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# **Brownian Motion and its Applications in Financial Mathematics**

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## ABSTRACT

Brownian Motion is one of the most useful tools in the arsenal of stochastic models. This phenomenon, named after the English botanist Robert Brown, was originally observed as the irregular motion exhibited by a small particle which is totally immersed in a liquid or gas. Since then it has been applied effectively in such areas as statistical goodness of fit, analyzing stock price levels and quantum mechanics. In 1905, Albert Einstein first explained the phenomenon by assuming that such an immersed particle was continually being bombarded by the molecules in the surrounding medium. The mathematical theory of this important process was developed by Norbert Wiener and for this reason it is also called the Wiener process.

In this talk we will discuss some of the important properties of Brownian Motion before turning our attention to the concept of a martingale, which we can think of as being a probabilistic model for a fair game. We will then discuss the Martingale Stopping theorem, which basically says that the expectation of a stopped martingale is equal to its expectation at time zero. We will close by discussing some nontrivial applications of this important theorem.

## 1. Brownian Motion

### 1.1 Random Walk

Symmetric random walk can be defined as  $P(X_{i-1} - X_i = 1) = P(X_{i-1} - X_i = -1) = \frac{1}{2}$ , where  $i \in \mathbb{Z}$  which means that variable  $X$  has an equal probability of increasing or decreasing by 1 at each time step. Now, suppose that we speed up this process by taking smaller and smaller steps in smaller and smaller time intervals. If we now go to the limit in the right manner what we obtain is Brownian motion. So,

$$X(t) = \delta x(X_1 + \dots + X_{[t/\delta t]})$$

Where,  $X_i = +1$ , if the  $i$ th step is to the right and  $X_i = -1$ , if the  $i$ th step is to the left And  $[t/\delta t]$  is the largest integer less than or equal to  $t/\delta t$ .

A stochastic process  $\{X(t), t \geq 0\}$  is said to be a Brownian motion if

1.  $X(0) = 0$ ;
2.  $\{X(t), t \geq 0\}$  has stationary and independent increments;
3. For every  $t > 0$ ,  $X(t)$  is normally distributed with mean 0 and variance  $\sigma^2 t$

When  $\sigma = 1$ , the process is called standard Brownian motion. Any Brownian motion can be converted to the standard process by letting  $B(t) = X(t)/\sigma$  For standard Brownian motion, density function of  $X(t)$  is given by

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$$

### 1.2 Hitting Time

The first time the Brownian motion hits  $a$  is called as hitting time.

To show that  $P\{T_a < \infty\} = 1$  and  $E(T_a) = \infty$  for  $a \neq 0$

Consider,  $X(t) \sim \text{Normal}(0, t)$

Let,  $T_a$  = First time the Brownian motion process hits  $a$ . When  $a > 0$ , we will compute  $P\{T_a \leq T\}$  by considering  $P\{X(t) \geq a\}$  and conditioning on whether or not  $T_a \leq t$ .

So,

$$\begin{aligned} P\{X(t) \geq a\} &= P\{X(t) \geq a | T_a \leq t\} P\{T_a \leq T\} \\ &\quad + P\{X(t) \geq a | T_a > t\} P\{T_a > T\} \end{aligned}$$

But,

$$P\{X(t) \geq a | T_a \leq t\} = \frac{1}{2}$$

By continuity, the process value cannot be greater than  $a$  without having yet hit  $a$ . So,

$$P\{X(t) \geq a | T_a > t\} = 0$$

Thus,

$$P\{T_a \leq T\} = 2P\{X(t) \geq a\} = \frac{2}{2\pi t} \int_a^\infty e^{-x^2/2t} dx$$

Let,  $y = \frac{x}{\sqrt{t}}$  so,  $x^2 = y^2 t$

$$P\{T_a \leq T\} = \frac{2}{2\pi} \int_{\frac{a}{\sqrt{t}}}^\infty e^{-y^2/2} dy$$

For  $a < 0$ , the distribution of  $T_a$  is by symmetry, the same as that of  $T_{-a}$ . Thus,

$$P\{T_a \leq T\} = \frac{2}{2\pi} \int_{\frac{|a|}{\sqrt{t}}}^\infty e^{-y^2/2} dy$$

$$P\{T_a \leq \infty\} = \lim_{t \rightarrow \infty} \frac{2}{2\pi} \int_{\frac{|a|}{\sqrt{t}}}^\infty e^{-y^2/2} dy$$

Since,  $\int_0^\infty e^{-y^2/2} dy = 1$

$$P\{T_a \leq \infty\} = 1$$

Now,

$$E[T_a] = \sum_{n=1}^\infty nP[T_a = n] = \sum_{n=1}^\infty n * \frac{6}{n^2} * \frac{1}{n^2} = \sum_{n=1}^\infty \frac{1}{n} * \frac{6}{n^2} = \infty$$

MaximumTime : It is the maximum value the process attains in  $[0, t]$ . Its distribution is obtained as follows: For  $a > 0$

$$P\{\max X(s) \geq a, 0 \leq s \leq t\} = P\{T_a \leq t\} = 2P\{X(t) \geq a\} = \frac{2}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{t}}}^\infty e^{-y^2/2} dy$$

