning such that $p_i = p \; for \; all \; i = 1, 2, \dots, n$. Then our reliability function is as follows:

$$r(p, \dots, p) = P\{\phi(\boldsymbol{X}) = 1\}$$

$$= P\left\{\sum_{i=1}^{n} X_i \geq k\right\}$$

$$= \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

For this system, we are simply adding up the probabilities of all the possible cases that satisfy the condition of needing a minimum *k* out of a possible *n* components functioning. This is essentially the calculation of at least k successes out of n Bernoulli trials.

2.2 Reliability Functions and Duplication on Component vs. System Level

Let's look at a system where we have 3 components and we only need 2 components to function and each component has its own probability of functioning:

$p_i = p_1, p_2, p_3 \ for\ each\ component\ i = 1, 2, 3$. So, the reliability function is given by:

$$r(\boldsymbol{p}) = P\{\phi(\boldsymbol{X}) = 1\}$$

$$= P\{\boldsymbol{X} = (1,1,1)\} + P\{\boldsymbol{X} = (1,1,0)\} + P\{\boldsymbol{X} = (1,0,1)\} + P\{\boldsymbol{X} = (0,1,1)\}$$

$$= p_1p_2p_3 + p_1p_2(1-p_3) + p_1(1-p_2)p_3 + (1-p_1)p_2p_3$$

$$= p_1p_2 + p_1p_3 + p_2p_3 - 2p_1p_2p_3$$

Let us now consider a system where we have five components and the system will ONLY function if components 1, 2, and 3 are functioning and at least one of the last two components are functioning. Its reliability function is as follows:

$$r(\boldsymbol{p}) = P\{X_1 = 1, X_2 = 1, X_3 = 1, \max(X_4, X_5) = 1\}$$

$$= P\{X_1 = 1\}P\{X_2 = 1\}P\{X_3 = 1\}P\{\max(X_4, X_5) = 1\}$$

$$= p_1 p_2 p_3[1 - (1 - p_4)(1 - p_5)]$$

Another way to compute $r(\boldsymbol{p})$ is to notice that since it is a Bernoulli random variable (it consists of either 0s or1s) we can simply find its expectation. So the reliability of the system based on the probability of its independent components functioning is the same as the expectation of the individual components functioning.

$$r(\boldsymbol{p}) = P\{\phi(\boldsymbol{X}) = 1\} = E[\phi(\boldsymbol{X})]$$

A property that is crucial in remembering is what was declared in the very beginning of this paper. The fact that since we are working with a sensible, coherent, system it is reasonable to understand that we are working with an increasing function. Specifically, $r(\boldsymbol{p})$ is the reliability function of a system of independent components and is therefore an increasing function of $\boldsymbol{p}$. This is a way of saying that we are working with a system that makes sense. It wouldn't make sense to be able to turn the system on (if it had been off) by turning a component off (had it been on). So, the reliability of the system should only increase given that more components are turned on.

When considering the reliability of a system, it is often questioned as to whether or not it would make sense to duplicate components in a system or to duplicate systems of components to obtain greater reliability. Logically speaking, it seems that working with the parts of a system and improving their reliability would be much more effective than duplicating an entire system in order to improve its reliability. We can actually show this to be correct mathematically with an example.

Consider having a system with two of each type of *n* different components. If we build two separate systems our probability of having a functioning system is as follows:

$$P\{at\ least\ one\ of\ the\ two\ systems\ functioning\}$$

$$= 1 - P\{neither\ of\ the\ systems\ function\}$$

$$= 1 - \left[\left(1 - r(\boldsymbol{p})\right)\left(1 - r(\boldsymbol{p}')\right)\right]$$

We can see that $r(\boldsymbol{p})$ is the reliability function of the first system, and $r(\boldsymbol{p}')$ is the reliability function of the second system. Now let's look at how it is written when we are duplicating on the component level (not the system level). Consider that since $r(\boldsymbol{p})$ is the reliability function of the probabilities of the components $(p_1, p_2, \ldots, p_i)$, then we have:

$$P\{at\ least\ one\ of\ the\ pairs\ of\ components\ functions\}$$

$$= 1 - P\{neither\ of\ the\ pair\ of\ the\ components\ functions$$

$$= 1 - \left[(1 - p_i)\left(1 - p'_i\right)\right]$$

Because we are referring to the reliability of the system, the probability of the system functioning, we must write this as:

$$r[\boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{p})(\boldsymbol{1} - \boldsymbol{p}')]$$

