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Rank numbers for graphs with paths and cycles

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Rank Numbers for Graphs with Paths and Cycles

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H. Shahmohamad July 2007

Acknowledgements

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Table of Contents

Table of Figures

Abstract

A *coloring* of a graph, G, is an assignment of positive integers to the vertices of the graph with one number assigned to each vertex, so that adjacent vertices are assigned different numbers. A *k-ranking* of a graph is a coloring that uses $\{1, 2, ..., k\}$ with the requirement that every path between any two vertices labeled with the same number contains a vertex with a higher label. The *rank number* of a graph, denoted $\chi_r(G)$, is the smallest k such that G has a k-ranking. In this paper we seek rank numbers for four specific families of graphs. Each of these families of graphs contains at least one cycle as a subgraph.

1 Introduction

A *graph* is a finite, nonempty set of objects called *vertices* with a set of unordered pairs of distinct vertices of the graph called *edges*. The *degree* of a vertex in a graph is equal to the number of edges which are incident with the vertex. A *pendant vertex* is a vertex of degree 1. A *cycle*, denoted C_n, is a graph, consisting of at least three vertices, whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. A graph is said to be *complete* if every two vertices of the graph are adjacent. A *star* is a graph which has $n-1$ vertices of degree 1 and one vertex is of degree $n-1$. A *u-v path*, denoted P_k, is a finite alternating sequence $u = u_0, e_1, u_1, e_2, u_2, ..., u_{k-1}, e_k, u_k = v$ such that $e_i = u_{i-1}u_i$ and no vertices are repeated. The number $k-1$ is called the length of the path.

A *coloring* of a graph, G, is an assignment of positive integers to the vertices of the graph with one number assigned to each vertex, so that adjacent vertices are assigned different numbers. A *k-ranking* of a graph is a coloring that uses labels $\{1, 2, ..., k\}$ with the requirement that every path between any two vertices labeled with the same number contains a vertex with a higher label. The *rank number* of a graph, denoted $\chi_r(G)$, is the smallest k such that G has a k-ranking.

2 Background

A vertex coloring of a graph is a labeling of the vertices so that adjacent labels are assigned different colors. The fewest number of colors that can be used in a vertex coloring of a graph G is known as the chromatic number, and is denoted $\chi(G)$. In this thesis we investigate k-rankings, which are vertex colorings with an additional condition imposed. A k-ranking is a labeling $f : V(G) \to \{1, 2, ..., k\}$ where every path connecting two vertices with the same label contains a vertex with a larger label. Following along the lines of the chromatic number, the rank number of a graph $\chi_r(G)$, is the minimum k such that G has a k-ranking. A k-ranking that uses $\chi_r(G)$ labels will be referred to as a χ_r-ranking. When the value of k is unimportant we will refer to a k-ranking as simply a ranking.

It was shown by Bodlaender et al. [2] that the problem: Given a bipartite graph G and a positive integer t, deciding whether $\chi_r(G) <= t$ is NP Complete. Rank numbers have been determined for several families of graphs including: paths, cycles, split graphs, complete multipartite graphs, Möbius graphs, powers of paths and cycles, and some grid graphs [2],[6],[3],[5],[10],[1], and [12].

Research on rank numbers was sparked by its applications to scheduling of manufacturing systems, Cholesky factorizations of matrices and VLSI layout [9] and [11]. The optimal tree node ranking problem is identical to the problem of generating a minimum height node separator tree for a tree graph. Node separator trees are extensively used in VLSI layout [9]. These models are suitable for communication networks design where information flow between nodes needs

to be monitored. Similar models are applicable in the design of management organizational structures. A matrix application was observed by Kloks, Müller, and Wong [8].

We will use P_n to denote the path on vertices $v_1, v_2, ..., v_n$ and we will use $\langle f(v_1), f(v_2), ..., f(v_n)\rangle$ to explicitly describe the labels in a ranking f. Bodlaender et al. showed that $\chi_r(P_n) = \lfloor \log_2(n) \rfloor + 1$ and that optimal rankings of $P_n = v_1, v_2, \ldots, v_n$ can be constructed by labeling v_i with $\alpha + 1$ where 2^α is the largest power of 2 such that 2^α divides i [2]. We will refer to this particular ranking as the *standard ranking of a path.* Furthermore, it was shown that the standard ranking of a path on $2^n - 1$ vertices is unique [2].

It was proven by Bruoth and Horňák [3] that $\chi_r(C_n) = \lceil \log_2 n \rceil + 1$.

Here we will build upon known results for rank numbers of paths and cycles to investigate rank numbers of graphs that are the union of a path and two cycles.

We restate a lemma from [5] that gives the monotonicity of the rank number.

Lemma 1 *Let H be a subgraph of a graph G. Then $\chi_r(H) \leq \chi_r(G)$.*

3 Results

3.1 We now consider a graph, H, of the form shown in Figure 1. This graph, H, is composed of a complete graph, K_m, where each of the m vertices on the complete graph are also on a cycle.