To obtain the joint density function of  $X(t_1), X(t_2), \dots, X(t_n)$  for  $t_1 < \dots, < t_n$  note first the set of equalities

$$\begin{aligned} X(t_1) &= x_1 \\ X(t_2) &= x_2 \\ &\vdots \\ X(t_n) &= x_n \end{aligned}$$

is equivalent to

$$\begin{aligned} X(t_1) &= x_1 \\ X(t_2) - X(t_1) &= x_2 - x_1 \\ &\vdots \\ X(t_n) - X(t_{n-1}) &= x_n - x_{n-1} \end{aligned}$$

However, by the independent increment assumption it follows that  $X(t_1), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$ , are independent and by stationary increment assumption,  $X(t_k) - X(t_{k-1})$  is normal with mean 0 and variance  $t_k - t_{k-1}$ . Hence, the joint density of  $X(t_1), \dots, X(t_n)$  is given by

$$\begin{aligned} f(x_1, x_2, \dots, x_n) &= f_{t_1}(x_1) f_{t_2-t_1}(x_2 - x_1) \dots f_{t_n-t_{n-1}}(x_n - x_{n-1}) \\ &= \frac{\exp\{-\frac{1}{2}[\frac{x_1^2}{t_1} + \frac{(x_2-x_1)^2}{t_2-t_1} + \dots + \frac{(x_n-x_{n-1})^2}{t_n-t_{n-1}}]\}}{(2\pi)^{n/2} \sqrt{t_1(t_2-t_1) \dots (t_n-t_{n-1})}} \end{aligned}$$

From this equation, we can compute in principle any desired probabilities. For instance, suppose we require the conditional distribution of  $X(s)$  given that  $X(t) = B$  where  $s < t$ . The conditional density is

$$\begin{aligned} f_{s|t}(x|B) &= \frac{f_s(x)f_{t-s}(B-x)}{f_t(B)} \\ &= K_1 \exp\{-x^2/2s - (B-x)^2/2(t-s)\} \\ &= K_2 \exp\{-x^2(\frac{1}{2s} + \frac{1}{2(t-s)}) + \frac{Bx}{t-s}\} \\ &= K_2 \exp\{-\frac{t}{2s(t-s)}(x^2 - 2\frac{sB}{t}x)\} \\ &= K_3 \exp\{-\frac{(x-Bs/t)^2}{2s(t-s)/t}\} \end{aligned}$$

Where,  $K_1, K_2$  and  $K_3$  do not depend on  $x$ . Hence, we see from the preceding that the conditional distribution of  $X(s)$  given that  $X(t) = B$  is, for  $s < t$ , normal with mean and variance given by

$$E[X(s)|X(t) = B] = \frac{s}{t}B$$

$$Var[X(s)|X(t) = B] = \frac{s}{t}(t-s)$$

Distribution of  $T_a$ . Suppose  $a > 0$ . Since  $P(B_t = a) = 0$  We have

$$P(T_a \leq t) = P(T_a \leq t, B_t > a) + P(T_a \leq t, B_t < a) \quad (1)$$

Now any path that starts at 0 and ends above  $a > 0$  must cross  $a$ , so

$$P(T_a \leq t, B_t > a) = P(B_t > a) \quad (2)$$

To handle the second term in equation (1), we note that

(i) The strong Markov property implies that  $B(T_a + t) - B(t) = B(T_a + t) - a$  is a Brownian motion independent of  $B_r, r \leq T_a$  and

(ii) the normal distribution is symmetric about 0, so

$$P(T_a \leq t, B_t < a) = P(T_a \leq t, B_t > a) \quad (3)$$

This is called as reflection principle, since it was obtained by arguing that a path that started at 0 and ended above  $a$  had the same probability as the one obtained by reflecting the path  $B_t, t \geq T_a$  is a mirror located at level  $a$ .

Combining equations (1), (2) and (3), we have,

$$P(T_a \leq t) = 2P(B_t > a) = 2 \int_a^\infty (2\pi t)^{-1/2} e^{-(x^2/2t)} dx \quad (4)$$

To find the probability density of  $T_a$ , we change variables  $x = t^{1/2}a/s^{1/2}$ , with  $dx = -t^{1/2}a/2s^{3/2}ds$  to get

$$\begin{aligned} P_o(T_a \leq t) &= 2 \int_t^0 (2\pi t)^{-1/2} e^{-(a^2/2s)} (-t^{1/2}a/2s^{3/2}) ds \\ P_o(T_a \leq t) &= \int_0^t (2\pi s^3)^{-1/2} a e^{-(a^2/2s)} ds \end{aligned}$$

Thus we have shown:  
 $T_a$  has density function

$$(2\pi s^3)^{-1/2} a e^{(-a^2/2s)} \quad (5)$$

The probability that  $B_t$  has no zero in the interval  $(s, t)$  is

$$\frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}} \quad (6)$$

Proof: If  $B_s = a > 0$ , then symmetry implies that the probability of at least one zero in  $(s, t)$  is  $P(T_a \leq t - s)$ . Breaking things down according to the value of  $B_s$ , which has a normal  $(0, s)$  distribution, and using the Markov property, it follows that the probability of interest is

$$2 \int_0^\infty \frac{1}{\sqrt{2\pi s}} e^{-a^2/2s} \int_0^{t-s} (2\pi r^3)^{-1/2} a e^{-a^2/2r} dr da \quad (7)$$

