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### Analysis of a queueing model with service threshold

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# **Analysis of a Queueing Model with Service Threshold**

by

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A Thesis Submitted in Partial Fulfillment of the Requirements for the

Degree of Master of Science

in

Applied and Computational Mathematics

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December 17, 2010

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## Abstract

We study a variation of the M/G/1 queueing model in which service time of customers is modified depending on the class to which the customers belong. Specifically, apart from the regular service time, we consider different service times for every customer who starts service after an idle and also for every  $(m+1)^{\text{st}}$  customer. The model represents situations such as when the system requires a warm up time from a cold start (i.e.) after being idle for some time and also a system that is taken down at regular intervals perhaps for maintenance. Such systems come under the general class of vacation models. The system is modeled as a Markov process with a transition matrix of the M/G/1 type. Matrix-analytic results are utilized to compute some performance measures of interest.

*Subject Classification: Markov processes; Queueing models; Matrix-Analytic Solutions; Vacation models;*

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## References

## 1. Introduction

The mathematical study of waiting lines is known as Queueing Theory. Queueing Theory was originally developed mostly in the context of telephone traffic engineering but it has found applications in several disciplines such as engineering, operation research, and computer science, with practical applications in such areas as layout of manufacturing systems, airport traffic modeling, measurement of computer performances, analysis of traffic control, study of telecommunications systems and even to model decision-making to replace a goalie in a hockey game. The earliest mention of Queueing Theory was made in 1909 in a paper by A.K. Erlang. In 1951 David G. Kendall provided a systematic treatment of the study of basic queues and included in his paper the first mention ever of the term “queueing systems”. Later in 1953, Kendall also introduced a formal classification of queueing systems. Since then, various queueing models and their analyses have occupied a voluminous part of the operations research literature.

## 2. The Notation

As mentioned earlier, Queueing Theory has been successfully used to model, analyze, and solve complex systems, using analytical, numerical, and simulation techniques. A basic queueing system is specified by identifying the essential components that make up such a system- arrival process, service process, number of servers, buffer size to hold waiting entities, size of the calling population, and service priority. A system is indicated in a notational form  $A/S/N/C/P/D$ . The arrival process, A, is denoted by specifying the

distributions of inter-arrival times. For example, if the inter-arrival times are exponentially distributed, the letter **M** is used. This is due to the Markovian, or memory-less property of the exponential distribution. If the inter-arrival times are assumed to be independent and have an arbitrary, general distribution, the notation of **GI** is used. Another common inter-arrival distribution is an Erlang distribution of the order  $k$ . This is the distribution of the sum of  $k$  independent and identically distributed (i.i.d) exponential random variables. This distribution is denoted by the symbol  $E_k$ . A generalization of the Erlang distribution in which the inter-arrival times are associated with the times of absorption in a finite-state Markov process with one absorbing state, is known as the phase-type distribution [5]. The notation used to indicate such a distribution is **PH**. Various other inter-arrival time distributions have been used in the literature to model specific queueing systems. The second parameter **S** stands for the service process and is indicated in a way similar to the arrival process described above. The parameter **N** indicates the number of servers, **C** represents the maximum system capacity, the population size is denoted by **P** and **D** stands for the service discipline such as first in first out, last in first out, random. By default, system capacity and population size are taken to be infinite and the service process is assumed to be first-in-first out. A good review of a variety of queueing models can be found in [1] and [2].

### 3. M/G/1 Model

The M/G/1 model assumes that the system consists of Poisson arrivals with average arrival rate of  $\lambda$ , general service time distribution  $H(\cdot)$  with an average service time of  $1/\mu$ , a single server, and customers are served in the same order as they arrive. The stochastic

process of interest is  $\{X_t\}$ , where  $X_t$  is the number of customers in the system in steady state at time  $t$ . For the simple M/M/1 model and most related queues with Poisson arrivals and exponential services, due to the memory-less property of exponential distribution the process  $\{X_t\}$  is a Markov process and its stationary distribution as well as relevant performance measures of the model can be given in closed, tractable form.

When the service time distribution is not exponential, the process  $\{X_t\}$  is non-Markovian, since the number of customers at time  $t$  depends on the number of arrivals during the time the current customer was in service. For queues with exponential service times, the distribution of the remaining time of service is independent of the elapsed service time and is also an exponential distribution with the same service rate. This advantage is lost in queues with general service time distributions.

However, by observing the system at departures we can define a Markov chain. Consider the sequence of random variables  $\{X_n, n \geq 1\}$ , where  $X_n$  is the number of customers in the system at the  $n$ th departure. If the  $n$ th customer leaves behind an empty system, then the number of customers in the system at the next departure will precisely be the number of customers who arrived during the service time of the first customer who came after the  $n$ th. On the other hand, if the  $n$ th customer leaves behind a non-empty system, then during the service time of the customer at the head of the line,  $Y$  customers arrived and the number at the next departure will be those who were there at the last departure, plus those who arrived recently, minus the customer who is leaving. Thus, we have the following recurrence relation:



$$\begin{aligned}
X_{n+1} &= (X_n - 1)^+ + Y, \text{ where} \\
W^+ &= \begin{cases} W, & \text{if } W \geq 0 \\ 0, & \text{otherwise} \end{cases}
\end{aligned} \tag{1}$$

and  $Y$  is the number of arrivals during one service time. Since each value in the sequence depends only on the previous value, we can see that the sequence  $\{X_n, n \geq 1\}$  forms a Markov chain, known as the embedded Markov chain.