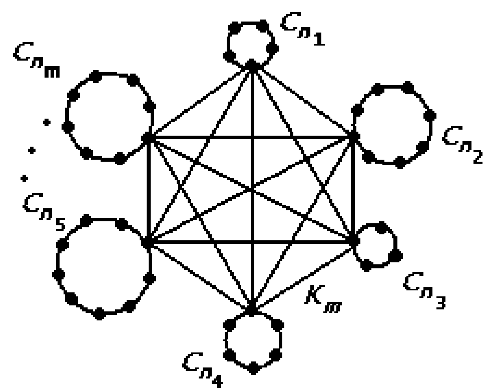

Figure 1

Theorem 3.1.1 If the rank of each cycle, C_{n_i}, is unique, then the graph H has rank equal to the rank of the largest cycle.

Proof. If $\lceil \log_2 n_1 \rceil \neq \lceil \log_2 n_2 \rceil \neq ... \neq \lceil \log_2 n_m \rceil$, label each vertex on K_m with the values of the ranks of each cycle (The maximum label of each cycle is placed on the vertices of K_m.). Label all other vertices on in cycle in the optimal way (1 2 1 3 1 2 . . .). It is known that any cycle cannot have a rank number less than 3, so each vertex on K_m will have a value greater than or equal to 3. As it is assumed that each cycle has a unique rank number, the requirements of the ranking are satisfied for H. As each C_{n_i} is a subgraph of H, it is impossible to have a rank for H which is less than the rank of any of its subgraphs.

In this way, $\chi_r(H) = \max(\chi_r(C_{n_i})) = \max(1+\lceil \log_2 n_i \rceil)$, $i = 1, 2, ..., m$. ■

Theorem 3.1.2 If the rank of each cycle, C_{n_i}, is not unique, then it is possible to find the rank number of H.

Proof. Begin similar to the above construction. Label each vertex on K_m with the ranks of each cycle. Label all other vertices on in cycle in the optimal way (1 2 1 3 1 2 . . .), as is shown for a specific example in Figure 2.1.

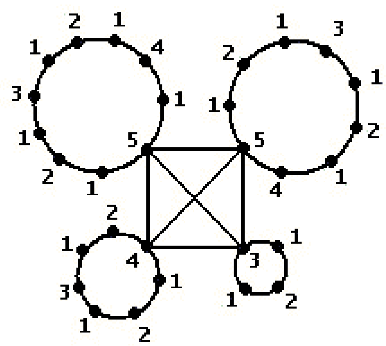

Figure 2.1

If the rank number of each cycle is not unique, the labels must be adjusted so that every vertex of degree $m+1$ has a unique label. Suppose C_{n_i} and C_{n_j} have the same rank, say r. Without loss of generality, label the vertex of degree $m+1$ on C_{n_i} with the minimum value greater than r which is not already used on K_m. Repeat this process until all vertices on K_m have unique labels, as is shown for the specific example in Figure 2.2.

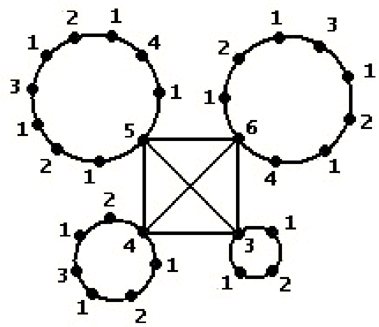

Figure 2.2

Following this process, the ranking is obtained. It may seem that it is possible to find a ranking which is not greater than the maximum rank of any cycle in H. Perhaps one could take the approach of 'rotating' one or more cycles which have the same rank. This can quickly cause a problem, as is shown in the specific example in Figure 2.3.

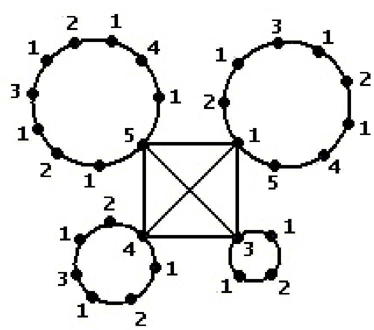

Figure 2.3

As above, if C_{n_i} and C_{n_j} have the same rank, say r, and instead of labeling the vertex on C_{n_i} which is also on K_m with a higher label, we 'rotate' the cycle so that a lower value than its rank, which is unused as of yet on K_m, without loss of generality, perhaps $r-1$. Remember, there is still a vertex labeled with

a value of r on C_{n_i}. In this way, we can now create a path beginning with the vertex labeled r on C_{n_i} and ending with the vertex labeled r on C_{n_j} which has no vertices labeled with a value greater than $r-1$. It is clear that this problem is not solved by 'rotating' both cycles. Therefore, the original construction holds for the ranking of H. ∎

3.2 Consider the graph J shown in Figure 3. The graph, J, is composed of a star where each pendant vertex is also on a cycle.