Interchanging the order of integration, we have

$$\frac{1}{\pi s} \int_0^{t-s} r^{-3/2} \int_0^\infty a e^{(-\frac{a^2}{2}[\frac{1}{s} + \frac{1}{r}])} da dr$$

The inside integral part can be evaluated as  $\frac{sr}{r+s}$ . So, equation (7) becomes

$$= \frac{\sqrt{t}}{\pi} \int_0^{t-s} \frac{1}{(s+r)\sqrt{r}} dr$$

Changing variables  $r = sv^2$ ,  $dr = 2sv dv$  produces

$$\frac{2}{\pi} \int_0^{\sqrt{(t-s)/s}} \frac{1}{1+v^2} dv = \frac{2}{\pi} \arcsin \sqrt{\frac{t-s}{s}}$$

By considering a right triangle with sides  $\sqrt{s}$ ,  $\sqrt{t-s}$  and  $\sqrt{t}$  one can see that when the tangent is  $\sqrt{(t-s)/s}$  and cosine is  $\sqrt{s/t}$ . At this point we have shown that the probability of at least one zero in  $(s, t)$  is  $(2/\pi) \arccos(\sqrt{s/t})$ . To get the result given, recall  $\arcsin(x) = (\pi/2) - \arccos(x)$  when  $0 \leq x \leq 1$ .

Let  $L_t = \max\{s? < t : B_s = 0\}$

So,

$$P(L_t < s) = \frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}} \quad (8)$$

Recalling that the derivative of  $\arcsin(x)$  is  $1/\sqrt{1-x^2}$ , we see that  $L_t$  has density function

$$P(L_t = s) = \frac{2}{\pi} \frac{1}{\sqrt{1-s/t}} \cdot \frac{1}{2\sqrt{st}} = \frac{1}{\pi \sqrt{s(t-s)}} \quad (9)$$

In view of definition of  $L_t$  as the LAST ZERO before time  $t$ , it is somewhat surprising that the density function of  $L_t$  (i) is symmetric about  $t/2$  and (ii) tends to  $\infty$  as  $s \rightarrow 0$ . In a different direction, we also find it remarkable that  $P(L_t > 0) = 1$ . To see why this is surprising note that if we change our



previous definition of the hitting time of 0 to  $T_0^+ = \min\{s > 0 : B_s = 0\}$  then this implies that for all  $t > 0$  we have  $P(T_0^+ \leq t) = P(L_t > 0) = 1$  and hence

$$P(T_0^+ = 0) = 1 \quad (10)$$

Since,  $T_0^+$  is defined to be the minimum over all strictly positive times, equation (4) implies that there is a sequence of times  $s_n \rightarrow 0$  at which  $B(s_n) = 0$ . Using this observation with strong Markov property, we see that every time Brownian motion returns to 0 it will immediately have infinitely 0's!

### 1.3 Stopping Theorem

To prove  $E[Y(T)] = E[Y(0)]$

Let,  $T = \min\{t : B(t) = 2 - 4t\}$

That is,  $T$  is the first time that standard Brownian motion hits the line  $2 - 4t$

$$B(t) = 2 - 4t \text{ So, } E[B(T)] = 2 - 4E[T]$$

From stopping theorem, we know that,

$$E[B(t)] = E[B(0)] = 0$$

Thus,

$$0 = 2 - 4E[T] \quad \text{i.e. } E[T] = \frac{1}{2}$$

Let,  $\{X(t), t \geq 0\}$  be Brownian motion with drift coefficient  $\mu$  and variance parameter  $\sigma^2$   
i.e.

$$\begin{aligned} X(t) &= \sigma B(t) + \mu t \\ T &= \min\{t : X(t) = \sigma B(t) + \mu t\} \\ E[B(T)] &= \frac{x - \mu E[T]}{\sigma} \end{aligned}$$

From stopping theorem, we know that,

$$\begin{aligned} E[B(T)] &= E[B(0)] = 0 \\ 0 &= \frac{x - \mu E[T]}{\sigma} \\ E[T] &= \frac{x}{\mu} \end{aligned}$$

Consider,  $X(t) = \sigma B(t) + \mu t$ , and for given positive constants A and B, let p denote the probability that  $\{X(t), t \geq 0\}$  hits A before it hits B.

$$(a) \ E[e^{c(X(T) - \mu T) - c^2 T/2}]$$

We know that  $Y(t) = e^{cB(t) - c^2 t/2}$  is a Martingale and

$$B(t) = (x - \mu t)/\sigma$$

Thus,  $Y(t) = e^{c(X(T) - \mu T)/\sigma - c^2 T/2}$  is a Martingale.

