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Turing Instabilities in a S-I-R Model

A thesis presented

by

Hye Yon Yi

to

The School of Mathematical Sciences

in partial fulfillment of the requirements

for the degree of

Master's of Science

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Applied Mathematics

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Abstract

With the increasing threat of biological warfare and the fear of an epidemic outbreak of influenza, smallpox, and other deadly diseases, the field of epidemic modeling is becoming increasingly important in the scientific fields. The focus of this thesis will be to create a model to study the effects of the rates of reaction and the rates of diffusion within a network based on the different parameters used in the modeling of any disease. For this model, the exact parameters of a specific disease are not as crucial as the qualitative behaviors that occur from the changing parameters. The model is linearly stable when diffusion does not exist. As diffusion is incorporated, Turing instabilities occur.

Acknowledgments

“... We also rejoice in our sufferings, because we know that suffering produces perseverance; perseverance, character; and character, hope.” Romans 5:3-4

This thesis is dedicated to my dad, David Laida. I stuck to the plan and finished.

I owe many thanks to God and several people for their support in helping me finish the thesis. My parents, David Martin Laida and Nam Yon Laida, and my sister Hye Lim Yi, for always telling it like it is, and CONSTANTLY pushing me to finish. My advisor Dr. Bernard Brooks for his endless patience and having the vision for the thesis before we even began, and being understanding during my searching process for a thesis topic. Thank you to my committee members Dr. Carl Lutzer, Dr. David Ross, and Dr. Tamas Wiandt. Dr. Lutzer for challenging me to approach problems in terms of linear algebra. Dr. Ross for showing me that we really can apply math in real world problems. Dr. Wiandt for always seeing another solution. Ms. Catherine Washington for her endless encouragement and faith in my ability from the very beginning. The staff of the School of Mathematical Sciences for making RIT feel like home and being my family away from home. My friends, for keeping me grounded during the stressful times and inspiring me to always go the extra mile.

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Introduction

In this thesis, we derive and analyze a system of equations that model the transmission of diseases among three pools of population: the susceptible, contagious, and recovered population, set in a network. We have created a model that allows us to compare the difference between reacting the population first, and then diffusing them amongst other populations in a network, or diffusing the population and then reacting. The network is a cycle graph, where each town is connected to exactly two towns. We are searching for the parameters in which the slightest change in the variable, whether it is the parameters for the reaction, or the diffusion constants, that drastically shifts the system from being stable, to become unstable. Instability induced by diffusion is known as a Turing instability (Turing, 1952).

Our model combines aspects of the predator-prey and the $S-I-R$ (Susceptible, Infected, and Recovered) systems to create a first order system of difference equations. These equations determine the current value of a variable using the previous iteration. The $S-I-R$ system, the $S-C-R$ (Susceptible, Contagious, and Recovered), for this specific model, is a first order nonlinear system of difference equations, in which each of the equations uses all three values, S, C, R , from time step $n - 1$ to determine the value at time step n . That is

$$S_n = g(S_{n-1}, C_{n-1}, R_{n-1})$$

$$C_n = h(S_{n-1}, C_{n-1}, R_{n-1})$$

$$R_n = k(S_{n-1}, C_{n-1}, R_{n-1}).$$

The $S - C - R$ system

$$S_{n+1} = S_n - dS_nC_n + bR_n$$

$$C_{n+1} = C_n + dS_nC_n - eC_n - fC_n$$

$$R_{n+1} = R_n + eC_n - bR_n$$

will be derived in Section 1.1: The Reaction Model. This will be the reaction term for the reaction diffusion equations. Linear stability conditions for the reaction term at an equilibrium point will be determined. If this equilibrium point is linearly stable then the system is said to be temporally stable. If this equilibrium point is linearly stable with both the reaction and diffusion terms included then the equilibrium point is said to be spatially stable. Thus a Turing instability occurs when a temporally stable equilibrium is not spatially stable.

The population in each town is partitioned into susceptible, contagious, and recovered subpopulations. The total population at each time step is the sum of the three subpopulations at the current time step. Members of the population move from being susceptible to contagious, contagious to recovered, and recovered to susceptible based on the rates of infection, recovery, and complete healing.

The model is derived similarly to models such as mass-action reaction models from chemistry and predator-prey models. The number of people that get sick in each time step is formulated the same way the number of prey are killed in each time step. In the classic predator-prey model, this is done by taking into account the number of ways a predator interacts with prey, multiplied by the percentage of interactions that result in killings. In

the S-C-R model derived in this paper the increase in the number of contagious people is proportional to the product of contagious and susceptible. For our model, the number of people that are contagious is formed by calculating the total number of possible interactions between the susceptible and the contagious, multiplied by the percentage of those interactions that result in the transmission of the disease.

Chapter 1

The Reaction Model

1.1 Building the Model

The S - C - R model is set in a network. The network consists of ten towns equidistant from each other, forming a single ring where each town is connected to exactly two towns. We will first focus on the reaction portion of the model, and then incorporate the diffusion.

The reaction occurs in the system of difference equations below:

$$S_{n+1} = S_n - dS_nC_n + bR_n$$

$$C_{n+1} = C_n + dS_nC_n - eC_n - fC_n$$

$$R_{n+1} = R_n + eC_n - bR_n$$

The Variables:

S : The susceptible population (people).

C : The contagious population (people).

R : The recovered population (people).

b : The fraction of the recovered population that becomes susceptible in a time step.

d : The fraction of the susceptible population that becomes contagious in a time step.

e : The fraction of the contagious population that becomes recovered in a time step.

f : The rate at which the contagious people die.

The units for the rates of the population are in terms of the percentage of the population.

The susceptible population (S) is formed by taking the previous time unit's susceptible population, plus the population that become susceptible again after they have recovered from being sick/contagious (bR), minus the population that become contagious (dSC). To calculate the number of people that become contagious, we multiply the susceptible and the contagious populations together and take a fraction of the total. This is the same method used in the predator-prey model. SC is the number of ways two people, one from the susceptible population and one from the contagious population, can be selected. When we multiply this by the rate at which people become infected, we get the number of people that become contagious (dSC).

The contagious population (C) is formed by taking the previous time unit's contagious population, plus the people who became contagious (dSC), minus the people who recover (eC), minus the people who die (fC).

The recovered population is formed by taking the previous time unit's recovered population (R), plus the population who recover from the disease (eR), minus the people who become susceptible to the disease again (bR).

The time step is one day. Because we assume that the disease strikes too quickly for births to occur, we do not account for births in this system of equations. Therefore the total population never exceeds the original population. Note that our model does not account for the death from causes other than the disease; we assume that the effect of such deaths on the population is insignificant. The population can decrease because of the deaths that occur from the disease, therefore we have the death rate f only in the contagious population. To make the analysis of the model more interesting, we will make f equal to zero.

$$S_{n+1} = S_n - dS_n C_n + bR_n$$

$$C_{n+1} = C_n + dS_n C_n - eC_n - fC_n$$

$$R_{n+1} = R_n + eC_n - bR_n$$

$$T_{n+1} = S_{n+1} + C_{n+1} + R_{n+1}$$

$$T_{n+1} = S_n + C_n + R_n - dS_n C_n + dS_n C_n + bR_n - bR_n - eC_n + eC_n - fC_n$$

$$T_{n+1} = S_n + C_n + R_n - fC_n$$

$$T_{n+1} = T_n - fC_n$$

$$T_{n+1} - T_n = -fC_n$$

$$\Delta T = -fC$$

There are two cases when looking at changes in the total population: when $C = 0$ and $C > 0$. When $C = 0$, there are no contagious people, resulting in no recovered population either. Thus the total population consists of only the susceptible population and the resulting point $(S^*, 0, 0)$ is an equilibrium. If $C > 0$, as time increases, the change in the total population is negative, that is, $\Delta T = -fC$. Thus the total population will approach zero in the limit. Since Turing instabilities cannot occur with a total population of zero, we do not want the total population to approach zero for $C > 0$. Therefore we will assume that $f = 0$.

1.2 The Equilibrium Points

Equilibrium points are the values where no further change occurs in the system from time n to $n + 1$, denoted by (S^*, C^*, R^*) . To find the equilibrium points, we set all of the S_{n+1} and S_n to S , since $S_{n+1} = S_n = S$. The same is done for all of the C 's and R 's. Thus we solve the system below for the equilibrium points.

$$S = S - dSC + bR$$

$$C = C + dSC - eC - fC$$

$$R = R + eC - bR$$

This simplifies to:

$$0 = -dSC + bR$$

$$0 = dSC - eC - fC$$

$$0 = eC - bR$$

We solve the third equation for R :

$$R = \left(e \frac{C}{b} \right)$$

Then substitute the R into the first equation to obtain:

$$0 = -dSC + b \left(e \frac{C}{b} \right)$$

$$0 = dSC - eC$$

Thus:

$$S = \frac{e}{d}$$

We find that $S = \frac{e}{d}$ for any value of C where $C \neq 0$. Therefore, our equilibrium points are as follows:

$$\mathbf{E1} : \left(\frac{e}{d}, C, e\frac{C}{b} \right)$$

for $C \neq 0$ and

$$\mathbf{E2} : (S, 0, 0)$$

for $C = 0$ where S can be any positive real value. We will focus on equilibrium $E2$. When the system goes to the equilibrium point $E2$, the disease dies out. If the system does not go to the equilibrium, the disease exists. Therefore when Turing instabilities, instabilities caused by diffusion, exist in the system, the disease also exists.

1.3 The Jacobian

We can write each equation as a function for clarity.

$$S_{n+1} = S_n - dS_nC_n + bR_n = g(S_n, C_n, R_n)$$

$$C_{n+1} = C_n + dS_nC_n - eC_n - fC_n = h(S_n, C_n, R_n)$$

$$R_{n+1} = R_n + eC_n - bR_n = k(S_n, C_n, R_n)$$

The Jacobian matrix is defined by the partial derivatives of the system of equations as follows:

$$J = \begin{bmatrix} \frac{\partial g}{\partial S_n} & \frac{\partial g}{\partial C_n} & \frac{\partial g}{\partial R_n} \\ \frac{\partial h}{\partial S_n} & \frac{\partial h}{\partial C_n} & \frac{\partial h}{\partial R_n} \\ \frac{\partial k}{\partial S_n} & \frac{\partial k}{\partial C_n} & \frac{\partial k}{\partial R_n} \end{bmatrix}$$

For our system, if we take the partial derivatives to get the Jacobian, we get the following:

$$J = \begin{bmatrix} 1 - dC & -dS & b \\ dC & 1 + dS - e - f & 0 \\ 0 & e & 1 - b \end{bmatrix}$$

Jacobian at the equilibrium points of the form $E2$:

$$J_{C=0} = \begin{bmatrix} 1 & -dS & b \\ 0 & 1 + dS - e - f & 0 \\ 0 & e & 1 - b \end{bmatrix}$$

The Trace of $J_{C=0}$:

$$3 + dS - e - f - b$$

The Determinant of $J_{C=0}$:

$$(1 + dS - e - f)(1 - b)$$

Eigenvalues of $J_{C=0}$:

$$\lambda_1 = 1$$

$$\lambda_2 = 1 - b$$

$$\lambda_3 = 1 + dS - e - f$$

The trace, determinant, and eigenvalues are common concepts used in linear algebra; the definitions are as follows:

Definition1 An *eigenvalue* of an $n \times n$ matrix A is a constant λ such at $Av = \lambda v$ where v is a vector.

$$A \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}$$

In the discrete system, when the absolute value of the eigenvalues are real and greater than 1, the equilibrium points of the system are unstable. If the absolute value of the eigenvalues

are real and less than 1, then the equilibrium point of the system are stable. The stability of the equilibrium point of the system is cannot be determined using the linearization methods to follow when the eigenvalues are real and equal to 1 or -1.

Definition2 The *trace* of an $n \times n$ matrix A is the sum of the diagonal elements, which is also the sum of the eigenvalues.

$$tr(A) = \sum_i^n A_{ii} = \sum_i^n \lambda_i$$

Definition3 The *determinant* of an $n \times n$ matrix A is scalar product of the eigenvalues.

$$\det(A) = \prod_i^n \lambda_i$$

If a determinant is equal to zero, then the matrix does not have an inverse.

1.4 The Sum of Principal Minors:

The sum of principal minors, along with the trace and determinant of a matrix, can be used to determine the stability of an equilibrium of a system when finding the eigenvalues becomes too difficult.

Definition4 The sum of the principal minors is the determinants along the diagonal. In a 3×3 matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

the sum of principal minors is

$$\sum M(A) = \det \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \det \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \det \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}.$$

The sum of principal minors can also be written in terms of the trace and the determinant.