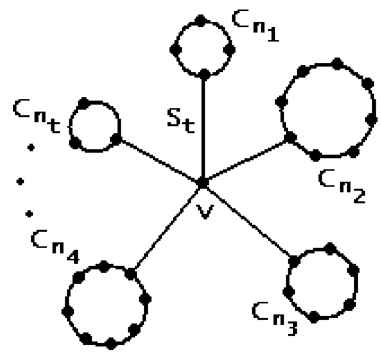

Figure 3

Theorem 3.2.1 If the rank number of each cycle, C_{n_i} with $i = 1, 2, ..., t$, is unique, the rank number of the graph J is equivalent to the rank number of the largest cycle in the graph.

Proof. If $\lceil \log_2 n_1 \rceil \neq \lceil \log_2 n_2 \rceil \neq ... \neq \lceil \log_2 n_t \rceil$, label each cycle in the optimal way (1 2 1 3 1 2 1 4 . . .), making sure that the highest value assigned to any label on any cycle is placed on its vertex of degree 3. Now, the vertex at the center of the star, v, can be labeled with a value of 1. The vertex, v, can take on such a label as it is known that all cycles cannot have a rank less than

3, meaning all requirements of labeling and ranking are satisfied.

Therefore, $\chi_r(J) = \max(\chi_r(C_{n_i})) = \max(1 + \lceil \log_2 n_i \rceil)$, $i = 1, 2, ..., t$. ∎

Theorem 3.2.2 If there are at least 2 cycles in J with the same rank, say r, where r is the largest of the repeated rank values for the cycles, and if there exists a number, s, with $r < s < \max(\chi_r(C_{n_i}))$, $i = 1, 2, 3, ...t$, for which there is no cycle with rank equal to s, then the rank of J is equivalent to the rank of the largest cycle in the graph.

Proof. If multiple cycles in J have the same rank number, say r, begin by labeling each cycle in the optimal way, with the rank of each cycle being used for the labels for each of the vertices of degree 3. The label for vertex v is created by finding the minimum value between r and $\max(\chi_r(C_{n_i}))$, $i = 1, 2, 3, ...t$, which is not the value of the rank number of any cycle in J. Call this value s and use it as the label for v. Therefore, $\chi_r(J) = \max(\chi_r(C_{n_i})) = \max(1 + \lceil \log_2 n_i \rceil)$, $i = 1, 2, 3, ...t$. This is also true for forms of J that have multiple sets of repeated rank numbers for cycles. If there are multiple sets of repeated cycles, take r to be the largest repeated value, and proceed as above. ∎

Theorem 3.2.3 If there are at least 2 cycles in J with the same rank, say r, where r is the largest of the repeated rank values for the cycles, and if for every value between r and $\max(\chi_r(C_{n_i}))$, $i = 1, 2, 3, ...t$, there exist cycles with ranks equal to $r + 1, r + 2, ..., \max(\chi_r(C_{n_i})) - 1$, then the rank number of J is equal to 1 more than the rank of the maximum cycle in the graph.

Proof. If multiple cycles in J have the same rank number, say r, begin by labeling each cycle in the optimal way, with the rank of each cycle being used

for the labels for each of the vertices of degree 3. Given that for every value between r and $\max(\chi_r(C_{n_i}))$, $i = 1, 2, 3, ...t$, there exist cycles with ranks equal to $r+1, r+2, ..., \max(\chi_r(C_{n_i})) - 1$, the label for v must be $1 + \max(\chi_r(C_{n_i}))$, meaning $\chi_r(J) = 1 + \max(\chi_r(C_{n_i})) = 2 + \max(\lceil \log_2 n_i \rceil)$, $i = 1, 2, 3...t$. ■

3.3: Consider the graph G shown in Figure 4. The graph, G, is composed of a path, P_k, the ends of which are on each of two cycles, C_n and C_m. The first four theorems of this section focus on the situation where the rank numbers of C_n and C_m are equal, and theorems numbered five through seven in this section are concerned with the situations what occur when C_n and C_m do not have the same rank number, but one of the cycles has an equal rank number to that of P_{k-2}. Finally, theorems eight, nine, and ten in this section discuss the situation where no two subgraphs, among C_n, C_m, and P_{k-2}, have the same rank number.

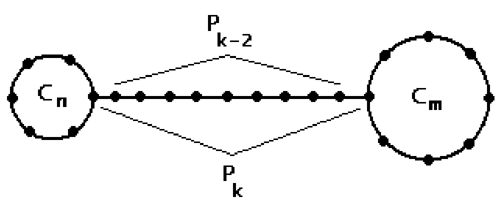

Figure 4

Theorem 3.3.1 If P_{k-2}, C_n, and C_m all have the same rank number, say r, then the rank number of G is $r + 1$.