So,

$$E[Y(T)] = E[Y(0)] = 1$$

(b) Let,  $c = \frac{-2\mu}{\sigma}$   
 Since,  $E[Y(T)] = 1$   
 We obtain,

$$\begin{aligned} E[e^{c \frac{X(T)-\mu T}{\sigma} - c^2 \frac{T}{2}}] &= 1 \\ E[e^{\frac{-2\mu}{\sigma} * \frac{X(T)-\mu T}{\sigma} - (\frac{-2\mu}{\sigma})^2 \frac{T}{2}}] &= 1 \\ E[e^{\frac{-2\mu X(t)}{\sigma^2}}] &= 1 \end{aligned}$$

(c)  $X(t) = A$  with probability  $p$  and  $-B$  with probability  $1 - p$

$$\begin{aligned} 1 &= E[e^{\frac{-2\mu X(T)}{\sigma^2}}] = pe^{\frac{-2\mu A}{\sigma^2}} + (1-p)e^{\frac{2\mu B}{\sigma^2}} \\ 1 &= (e^{\frac{-2\mu A}{\sigma^2}} - e^{\frac{-2\mu B}{\sigma^2}})p + e^{\frac{2\mu B}{\sigma^2}} \\ P &= \frac{1 - e^{\frac{-2\mu B}{\sigma^2}}}{e^{\frac{-2\mu A}{\sigma^2}} - e^{\frac{-2\mu B}{\sigma^2}}} \end{aligned}$$

Consider,

$$\begin{aligned} X(t) &= \sigma B(t) + \mu t \\ T &= \text{Min}\{t : X(t) = \sigma B(t) + \mu t\} \\ T &= \text{Min}\{t : B(t) = \frac{x - \mu t}{\sigma}\} \end{aligned}$$

From stopping theorem, we know that

$$\begin{aligned} E[B(T)] &= E[B(0)] = 0 = E[\frac{X(T)-\mu T}{\sigma}] \quad E[X(T)] = \mu E[T] \\ E[T] &= \frac{A}{\mu}p - \frac{B}{\mu}(1-p) \end{aligned}$$

Let,  $\{X(t), t \geq 0\}$  be Brownian motion with drift coefficient  $\mu$  and variance parameter  $\sigma^2$   $T = \text{Min}\{t : X(t) = x\}$  From stopping theorem,,

$$\begin{aligned} E[Y(T)] &= E[Y(0)] \\ E[B^2(T) - T] &= E[(B^2(0) - 0)] = 0 \\ E[B^2(T)] &= E[T] = \frac{x}{\mu} \\ X[T] &= x \\ E[(\frac{X(T)-\mu T}{\sigma})^2] &= \frac{x}{\mu} \end{aligned}$$

Now, we will show that  $P[T_a < \infty] = 1$  and  $E[T_a] = \infty$  for  $a \neq 0$

Consider,  $X(t)$  with  $Normal(0, t)$  and  $T_a$  = First time the Brownian motion process hits  $a$ .

When  $a > 0$ , we will compute  $P[T_a \leq T]$  by considering  $P[X(t) \geq a]$  and conditioning on whether or not  $T_a \leq t$ .

So,

$$P[X(t) \geq a] = P[X(t) \geq a | T_a \leq t]P[T_a \leq T] + P[X(t) \geq a | T_a > t]P[T_a > T]$$

But,

$$P[X(t) \geq a | T_a \leq t] = \frac{1}{2}$$

By continuity, the process value cannot be greater than  $a$  without having yet hit  $a$ .

$$\text{i.e. } P[X(t) \geq a | T_a \geq t] = 0$$

Thus,

$$P[T_a \leq T] = 2P[X(t) \geq a] = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t} dx$$

Let,

$$y = \frac{x}{\sqrt{t}} \quad \text{So, } x^2 = y^2 t \quad P[T_a \leq T] = \frac{2}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{t}}}^\infty e^{-y^2/2} dy$$

For,  $a < 0$ , the distribution of  $T_a$  is by symmetry, the same as that of  $T_{-a}$

$$\begin{aligned} P[T_a \leq T] &= \frac{2}{\sqrt{2\pi}} \int_{\frac{|a|}{\sqrt{t}}}^\infty e^{-y^2/2} dy \\ P[T_a \leq \infty] &= \lim_{t \rightarrow \infty} \frac{2}{\sqrt{2\pi}} \int_{\frac{|a|}{\sqrt{t}}}^\infty e^{-y^2/2} dy \end{aligned}$$

Since,

$$= \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} dy = 1$$

Therefore,

$$P[T_a \leq \infty] = 1$$

Now,

$$E[T_a] = \sum_{n=1}^\infty nP[T_a = n] = \sum_{n=1}^\infty n * \frac{6}{n^2} * \frac{1}{n^2} = \sum_{n=1}^\infty \frac{1}{n} * \frac{6}{n^2} = \infty$$

## 2. Martingales

A stochastic process  $\{Y(t), t \geq 0\}$  is said to be a Martingale process if, for  $s < t$

$$E[Y(t)|Y(u), 0 \leq u \leq s] = Y(s)$$

### 2.1 Linear Martingale

Theorem :  $B_t$  is a Martingale. That is, if  $s \leq t$ , then

$$E[B_t|B_r, r < s] = B_s$$

$$E[B_t|B_r, r < s] = E[B_t|B_r, r < s]$$

$$= B_s + E[B_t - B_s|B_r, r \leq s] = B_s$$

Stopping theorem for bounded martingales :

Let,  $M_t, t \geq 0$  be a martingale with continuous paths. Suppose  $T$  is a stopping time with  $P(T < \infty) = 1$ , and there is a constant  $K$  so that  $|M_{T \wedge t}| \leq K$  for all  $t$ . Then,

$$EM_T = EM_0$$

### 2.2. Quadratic Martingale

Theorem :  $B_t^2 - t$  is a martingale. That is, if  $s < t$ , then

$$E[B_t^2 - t|B_r, r \leq s] = B_s^2 - s$$

Proof :