The theory is that duplicating on the component level is much more effective in increasing reliability than is duplicating the system as a whole as expressed in the following inequality:

$$r[\boldsymbol{1} - (\boldsymbol{1} - \boldsymbol{p})(\boldsymbol{1} - \boldsymbol{p}')] \geq 1 - \left[\left(1 - r(\boldsymbol{p})\right)\left(1 - r(\boldsymbol{p}')\right)\right]$$

Let's now compare the two sides of the inequality by using a numerical example. Suppose we want to build a series system of two different types of components from a stockpile containing three of each kind of component. Essentially, we have type A and type B component, and we have 3 of each type. Let us say that the reliability of each component is $\frac{1}{2}$. If we build two separate systems, the probability of having a functioning system is:

$$= 1 - \left[\left(1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right)\left(1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right)\right]$$

$$= 1 - \left[\left(1 - \left(\frac{1}{8}\right)\right)\right]^2$$

$$= 1 - \left(\frac{7}{8}\right)^2$$

$$= \frac{15}{64}$$

If we instead build a single system duplicating components, the probability of having a functioning system is:

$$(r[\mathbf{1} - (\mathbf{1} - \boldsymbol{p})(\mathbf{1} - \boldsymbol{p}')])^3$$

$$= \left[1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\right]^3$$

$$= \left[1 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\right]^3$$

$$= \left[1 - \frac{1}{4}\right]^3$$

$$= \frac{27}{64}$$

It is clear that duplicating on the component level leads to a higher reliability than does duplicating systems!

Chapter Three

Failure Rate

3.1 Notation and Properties

We are now ready to discuss failure rate. In this final section, we will discuss some interesting concepts related to increasing failure rate (IFR) and increasing failure rate on average (IFRA). First, we must learn some of the notation involved with failure rate itself.

Let T be a positive random variable of the continuous type with cumulative distribution function (cdf) defined by $F(t) = P(T \leq t)$ and a probability density function (pdf) $f(t) = F'(t)$. Essentially $F(t)$ is the probability that the lifetime of the system has lived to at least time t. The failure rate of this distribution, $\lambda(t)$, can be calculated by using the following formula:

$$\lambda(t) = \frac{f(t)}{1-F(t)} \tag{1}$$

For small Δt, we notice that $\lambda(t) \approx P(T < t + \Delta t | T > t)$. Looking at T as the system lifetime, we can regard $\lambda(t)$ as the conditional probability density that a t-year old system will fail. We can explore how to utilize this formula in our next example.

Suppose that T has the exponential distribution with parameter $\lambda > 0$ and cdf:

$$F(t) = \begin{cases} 1 - e^{-\lambda t}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

then by using formula (1), we obtain:

$$\lambda(t) = \frac{\lambda e^{-\lambda t}}{1 - (1 - e^{-\lambda t})} = \lambda$$

We see that the failure rate happens to be a constant in this case. This is indicative of the type of distribution function, the exponential distribution, which is the only distribution of the continuous type with constant failure rate. We should recall that the exponential distribution has the memoryless property in that the probability of the system continuing to function for additional time doesn't depend on how "old" the system is. In other words, the distribution of the additional system lifetime doesn't depend on the age of the components.