Proof. If $\lceil \log_2 n \rceil + 1 = \lceil \log_2 m \rceil + 1 = \lfloor \log_2 (k-2) \rfloor + 1 = r$, then a basic structure for a labeling of G would begin as is shown in Figure 5.1.

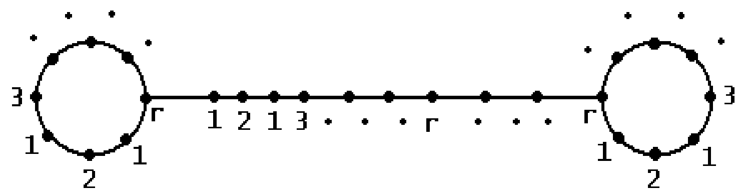

Figure 5.1

Begin by labeling C_n and C_m, ensuring that both vertices of degree 3 are labeled with a value of r. Label the rest of each cycle (either clockwise or counter-clockwise) and the path in the optimal way (1 2 1 3 1 2 1 4 . . .). From there, it can be seen that we need only to replace the label of vertex with a label of r on P_{k-2} with a label of $r+1$, as shown in Figure 5.2.

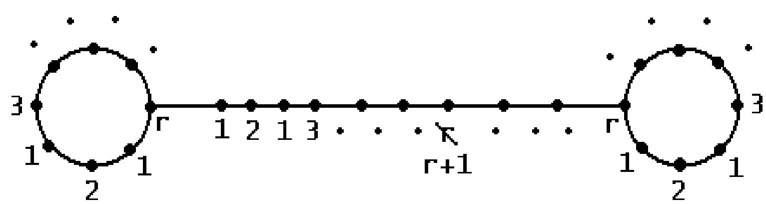

Figure 5.2

Therefore, $\chi_r(G) = r + 1 = \lceil \log_2 n \rceil + 2 = \lceil \log_2 m \rceil + 2 = \lfloor \log_2(k-2) \rfloor + 2$. ■

Theorem 3.3.2 If C_n and C_m have the same rank number, say r, and P_{k-2} has rank $r+1$, then the rank number of G is $r+2$.

Proof. If $\lceil \log_2 n \rceil + 1 = \lceil \log_2 m \rceil + 1 = r$ and $\lfloor \log_2(k-2) \rfloor + 1 = r + 1$, then a basic structure for labeling G would being as shown in Figure 6.

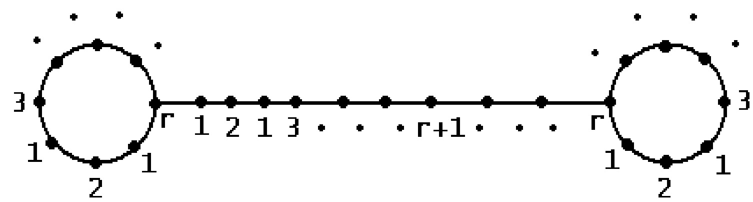

Figure 6

Being by labeling C_n and C_m, ensuring that both vertices of degree 3 are labeled with a value of r. Label the rest of each cycle (either clockwise or counter-clockwise) and the path in the optimal way (1 2 1 3 1 2 1 4 . . .). From here, change the labels on each of the degree 3 vertices to a value of $r+1$. Finally, change the highest label on P_{k-2} from $r+1$ to $r+2$.

Therefore, $\chi_r(G) = r + 2 = \lceil \log_2 n \rceil + 3 = \lceil \log_2 m \rceil + 3 = \lfloor \log_2 (k-2) \rfloor + 2$. ■

Theorem 3.3.3 If C_n and C_m have the same rank number, say r, and P_{k-2} has a rank of any value less than r, then the rank number of G is $r+1$.

Proof. If $\lceil \log_2 n \rceil = \lceil \log_2 m \rceil > \lfloor \log_2 (k-2) \rfloor$, then the ranking is nearly complete. The only thing to consider is that there is now a label of r on each end of P_k. It is necessary to replace the label of any vertex on P_{k-2} with a value of $r+1$, and the ranking is complete, as is shown in Figure 7.

Therefore, $\chi_r(G) = r + 1 = \lceil \log_2 n \rceil + 2 = \lceil \log_2 m \rceil + 2$. ■

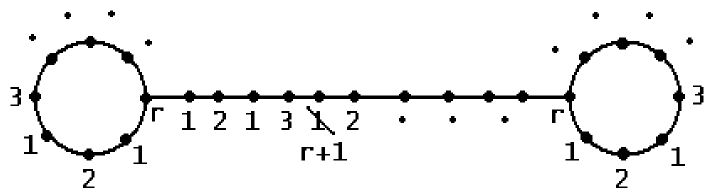

Figure 7

Theorem 3.3.4 If C_n and C_m have the same rank number, say r, and P_{k-2} has a rank of at least $r+2$, say s, then if $k-2 < 2^s - 2^{r+1} + 2^{r-1}$, then the rank number of G is s, or if $k-2 \geq 2^s - 2^{r+1} + 2^{r-1}$, then the rank number of G is $s+1$.