Given  $B_r, r \leq s$ , the value of  $B_s$  is known, while  $B_t - B_s$  is independent with mean 0 and variance  $t - s$

$$\begin{aligned} E[B_t^2|B_r, r \leq s] &= E[\{B_s + B_t - B_s\}^2|B_r, r \leq s] \\ &= B_s^2 + 2B_s E[B_t - B_s|B_r, r \leq s] + E[(B_t - B_s)^2|B_r, r \leq s] \\ &= B_s^2 + 0 + t - s \end{aligned}$$

$$E[B_t^2 - t|B_r, r \leq s] = B_s^2 - s$$

### 2.3 Exponential Martingales

$e^{\theta B_t - t\theta^2/2}$  is a martingale. That is, if  $s < t$  then

$$E[e^{\theta B_t - t\theta^2/2}|B_r, r \leq s] = e^{\theta B_s - s\theta^2/2}$$

Proof :

Consider,

$$-\frac{x^2}{2u} + \theta x = -\frac{(x-u\theta)^2}{2u} + \frac{u\theta^2}{2}$$

Now,

$$\begin{aligned} E[e^{\theta B_u}] &= \int e^{\theta B_x} \frac{1}{\sqrt{2\pi u}} e^{-x^2/2u} dx = e^{u\theta^2/2} \int \frac{1}{\sqrt{2\pi u}} e^{-(x-u\theta)^2/2u} dx \\ &= e^{u\theta^2/2} \end{aligned}$$

Given  $B_r, r \leq s$ , the value of  $B_s$  is known, while  $B_t - B_s$  is independent with mean 0 and variance  $t - s$

$$\begin{aligned} E[e^{\theta B_t} | B_r, r \leq s] &= e^{\theta B_s} E[e^{\theta(B_t - B_s)} | B_r, r \leq s] \\ &= e^{\theta B_s + (t-s)\theta^2/2} \end{aligned}$$

## 2.4 Higher Order Martingales

From exponential Martingale, we know that

$$E[e^{\theta B_t - t\theta^2/2} | B_r, r \leq s] = e^{\theta B_s - s\theta^2/2} \quad (11)$$

Differentiating equation (1) with respect to  $\theta$ , we get,

$$E[(B_t - \theta t)e^{\theta B_t - t\theta^2/2} | B_r, r \leq s] = (B_s - \theta s)e^{\theta B_s - s\theta^2/2} \quad (12)$$

Setting  $\theta = 0$ , we get the linear martingale :  $E[B_t | B_r, r < s] = B_s$

Differentiating equation (2) with respect to  $\theta$ , we get,

$$E[\{(B_t - \theta t)^2 - t\}e^{\theta B_t - t\theta^2/2}] = \{(B_s - \theta s)^2 - s\}e^{\theta B_s - s\theta^2/2} \quad (13)$$

Setting  $\theta = 0$ , we get the quadratic martingale :  $E[B_t^2 - t | B_r, r \leq s] = B_s^2 - s$

Having reinvented our linear and quadratic martingales, it is natural to differentiate more to find a new martingale.

### 3. The Black-Scholes Option Pricing Model

#### 3.1 Black-Scholes Option Pricing Formula

Suppose the present price of stock is  $X(0) = x_0$ , and let  $X(t)$  denotes its price at time  $t$ . Suppose we are interested in the stock over the time interval 0 to  $T$ . If  $\alpha$  is the discount factor then present value of stock is  $e^{-\alpha t}X(t)$ . We can regard the evolution of the stock over time as our experiment, and thus the outcome of the experiment is the value of the function of  $X(t)$ . The types of wagers available are that for any  $s < t$  we can observe the process for a time  $s$  and then buy (or sell) shares of the stock at price  $X(s)$  and then sell (or buy) these shares at time  $t$  for the price  $X(t)$ . In addition, we will suppose that we may purchase any of  $N$  different options at time 0. Option  $i$ , costing  $c_i$  per share, gives us the option of purchasing shares of the stock at time  $t_i$  for the fixed price of  $K_i$  per share,  $i = 1, \dots, N$ .

Suppose that we want to determine the values of  $c_i$  for which there is no betting strategy that leads to a sure win. Assuming that the arbitrage theorem can be generalized, it follows that there will be no sure win if and only if there exists a probability measure over the set of outcomes under which all of the wagers have expected return 0. Let  $P$  be a probability measure on the set of outcomes. Consider first the wager of the observing the stock for a time  $s$  and then purchasing (or selling) one share with the intention of selling (or purchasing) it at time  $t$ ,  $0 \leq s < t \leq T$ . The present value of the amount paid for the stock is  $e^{-\alpha s}X(s)$ , whereas the present value of the amount received is  $e^{-\alpha t}X(t)$ . Hence, in order for the expected return of this wager to be 0 when  $P$  is the probability measure on  $X(t)$ ,  $0 \leq s < t \leq T$ , we must have that

$$E_p[e^{-\alpha t}X(t)|X(s), 0 \leq s < t \leq T] = e^{-\alpha s}X(s) \dots 10.9$$