The trace :

$$3 + dS - e - f - b$$

The determinant :

$$(1 + dS - e - f)(1 - b)$$

The sum of principal minors when one of the eigenvalues is 1:

$$\sum M + 1 = \text{determinant} + \text{trace}$$

The Sum of Principal Minors for $J_{C=0}$:

$$\begin{aligned}
 \sum M &= (\lambda_1 \times \lambda_3) + (\lambda_2 \times \lambda_3) + (\lambda_1 \times \lambda_2) \\
 \sum M &= 1 \times (1 - b) + (1 - b)(1 + dS - e - f) \\
 &\quad + 1 \times (1 + dS - e - f) \\
 \sum M &= (1 - b) + (1 - b)(1 + dS - e - f) \\
 &\quad + (1 + dS - e - f) \\
 \sum M &= 1 + (1 - b)(1 + dS - e - f) \\
 &\quad + 1 + dS - e - f - b \\
 \sum M &= (1 - b)(1 + dS - e - f) \\
 &\quad + (2 + dS - e - f - b) \\
 \sum M + 1 &= (1 - b)(1 + dS - e - f) \\
 &\quad + (2 + dS - e - f - b) + 1 \\
 \sum M + 1 &= (1 - b)(1 + dS - e - f) \\
 &\quad + (3 + dS - e - f - b)
 \end{aligned}$$

Therefore we see that the sum of principal minors for $J_{C=0}$ is equal to

$$\sum M + 1 = \text{determinant} + \text{trace}, \text{ since } \lambda_1 = 1.$$

1.4.1 Deriving the Sum of Principal Minors

We will derive the sum of principal minors to show that in a 3×3 matrix A , the sum of principal minors is

$$\sum M + 1 = \text{determinant} + \text{trace}$$

In a 3×3 matrix A , the characteristic polynomial is

$$P(\lambda) = \lambda^3 - \text{tr}(A)\lambda^2 + M(A)\lambda - \det(A)$$

By the Fundamental Theorem of Algebra, the characteristic polynomial can be factored into

$$\begin{aligned} P(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \\ &= (\lambda^2 - \lambda\lambda_2 - \lambda\lambda_1 + \lambda_1\lambda_2)(\lambda - \lambda_3) \\ &= \lambda^3 - \lambda^2\lambda_2 - \lambda^2\lambda_1 + \lambda\lambda_1\lambda_2 - \lambda_3(\lambda^2 - \lambda\lambda_2 - \lambda\lambda_1 + \lambda_1\lambda_2) \\ &= \lambda^3 - \lambda^2\lambda_2 - \lambda^2\lambda_1 + \lambda\lambda_1\lambda_2 - \lambda^2\lambda_3 + \lambda\lambda_2\lambda_3 + \lambda\lambda_1\lambda_3 - \lambda_1\lambda_2\lambda_3 \\ &= \lambda^3 - \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) + \lambda(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3) - \lambda_1\lambda_2\lambda_3 \\ &= \lambda^3 - \lambda^2\text{tr}(A) + \lambda \sum M(A) - \det(A) \end{aligned}$$

Since one of the eigenvalues is equal to one ($\lambda_3 = 1$), when we set $\lambda = 1$, and solve the characteristic polynomial, we see that

$$P(1) = (1 - \lambda_1)(1 - \lambda_2)(1 - \lambda_3)$$

$$P(1) = (1 - \lambda_1)(1 - \lambda_2)(1 - 1)$$

$$P(1) = (1 - \lambda_1)(1 - \lambda_2)(0)$$

$$P(1) = 0$$

Therefore,

$$\begin{aligned}
 P(1) &= \lambda^3 - \lambda^2 \text{tr}(A) + \lambda \sum M(A) - \det(A) = 0 \\
 0 &= 1^3 - 1^2 \text{tr}(A) + 1 \sum M(A) - \det(A) \\
 0 &= 1 - \text{tr}(A) + \sum M(A) - \det(A) \\
 \sum M(A) + 1 &= \text{tr}(A) + \det(A)
 \end{aligned}$$

Determining Stability using the Trace, Determinant, and the Sum of Principal Minors

$$\begin{aligned}
 \lambda_1 + \lambda_2 + \lambda_3 &= \text{tr}(A) \\
 \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 &= \sum M(A) \\
 \lambda_1 \lambda_2 \lambda_3 &= \det(A)
 \end{aligned}$$

For the discrete system, we must use the following set of conditions from the paper "Linear Stability Conditions for a First Order 3-Dimensional Discrete Dynamic" by Dr. B.P. Brooks to determine the stability of the system at an equilibrium:

$$\begin{aligned}
 |\det(A)| &< 1 \\
 1 &> \left| \sum M(A) \right| - |\text{tr}(A)| |\det(A)| + |\det(A)|^2 \\
 - \left(\sum M(A) + 1 \right) &< \text{tr}(A) + \det(A) < \left(\sum M(A) + 1 \right)
 \end{aligned}$$

(Brooks, 2004)

This method can be used easily to determine the stability of the system at an equilibrium when finding the eigenvalues become too difficult.

Chapter 2

The Diffusion Models

2.1 The Network

To apply our model, we focus on the Yucatan peninsula where the Mayan people resided during the 1500s. This was about the time when the Spanish arrived, bringing diseases such as smallpox, influenza, and measles. There were approximately 7 million Mayan people when the Spanish arrived during the 1500s, and of that, nearly 90 percent died from the European diseases.

We first consider the network of the 10 largest towns located in the Yucatan Peninsula. Once the towns are selected, we connect each town to two towns, creating a ring with ten nodes. Even though the towns are actually different distances apart from each other, for the model we ignore these differences.



Fig. 2.1. 10 Towns In a Network

Chapter 3

Reaction-Diffusion versus Diffusion-Reaction

We will look at two methods of incorporating diffusion into this model. The first method is to have the diffusion and then a reaction at each time step, and the second method is to have reaction and then diffuse the population in a time step. The diffusion-reaction models a situation in which the communication between towns is very poor (there is a delay in the data), causing the population to move based on the previous data. The reaction-diffusion models a situation in which communication is good. The population will know right away, before the end of each time step, how many people got sick that day and move to the another town. Our simulation uses the concentration gradients as the decision making process. The concentration gradient is the difference in the number of people in the sub-populations. If one town has a higher population concentration, the people will move to the less populated town.

3.1 Deriving the Equation for Diffusion

To incorporate diffusion into the model, we want to calculate the number of people in each town based on the number of people that move from the two neighboring towns. We will discuss two different methods to derive the equation for diffusion, known as the discrete analog of the Laplacian. The first method is using the concentration gradients, and the second method uses the Taylor series.

Method 1: Using the Concentration Gradient

To calculate the diffusion, we must create a discrete analog of the Laplacian. We take the concentration gradient (difference) from the left of the current town, and add it to the concentration gradient from the right of the town.

$$(S^{j-1} - S^j) + (S^{j+1} - S^j)$$

$$S^{j-1} - 2S^j + S^{j+1}$$

This is referred to as the Bernian (\square^2).

$$\square^2 = S^{j-1} - 2S^j + S^{j+1}$$

(Brooks, 2000)

We can also derive this from the Taylor series (see method 2 below).

Method 2: Using the Taylor Series

The number of people moving from high concentration to low concentration can also be represented by the estimation of the second derivative used in the Taylor series.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$

The values of x represents the town, while $f(x)$ represents the population in the town x . Therefore, we will let $x - x_0 = \Delta x$, which represents the distance between the two towns.

We can estimate the value of a function one step of Δx away, both in the positive and negative direction as well:

$$\begin{aligned} f(x_{-1}) &= f(x_0) - f'(x_0)\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 - \dots \\ f(x_1) &= f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 + \dots \end{aligned}$$

When we add these functions together, we get the following:

$$f(x_{-1}) + f(x_1) \approx 2f(x_0) + 2 \left[\frac{f''(x_0)}{2!}(\Delta x)^2 \right]$$

Which simplifies to:

$$f(x_{-1}) + f(x_1) \approx 2f(x_0) + f''(x_0)(\Delta x)^2$$

We solve for $f''(x_0)$.

$$\begin{aligned} f(x_{-1}) + f(x_1) &\approx 2f(x_0) + f''(x_0)(\Delta x)^2 \\ -f''(x_0)(\Delta x)^2 &\approx 2f(x_0) - f(x_{-1}) - f(x_1) \\ -f''(x_0) &\approx \frac{2f(x_0) - f(x_{-1}) - f(x_1)}{(\Delta x)^2} \\ -f''(x_0) &\approx - \left(\frac{f(x_{-1}) - 2f(x_0) + f(x_1)}{(\Delta x)^2} \right) \\ f''(x_0) &\approx \frac{f(x_{-1}) - 2f(x_0) + f(x_1)}{(\Delta x)^2} \end{aligned}$$

The second derivative of the function, $f''(x_0)$, multiplied by a diffusion coefficient, is the diffusion term for the function $f(x)$. For our model, it is the closest discrete approximation we can get for the second derivative.

$$f''(x_0) \approx \frac{f(x_{-1}) - 2f(x_0) + f(x_1)}{(\Delta x)^2}$$

If the second derivative is negative,

$$f(x_{-1}) - 2f(x_0) + f(x_1) < 0$$

$$2f(x_0) > f(x_{-1}) + f(x_1)$$

we know that the function is concave down, and the function diffuses to become more evenly spread amongst the towns, as seen below.

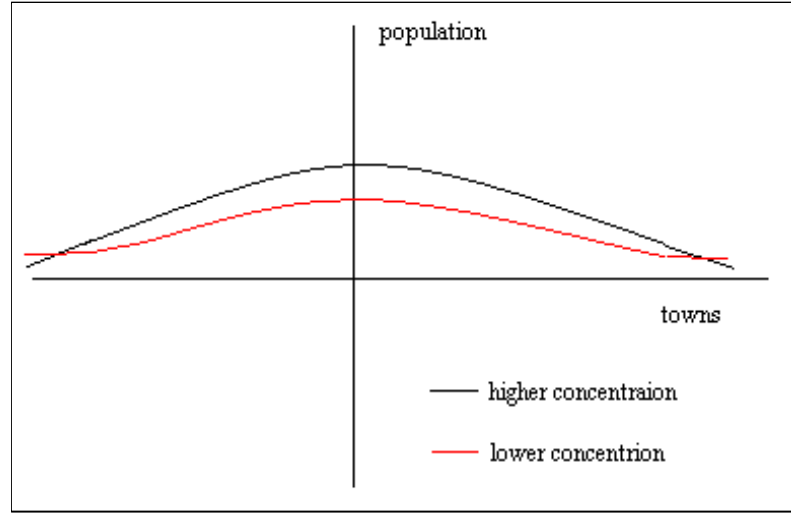


Fig. 3.2. High Concentration to Lower Concentration of Populations in the Network

The diffusion term for the susceptible population in town j at time step $n + 1$ is the following:

$$\frac{S_n^{j-1} - 2S_n^j + S_n^{j+1}}{(\Delta x)^2}$$

The diffusion term for the contagious population in town j at time step $n + 1$ is the following:

$$\frac{C_n^{j-1} - 2C_n^j + C_n^{j+1}}{(\Delta x)^2}$$

The diffusion term for the recovered population in town j at time step $n + 1$ is the following:

$$\frac{R_n^{j-1} - 2R_n^j + R_n^{j+1}}{(\Delta x)^2}$$

We will assume that $(\Delta x)^2 = 1$ for our model. We allow the populations to diffuse at different rates by multiplying each diffusion term by a constant D_S , D_C , and D_R :

$$\begin{aligned} D_S \left(S_n^{j-1} - 2S_n^j + S_n^{j+1} \right) \\ D_C \left(C_n^{j-1} - 2C_n^j + C_n^{j+1} \right) \\ D_R \left(R_n^{j-1} - 2R_n^j + R_n^{j+1} \right) \end{aligned}$$

Each town has two neighboring towns. We assume that each day, a fraction D_S of the susceptible population in each town migrates to each neighboring town. We assume that fractions of D_C and D_R of the contagious and recovered populations, respectively, do the same. The fractions must be less than or equal to $\frac{1}{2}$. If half the population moves to one neighbor and the other half moves to the other, this equals the whole population.

3.2 Creating the Diffusion-Reaction Model

The diffusion-reaction system of equations is the following:

$$\begin{aligned} S_{n+1}^j &= S_n^j - dS_n^j C_n^j + bR_n^j + D_s \left(S_n^{j-1} - 2S_n^j + S_n^{j+1} \right) \\ C_{n+1}^j &= C_n^j + dS_n^j C_n^j - eC_n^j - fC_n^j + D_c \left(C_n^{j-1} - 2C_n^j + C_n^{j+1} \right) \\ R_{n+1}^j &= R_n^j + eC_n^j - bR_n^j + D_R \left(R_n^{j-1} - 2R_n^j + R_n^{j+1} \right) \end{aligned}$$

We can linearize the system by taking the Jacobian of the reaction system and substituting the nonlinear components with the linearized components of the Jacobians as follows:

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} = \begin{bmatrix} 1 - dC & -dS & b \\ dC & 1 + dS - e - f & 0 \\ 0 & e & 1 - b \end{bmatrix}$$

When the Fourier transform is applied to this system, the C and S will be replaced with the values from the equilibrium point $E2$.

$$\begin{aligned} S_{n+1}^j &= J_{11}S_n^j + J_{12}C_n^j + J_{13}R_n^j + D_s \left(S_n^{j-1} - 2S_n^j + S_n^{j+1} \right) \\ C_{n+1}^j &= J_{21}S_n^j + J_{22}C_n^j + J_{23}R_n^j + D_c \left(C_n^{j-1} - 2C_n^j + C_n^{j+1} \right) \\ R_{n+1}^j &= J_{31}S_n^j + J_{32}C_n^j + J_{33}R_n^j + D_R \left(R_n^{j-1} - 2R_n^j + R_n^{j+1} \right) \end{aligned}$$

The system of equations consists of a reaction term and the diffusion term from time step n . The reaction term occurs using time n data. When we take the diffusion term, we also take time n data. Therefore the diffusion is occurring with old data, hence the term diffusion-reaction. A problem with this model occurs in the diffusion. If diffusion occurs using time n data, when the reaction occurs for the $n + 1$ time step, the population from time n may not actually be in the population pool to be moved, which could result in a negative population. Therefore even though we are able to create this model, we will focus on the reaction-diffusion model.