Proof. Begin by labeling each of the 3 main subgraphs independently. Begin with 2 examples. For example 1, consider the graph where $r = 3$, $s = 6$, and $k - 2 = 50$. What must be considered is the number of labels we can use which had a higher value than r, or 3 in this case. These higher values can 'protect' all of the vertices labeled with a value of 3. With $s = 6$, we can think about labels of 4, 5, and 6. Adhering to the rules of ranking, this means that somewhere on P_{k-2}, it is permissible to have vertices labeled 4, 5, 4, 6, 4, 5, 4. If all 7 of these labels are to be used, their order must be maintained. These higher labels must be placed between each pair of vertices labeled with 3, or they can be used to replace a vertex labeled with 3 or lower. In this first example, there are 8 vertices labeled with a value of 3. In Figure 8, it can be seen how to relabel the vertices so that we obtain a 6-ranking using the ideas above. In Figure 8, the letter X represents a portion of the graph which has been omitted and contains vertices with the labels of 1, 2, 1, 3, 1, 2, 1.

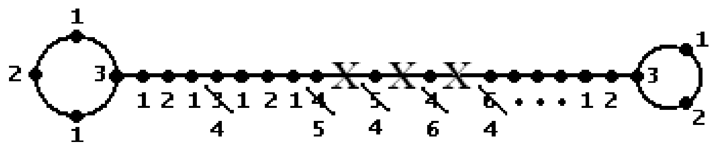

Figure 8

It can be seen that these labels are really only being shifted to the left, in order to solve the conflict caused by the two vertices with a label of 3 on the left-most portion of the graph.

We must find the general path length which separates the ability or inability to relabel the graph without increasing its rank number. Now, to turn to the general case, it is clear that the number of 'protector' vertices possible, or vertices with a label value of more than r on P_{k-2} is equivalent to $2^{s-r}-1$. In the process of relabeling, nearly all of these 'protection' values are used to replace another value which is greater than r. The only exception to this is the label which is placed, in Figure 8.2, on the left-most vertex of P_{k-2} which has a value of 3. In this way, as we must replace the value on one vertex which previously had a label of r or less, it does not contribute to the length of the path. Truly, we can only use $2^{s-r}-1-1$ labels to increase the critical value. In general, each label which is greater than r will be placed at a distance of 2^r from any other vertex label which is greater than r. From both of these qualities, it is known that the length of path we are looking for is at least $2^r*(2^{s-r}-1-1)$. From here, we can add on whatever length of path which will not cause the value of r to appear again. Therefore, add on 2^{r-1}, and the critical value of the path length

is $2^r * (2^{s-r} - 1 - 1) + 2^{r-1}$, which simplifies to $2^s - 2^{r+1} + 2^{r-1}$. In summary, given a graph, G, with parameters r, s, and $k-2$, if $k-2 < 2^s - 2^{r+1} + 2^{r-1}$, then the rank number of G is $s = 1 + \lfloor \log_2(k-2) \rfloor$, and if $k-2 \geq 2^s - 2^{r+1} + 2^{r-1}$, then the rank number of G is $s+1 = 2 + \lfloor \log_2(k-2) \rfloor$. ■

Theorem 3.3.5 If, without loss of generality, C_m has a larger rank number than C_n and if C_m and P_{k-2} have the same rank number, say r, then the rank number of G is $r+1$.

Proof. If $\lceil \log_2 n \rceil + 1 < \lceil \log_2 m \rceil + 1 = \lfloor \log_2(k-2) \rfloor + 1 = r$, then there are exactly two vertices (one on C_m and one on P_{k-2}) which must be labeled with a value of r. As it is known that G is connected, there must exist at least one path between these two vertices labeled with a value of r. This implies that the smallest possible rank is $r+1$. Unlike Theorem 3.3.4, the two cycles are of different rank numbers. As C_n has a rank which is any value less than r, there is more 'cushion' built into our ranking. Consider the fact that the maximum possible value for the rank of C_n is $r-1$. This is the worst-case scenario. Using the reasoning stated in Theorem 3.3.4, the maximum number of vertices on P_{k-2} with a label that is greater than $r-1$ is $2^{(r+1)-(r-1)} - 1 = 2^2 - 1 = 3$, if the rank number of G is to be less than or equal to $r+1$. Due to the need to relabel one of the vertices which has a label of $r-1$, only $3-1=2$ of these labels actually can contribute to the possible length of P_{k-2}. Each label that is greater than $r-1$ is placed at a distance of 2^{r-1}. We then add on 2^{r-2}, or the number of vertices that can be added to the path before the label of $r-1$ must appear again. Therefore, the length of P_{k-2} can be no more than