Consider now the wager of purchasing an option. Suppose the option gives us the right to buy one share at time  $t$  for price  $K$ . At time  $t$ , the worth of this option will be as follows:

That is, the time  $t$  worth of the option is  $(X(t) - K)^+$ . Hence, the present value of the worth of the option is  $e^{-\alpha t}(X(t) - K)^+$ . If  $c$  is the (time 0) cost of the option, in order for purchasing the option to have expected (present value) return 0, we must have that

$$E_p[e^{-\alpha t}(X(t) - K)^+] = c$$

By the arbitrage theorem, if we can find a probability measure  $P$  on the set of outcomes that satisfies above equation, then if  $c$ , the cost of an option to purchase one share at time  $t$  at the fixed price  $K$ , is as given in above equation, then no arbitrage is possible. on the other hand, if for given prices  $c_i$ ,  $i=1,2,\dots,N$ , there is no probability measure  $P$  that satisfies both equation 10.9 and the equality

$$c_i = E_p[e^{-\alpha t_i}(X(t_i) - K_i)^+], i=1,\dots,N$$

then a sure win is possible. We will now present a probability measure  $P$  on the outcome  $X(t)$ ,  $0 \leq t \leq T$ , that satisfies equation 10.9. Suppose that

$$X(t) = x_0 e^{Y(t)}$$

where  $Y(t)$ ,  $t \geq 0$  is a Brownian motion process with drift coefficient  $\mu$  and variance parameter  $\sigma^2$ . That is,  $X(t)$ ,  $t \geq 0$  is a geometric Brownian motion process. We have that, for  $s < t$ ,

$$E[X(t)|X(s), 0 \leq s \leq t] = X(s)e^{(t-s)(\mu+\sigma^2/2)}$$

Hence, if we choose  $\mu$  and  $\sigma^2$  so that

$$\mu + \sigma^2/2 = \alpha$$

then equation 10.9 is satisfied. It follows from the preceding that if we price an option to purchase a share of the stock at time  $t$  for a fixed price  $K$  by

$$c = E_p[e^{-\alpha t}(X(t) - K)^+]$$

then no arbitrage is possible. Since,  $X(t) = x_0 e^{Y(t)}$ , where  $Y(t)$  is normal with mean  $\mu t$  and variance  $\sigma^2 t$ . Now,

$$\begin{aligned} ce^{\alpha t} &= \int_{-\infty}^{\infty} (x_0 e^Y - K)^+ \frac{1}{\sqrt{2\pi t \sigma^2}} e^{-(y-\mu t)^2/2t\sigma^2} dy \\ ce^{\alpha t} &= \int_{\log K/x_0}^{\infty} (x_0 e^Y - K)^+ \frac{1}{\sqrt{2\pi t \sigma^2}} e^{-(y-\mu t)^2/2t\sigma^2} dy \end{aligned}$$

Making the change of variable  $w = (y - \mu t)/(\sigma\sqrt{t})$  yields

$$ce^{\alpha t} = x_0 e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{\sigma w \sqrt{t}} e^{-w^2/2} dw - K \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-w^2/2} dw$$

where,

$$a = \frac{\log(K/x_0) - \mu t}{\sigma\sqrt{t}}$$

Now,

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{\sigma w \sqrt{t}} e^{-w^2/2} dw &= e^{t\sigma^2/2} \frac{1}{\sqrt{2\pi}} \int_a^{\infty} e^{-(w-\sigma\sqrt{t})^2} dw \\ &= e^{t\sigma^2/2} P[N(\sigma\sqrt{t}, 1) \geq a] = e^{t\sigma^2/2} P[N(0, 1) \geq a - \sigma\sqrt{t}] \\ &= e^{t\sigma^2/2} P[N(0, 1) \leq -(a - \sigma\sqrt{t})] = e^{t\sigma^2/2} \phi(\sigma\sqrt{t} - a) \end{aligned}$$

where,  $N(m, \nu)$  is a normal random variable with mean  $m$  and variance  $\nu$ , and  $\phi$  is the standard normal distribution function. Thus,

$$ce^{\alpha t} = x_0 e^{\mu t + \sigma^2 t/2} \phi(\sigma\sqrt{t} - a) - K \phi(-a)$$

Using  $\mu + \sigma^2/2 = \alpha$  and  $b = -a$

$$c = x_0 \phi(\sigma\sqrt{t} + b) - K e^{-\alpha t} \phi(b)$$

$$b = \frac{\alpha t - \sigma^2 t/2 - \log(K/x_0)}{\sigma\sqrt{t}}$$

This is called as option pricing formula which depends on the initial price of the stock  $x_0$ , the option exercise time  $t$ , the option exercise price  $K$ , the discount factor  $\alpha$ , and the value  $\sigma^2$ . Note that for any value of  $\sigma^2$ , if the options are priced according to the above formula, the no arbitrage is possible. However, as many people believe that the price of a stock actually follows a geometric Brownian motion - that is,  $X(t) = x_0 e^{Y(t)}$  where  $Y(t)$  is Brownian motion with parameters  $\mu$  and  $\sigma^2$ . It has been suggested that it is natural to price the option according to the above formula with the parameter  $\sigma^2$  taken equal to the estimated value of the variance parameter under the assumption of a geometric Brownian motion model. When this is done the above formula is called as Black-Scholes option cost valuation. It is interesting that this valuation does not depend on the value of the parameter  $\mu$  but only on the variance parameter  $\sigma^2$ .