3.3 Creating the Reaction-Diffusion Model

To determine the reaction diffusion equation, we break down the equation into two parts, one to represent the reaction, and the second to represent the diffusion. We will use two time steps to represent the reaction and diffusion happening. We will react the equation using time n data, resulting in the population at time step $n + .5$. This data will then be diffused, resulting in the population after the reaction and diffusion at time $n + 1$. We are splitting the time interval into two moments of time, $n + .5$ and $n + 1$, to represent the difference of using the data that has already been reacted.

$$f''(S_{n+.5}^j) = (S_{n+.5}^{j-1} - 2S_{n+.5}^j + S_{n+.5}^{j+1})$$

$$S_{n+.5}^j = S_n^j - dS_n^j C_n^j + bR_n^j$$

$$S_{n+1}^j = S_{n+.5}^j + D_S f''(S_{n+.5}^j)$$

At $S_{n+.5}^j$, time step $n + .5$, the reaction occurs. At the second time step $n + 1$, S_{n+1}^j , the diffusion used the reacted data found in time step $n + .5$. Note, unlike in the diffusion-reaction equation, for the reaction-diffusion equation, we add the reaction term to the diffusion equation, where the diffusion term uses the reacted values instead of the unreacted values for the population. We define the diffusion that occurs at $S_{n+.5}^j$ as the Bernian:

$$\square^2 S_{n+.5}^j = S_{n+.5}^{j-1} - 2S_{n+.5}^j + S_{n+.5}^{j+1}$$

$$S_{n+1}^j = S_n^j - dS_n^j C_n^j + bR_n^j + D_S (S_{n+.5}^{j-1} - 2S_{n+.5}^j + S_{n+.5}^{j+1})$$

$$S_{n+1}^j = S_n^j - dS_n^j C_n^j + bR_n^j + D_S (\square^2 S_{n+.5}^j)$$

Since $S_{n+.5}^j = S_n^j - dS_n^j C_n^j + bR_n^j$ is nonlinear, in terms of S , C , and R , we need to linearize $S_{n+.5}^j$ along with the reaction portion of S_{n+1}^j as follows:

$$\begin{aligned} S_{n+1}^j &= S_n^j - dS_n^j C_n^j + bR_n^j + D_S \square^2 (S_n^j - dS_n^j C_n^j + bR_n^j) \\ S_{n+1}^j &= S_n^j - dS_n^j C_n^j + bR_n^j + D_S \square^2 (J_{11}S_n^j - J_{12}C_n^j + J_{13}R_n^j) \end{aligned}$$

We can apply the same method to both the contagious and the recovered population to get the complete system for the Reaction-Diffusion system.

$$\begin{aligned} S_{n+1}^j &= J_{11}S_n^j + J_{12}C_n^j + J_{13}R_n^j + D_S (J_{11}\square^2 S_n^j + J_{12}\square^2 C_n^j + J_{13}\square^2 R_n^j) \\ C_{n+1}^j &= J_{21}S_n^j + J_{22}C_n^j + J_{23}R_n^j + D_C (J_{11}\square^2 S_n^j + J_{12}\square^2 C_n^j + J_{13}\square^2 R_n^j) \\ R_{n+1}^j &= J_{31}S_n^j + J_{32}C_n^j + J_{33}R_n^j + D_R (J_{11}\square^2 S_n^j + J_{12}\square^2 C_n^j + J_{13}\square^2 R_n^j) \end{aligned}$$

This system is linearized, but due to the diffusion term (the discrete Laplacian), the system is still not decoupled, where each town relies only on its current town to determine the data. The Bernian (the discrete Laplacian) requires the use of data from town $j - 1$, j , and $j + 1$. We will use the discrete Fourier transforms to make the population pool in town j to only use town j data.

Chapter 4

Fourier Transform of the System

The discrete Fourier transform will be applied to decouple the system, where the cells no longer use other cells to determine the value of the population at time n .

4.1 The Discrete Fourier Transform

The Fourier transform for the discrete case is

$$s_t^k = \sum_{r=0}^{N-1} e^{\frac{2\pi i r k}{N}} S_t^r$$

where we are taking the transform of a sequence of values of the susceptible population, $[S_t^0, S_t^1, S_t^2, \dots, S_t^{N-1}]$. The inverse Fourier transform for the discrete case is

$$S_t^r = \frac{1}{N} \sum_{k=1}^N e^{\frac{-2\pi i r k}{N}} s_t^k$$

4.1.1 Showing the Fourier Transforms are Inverses

To show that these transforms are inverses of each other, if we plug in $s_t^k = \sum_{a=0}^{N-1} e^{\frac{2\pi i a k}{N}} S_t^a$ into S_t^a , we should get S_t^a in return.

$$\begin{aligned}
 S_t^a &= \frac{1}{N} \sum_{k=1}^N e^{\frac{-2\pi i r k}{N}} s_t^k \\
 &= \frac{1}{N} \sum_{k=1}^N e^{\frac{-2\pi i r k}{N}} \left(\sum_{a=0}^{N-1} e^{\frac{2\pi i a k}{N}} S_t^a \right) \\
 &= \frac{1}{N} \sum_{k=1}^N \left(\sum_{a=0}^{N-1} e^{\frac{2\pi i k(a-r)}{N}} S_t^a \right) \\
 &= \frac{1}{N} \sum_{a=0}^{N-1} \left(\sum_{k=1}^N e^{\frac{2\pi i k(a-r)}{N}} S_t^a \right) \\
 &= \frac{1}{N} \sum_{a=0}^{N-1} \left(S_t^a \sum_{k=1}^N e^{\frac{2\pi i k(a-r)}{N}} \right) \\
 &= \frac{1}{N} (S_t^a N) \\
 &= S_t^a
 \end{aligned}$$

We can find the conditions for the summation used above.

$$\sum_{k=1}^N e^{\frac{2\pi i k(a-r)}{N}} = \begin{cases} N & a = r \\ 0 & a \neq r \end{cases}$$

When $a = r$

$$\begin{aligned}
 \sum_{k=1}^N e^{\frac{2\pi i k(a-r)}{N}} &= \sum_{k=1}^N e^{\frac{2\pi i k(r-r)}{N}} \\
 &= \sum_{k=1}^N e^0 \\
 &= \sum_{k=1}^N 1 \\
 &= N
 \end{aligned}$$

We can prove that the sum is zero at every value except for when $a = r$ (as shown above) by using properties of geometric series.

The sum can be written out as the following:

$$\sum_{k=1}^N e^{\frac{2\pi i k(a-r)}{N}} = e^{\frac{2\pi i(a-r)}{N}} + e^{\frac{2\pi i 2(a-r)}{N}} + \dots e^{\frac{2\pi i(a-r)N}{N}}$$

We can factor out a $e^{\frac{2\pi i(a-r)}{N}} \neq 1$ ($a - r$ is an integer less than N)..

$$\sum_{k=1}^N e^{\frac{2\pi i k(a-r)}{N}} = e^{\frac{2\pi i(a-r)}{N}} \left(1 + e^{\frac{2\pi i(a-r)}{N}} + \dots e^{\frac{2\pi i(a-r)(N-1)}{N}} \right)$$

If we set $y = e^{\frac{2\pi i(a-r)}{N}}$, we obtain:

$$\sum_{k=1}^N e^{\frac{2\pi i k(a-r)}{N}} = y (1 + y + \dots y^{N-1})$$

By properties of geometric series, $1 + y + \dots y^{N-1} = \frac{y^N - 1}{y - 1}$. If we substitute $y = e^{\frac{2\pi i(a-r)}{N}}$ once again:

$$\begin{aligned} \sum_{k=1}^N e^{\frac{2\pi i k(a-r)}{N}} &= e^{\frac{2\pi i(a-r)}{N}} \left(\frac{e^{\frac{2\pi i(a-r)}{N}N} - 1}{e^{\frac{2\pi i(a-r)}{N}} - 1} \right) \\ &= e^{\frac{2\pi i(a-r)}{N}} \left(\frac{e^{2\pi i(a-r)} - 1}{e^{\frac{2\pi i(a-r)}{N}} - 1} \right) \end{aligned}$$

At $a \neq r$, $e^{2\pi i(a-r)} = \cos(2\pi(a-r)) + i \sin(2\pi(a-r)) = 1$. Substitute $e^{2\pi i(a-r)} = 1$:

$$\begin{aligned} \sum_{k=1}^N e^{\frac{2\pi i k(a-r)}{N}} &= e^{\frac{2\pi i(a-r)}{N}} \left(\frac{1 - 1}{e^{\frac{2\pi i(a-r)}{N}} - 1} \right) \\ &= 0 \end{aligned}$$

Therefore for any value when $a \neq r$, the sum is zero.

4.1.2 Applying the Discrete Fourier Transform to the Reaction-Diffusion Model

By taking the Fourier transform of the system, we are decoupling the system to make each the equations for each town and population depend only on itself. To take the Fourier transform of our equations:

$$\begin{aligned}
 S_{n+1}^j &= J_{11}S_n^j + J_{12}C_n^j + J_{13}R_n^j \\
 &\quad + D_S (J_{11}\square^2 S_n^j + J_{12}\square^2 C_n^j + J_{13}\square^2 R_n^j) \\
 C_{n+1}^j &= J_{21}S_n^j + J_{22}C_n^j + J_{23}R_n^j \\
 &\quad + D_C (J_{21}\square^2 S_n^j + J_{22}\square^2 C_n^j + J_{23}\square^2 R_n^j) \\
 R_{n+1}^j &= J_{31}S_n^j + J_{32}C_n^j + J_{33}R_n^j \\
 &\quad + D_R (J_{31}\square^2 S_n^j + J_{32}\square^2 C_n^j + J_{33}\square^2 R_n^j)
 \end{aligned}$$

we use the discrete version of the inverse Fourier transform:

$$\tilde{S}_n^r = \frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi i r j}{N}} S_n^j$$

Note that the J elements are from the Jacobian from the reaction system of equations. We will use \tilde{S} , \tilde{C} , and \tilde{R} to represent the transformed population pools.

Fourier Transform of $J_{11}\square^2 S_n^j$

To investigate the behavior of the whole system when the Fourier transform is applied, we will first focus on the Fourier transform of:

$$J_{11}\square^2 S_n^j = J_{11}(S_n^{j-1} - 2S_n^j + S_n^{j+1})$$

We will first multiply the function by $e^{\frac{-2\pi irj}{N}}$:

$$e^{\frac{-2\pi irj}{N}} J_{11}\square^2 S_n^j = J_{11} \left(e^{\frac{-2\pi irj}{N}} S_n^{j-1} - 2e^{\frac{-2\pi irj}{N}} S_n^j + e^{\frac{-2\pi irj}{N}} S_n^{j+1} \right)$$

When we sum this throughout N towns, we get the transform in terms of summations.

$$\begin{aligned} \frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi irj}{N}} J_{11}\square^2 S_n^j &= J_{11} \left(-2\frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi irj}{N}} S_n^j + \frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi irj}{N}} S_n^{j+1} \right) \\ &= J_{11} \left(\begin{aligned} &e^{\frac{-2\pi ir}{N}} \frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi ir(j-1)}{N}} S_n^{j-1} \\ &- 2\tilde{S}_n^r + e^{\frac{2\pi ir}{N}} \frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi ir(j+1)}{N}} S_n^{j+1} \end{aligned} \right) \\ &= J_{11} \left(e^{\frac{-2\pi ir}{N}} \tilde{S}_n^r - 2\tilde{S}_n^r + e^{\frac{2\pi ir}{N}} \tilde{S}_n^r \right) \\ &= J_{11} \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \tilde{S}_n^r \end{aligned}$$

We will now use this method and apply it to the full reaction-diffusion equation.

$$\begin{aligned} S_{n+1}^j &= J_{11}S_n^j + J_{12}C_n^j + J_{13}R_n^j \\ &\quad + D_S (J_{11}\square^2 S_n^j + J_{12}\square^2 C_n^j + J_{13}\square^2 R_n^j) \\ e^{\frac{-2\pi irj}{N}} S_{n+1}^j &= J_{11}e^{\frac{-2\pi irj}{N}} S_n^j + J_{12}e^{\frac{-2\pi irj}{N}} C_n^j + J_{13}e^{\frac{-2\pi irj}{N}} R_n^j \\ &\quad + D_S \left(J_{11}e^{\frac{-2\pi irj}{N}} \square^2 S_n^j + J_{12}e^{\frac{-2\pi irj}{N}} \square^2 C_n^j + J_{13}e^{\frac{-2\pi irj}{N}} \square^2 R_n^j \right) \\ \frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi irj}{N}} S_{n+1}^j &= J_{11} \frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi irj}{N}} S_n^j + J_{12} \frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi irj}{N}} C_n^j + J_{13} \frac{1}{N} \sum_{j=1}^N e^{\frac{-2\pi irj}{N}} R_n^j \\ &\quad + D_S \left(\begin{aligned} &J_{11}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}}) \tilde{S}_n^r + \\ &J_{12}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}}) \tilde{C}_n^r + J_{13}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}}) \tilde{R}_n^r \end{aligned} \right) \\ \tilde{S}_{n+1}^r &= J_{11}\tilde{S}_n^r + J_{12}\tilde{C}_n^r + J_{13}\tilde{R}_n^r \\ &\quad + D_S \left(\begin{aligned} &J_{11}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}}) \tilde{S}_n^r + \\ &J_{12}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}}) \tilde{C}_n^r + J_{13}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}}) \tilde{R}_n^r \end{aligned} \right) \end{aligned}$$

Therefore the whole system for the reaction-diffusion, when the same method is applied will be:

$$\begin{aligned}
\tilde{S}_{n+1}^r &= J_{11}\tilde{S}_n^r + J_{12}\tilde{C}_n^r + J_{13}\tilde{R}_n^r \\
&\quad + D_S \left(\begin{array}{c} J_{11}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}})\tilde{S}_n^r + \\ J_{12}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}})\tilde{C}_n^r + J_{13}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}})\tilde{R}_n^r \end{array} \right) \\
\tilde{C}_{n+1}^r &= J_{21}\tilde{S}_n^r + J_{22}\tilde{C}_n^r + J_{23}\tilde{R}_n^r \\
&\quad + D_C \left(\begin{array}{c} J_{21}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}})\tilde{S}_n^r + \\ J_{22}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}})\tilde{C}_n^r + J_{23}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}})\tilde{R}_n^r \end{array} \right) \\
\tilde{R}_{n+1}^r &= J_{31}\tilde{S}_n^r + J_{32}\tilde{C}_n^r + J_{33}\tilde{R}_n^r \\
&\quad + D_R \left(\begin{array}{c} J_{31}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}})\tilde{S}_n^r + \\ J_{32}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}})\tilde{C}_n^r + J_{33}(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}})\tilde{R}_n^r \end{array} \right)
\end{aligned}$$

The Jacobian applied to the transformed system for the reaction-diffusion will be represented by Γ_{r-d} :

Γ_{r-d} Column 1 :

$$\begin{aligned}
&J_{11} \left(1 + D_S \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \right) \\
&J_{21} \left(1 + D_C \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \right) \\
&J_{31} \left(1 + D_R \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \right)
\end{aligned}$$

Γ_{r-d} Column 2 :

$$\begin{aligned}
&J_{12} \left(1 + D_S \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \right) \\
&J_{22} \left(1 + D_C \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \right) \\
&J_{32} \left(1 + D_R \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \right)
\end{aligned}$$

Γ_{r-d} Column 3 :

$$\begin{aligned}
&J_{13} \left(1 + D_S \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \right) \\
&J_{23} \left(1 + D_C \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \right) \\
&J_{33} \left(1 + D_R \left(e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} \right) \right)
\end{aligned}$$