$2 * 2^{r-1} + 2^{r-2} = 2^r + 2^{r-2}$. As the rank of P_{k-2} is known to be r, its length must be less than 2^r. Clearly this is within the bound. In this way, the rank number of G is $r + 1 = \lceil \log_2 m \rceil + 2 = \lfloor \log_2 (k-2) \rfloor + 2$. ■

Theorem 3.3.6 If, without loss of generality, the rank number of C_m, say $r+1$, is exactly one more than the rank number of C_n and if C_n and P_{k-2} have the same rank number, the rank number of G is $r+2$, or one more than the rank number of C_m.

Proof. Begin by labeling C_n and C_m, ensuring that the vertices of degree 3 are labeled with value of r and $r+1$, respectively. Label the rest of each cycle (either clockwise or counter-clockwise) and the path in the optimal way (1 2 1 3 1 2 1 4 . . .). From there, we can see that we must have two vertices with a label value of r on P_k (vertices A and B in Figure 9.1).

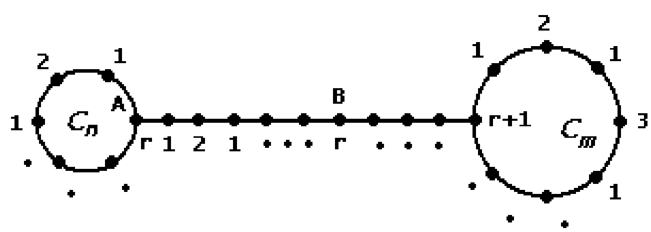

Figure 9.1

Therefore, we must relabel a vertex with a label of $r+1$ between vertices A and B. To complete the ranking, relabel vertex B with a value of $r+2$, as is shown in Figure 9.2.

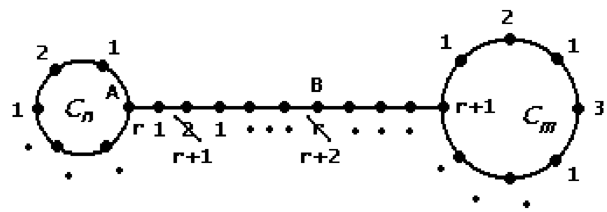

Figure 9.2

The rank number of G is $r + 2 = \lceil \log_2 m \rceil + 2 = \lceil \log_2 n \rceil + 3$. ■

Theorem 3.3.7 Without loss of generality, if the rank number of C_m is at least 2 more than the rank number of C_n and if the rank number of C_n is equal to the rank number of P_{k-2}, then the rank number of G is equal to the rank number of C_m.

Proof. Begin by labeling C_m in the optimal way with its rank number being assigned to the vertex of degree 3. As the rank of C_m is sufficiently large, it does not interfere with the labeling of C_n or P_{k-2}. Labeling C_n and P_{k-2} it can be seen that there are now two vertices with a label of r, one on the vertex of degree 3 of C_n and one that is located somewhere on P_{k-2}. Between these two vertices, there must be a vertex labeled with $r + 1$. The ranking, shown in Figure 10, is now complete and $\chi_r(G) = \chi_r(C_m) = \lceil \log_2 m \rceil + 1$. ■

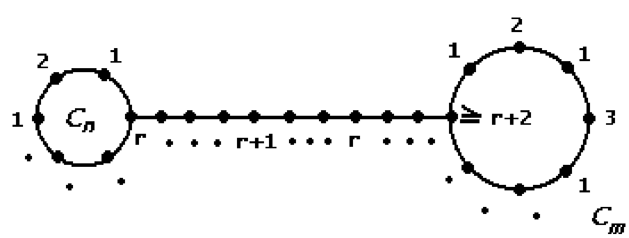

Figure 10

Theorem 3.3.8 Without loss of generality, if the rank number of C_m is greater than the rank number of C_n, and if the rank number of C_n is greater than the rank number of P_{k-2}, then the rank number of G is equal to the rank number of C_m.

Proof. Given that $\chi_r(P_{k-2}) = r$, $\chi_r(C_n) = r + s$, and $\chi_r(C_m) = r + s + t$, where r, s, and t are positive integers, it is clear that each of the three subgraphs can be labeled optimally and independently and all requirements of ranking are satisfied. As the values on the labels of the vertices of degree 3 are not equal to each other, there is no need to increase the label of any vertex on P_{k-2}. Therefore, $\chi_r(G) = r + s + t = \chi_r(C_m) = 1 + \lceil \log_2 m \rceil$. ■

Theorem 3.3.9 Without loss of generality, if the rank number of C_m, say $r + s + t$, is greater than the rank number of P_{k-2}, say $r + s$,and if the rank number of P_{k-2} is greater than the rank number of C_n, say r, and

a. if $t \geq 2$, then the rank number of G is equal to the rank number of C_m.

b. if $t = 1$, and if $P_{k-2} \leq 2^{r+s} - 2^{r-1} - 1$, then the rank number of G is equal to the rank number of C_m.

c. if $t = 1$, and if $P_{k-2} > 2^{r+s} - 2^{r-1} - 1$, then the rank number of G is equal to one more than the rank number of C_m.