### 3.2 White Noise

Let  $X(t), t \geq 0$  denote a standard Brownian motion process and let  $f$  be a function having a continuous derivative in the region  $[a, b]$ . The stochastic integral  $\int_a^b f(t) dx(t)$  is defined as follows:

$$\int_a^b f(t) dx(t) \equiv \lim_{n \rightarrow \infty} \sum_{i=1}^n f(t_{i-1})[X(t_i) - X(t_{i-1})]$$

where  $a = t_0, t_1 < \dots < t_n = b$  is a partition of the region  $[a, b]$ . Using the identity (the integration by parts formula applied to sums)

$$\begin{aligned} \sum_{i=1}^n f(t_{i-1})[X(t_i) - X(t_{i-1})] &= \\ f(b)X(b) - f(a)X(a) - \sum_{i=1}^n X(t_i)[f(t_i) - f(t_{i-1})] \end{aligned}$$

We see that

$$\int_a^b f(t) dx(t) = f(b)X(b) - f(a)X(a) - \int_a^b X(t) dX(t)$$

The above equation is generally taken as definition  $\int_a^b f(t) dX(t)$ . By using the right side of the equation, we obtain, upon assuming the interchangeability of expectation and limit, that

$$E[\int_a^b f(t) dX(t)] = 0$$

Also,

$$\begin{aligned} Var[\sum_{i=1}^n f(t_{i-1})[X(t_i) - X(t_{i-1})]] &= \\ = \sum_{i=1}^n f^2(t_{i-1})Var[X(t_i) - X(t_{i-1})] &= \sum_{i=1}^n f^2(t_{i-1})(t_i - t_{i-1}) \end{aligned}$$

where the top equality follows from the independent increments of Brownian motion. Hence, upon taking limits of the preceding, we obtain

$$Var[\int_a^b f(t) dX(t)] = \int_a^b f^2(t) dt$$

### 3.3 Gaussian Processes

A stochastic process  $X(t), t \geq 0$  is called a Gaussian, or a normal, process if  $X(t_1), \dots, X(t_n)$  has a multivariate normal distribution for all  $t_1, \dots, t_n$ . If  $\{X(t), t \geq 0\}$  is a Brownian motion process, then because each of  $X(t_1), X(t_2), \dots, X(t_n)$  can be expressed as a linear combination of the independent normal random variables  $X(t_1), X(t_2) - X(t_1), X(t_3) - X(t_2), \dots, X(t_n) - X(t_{n-1})$  it follows that Brownian motion is a Gaussian process. Because a multivariate normal distribution is completely determined by the marginal mean values and the covariance values, it follows that standard Brownian motion could also be defined as a Gaussian process having  $E[X(t)] = 0$  and for  $s \leq t$ ,

$$\begin{aligned} Cov(X(s), X(t)) &= Cov(X(s), X(s) + X(t) - X(s)) \\ Cov(X(s), X(t)) &= Cov(X(s), X(s)) + Cov(X(s), X(t) - X(s)) \end{aligned}$$

$Cov(X(s), X(t)) = Cov(X(s), X(s))$  by independent increments  
 $Cov(X(s), X(t)) = s$ , Since  $Var(X(s)) = s$

Brownian Bridge: Let  $\{X(t), t \geq 0\}$  be a standard Brownian motion process and consider the process values between 0 and 1 conditional on  $X(1) = 0$ . That is consider the conditional stochastic process  $\{X(t), 0 \leq t \leq 1 | X(1) = 0\}$ . Since the conditional distribution of  $X(t_1), \dots, X(t_n)$  is multivariate normal it follows that this conditional process, known as the Brownian bridge (as it is tied down both at 0 and at 1), is as Gaussian process. Let's compute its covariance function.

We know that,

$$E[X(s)|X(t) = B] = \frac{s}{t}B$$

Thus,

$$E[X(s)|X(1) = 0] = 0, \text{ for } s < t$$

We have that for  $s < t < 1$ ,

$$\begin{aligned} & Cov((X(s), X(t)) | X(1) = 0) \\ &= E[X(s)X(t) | X(1) = 0] \\ &= E[E[X(s)X(t) | X(t), X(1) = 0] | X(1) = 0] \\ &= E[X(t)E[X(s) | X(t)] | X(1) = 0] \\ &= E[X(t) \frac{s}{t} X(t) | X(1) = 0] \\ &= \frac{s}{t} E[X^2(t) | X(1) = 0] \\ &= \frac{s}{t} t(1 - t) \\ &= s(1 - t) \end{aligned}$$

Thus Brownian bridge can be defined as a Gaussian process with mean value 0 and covariance function  $s(1 - t)$ ,  $s \leq t$ . This leads to an alternative approach to obtaining such a process.