We can simplify $e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}}$ using the definitions:

$$e^{i\theta} = \cos(\theta) + i \sin \theta$$

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$$

$$\begin{aligned} e^{\frac{-2\pi ir}{N}} - 2 + e^{\frac{2\pi ir}{N}} &= e^{i\left(\frac{-2\pi r}{N}\right)} - 2 + e^{i\left(\frac{2\pi r}{N}\right)} \\ &= \cos\left(\frac{-2\pi r}{N}\right) + i \sin\left(\frac{-2\pi r}{N}\right) - 2 + \cos\left(\frac{2\pi r}{N}\right) + i \sin\left(\frac{2\pi r}{N}\right) \\ &= \cos\left(\frac{2\pi r}{N}\right) + i \sin\left(\frac{-2\pi r}{N}\right) - 2 + \cos\left(\frac{2\pi r}{N}\right) + i \sin\left(\frac{2\pi r}{N}\right) \\ &= \cos\left(\frac{2\pi r}{N}\right) - 2 + \cos\left(\frac{2\pi r}{N}\right) \\ &= -2 \left(1 - \cos\left(\frac{2\pi r}{N}\right)\right) \\ &= -2(2) \left(\frac{1 - \cos\left(\frac{2\pi r}{N}\right)}{2}\right) \\ &= -4 \sin^2 \frac{\pi r}{N} \end{aligned}$$

Therefore we can rewrite the Γ matrix as follows:

$$\Gamma_{r-d} = \begin{bmatrix} J_{11} \left(1 - 4D_S \sin^2 \frac{\pi r}{N}\right) & J_{12} \left(1 - 4D_S \sin^2 \frac{\pi r}{N}\right) & J_{13} \left(1 - 4D_S \sin^2 \frac{\pi r}{N}\right) \\ J_{21} \left(1 - 4D_C \sin^2 \frac{\pi r}{N}\right) & J_{22} \left(1 - 4D_C \sin^2 \frac{\pi r}{N}\right) & J_{23} \left(1 - 4D_C \sin^2 \frac{\pi r}{N}\right) \\ J_{31} \left(1 - 4D_R \sin^2 \frac{\pi r}{N}\right) & J_{32} \left(1 - 4D_R \sin^2 \frac{\pi r}{N}\right) & J_{33} \left(1 - 4D_R \sin^2 \frac{\pi r}{N}\right) \end{bmatrix}$$

We can substitute the components of the original Jacobian into the gamma matrix:

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} = \begin{bmatrix} 1 - dC & -dS & b \\ dC & 1 + dS - e - f & 0 \\ 0 & e & 1 - b \end{bmatrix}$$

$$\Gamma_{r-d,J} = \begin{bmatrix} (1-dC) * (1-4D_S \sin^2 \frac{\pi r}{N}) & -dS (1-4D_S \sin^2 \frac{\pi r}{N}) & b (1-4D_S \sin^2 \frac{\pi r}{N}) \\ dC (1-4D_C \sin^2 \frac{\pi r}{N}) & (1+dS-e-f) * (1-4D_C \sin^2 \frac{\pi r}{N}) & 0 \\ 0 & e (1-4D_R \sin^2 \frac{\pi r}{N}) & (1-b) * (1-4D_R \sin^2 \frac{\pi r}{N}) \end{bmatrix}$$

We will investigate the stability of the system at $E2$, where we set $C = 0$.

$$\Gamma_{r-d,J,C=0} = \begin{bmatrix} (1-4D_S \sin^2 \frac{\pi r}{N}) & -dS (1-4D_S \sin^2 \frac{\pi r}{N}) & b (1-4D_S \sin^2 \frac{\pi r}{N}) \\ 0 & (1+dS-e-f) * (1-4D_C \sin^2 \frac{\pi r}{N}) & 0 \\ 0 & e (1-4D_R \sin^2 \frac{\pi r}{N}) & (1-b) * (1-4D_R \sin^2 \frac{\pi r}{N}) \end{bmatrix}$$

Eigenvalues of $\Gamma_{r-d,J,C=0}$:

$$\begin{aligned} \lambda_1 &= 1 - 4D_S \sin^2 \frac{\pi r}{N} \\ \lambda_2 &= 1 - 4D_R \sin^2 \frac{\pi r}{N} - b + 4bD_R \sin^2 \frac{\pi r}{N} \\ \lambda_3 &= 1 - 4D_C \sin^2 \frac{\pi r}{N} + dS - 4dSD_C \sin^2 \frac{\pi r}{N} - e \\ &\quad + 4eD_C \sin^2 \frac{\pi r}{N} - f + 4fD_C \sin^2 \frac{\pi r}{N} \end{aligned}$$

We will evaluate these eigenvalues to determine the stability of the system

4.2 Stability Conditions for $\Gamma_{r-d,J,C=0}$

Eigenvalues of $\Gamma_{r-d,J,C=0}$:

Eigenvalue λ_1 :

$$-1 < 1 - 4D_S \sin^2 \frac{\pi r}{N} < 1$$

$$-2 < -4D_S \sin^2 \frac{\pi r}{N} < 0$$

$$\frac{1}{2} > D_S \sin^2 \frac{\pi r}{N} > 0$$

$$0 < D_S < \frac{1}{2 \sin^2 \frac{\pi r}{N}}$$

By maximizing $\sin^2 \frac{\pi r}{N}$ to be 1, we will find the most stringent condition on D_S for spatial stability. Therefore we know if the previous condition is met at $\sin^2 \frac{\pi r}{N} = 1$, then the system will be stable for any value of $\sin^2 \frac{\pi r}{N}$. As $\sin^2 \frac{\pi r}{N}$ decreases from 1, $\frac{1}{2 \sin^2 \frac{\pi r}{N}}$ will become greater than $\frac{1}{2}$. Therefore, if $D_S < \frac{1}{2}$, we know that for any value of $\sin^2 \frac{\pi r}{N}$, the stability condition is met. Since the condition on the diffusion coefficient is already D_S must be less than or equal to $\frac{1}{2}$, the diffusion of the susceptible population does not cause a Turing instability.

Eigenvalue λ_2 :

$$-1 < 1 - 4D_R \sin^2 \frac{\pi r}{N} - b + 4bD_R \sin^2 \frac{\pi r}{N} < 1$$

$$-2 + b < -4D_R \sin^2 \frac{\pi r}{N} + 4bD_R \sin^2 \frac{\pi r}{N} < b$$

$$-2 + b < D_R \sin^2 \frac{\pi r}{N} (-4 + 4b) < b$$

Since $(-4 + 4b) = 4(-1 + b)$ and b is always less than 1, $(-1 + b) < 0$, therefore $(-4 + 4b) < 0$.

$$\frac{-2+b}{(-4+4b)\sin^2\frac{\pi r}{N}} > D_R > \frac{b}{(-4+4b)\sin^2\frac{\pi r}{N}}$$

$$\frac{b-2}{(4b-4)\sin^2\frac{\pi r}{N}} > D_R > \frac{b}{(4b-4)\sin^2\frac{\pi r}{N}}$$

Similar to the maximization of $\sin^2\frac{\pi r}{N}$ in finding the constraints for D_S , we can do the same for the $\sin^2\frac{\pi r}{N}$ in the conditions above. We set the $\sin^2\frac{\pi r}{N} = 1$ to get the most stringent conditions, which follows:

$$\frac{b-2}{(4b-4)} > D_R > \frac{b}{(4b-4)}$$

If we graph b versus $\frac{b-2}{4b-4}$, we see the following graph.

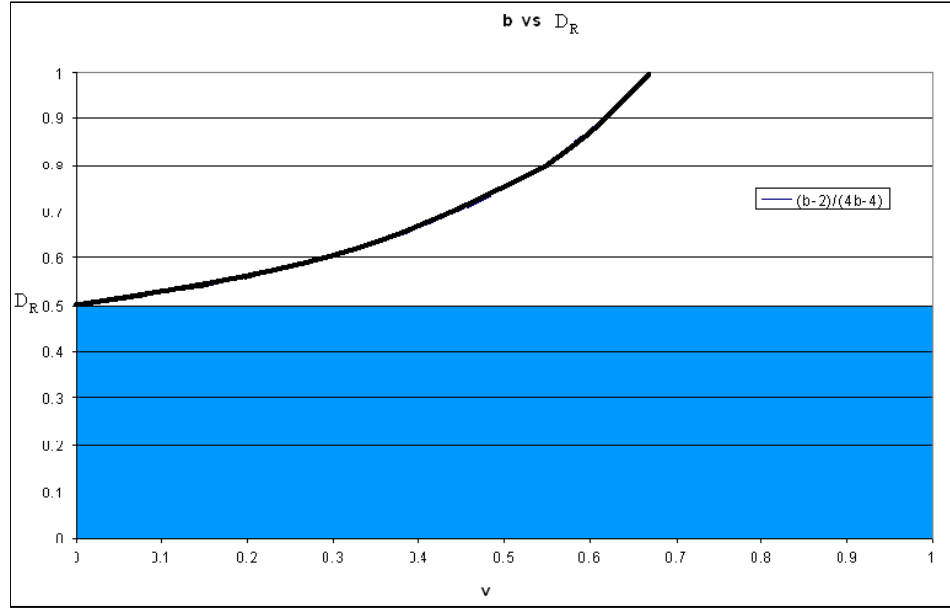


Fig. 4.3. The graph shows where the stability of the recovered population occurs when diffusion is incorporated.

For the system to be stable, the diffusion constant must lie in the shaded region above, which is below the graph $\frac{b-2}{4b-4}$ and D_R must be less than $\frac{1}{2}$. Since D_R cannot be greater than $\frac{1}{2}$, the system is always stable with diffusion of the recovered population. Therefore a Turing instability does not occur with the diffusion of the recovered population

Eigenvalue λ_3 :

$$-1 < \left(\frac{1 - 4D_C \sin^2 \frac{\pi r}{N} + dS - 4dSD_C \sin^2 \frac{\pi r}{N} - e}{+4eD_C \sin^2 \frac{\pi r}{N} - f + 4fD_C \sin^2 \frac{\pi r}{N}} - e \right) < 1$$

Similar to the first two stability condition analysis, we will maximize the $\sin^2 \frac{\pi r}{N}$ to give us the most stringent conditions for D_C .

$$-1 < 1 - 4D_C + dS - 4dSD_C - e + 4eD_C - f + 4fD_C < 1$$

$$-2 < -4D_C + dS - 4dSD_C - e + 4eD_C - f + 4fD_C < 0$$

$$-2 - dS + e + f < -4D_C - 4dSD_C + 4eD_C + 4fD_C < -dS + e + f$$

$$-2 - dS + e + f < D_C(-4 - 4dS + 4e + 4f) < -dS + e + f$$

If we assume that $(-4 - 4dS + 4e + 4f)$ is a positive value,

$$\frac{-2 - dS + e + f}{-4 - 4dS + 4e + 4f} < D_C < \frac{-dS + e + f}{-4 - 4dS + 4e + 4f}$$

For notational convenience, we will set $Q = -dS + e + f$,

$$\frac{-2 + Q}{-4 + 4Q} < D_C < \frac{Q}{-4 + 4Q}$$

Since one of the eigenvalues for the original reaction system is $1 + dS - e - f$, which is $1 - Q$, when we find the stability conditions, we notice that

$$0 < 1 - Q < 1$$

$$0 < Q < 2$$

Therefore, for the temporal stability to occur, where the reaction system has no diffusion, Q must be less than 2. Since $-4 - 4dS + 4e + 4f > 0$, we know that

$$-4 - 4dS + 4e + 4f > 0$$

$$-dS + e + f > 1$$

$$Q > 1$$

Therefore, as long as $1 < Q < 2$, the system is stable at the equilibrium point E_2 .

If $-4 - 4dS + 4e + 4f < 0$,

$$\begin{aligned} \frac{-2 - dS + e + f}{-4 - 4dS + 4e + 4f} &> D_C > \frac{-dS + e + f}{-4 - 4dS + 4e + 4f} \\ \frac{-2 + Q}{-4 + 4Q} &> D_C > \frac{Q}{-4 + Q} \\ D_C &< \frac{-2 + Q}{-4 + 4Q} \end{aligned}$$

This is similar to the analysis of the D_R . We can conclude that the constraints for D_C are always met since the graph of $\frac{-2+Q}{-4+4Q}$ is always above the $D_C = \frac{1}{2}$ line for $0 < Q \leq 1$.

We will focus on the upper bound of the constraints for D_C for $-4 + 4Q > 0$, and the graph must be below the following condition for the system to be stable.

$$D_C < \frac{Q}{-4 + 4Q}$$

For the system to be stable, the value D_C must lie in the shaded area in Figure 4.5.

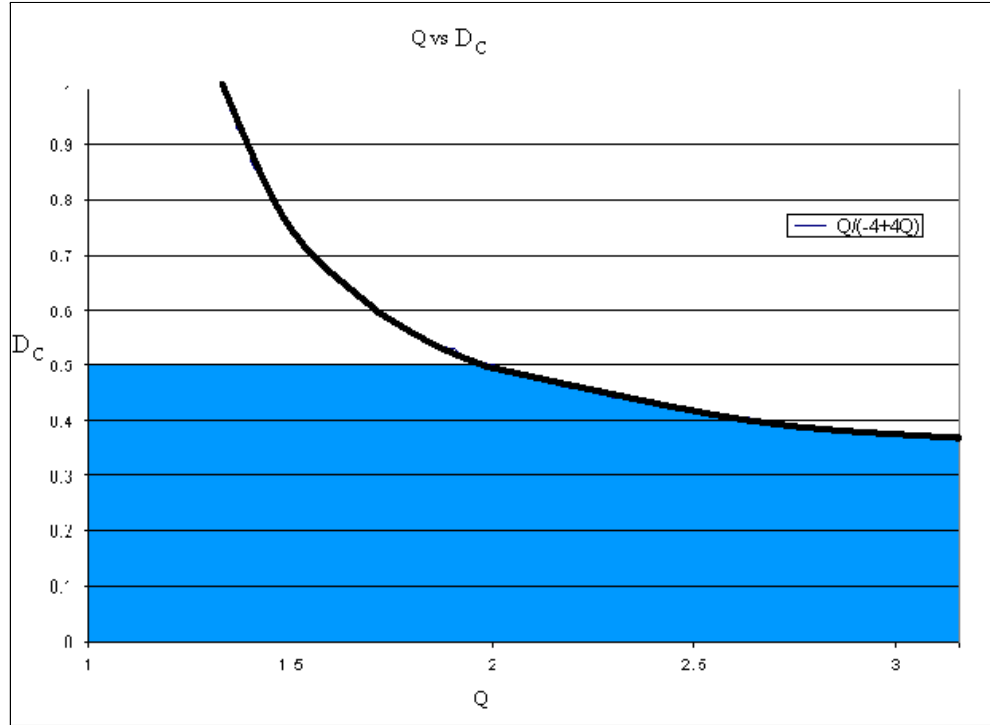


Fig. 4.4. The graph shows where the stability of the contagious population occurs when diffusion is incorporated.