Proof. Given that $\chi_r(C_n) = r$, $\chi_r(P_{k-2}) = r + s$, and $\chi_r(C_m) = r + s + t$, where r, s, and t are positive integers, begin by labeling the vertices of C_m, ensuring that its largest label is placed on its vertex of degree 3. Repeat this for C_n. Next, label P_{k-2}, beginning on the end of the path which is nearest

to the larger cycle. For part a, as the rank number of C_m is sufficiently larger than any other label, even if it is necessary to adjust the labels of some vertices on P_{k-2}, it is clearly impossible for these labels to exceed $\chi_r(C_m)$. For part b, $P_{k-2} \leq 2^{r+s} - 2^{r-1} - 1$. Again, it is necessary to discuss exactly where this critical value is from. First, we know the length of P_{k-2} must be at least 2^{r+s-1}, as this is required for its rank number to be $r + s$. Next, we go back to the idea of 'protector' vertices which was discussed in Theorem 3.3.4. From C_m to the vertex labeled with the rank number of P_{k-2}, there is no need to relabel any vertices as $\chi_r(C_m)$ exceeds all of these values. In this way, we have counted all of these vertices in our original number of 2^{r+s-1}. From the vertex labeled with the rank number of P_{k-2} to C_n, there are a maximum of $2^s - 1$ protector vertices with labels greater than or equal to the rank number of C_n. These vertices are placed at a distance of 2^{r-1} from each other. Finally, we must consider that one of these high labeled vertices will be in place already on the vertex of degree 3 on C_n. In this way, the critical value of the length of P_{k-2} is $2^{r+s-1} + (2^s - 1) * (2^{r-1}) - 1$. If the length of P_{k-2} is less than this value, there is no need to relabel any vertices between the vertex labeled with the rank number of P_{k-2} and C_n, meaning the rank still does not exceed that of C_m and the $\chi_r(G) = \chi_r(C_m)$. For part c, if the length of P_{k-2} exceeds the critical value, it is necessary to relabel vertices between the vertex labeled with the rank number of P_{k-2} and C_n, at which point, there will be a need for a label which is greater than the rank of P_{k-2}, which then creates the need to label exactly one vertex with a value of $r+s+t+1$, and $\chi_r(G) = r+s+t+1 = \chi_r(C_m)+1 = 2+\lceil \log_2 m \rceil$.

■

Theorem 3.3.10 Without loss of generality, if the rank number of C_m, say $r+s$, is greater than the rank number of C_n, say r, and if the rank number of P_{k-2}, say $r+s+t$, is greater than the rank number of C_m, then

a. the rank number of G is equal to the rank number of P_{k-2} if the length of P_{k-2} is less than or equal to

$$(2^{t+1}-2)*(2^{r+s-1})+\left(\sum_{i=0}^{s-1}2^{r+s-2-i}\right)-1$$

b. the rank number of G is equal to one more than the rank number of P_{k-2} if the length of P_{k-2} is greater than

$$(2^{t+1}-2)*(2^{r+s-1})+\left(\sum_{i=0}^{s-1}2^{r+s-2-i}\right)-1$$

Proof. Given that $\chi_r(C_n)=r$, $\chi_r(C_m)=r+s$, and $\chi_r(P_{k-2})=r+s+t$, where r, s, and t are positive integers, begin by labeling the vertices of C_m, ensuring that its largest label is placed on its vertex of degree 3. Repeat this for C_n. Next, label P_{k-2}, beginning on the end of the path which is nearest to the larger cycle, C_m. Consider the example shown in figure 11.1, where $r=3$, $s=1$, and $t=2$ and again, the symbol X represents a segment of the path that has been omitted containing vertices labeled with values of 1, 2, 1, 3, 1, 2, 1. Specifically, the path length of P_{k-2} is 42.

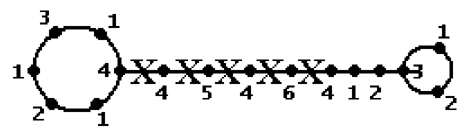

Figure 11.1

As has been discussed in multiple theorems before this, we look to see which labels will be high enough to 'protect' the labels on the vertices of degree 3. In this way, for any given graph, we can have a maximum of $2^{t+1}-1$ vertices which have a label higher than the rank number of C_n. However, one of these labels has already been used to label the vertex of degree 3 on C_m. Therefore, there are a total of $2^{t+1}-2$ labels which can be used to relabel any vertices which do not follow the requirements of rankings and thus extend the critical value of the length of P_{k-2} without the need to increase the rank number of the graph. This relabeling is shown for our example in Figure 11.2, and again, the symbol X represents a segment of the path that has been omitted containing vertices labeled with values of 1, 2, 1, 3, 1, 2, 1.