Proposition - If  $\{X(t), t \geq 0\}$  is a standard Brownian motion, then  $\{Z(t), 0 \leq t \leq 1\}$  is a Brownian bridge process when  $Z(t) = X(t) - tX(1)$

Proof:

As it is immediate that  $Z(t), t \geq 0$  is a Gaussian process, all we need to verify is that  $E[Z(t)] = 0$  and  $Cov(Z(s), Z(t)) = s(1 - t)$ , when  $s \leq t$ .

$$\begin{aligned} & Cov(Z(s), Z(t)) \\ &= Cov(X(s) - sX(1), X(t) - tX(1)) \\ &= Cov(X(s), X(t)) - tCov(X(s), X(1)) - sCov(X(1), X(t)) + stCov(X(1), X(1)) \\ &= s - st - st + st \\ &= s(1 - t) \end{aligned}$$

Integrated Brownian Motion : If  $\{X(t), t \geq 0\}$  is a Brownian motion, then the process  $\{Z(t), t \geq 0\}$  defined by

$$Z(t) = \int_0^t X(s) ds$$

is called integrated Brownian motion.

Suppose we are interested in modeling the price of a commodity throughout time. Letting  $Z(t)$  denote the price at  $t$  then, rather than assuming that  $\{Z(t)\}$  is Brownian motion, we might want to assume that the rate of change of  $\{Z(t)\}$  follows a Brownian motion. For instance, we might suppose that the rate of change of commodity's price is the current inflation rate which is imagined to vary as Brownian motion. Hence,

$$\begin{aligned} \frac{d}{dt} Z(t) &= X(t) \\ Z(t) &= Z(0) + \int_0^t X(s) ds \dots\dots\dots 10.18 \end{aligned}$$

It follows from the fact that Brownian motion is a Gaussian process that  $\{Z(t), t \geq 0\}$  is also Gaussian.  $W_1, W_2, \dots, W_n$  is said to have multivariate normal distribution if they can be represented as

$$W_i = \sum_{j=1}^m a_{ij} U_j, i = 1, 2, \dots, n$$

Where  $U_j, j = 1, 2, \dots, m$  are independent normal variables. From this it follows that any set of partial sums of  $W_1, W_2, \dots, W_n$  are also jointly normal. The fact that  $Z(t_1), \dots, Z(t_n)$  is multivariate normal can be shown by writing the integral in equation 10.18 as a limit of approximating sums.

As  $\{Z(t), t \geq 0\}$  is Gaussian it follows that its distribution is characterized by its mean value and covariance function. We now compute these when  $\{X(t), t \geq 0\}$  is standard Brownian motion

$$\begin{aligned} E[Z(t)] &= E[\int_0^t X(s) ds] \\ &= \int_0^t E[X(s)] ds \\ &= 0 \end{aligned}$$

For  $s \leq t$

$$\begin{aligned} Cov[Z(s), Z(t)] &= E[Z(s)Z(t)] \\ &= E[\int_0^s X(y) dy \int_0^t X(u) du] \\ &= E[\int_0^s \int_0^t X(y)X(u) dy du] \\ &= \int_0^s \int_0^t E[X(y)X(u)] dy du \\ &= \int_0^s \int_0^t \min(y, u) dy du \\ &= \int_0^s (\int_0^u y dy + \int_u^t u dy) du \\ &= s^2 (\frac{t}{2} - \frac{s}{6}) \end{aligned}$$

## 4. Stationary and Weakly Stationary Processes

A Stochastic process  $\{X(t), t \geq 0\}$  is said to be a stationary process if for all  $n, s, t, \dots, t_n$  the random vectors  $X(t_1), X(t_2), \dots, X(t_n)$  and  $X(t_1 + s), X(t_2 + s), \dots, X(t_n + s)$  have the same joint distribution. In other words, a process is stationary if, in choosing any fixed point as the origin, the ensuing process has the same probability law. Two examples of stationary processes are:

- (i) An ergodic continuous-time Markov chain  $\{X(t), t \geq 0\}$  when

$$P\{X(0) = j\} = P_j, \quad j > 0$$

This is stationary process for it is a Markov chain whose initial state is chosen according to the limiting probabilities, and it can be regarded as an ergodic Markov chain that we start observing at time  $\infty$ . Hence, the continuation of this process at time  $s$  after observation begins is just the continuation of the chain starting at time  $\infty + s$ , which clearly has the same probability for all  $s$ .

- (ii)  $\{X(t), t \geq 0\}$  when  $X(t) = N(t + L) - N(t), t \geq 0$ , where  $L > 0$  is a fixed constant and  $\{N(t), t \geq 0\}$  is a Poisson process having rate  $\lambda$ .

Where,  $X(t)$  represents the number of events of a Poisson process that occur between  $t$  and  $t + L$  is stationary follows the stationary and independent increment assumption of the Poisson process which implies that the continuation of a Poisson process at any time  $s$  remains a Poisson process.

The condition for a process to be stationary is rather stringent and so we define the process  $\{X(t), t \geq 0\}$  to be a second-order or a weakly stationary process if  $E[X(t)] = c$  and  $Cov[X(t), X(t + s)]$  does not depend on  $t$ . That is, a process is stationary if first two moments of  $X(t)$  are the same for all  $t$  and the covariance between  $X(s)$  and  $X(t)$  depends only on  $|t - s|$ . For a second-order stationary process, let

$$R(s) = Cov[X(t), X(t + s)]$$

As the finite dimensional distribution of a Gaussian process (being multivariate normal) are determined by their means and covariance, it follows that a second order stationary Gaussian process is stationary.

## REFERENCES

1. A First Course in Probability by Sheldon Ross, Fifth Edition
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