Since we showed that Q cannot be greater than 2 for stability in the temporal state of the system, the shaded area in Figure 4.5 where $Q > 2$ represents the cases when the system is unstable at the equilibrium when there is no diffusion, and then becomes stable when diffusion is applied. Figure 4.6 represents the values of the diffusion coefficients which are less than $\frac{1}{2}$, but do not meet the conditions of $Q < 2$ and $D_C < \frac{Q}{-4+4Q}$, therefore we do not need to take into account the shaded region.

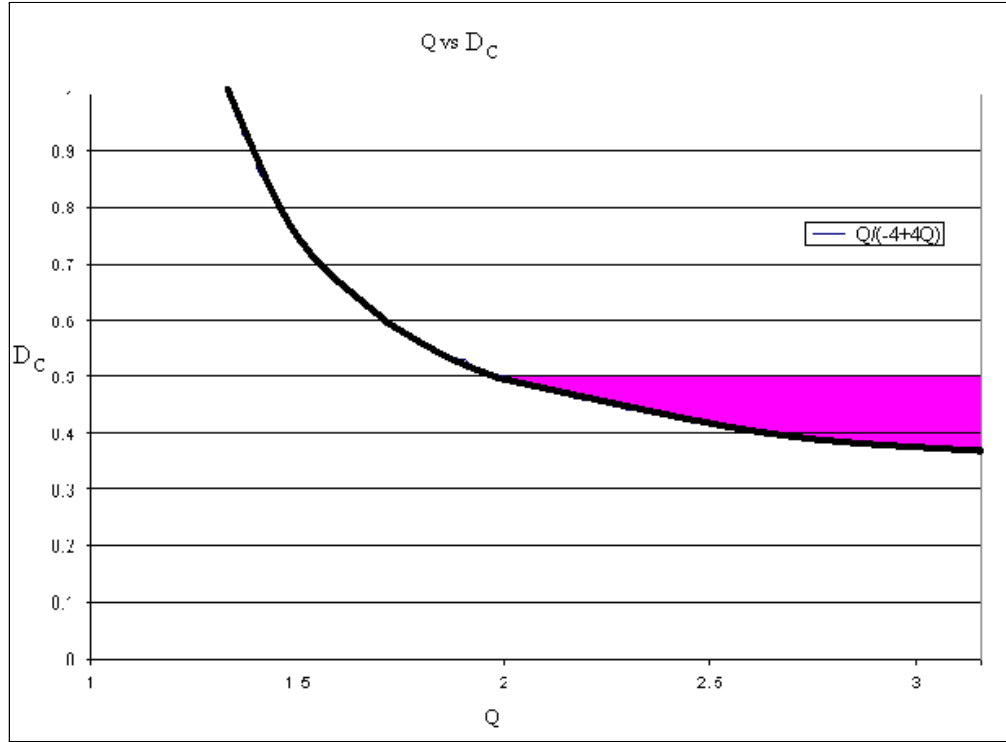


Fig. 4.5. The graph shows where the instability of the contagious population occurs when diffusion is incorporated.

Therefore we see that we do not get Turing instabilities when diffusion is included since all of the conditions on the diffusion coefficients are always met.

4.3 Fourier Transformed Diffusion-Reaction Model

The following system represents the diffusion-reaction system before the Fourier transform.

The reaction occurs, and at the same time, the diffusion uses the terms before the reaction occurs to calculate the diffusion.

$$S_{n+1}^j = J_{11}S_n^j + J_{12}C_n^j + J_{13}R_n^j + D_S (S_n^{j-1} - 2S_n^j + S_n^{j+1})$$

$$C_{n+1}^j = J_{21}S_n^j + J_{22}C_n^j + J_{23}R_n^j + D_C (C_n^{j-1} - 2C_n^j + C_n^{j+1})$$

$$R_{n+1}^j = J_{31}S_n^j + J_{32}C_n^j + J_{33}R_n^j + D_R (R_n^{j-1} - 2R_n^j + R_n^{j+1})$$

Using the methods from section 4.1.2, we can create the decoupled system for the diffusion-reaction model:

$$\begin{aligned}\tilde{S}_{n+1}^r &= J_{11}\tilde{S}_n^r + J_{12}\tilde{C}_n^r + J_{13}\tilde{R}_n^r + D_S \left(-4 \sin^2 \left(\frac{\pi r}{N} \right) \right) \tilde{S}_n^r \\ \tilde{C}_{n+1}^r &= J_{21}\tilde{S}_n^r + J_{22}\tilde{C}_n^r + J_{23}\tilde{R}_n^r + D_C \left(-4 \sin^2 \left(\frac{\pi r}{N} \right) \right) \tilde{C}_n^r \\ \tilde{R}_{n+1}^r &= J_{31}\tilde{S}_n^r + J_{32}\tilde{C}_n^r + J_{33}\tilde{R}_n^r + D_R \left(-4 \sin^2 \left(\frac{\pi r}{N} \right) \right) \tilde{R}_n^r\end{aligned}$$

The Γ_{d-r} matrix is the Jacobian of the diffusion-reaction system above after it has gone through a coordinate shift and a Fourier transformation:

$$\Gamma_{d-r} = \begin{bmatrix} J_{11} - 4D_1 \sin^2 \left(\frac{\pi r}{N} \right) & J_{12} & J_{13} \\ J_{21} & J_{22} - 4D_2 \sin^2 \left(\frac{\pi r}{N} \right) & J_{23} \\ J_{31} & J_{32} & J_{33} - 4D_3 \sin^2 \left(\frac{\pi r}{N} \right) \end{bmatrix}$$

By substituting the Jacobian values from our model, we get the new Γ_{r-d} matrix:

$$\Gamma_{d-r} = \begin{bmatrix} 1 - dC - 4D_1 \sin^2 \left(\frac{\pi r}{N} \right) & -dS & 0 \\ dC & 1 + dS - e - f - 4D_2 \sin^2 \left(\frac{\pi r}{N} \right) & 0 \\ 0 & e & 1 - b - 4D_3 \sin^2 \left(\frac{\pi r}{N} \right) \end{bmatrix}$$

Since the maximum the $\sin^2 \left(\frac{\pi r}{N} \right)$ is one, we will replace all of the $\sin^2 \left(\frac{\pi r}{N} \right)$ since the maximizing of the $\sin^2 \left(\frac{\pi r}{N} \right)$ will result in creating the most stringent constraints for the diffusion constants, as seen in section 4.2.

$$\Gamma_{d-r} = \begin{bmatrix} 1 - dC - 4D_1 & -dS & 0 \\ dC & 1 + dS - e - f - 4D_2 & 0 \\ 0 & e & 1 - b - 4D_3 \end{bmatrix}$$

Determinant of Γ_{d-r} :

$$- \begin{pmatrix} 1 - dC - 4D_2 + 4D_1f + 16D_1D_2 - f + dCf \\ -e - 4D_1 + dS + dCe + 4D_1e - 4D_1dS + 4dCD_2 \end{pmatrix} \\ * (-1 + b + 4D_3)$$

Trace of Γ_{d-r} :

$$3 - dC - 4D_1 + dS - e - f - 4D_2 - b - 4D_3$$

The eigenvalues for the Γ_{d-r} are very difficult to work with, so we will use the Gerschgorin disks to evaluate where the eigenvalues lie. Gerschgorin's Theorem:

Let $A = [a_{ij}]$ be a (real or complex) $n \times n$ matrix, and let r_i denote the sum of the absolute values of the off-diagonal entries in the i th row of A: $r_i = \sum_{j \neq i} |a_{ij}|$. The i th Gerschgorin disk is the disk D_i in the complex plane with center a_{ii} and radius r_i . That is,

$$D_i = \{z \text{ in } C : |z - a_{ii}| \leq r_i\}$$

Using this theorem, we know that the centers for the eigenvalues are as follows:

$$\text{Center1} : 1 - dC - 4D_1$$

$$\text{Center2} : 1 + dS - e - f - 4D_2$$

$$\text{Center3} : 1 - b - 4D_3$$

The corresponding radii are as follows:

$$\text{Radius1} : -dS$$

$$\text{Radius2} : dC$$

$$\text{Radius3} : e$$

Therefore, we know the eigenvalues lie in the disks with centers listed above and the radii of the disks are also listed above.

4.3.1 Using the Gerschgorin Theorem to Determine Stability

To evaluate the stability of the equilibrium points using the Gerschgorin theorem, we must look at where the union of the three disks are located to determine whether or not they are stable.

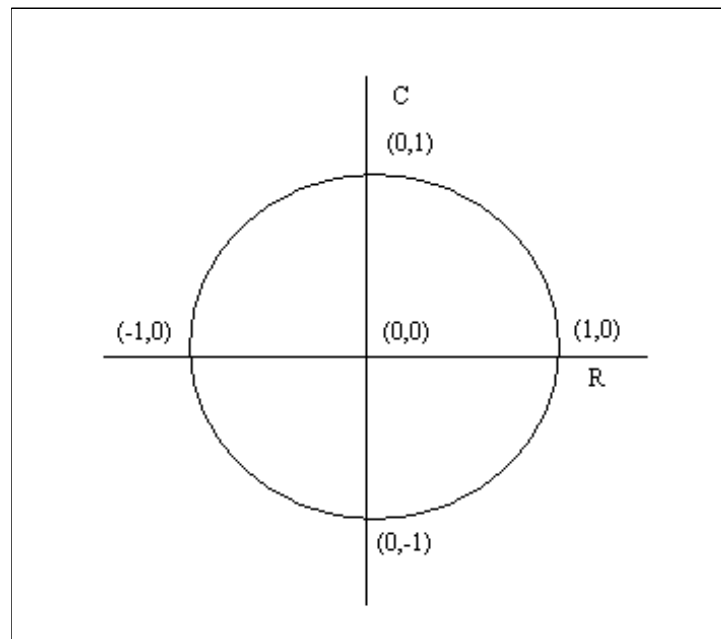


Fig. 4.6. The Unit Circle in Complex Coordinates

If the union of all three disks where the eigenvalues lie inside the unit circle of the complex plane, then we have stability.

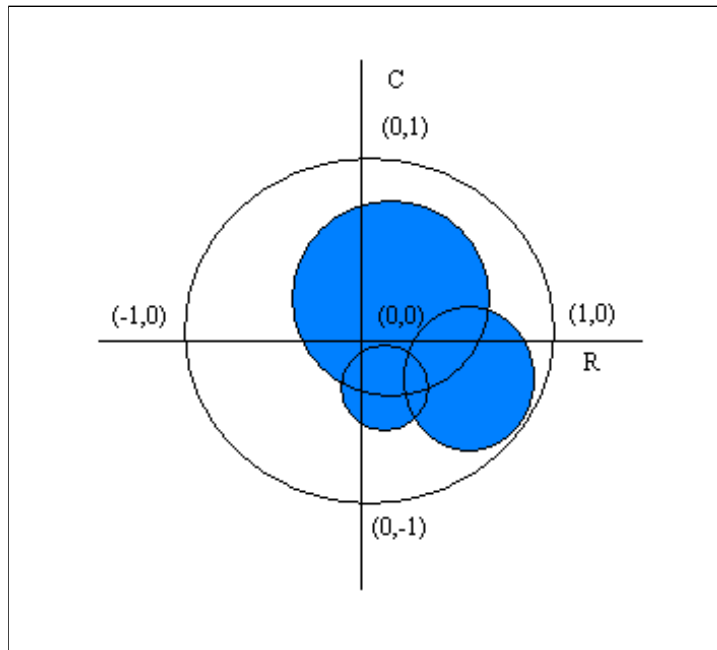


Fig. 4.7. Stable Eigenvalues

If the union of the three disks lie outside of the unit circle of the complex plane, then we have instability.

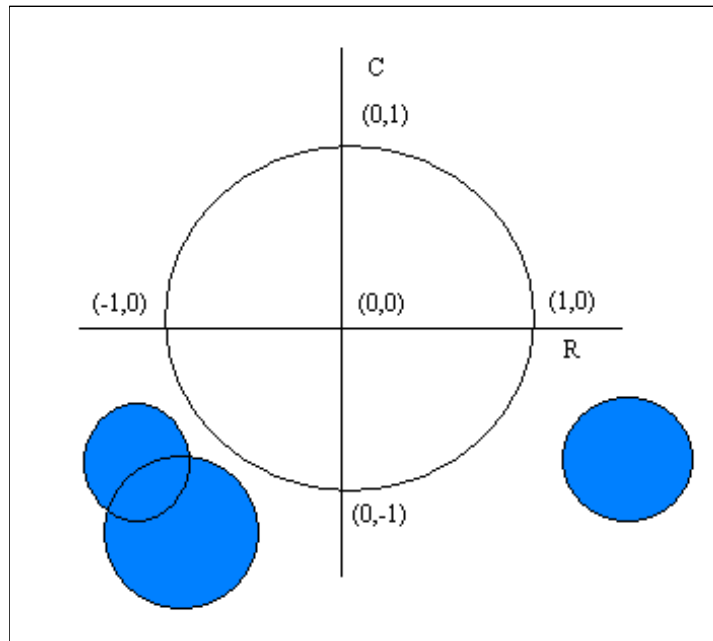


Fig. 4.8. Unstable Eigenvalues

If only a portion of the union is inside the unit circle and the rest of the union is outside of the unit circle, we cannot determine the stability of the system.