The transition probability  $P_{ij} = P(X_{n+1} = j \mid X_n = i)$  is then given by

$$p_{ij} = \begin{cases} P(Y = j - i + 1), & \text{if } i \geq 1, j \geq i - 1; \\ P(Y = j), & \text{if } i = 0, j \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

If  $T$  corresponds to the service time random variable then conditioning on  $T$  gives

$$P(Y = k \mid T = t) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}, k \geq 0 \tag{3}$$

since the arrivals form a Poisson process.

Unconditioning using the distribution of service times  $H(\cdot)$ , we get:

$$p_{ij} = \begin{cases} P(Y = j - i + 1) = \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^{j-i+1}}{(j-i+1)!} dH(t), & \text{if } i \geq 1, j \geq i - 1; \\ P(Y = j) = \int_0^\infty \frac{e^{-\lambda t} (\lambda t)^j}{j!} dH(t), & \text{if } i = 0, j \geq 0; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Note that  $P_{0j} = P_{1j}$  for  $j \geq 0$ , so that the first two rows of the transition probability matrix of the Markov chain are identical. For  $i \geq 1$ ,  $P_{ij} = P_{i+1, j+1}$  and in addition we have  $P_{ij} = 0$  if  $j < i - 1$ ,  $i \geq 1$ , so the matrix can be seen to be of upper Hessenberg form. Defining  $a_j$  to be the

probability that there are  $j$  arrivals during a service, we could write the transition probability matrix of the chain  $\{X_n\}$  as:

$$P = \begin{bmatrix} a_0 & a_1 & \dots & a_n & \dots \\ a_0 & a_1 & \dots & a_n & \dots \\ & a_0 & a_1 & \dots & \\ & & a_0 & a_1 & \\ & & & a_0 & \end{bmatrix} \quad (5)$$

The Markov chain  $\{X_n\}$  will be ergodic iff  $\rho = \lambda E(T) < 1$ , where  $T$  is the service time. The quantity  $\rho$  is known as the traffic intensity and measures the average number of arrivals during one service time. When  $\rho < 1$ , the invariant probability measure  $\underline{x}$ , of the embedded-Markov chain, is the unique non negative vector that satisfies the equations  $\underline{x} P = \underline{x}$  and  $\underline{x} \underline{e} = 1$  where the  $i$ th component  $x_i$  is the steady state probability that there are  $i$  customers in the system at a departure. Hence  $x_i = \lim_{n \rightarrow \infty} P(X_n = i)$ . It can be shown that  $x_i$  is also the same as the limiting value of the probability of  $i$  customers at an arbitrary time, so that  $x_i = \lim_{t \rightarrow \infty} P(X_t = i)$ . Many performance measures relevant to the queueing model may be expressed in terms of the invariant distribution  $\underline{x}$  ([1], [3], [7]).

## 4. M/G/1 Type Models

Many interesting applications lead to generalizations of M/G/1 models with a stationary probability matrix that has a structure similar to that of  $P$  in equation (5), except with matrix components instead of scalar values. Interesting examples, general analysis, and properties of such models can be found in ([6],[4]). The state of the process is defined

by (i,j), where j is the state at the ith level and the state space is the cross product of {0,1,2...} and {1,2...m}. A typical transition probability matrix is shown below.

$$P = \begin{bmatrix} B_0 & B_1 & \dots & B_n & \dots \\ A_0 & A_1 & \dots & A_n & \dots \\ & A_0 & A_1 & \dots & \\ & & A_0 & A_1 & \\ & & & A_0 & \end{bmatrix} \quad (6)$$

The matrix  $A = \sum_{n=0}^{\infty} A_n$  is stochastic and when the Markov chain with transition matrix P is irreducible, so is A. We let  $\underline{\pi}$  be the invariant vector of A and define  $\underline{\beta}$  to be the average of the A matrices as

$$\underline{\beta} = \sum_{n=1}^{\infty} n A_n \underline{e} \quad (7)$$

The unique invariant measure,  $\underline{x}$ , when it exists, again satisfies the equation  $\underline{x}P = \underline{x}$  and  $\underline{x}\underline{e} = 1$ . The Markov chain P is ergodic and hence admits an invariant vector  $\underline{x}$  if and only if the condition

$$\rho = \underline{\pi}\underline{\beta} < 1 \quad (8)$$

holds true. Note that this is similar to the stability condition that we had for the embedded Markov chain in the M/G/1 model. Partitioning the invariant vector  $\underline{x}$  into blocks of size m, we could write the steady-state equations as

$$\begin{aligned} \underline{x}_0(I - B_0) &= \underline{x}_1 A_1 \\ \underline{x}_0 B_1 + \sum_{k=1}^{i+1} \underline{x}_k A_{i+1-k} &= \underline{x}_i, i \geq 1 \end{aligned} \quad (9)$$

Since the chain is irreducible, the matrix  $(I - B_0)$  is invertible and hence we could write

$$\underline{x}_0 = \underline{x}_1 A_1 (I - B_0)^{-1} \quad (10)$$

Which relates the vectors  $\underline{x}_0$  and  $\underline{x}_1$ .