$$P(T \leq t + \Delta t | T \geq t) = P(0 \leq T \leq \Delta t)$$

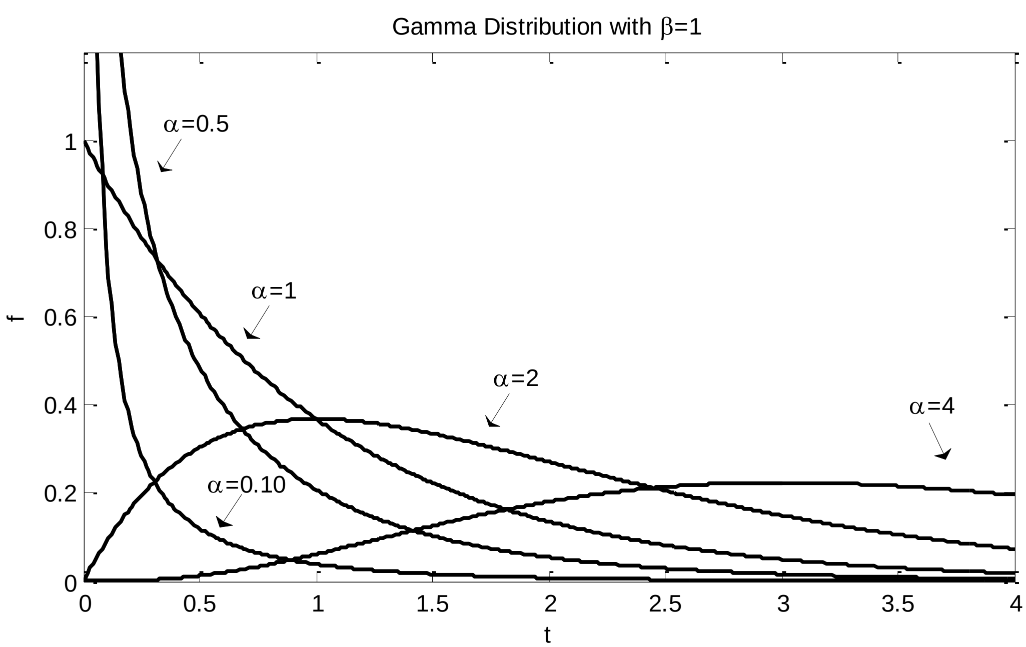
This may seem counterintuitive in the sense that as a system gets older, we expect the chances of failure become greater. Here's an example where it will make the concept of the exponential distribution's memoryless property clear. Let's say you are a business owner. You decide to open up shop at 8 a.m. After 10 minutes you notice that no customers have arrived. If you wait an additional 2 minutes, does the chance of a customer arriving become greater? The answer is clearly no. It doesn't matter how long the shop is open; the chances of someone walking in on the 11^{th} or 12^{th} minute is exactly the same as if no time had passed at all. Otherwise all businesses would thrive. This property is key in reliability theory in that it models the systems effectively in this way.

3.2 Exploring Differences between IFR and IFRA

We can now take this a step further and notice that the distribution of T is said to be IFR if $\lambda(t)$ is in an increasing function of t. Similarly, it will be DFR (decreasing failure rate) if $\lambda(t)$ is in a decreasing function of t.

Let us suppose that T has a gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$ with pdf:

$$f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\beta^{\alpha}} t^{\alpha-1} e^{-\frac{t}{\beta}}, & t > 0 \\ 0, & t \leq 0 \end{cases}$$



It can be easily proven that when $\alpha < 1$, the function has a DFR distribution, when $\alpha > 1$, the function has an IFR distribution, and when $\alpha = 1$, we are simply looking at the exponential distribution.

A distribution is said to have Increasing Failure Rate on Average (IFRA) if $\frac{\Lambda(t)}{t}$ is nondecreasing on $(0, \infty)$ where

$$\Lambda(t) = \int_0^t \lambda(u)du\,. \tag{2}$$

That is, the average failure rate on $(0, t)$ is nondecreasing in t. (A DFRA distribution is defined similarly.) This leads us to our first question: Is every IFR distribution also IFRA? The answer to this question will be given through the following proof.

Proposition: An IFR distribution is also IFRA.

Proof:

Assume that $\lambda(u)$ is increasing in u.