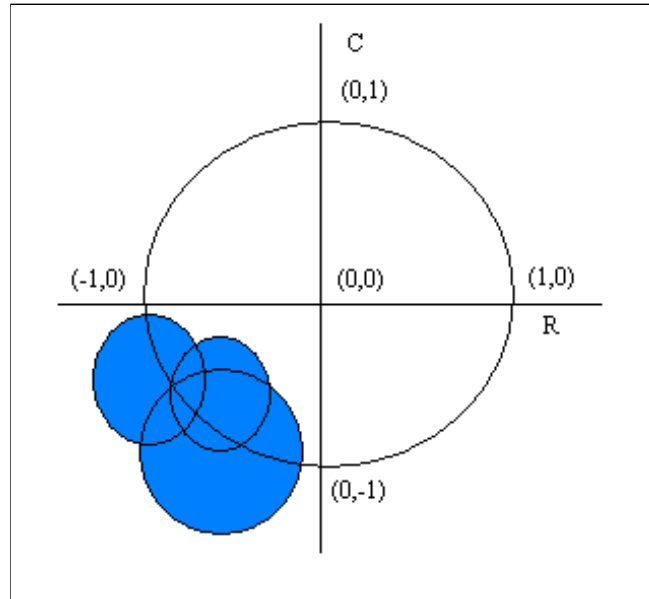


Fig. 4.9. Undefined Stability of Eigenvalues

(Gerschgorin, 1931)

Looking at the previous example, where the centers and the radii are the following:

$$Center1 : 1 - dC - 4D_1$$

$$Center2 : 1 + dS - e - f - 4D_2$$

$$Center3 : 1 - b - 4D_3$$

$$Radius1 : -dS$$

$$Radius2 : dC$$

$$Radius3 : e$$

Since all of the centers are real, they lie on the Real-axis, we can use the three different cases to determine the stability conditions.

Case1: $center < 0$

$$center - radius > -1$$

Case2: $center > 0$

$$center + radius < 1$$

Case3: $center = 0$

$$radius < 1$$

Case1: If the center is less than zero, then the difference between the center and the radius must be greater than -1 for the disk to lie in the unit circle.

$$center - radius > -1$$

$$(1 - dC - 4D_1) - (-dS) > -1$$

$$1 - dC - 4D_1 + dS > -1$$

$$D_1 < \frac{1}{4}(2 - dC + dS)$$

Case 2: If the center is greater than zero, then the sum of the center and the radius must be less than 1 for the disk to lie in the unit circle.

$$center + radius < 1$$

$$(1 - dC - 4D_1) + (-dS) < 1$$

$$1 - dC - 4D_1 - dS < 1$$

$$D_1 > \frac{dC + dS}{4}$$

Case 3: If the center is equal to zero, then the radius must be less than 1 for the disk to lie in the unit circle.

$$radius < 1$$

$$1 - dC - 4D_1 - dS < 1$$

$$-dC - 4D_1 - dS < 0$$

$$-4D_1 < d(C + D)$$

$$D_1 < -\frac{d}{4}(C + D)$$

We must take into account all three cases for all three eigenvalues/disks where the eigenvalues lie. That results in nine different scenarios that must be studied to come up with a conclusion of the stability. The down fall of this method in studying the stability of the system at an equilibrium by looking at the eigenvalue is that even with the analysis of all three cases, if the union of the disks does not lie completely inside or outside of the unit circle, then the stability cannot be determined. Therefore using the sum of principal minors, the trace, and the determinant to determine stability is much simpler to use for the analysis of the stability of eigenvalues.

4.4 Eigenvalues of Diffusion-Reaction System $\Gamma_{d-r,C=0}$

When we substitute the equilibrium value of $C = 0$, we get the following matrix:

$$\Gamma_{d-r,C=0} = \begin{bmatrix} 1 - 4D_1 & -dS & 0 \\ 0 & 1 + dS - e - f - 4D_2 & 0 \\ 0 & e & 1 - b - 4D_3 \end{bmatrix}$$

Determinant of $\Gamma_{d-r,C=0}$:

$$(1 - 4D_1)(1 + dS - e - f - 4D_2)(1 - b - 4D_3)$$

Trace of $\Gamma_{d-r,C=0}$:

$$3 - 4D_1 + dS - e - f - 4D_2 - b - 4D_3$$

Eigenvalues of $\Gamma_{d-r,C=0}$:

$$\lambda_1 = 1 - 4D_1$$

$$\lambda_2 = 1 + dS - e - f - 4D_2$$

$$\lambda_3 = 1 - b - 4D_3$$

4.5 Stability Conditions of the Diffusion-Reaction System

$\Gamma_{d-r,C=0}$:

Since we know that instability occurs when $-1 < \lambda < 1$, we can place conditions on our eigenvalues to determine the constraints on the other variables. The following finds the conditions on the diffusion constants for stability to occur.

Eigenvalue λ_1 :

$$-1 < 1 - 4D_1 < 1$$

$$-2 < -4D_1 < 0$$

$$\frac{1}{2} > D_1 > 0$$

$$0 < D_1 < \frac{1}{2}$$

If there is diffusion for the susceptible population, the system will always be stable as long as $D_1 < \frac{1}{2}$.

Eigenvalue λ_2 :

$$-1 < 1 + dS - e - f - 4D_2 < 1$$

$$-2 < dS - e - f - 4D_2 < 0$$

$$-2 - dS + e + f < -4D_2 < -dS + e + f$$

$$\frac{2 + dS - e - f}{4} > D_2 > \frac{dS - e - f}{4}$$

$$\frac{dS - e - f}{4} < D_2 < \frac{2 + dS - e - f}{4}$$

For this condition to be true, D_2 must be less than $\frac{2+dS-e-f}{4}$ which is equal to $\frac{1}{2} + \frac{dS-e-f}{4}$.

Since D_2 cannot be greater than $\frac{1}{2}$, therefore $\frac{dS-e-f}{4}$ must be less than or equal to zero, which is the same as $dS - e - f$ must be less than zero.

Eigenvalue λ_3 :

$$-1 < 1 - b - 4D_3 < 1$$

$$-2 < -b - 4D_3 < 0$$

$$-2 + b < -4D_3 < b$$

$$\frac{2-b}{4} > D_3 > \frac{-b}{4}$$

$$\frac{-b}{4} < D_3 < \frac{2-b}{4}$$

The maximum value for $b = 1$, therefore if b is maximized, the largest value for D_3 is $\frac{1}{4}$.

If b is minimized to zero, then the maximum value for D_3 is $\frac{1}{2}$. Therefore for all values of b , where $0 \leq b \leq 1$, the maximum value possible for D_3 will be $\left[\frac{1}{4}, \frac{1}{2}\right]$.

Chapter 5

Simulations

The following simulations will demonstrate 4 basic behavior.

Case 1. Stability without Diffusion

Case 2. Instability without Diffusion

Case 3. Stability with Diffusion

Case 4. Instability with Diffusion

5.1 Case 1: Stability Without Diffusion

With the variable set the following, we notice that the system is stable. See Figure 5.10.

The Reaction Variables	Values
d :	0.004
b :	0.08
e :	0.00189
f :	0
The Diffusion Coefficients	Values
D_S :	0
D_C :	0
D_R :	0

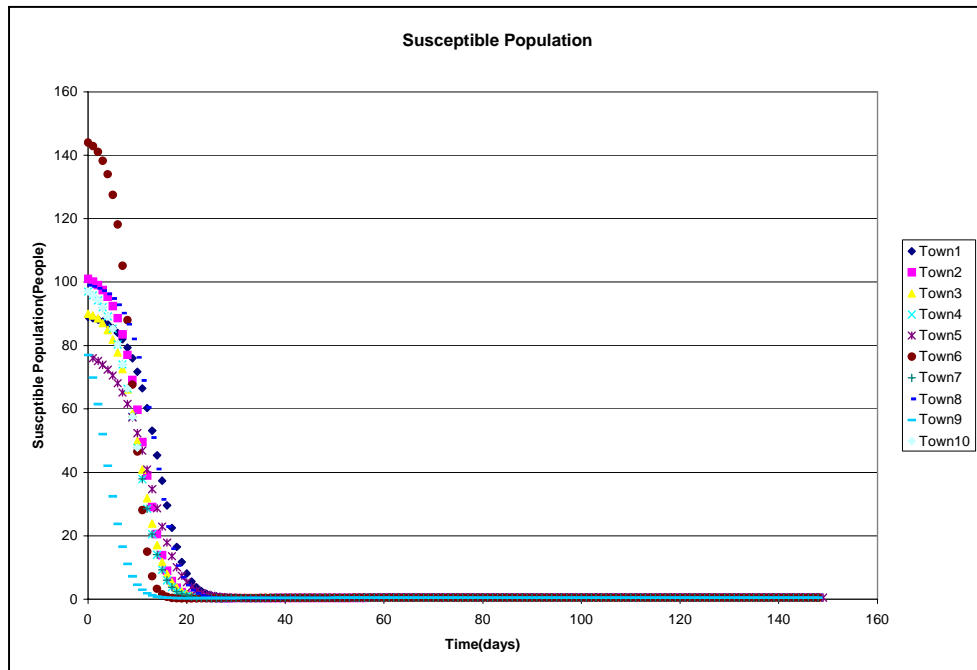


Fig. 5.10.

Figure 5.11 appears to converge. We can zoom in closer, Figure 5.12, to see that the variable actually converges to .4725, which is $\frac{e}{d}$, the equilibrium for the susceptible population, found in section 1.2, when there is no diffusion.

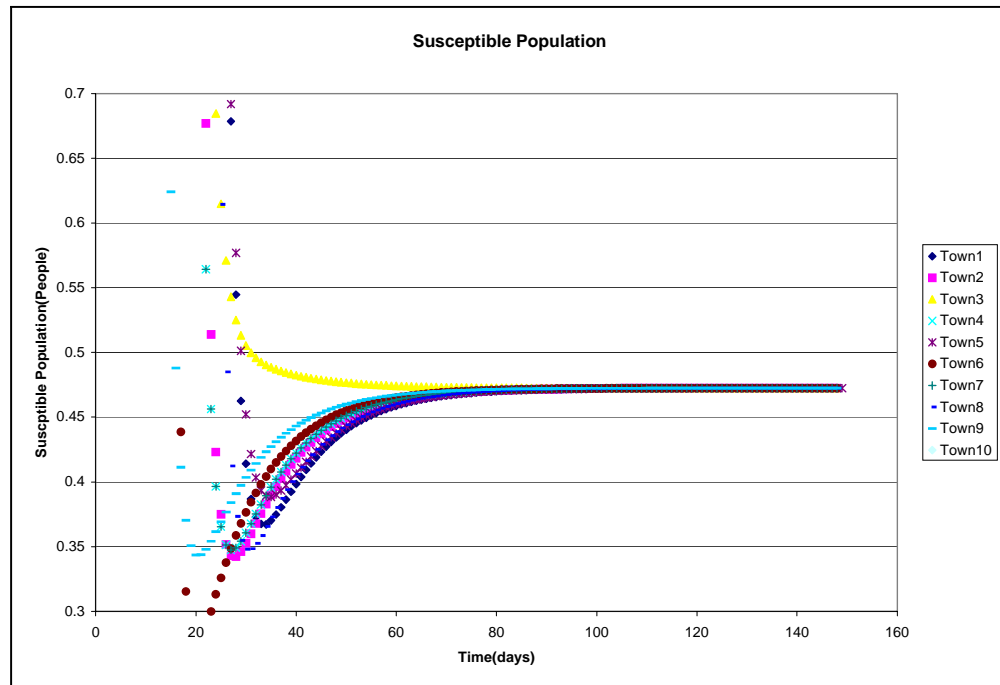


Fig. 5.11.

Since the equilibrium point of the contagious population is C , each town has a different equilibrium point. Even though they are different, since there is no diffusion, the equilibrium of the system is stable.

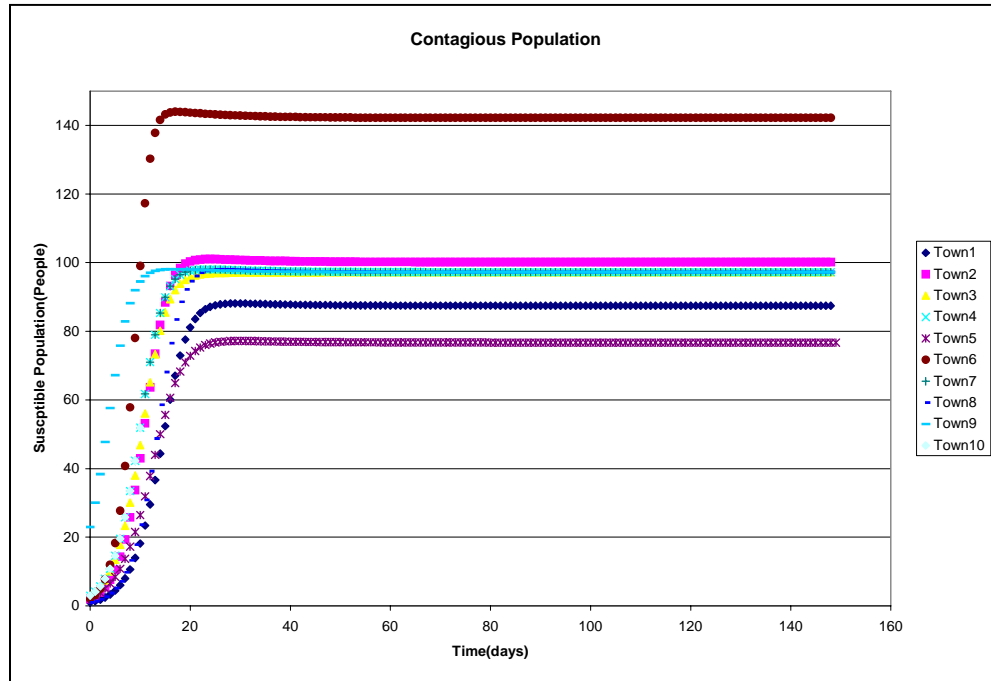


Fig. 5.12.

Similar to the contagious population, the equilibrium value for the recovered is dependent upon the contagious population, $\frac{e}{b}C$, resulting in different values for each town. Therefore the recovered population for each town is converging to the corresponding equilibrium point, which means the equilibrium points are all stable.