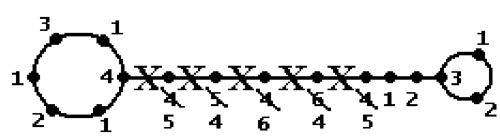

Figure 11.2

For any graph which satisfies the requirements stated in this theorem, each vertex label with a value which is greater than or equal to $r+s$ is shifted 2^{r+s-1} vertices. Now that we know the maximum number of vertex labels that

can be shifted and how far, we know that the length of P_{k-2} can be at least $(2^{t+1}-2)*(2^{r+s-1})$. However, the path can be longer still. After using all of the possible 'protector' vertices, the path can continue its regular pattern until there is a conflict caused between a vertex on the path and the vertex of degree 3 on C_n. How many such vertices can be added? For each integer, i, from $r-1$ to $r+s-2$, a power of 2^i can be added to the length of the path. We begin with 2^{r+s-2} because this is the largest portion of path we can add that is a power of 2 that does not cause conflict. In other words, if we were to attempt to add on as much as 2^{r+s-1}, this would most certainly require a vertex label which is greater than or equal to $r+s$, however, we have already used all possible vertex labels of this size, and adding another would require the rank number of the graph to increase. Running through all of the stated values, the smallest power of 2 we can add on is 2^{r-1}. This is due to the fact that after we have added on the intermittent values, we reach the vertices which are closest to C_n, and would like to add on the maximum number of vertices without having a label of r be needed again. In this way, the critical path length which separates the graphs which have a rank number equal to the rank number of P_{k-2} and those graphs which have a rank number which is one more than the rank number of P_{k-2} is $(2^{t+1}-2)*(2^{r+s-1})+\left(\sum_{i=0}^{s-1} 2^{r+s-2-i}\right)-1$. ■

3.4 Consider the graph F shown in Figure 12. The graph, F, is known as a Theta graph, given its resemblance to the Greek letter (θ). Theta graphs are composed of a cycle with a path connecting two vertices on the cycle.

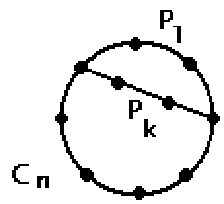

Figure 12

There are three parameters (C_n, P_k, P_l) necessary to appropriately characterize F. First, we must know n, the size of the largest cycle in F. Next, the length of the path connecting two vertices of the cycle must be given. This value is called k, as is shown in Figure 11. Finally, it is important to know l, or the length of the shortest path on the cycle between the vertices of degree 3. It is crucial to recognize that $k \leq l$. If $k > l$, then our value for n may be larger than we believe it to be. Also, $l \leq \lfloor \frac{n}{2} \rfloor + 1$. If we allow P_l to incorporate more than half of C_n, we will get rank numbers which were already found for smaller values of l. Specifically, it is true that $\chi_r(C_n, P_k, P_3) = \chi_r(C_n, P_k, P_{n-2})$, as these graphs are isomorphic.

Theorem 3.4.1 If P_k consists only of an edge ($k = 2$), the rank number of F is equal to the rank number of C_n.

Proof. Begin by labeling C_n in the optimal way. Now we must add one more edge. Locate the vertex, v, with the highest label. There can only be one vertex with the highest label, as if there were more than one, we would not be satisfying the properties of a ranking. Add the edge beginning at this vertex, and ending at a vertex on the circle, allowing for a path length of l between the vertex of highest degree and the other vertex to be chosen. If $l = \frac{n}{2} + 1$, there

will be only one choice for the vertex to be the other vertex on P_2, as the cycle is 'cut' exactly in half by this new edge. In any case, as we have chosen one end of P_2 to be the vertex with the highest rank, there is no danger in placing the other end on any other vertex in the cycle. This is due to the fact that, as was stated before, there is no other vertex which has the highest label, so there are no new paths created that cause a conflict with this vertex. Also, any new paths that are created for whichever other vertex is chosen must also follow all requirements of ranking, as all new paths that are created include the vertex with the highest label. In this way, it is guaranteed that after assigning one end of P_2 to v, the other end of P_2 is irrelevant, no vertices need to be relabeled, and $\chi_r(F) = \chi_r(C_n)$. ■

Having only completed the most simple of the theta graphs, there is great opportunity to investigate the rank number of these graphs with any path length.

4 Conclusion - Future Research Opportunities

There are an infinite number of families of graphs which are still yet to be researched, related to their rank numbers. A complete characterization of the rank numbers of all theta graphs would be the primary area of research that could extend the topics of this paper.

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