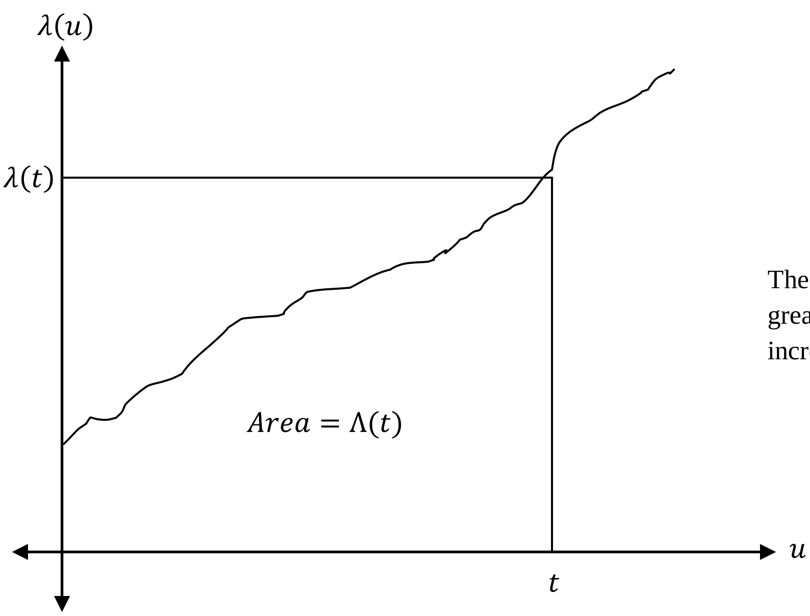
Show that $\frac{\Lambda(t)}{t}$ is increasing.

$$\frac{d}{dt}\left(\frac{\Lambda(t)}{t}\right) = \frac{t\Lambda'(t) - \Lambda(t)}{t^2} = \frac{t\lambda(t) - \Lambda(t)}{t^2}$$

Since $\lambda(t)$ is nondecreasing, we see that:

$$t\lambda(t) \geq \Lambda(t)$$

and therefore the derivative is nonnegative. This shows that every IFR distribution is also IFRA.



The area of the rectangle, $t\lambda(t)$, is greater than the area under the increasing function.

Our next big question is: If the component lifetimes are IFR, is the system lifetime IFR as well? Before we answer this, we should provide some additional notation. Let T_i be the lifetime of component i and let F_i be its cdf. Note that for $t > 0$, $\bar{F}_i(t) = 1 - F_i(t) = P(T_i > t)$ which is the reliability of this component at time t. If T is the system lifetime and F is its cdf, then the reliability of the system at time t is $\bar{F}(t) = r\left(\bar{F}_1(t), \bar{F}_2(t), \dots, \bar{F}_n(t)\right)$.

Let us now consider a parallel system of $n = 2$ components with exponentially distributed lifetimes and respective failure rates $\lambda_1 = 1$ and $\lambda_2 = 2$. Then,

$$T = \max(T_1, T_2)$$

$$\bar{F}(t) = 1 - P(T_1 \leq t \; and \; T_2 \leq t)$$

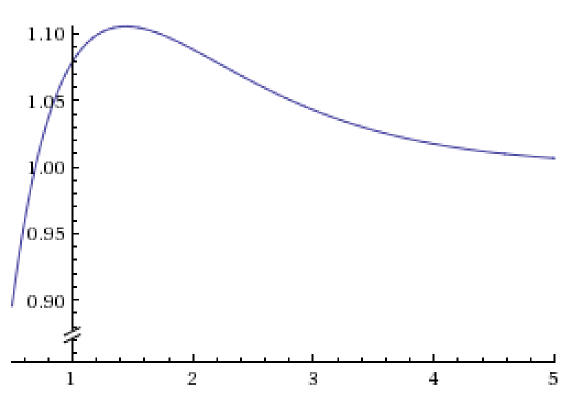
$$= 1 - (1 - e^{-t})(1 - e^{-2t})$$

$$= e^{-t} + e^{-2t} - e^{-3t}$$

Examining the graph of $\lambda(t)$ by computing formula (1), where

$$\lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{2e^{-2t} + e^{-t} - 3e^{-3t}}{e^{-t} + e^{-2t} - e^{-3t}}$$

we see that $\lambda'(t)$ is actually maximized when $t = \ln(2 + \sqrt{5}) \approx 1.44364$ and therefore the distribution of T is not IFR.



Our final big question is this: If the lifetimes of the components are IFRA, does this mean the system lifetime is also IFRA? In order to prove this, we need three lemmas. Additionally, we will consider a monotone system of n components whose lifetimes are independent random variables. Let F_i be the cdf of the lifetime of the i^{th} component and assume that F_i is IFRA for $i = 1,2, \dots, n$. We will prove that if F is the cdf of the system lifetime, then F is IFRA.