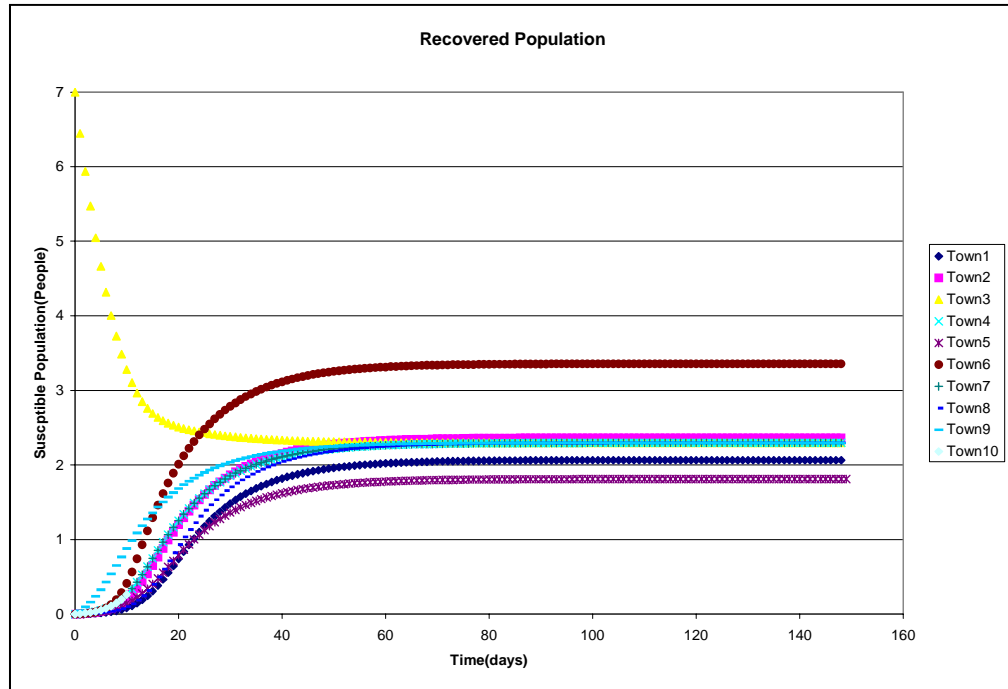


Fig. 5.13.

We can look at Town 1 to show that the system reaches equilibrium, therefore the system is stable at the equilibrium $(\frac{e}{d}, C, \frac{e}{b}C)$. This is true for all 10 towns in the case where there is no diffusion.

Example of Town1:	Starting Value	Ending Value	Equilibrium
Susceptible	89	0.4725	$\frac{e}{d}$
Contagious	1	87.46125	C
Recovered	0	2.066272	$\frac{e}{b}C$

Therefore for each town, all of the towns approach the equilibrium.

5.2 Case 2: Instability without Diffusion

The following case simply demonstrates that instability can exist in the system even though there is no diffusion.

The Reaction Variables	Values
d :	0.03302
b :	0.08
e :	0.00189
f :	0
The Diffusion Coefficients	Values
D_S :	0
D_C :	0
D_R :	0

If we look at the behavior in town 1, we notice that susceptible population is chaotic and the terms after the 16th iteration, becomes undefined. Resulting the equilibrium of the system to be unstable.

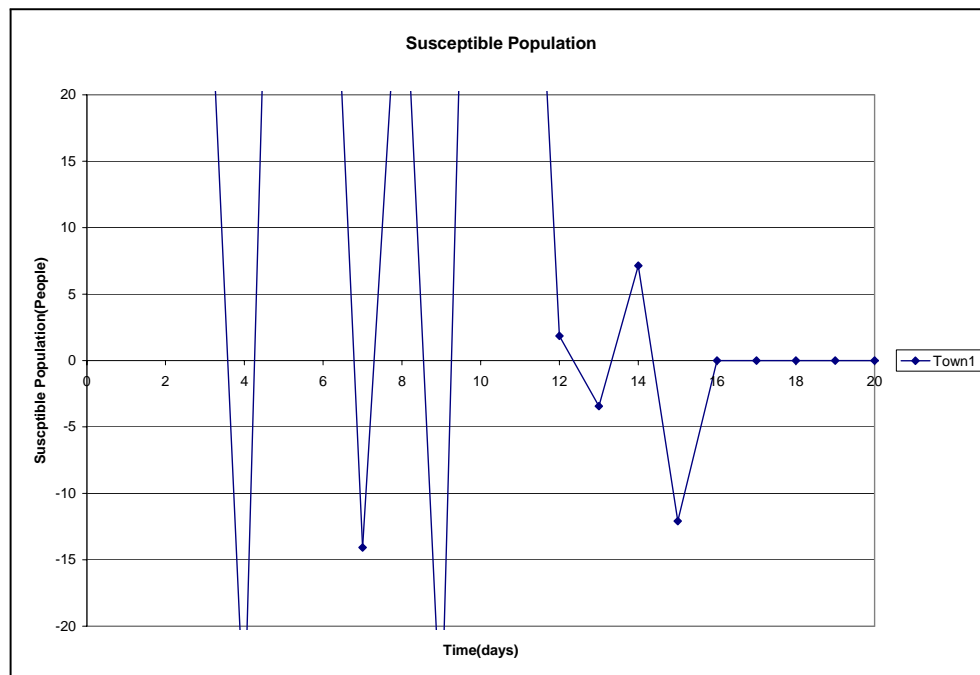


Fig. 5.14.

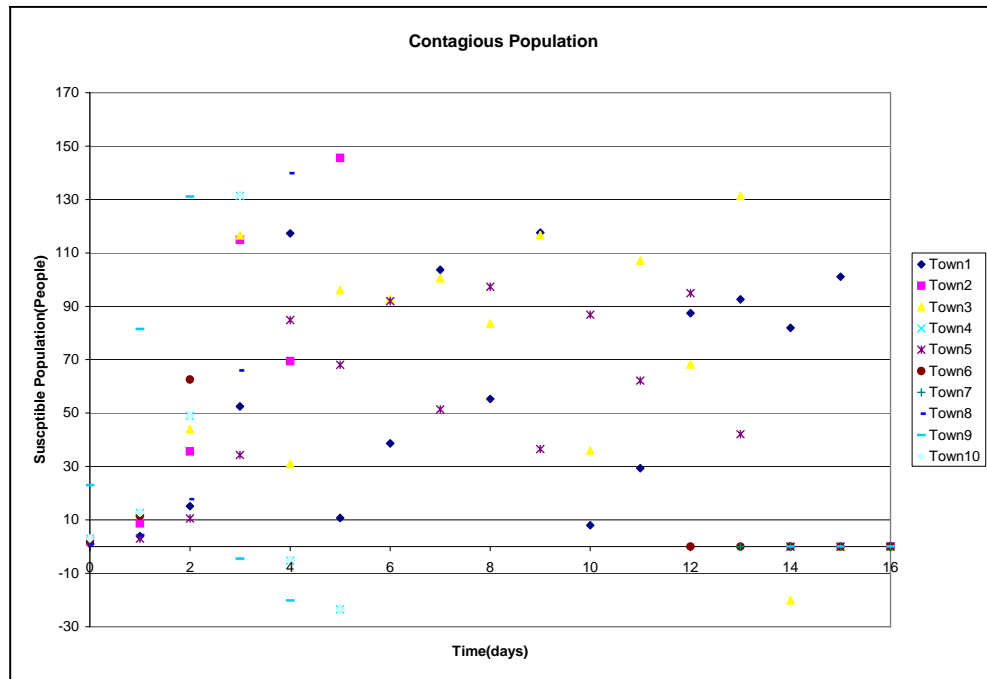


Fig. 5.15.

The contagious population does not go to an equilibrium point.

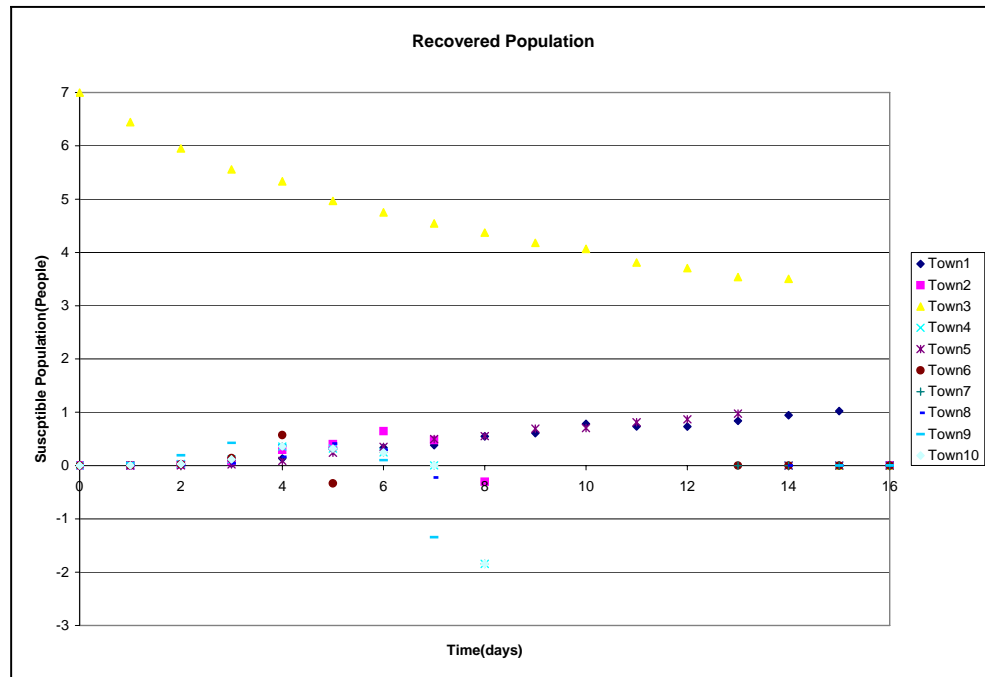


Fig. 5.16.

Even though the recovered population appears that it could be converging to an equilibrium past the 16th iteration based on this graph, in the spread sheet, the values after the 16th iterations are undefined as well. Therefore we can conclude that the equilibriums for the system are unstable, even though there is no diffusion.

5.3 Case 3: Stability with Diffusion:

The Reaction Variables	Values
d :	0.004
b :	0.08
e :	0.00189
f :	0

The Diffusion Coefficients	Values
D_S :	0.04
D_C :	0.4
D_R :	0.2

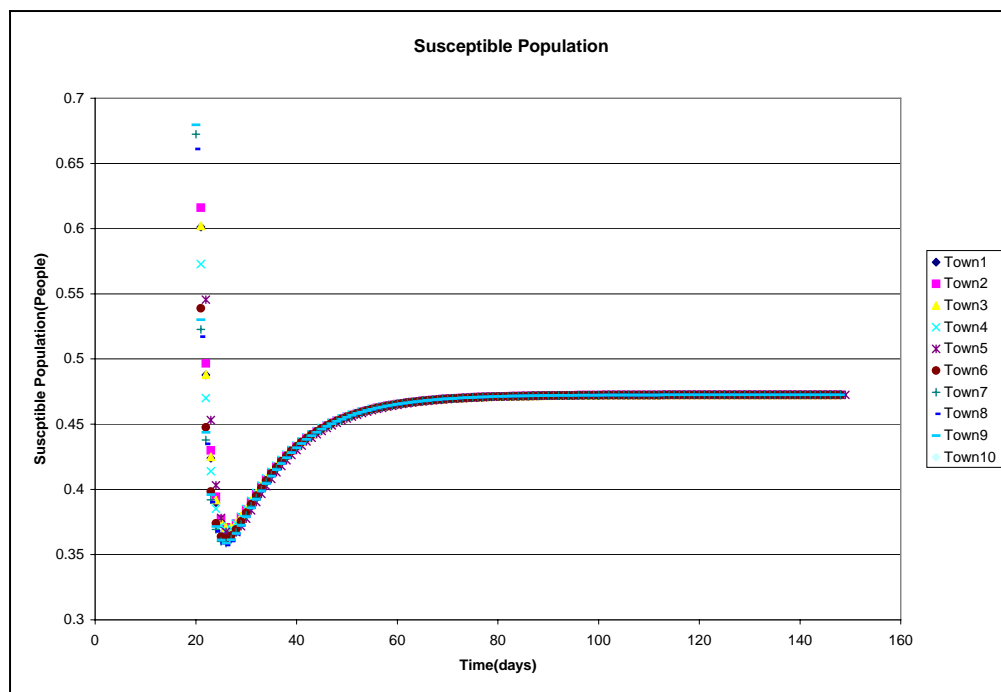


Fig. 5.17.

When the system is stable at the equilibrium and there is diffusion, the network of towns all go to the same equilibrium point $\frac{e}{d}$..

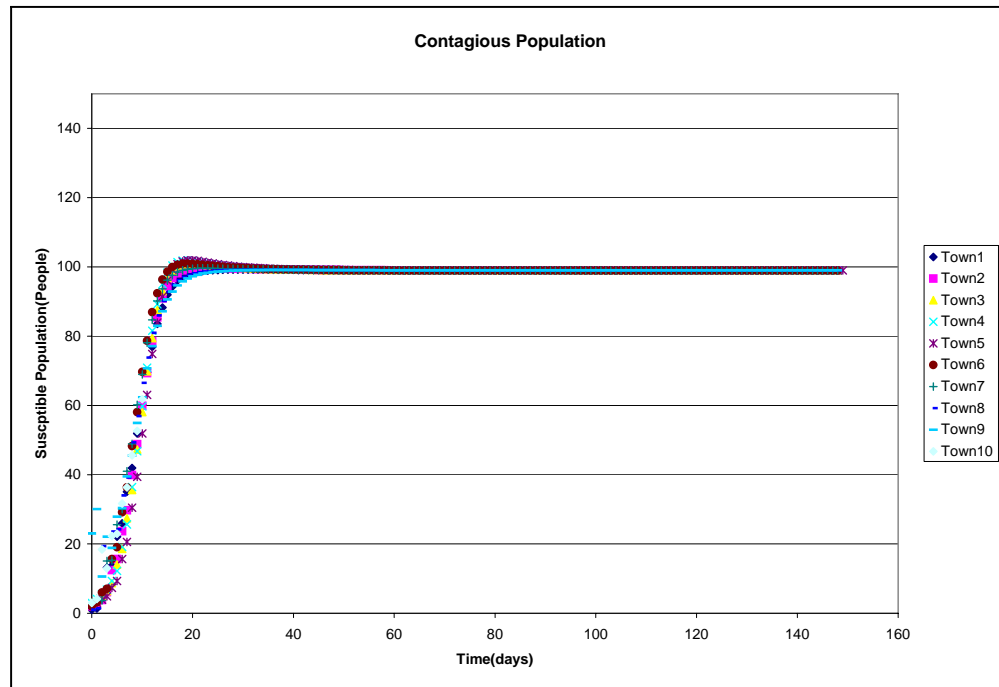


Fig. 5.18.

When there is stability in a system at the equilibrium when there is diffusion, all of the towns will go to the same equilibrium point even for the contagious population. All of the contagious population becomes evenly distributed amongst the 10 towns.

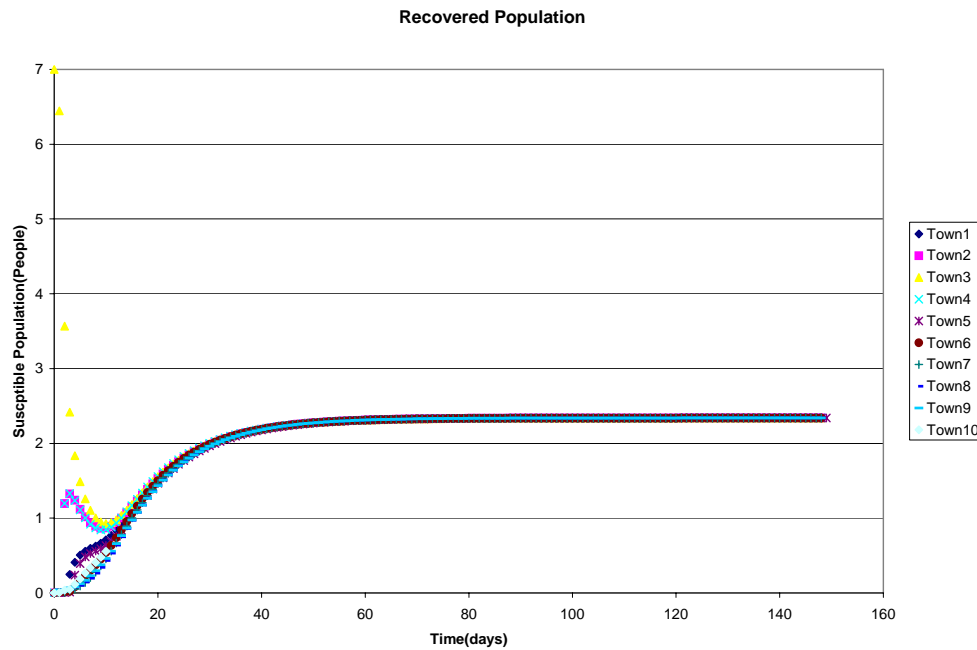


Fig. 5.19.

Since all of the contagious population is the same, the equilibrium for the recovered population will also be equal, $\frac{e}{b}C$.

Example of Town1 with Diffusion:

	Starting Value	Ending Value	Equilibrium
Susceptible	89	0.4725	$\frac{e}{d}$
Contagious	1	98.9889	C
Recovered	0	2.338613	$\frac{e}{b}C$

The limiting values found for Town 1 are the same values found for all of the towns in the system. This is due to the diffusion of the population from high concentration to low concentration, evenly distributing the population amongst all ten towns. .

5.4 Case 4: Instability with Diffusion:

The Reaction Variables	Values
d :	0.004
b :	0.08
e :	0.00189
f :	0
The Diffusion Coefficients	Values
D_S :	0.04
D_C :	0.4999
D_R :	0.2

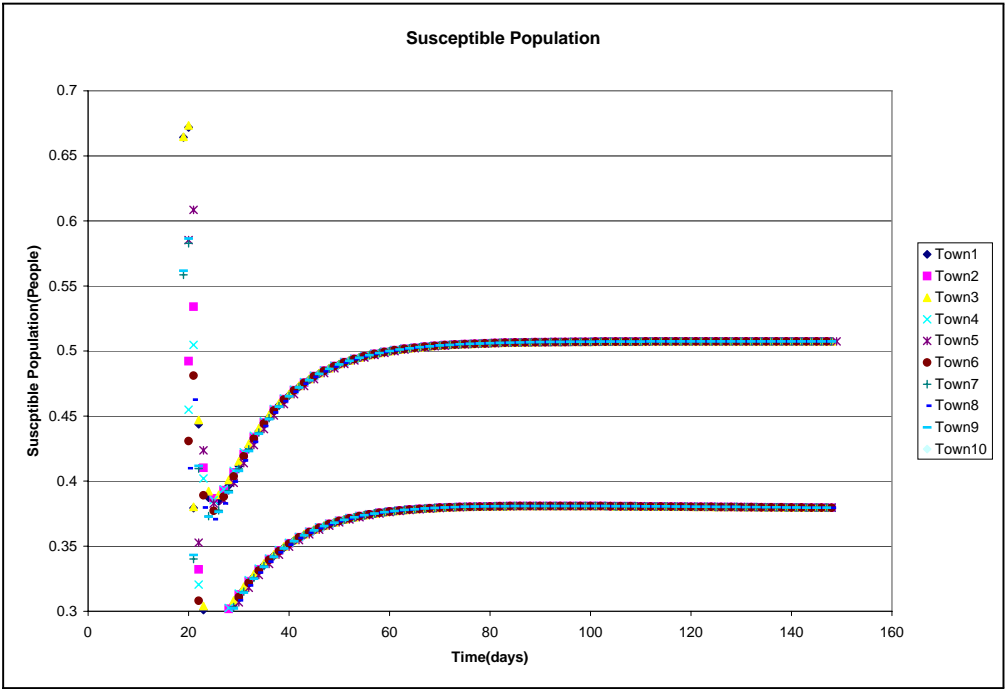


Fig. 5.20.

Even though this system may appear stable, the values for each town oscillates between two bounds. We can see an example of this when we graph only town 1.

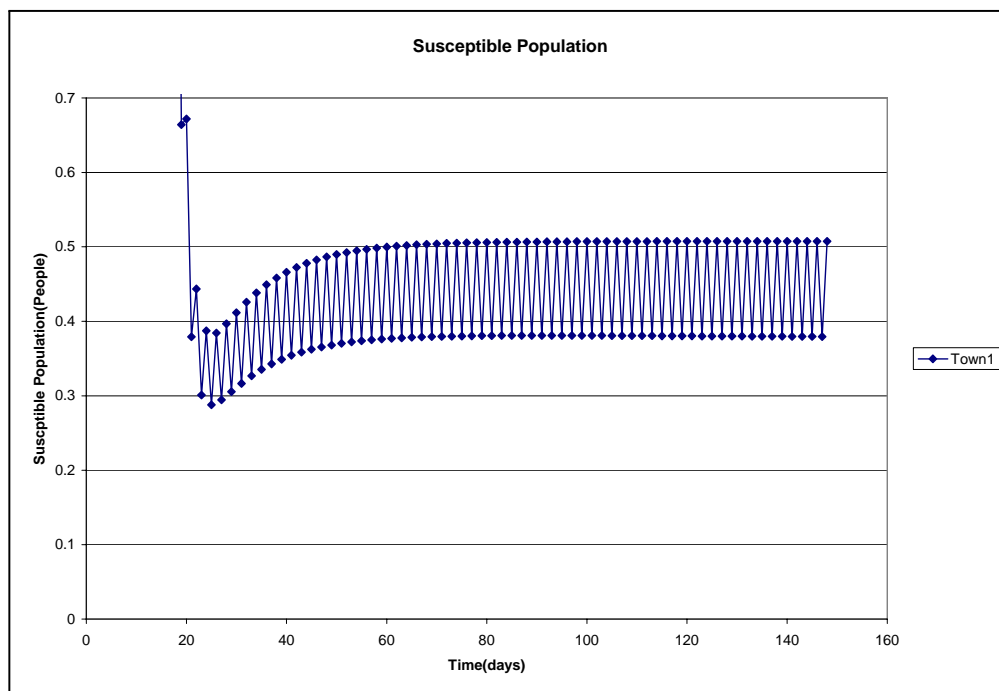


Fig. 5.21.

Therefore we see that the system is unstable at the equilibrium for the susceptible population. When we use the extreme condition for D_C , where $D_C \approx \frac{1}{2}$, we notice behavior that is similar to Turing instabilities.

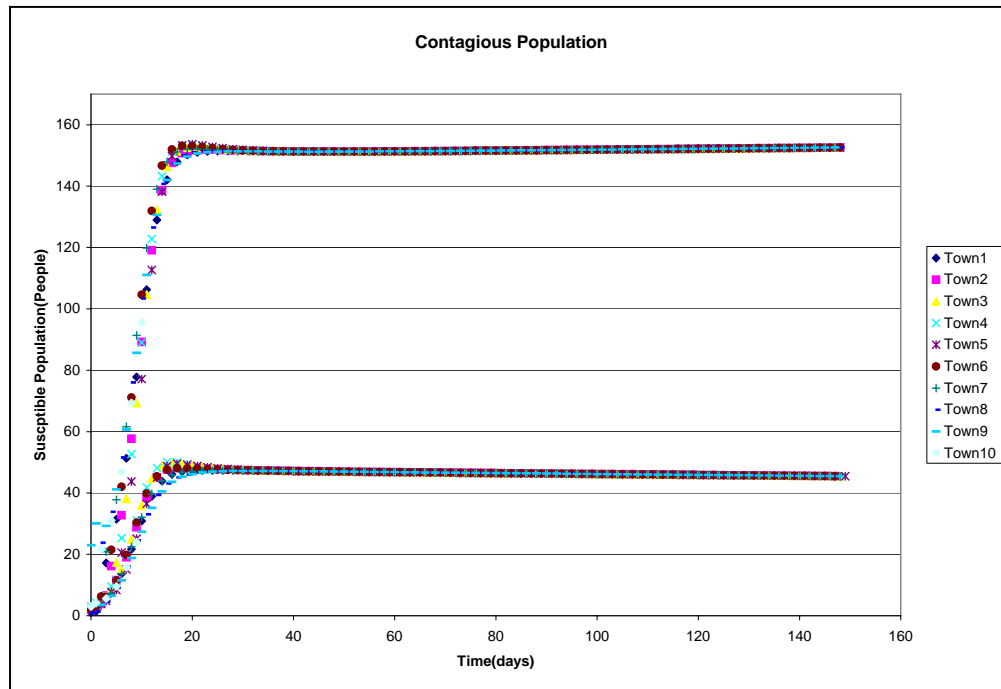


Fig. 5.22.

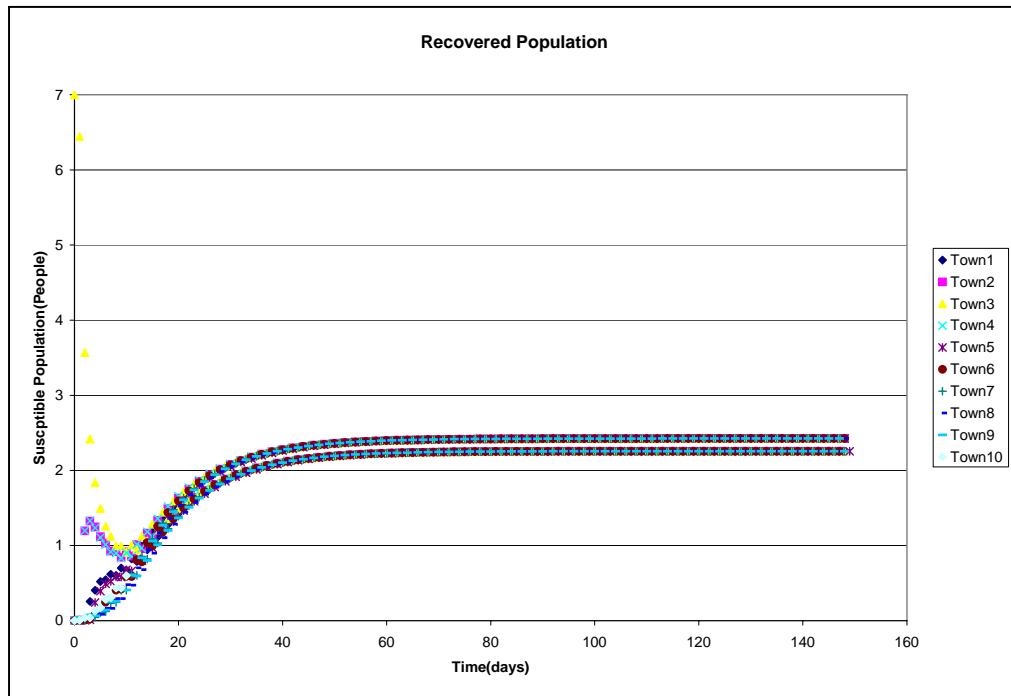


Fig. 5.23.

In all three of the population pools, we notice that the system does not converge to a single value. We can conclude the system at the equilibrium is unstable due to diffusion; a Turing instability.

Chapter 6

Conclusions

After creating both the diffusion-reaction and the reaction-diffusion models for the $S-C-R$ system in a network, we can conclude that reaction-diffusion model is better for modeling diffusion in a $S-C-R$ system. The diffusion-reaction model allows for the population to move without determining whether or not the population will be in the town and in the population pool is moving to, which could possibly result in negative populations. If we apply this method into a real life scenario, the diffusion-reaction would occur in locations where communication between each town is slow, resulting in people moving without knowing whether or not moving will cause a higher concentration of sick people in the town they are going to. In the reaction-diffusion model, the people moving always know the concentration of the sick people in the town next door.

We have also shown that using the determinant, trace and the sum of principal minors is a much easier method in determining the stability of the systems of questions compared to trying to use the Gerschgorin method. Even though we did not have to use either method for our cases since we were able to easily calculate the eigenvalues, we know we can always find the stability using the three conditions for discrete stability using the determinant, trace and sum of principal minors.

We were able to create a stable system with and without diffusion. The instabilities with diffusion only occurred when the values of the diffusion coefficients were at the upper possible limit. This resulted in a Turing instability. Even though mathematically

we showed that at value less than the upper bound for the diffusion coefficients, Turing instabilities should not occur, but numerically, in our simulations, Turing instabilities did occur due to numerical effects. In an actual biological system, due to the inherent fluctuation of the variables, the round off errors would be much greater, which could result in an increased probability of Turing instabilities to occur.

References

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