Lemma 1:

Let F be the cdf of a positive random variable of the continuous type. Then F is IFRA if and only if $\bar{F}(\alpha t) \geq \left(\bar{F}(t)\right)^{\alpha}$ $for\ 0 \leq \alpha \leq 1\ and\ t \geq 0$.

Proof:

F is IFRA if and only if

$$\frac{\Lambda(\alpha t)}{\alpha t} \leq \frac{\Lambda(t)}{t} \quad for\ o \leq \alpha \leq 1\ and\ t \geq 0.$$

Since $\Lambda'(t) = \frac{f(t)}{\bar{F}(t)}$ we have

$$\Lambda(t) = -ln\,\bar{F}(t)$$

so that

$$\frac{\Lambda(\alpha t)}{\alpha t} \leq \frac{\Lambda(t)}{t}$$

can be written as

$$-ln\,\bar{F}(\alpha t) \leq -\alpha\ ln\,\bar{F}(t)$$

which is equivalent to (by dividing out the negative and raising both sides of the inequality by base e):

$$\bar{F}(\alpha t) \geq \left(\bar{F}(t)\right)^{\alpha}.$$

Lemma 2:

If $0 \leq \alpha \leq 1$, $0 \leq \lambda \leq 1$, *and* $0 \leq y \leq x$, *then*

$$\lambda^\alpha x^\alpha + (1 - \lambda^\alpha) y^\alpha \geq (\lambda x + (1 - \lambda) y)^\alpha.$$

Proof:

Consider the function f *defined by* $f(t) = t^\alpha$ *for* $t \geq 0$. *Let*

$$t_1 = \lambda y$$
$$t_2 = \lambda x$$
$$t_3 = y$$
$$t_4 = \lambda x + (1 - \lambda) y.$$

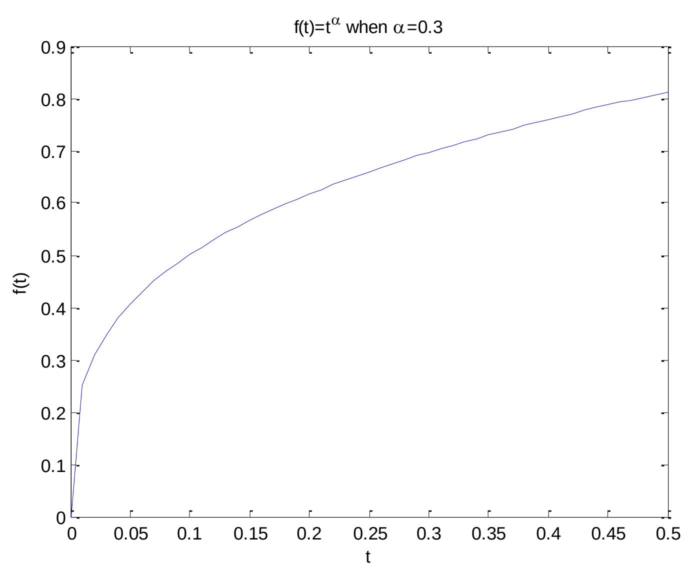
We observe that

$$t_1 \leq t_3$$
$$t_2 - t_1 = t_4 - t_3.$$

The desired conclusion is equivalent to the statement that

$$f(t_2) - f(t_1) \geq f(t_4) - f(t_3).$$

This follows from the fact that the graph of f *is concave down on* $[0, \infty)$.



Lemma 3:

If $r(p_1, p_2, \dots, p_n)$ *is the reliability function of a monotone system of* n *components and* $0 \le \alpha \le 1$*, then*

$$r(p_1^\alpha, p_2^\alpha, \dots, p_n^\alpha) \ge \big(r(p_1, p_2, \dots, p_n)\big)^\alpha.$$

Proof:

The proof is by induction on n*. Observe that the result is obvious if* $n = 1$*. Now assume that the result holds for all monotone systems of* $n - 1$ *components and consider a monotone system of* n *components having structure function* ϕ*. By conditioning on whether or not the* n^{th}*component is functioning, we obtain*

Equation (3):

$$r(p_1^\alpha, p_2^\alpha, \dots, p_n^\alpha) = p_n^\alpha r(p_1^\alpha, p_2^\alpha, \dots, p_{n-1}^\alpha, 1) + (1 - p_n^\alpha) r(p_1^\alpha, p_2^\alpha, \dots, p_{n-1}^\alpha, 0)$$

Now consider a system of components $1,2, \dots, n - 1$ *having a structure function*

$$\phi_1(X_1, X_2, \dots X_{n-1}) = \phi(X_1, \dots, X_{n-1}, 1).$$

The reliability function of this system is given by

$$r_1(p_1, p_2, \dots, p_{n-1}) = r(p_1, p_2, \dots, p_{n-1}, 1).$$

By the induction process, we have

$$r(p_1^\alpha, p_2^\alpha, \dots, p_{n-1}^\alpha, 1) \ge \big(r(p_1, p_2, \dots, p_{n-1}, 1)\big)^\alpha.$$

Similarly, by considering the system of components $1,2,\dots,n-1$ *with structure function*

$$\phi_0(X_1, X_2, \dots X_{n-1}) = \phi(X_1, \dots, X_{n-1}, 0),$$

we have

$$r(p_1^\alpha, p_2^\alpha, \dots, p_{n-1}^\alpha, 0) \geq \big(r(p_1, p_2, \dots, p_{n-1}, 0)\big)^\alpha.$$

From equation (3), we obtain

$$r(p_1^\alpha, p_2^\alpha, \dots, p_n^\alpha) \geq p_n^\alpha\big(r(p_1, p_2, \dots, p_{n-1}, 1)\big)^\alpha + (1 - p_n^\alpha)\big(r(p_1, p_2, \dots, p_{n-1}, 0)\big)^\alpha.$$

From Lemma 2, it then follows that

$$r(p_1^\alpha, p_2^\alpha, \dots, p_n^\alpha) \geq \big(p_n\, r(p_1, p_2, \dots, p_{n-1}, 1) + (1 - p_n)\, r(p_1, p_2, \dots, p_{n-1}, 0)\big)^\alpha$$
$$= \big(r(p_1, p_2, \dots, p_n)\big)^\alpha.$$

We are now able to prove our main theorem.

Main Theorem: If the component lifetimes are IFRA, is the system lifetime also IFRA?

Proof:

The distribution of the system lifetime is given by $\bar{F}(t) = r(\bar{F}_1(t), \bar{F}_2(t), \dots, \bar{F}_n(t))$. *Since r is monotone and since each* F_i *is IFRA, it follows from Lemma 1 that:*

$$\bar{F}(\alpha t) \geq r\big(\bar{F}_1^\alpha(t), \bar{F}_2^\alpha(t), \dots, \bar{F}_n^\alpha(t)\big)$$
$$\geq (r\big(\bar{F}_1(t), \bar{F}_2(t), \dots, \bar{F}_n(t)\big)^\alpha \quad (from\ Lemma\ 3)$$
$$= \big(\bar{F}(t)\big)^\alpha$$

Using Lemma 1 again, we now see that F *is IFRA.*

Chapter Four

Conclusion

In conclusion, we have seen a few of the different facets reliability theory has to offer. We have dissected different systems and learned how to calculate their structure and reliability functions which led us into being able to prove some interesting theorems regarding these systems and their components. We have seen visually why the exponential distribution is of such great importance in this field of study. Bearing witness to the difference in strictness of being IFR vs. being IFRA was the motivation for this paper ultimately bringing clarity to just some of the key concepts in reliability theory.

References

Barlow, Richard E, and Frank Proschan. *Mathematical Theory of Reliability.* New York: John Wiley & Sons, Inc., 1965.

Lyu, Michael R. "Handbook of Software Reliability Engineering ." 747-779. McGraw-Hill, 1996.

Ravindran, A Ravi. *Operations Research and Management Science Handbook.* Boca Raton : CRC Press, 2008.

Ross, Sheldon M. *Introduction to Probability Models.* Academic Press, 2010.