If the vectors  $\underline{x}_0$  and  $\underline{x}_1$ , are available, then the sequence  $\{\underline{X}_k\}, k \geq 2$  may be computed from equation (10) using an iterative process such as the block Gauss-Seidel method. We define  $G_{j,j'}(k)$  to be the conditional probability that this process starting at level  $i$  in state  $(i,j)$  takes exactly  $k$  steps to reach one level below and this level is reached at state  $(i-1,j')$ . Hence,  $G_{j,j'}(k)$  corresponds to the first passage time distribution of the chain to go from a level to the next lower level.

Defining the  $m \times m$  matrix  $G(k)$  as one with elements  $G_{j,j'}(k)$  and  $G(z)$  as its matrix of transforms so that

$$G(z) = \sum_{k=1}^{\infty} G(k) z^k, 0 \leq z \leq 1 \quad (11)$$

which can be shown to be equivalent to

$$G(z) = \sum z A_n G^n(z) \quad (12)$$

Defining the matrix  $G = G(1-)$ , it is known that under the stability condition of equation (8), the matrix  $G$  is the minimal non-negative, stochastic solution of the non-linear equation

$$G = \sum_{n=0}^{\infty} A_n G^n \quad (13)$$

We can compute the matrix  $G$  iteratively using the modified Gauss-Seidal method. The invariant vector  $\underline{g}$  associated with the matrix  $G$ , and the matrix  $G$  itself are useful in the calculation of the corresponding  $\underline{x}_0$  and  $\underline{x}_1$ , of the invariant vector  $\underline{x}$ , as well as in developing expressions for the moments of the distribution queue length. The probabilistic and algorithmic method also has the advantage of providing internal accuracy

checks on our computations. For example, the vectors  $\underline{x}_0$ ,  $\underline{x}_1$  are computed explicitly using the matrix  $G$  and the matrices  $A_n$ ,  $B_n$  and the two vectors must satisfy the condition of equation (10). Similarly defining the vector  $\underline{\mu}$  to represent the conditional mean first passage of time

$$\underline{\mu} = \sum k G(k) \underline{e} \quad (14)$$

the vectors  $\underline{g}$  and  $\underline{\mu}$  must satisfy the following condition.

$$\underline{g} \underline{\mu} = \frac{1}{(1 - \rho)} \quad (15)$$

Since the vectors  $\underline{g}$  and  $\underline{\mu}$  are computed iteratively and  $\frac{1}{(1 - \rho)}$  is a known scalar, the above equation is yet another accuracy check on our numerical methods. Detailed derivation of the above results can be found in [6].

## 5. Threshold Model

In this project, we look at a particular variation of the M/G/1 model. Consider a queueing system in which customers arrive according to a Poisson process with rate  $\lambda$ . The service times are deterministic but vary in the following way. Normally each customer spends  $c$  units of time in service. Every customer who starts service after an idle period needs  $c_0$  units of time and every  $m^{\text{th}}$  customer requires a special service time,  $c_1$ . This model represents the system in which extra time is required to initiate service from a “cold start”, as when the system requires a warm-up time. Similarly, the system is taken down at regular intervals perhaps for preventive maintenance, after a fixed number of services and hence the customer who arrives during this maintenance experiences a longer than usual

service time. Queueing models of this type are described in general as vacation models, with vacations occurring after an idle period or after a number of jobs are completed, and are solved by using transform methods. A review of vacation models and references can be found in [2].

The transition probability matrix for the embedded Markov chain of this model is of the M/G/1 type described in [6]. Defining the state space  $\{0,1,2,\dots\} \times \{1,2,\dots,m\}$  so that the system is in state  $(i,j)$  if there are  $i$  customers in the system and the customer in service is of type  $j$ ,  $j=1,\dots,m$ . We can write the transition matrix  $P$  as

$$P = \begin{bmatrix} B_0 & B_1 & \dots & B_n & \dots \\ A_0 & A_1 & \dots & A_n & \dots \\ & A_0 & A_1 & \dots & \\ & & A_0 & A_1 & \\ & & & A_0 & \end{bmatrix} \quad (16)$$

where  $A_\nu$  is the probability of  $\nu$  arrivals during one service when the queue is non-empty,

and  $B_\nu$  is the probability of  $\nu$  arrivals during a service when the queue is empty.  $A_\nu$  and

$B_\nu$  are  $m \times m$  matrices and have the following simple structure

$$A_\nu = \begin{bmatrix} 0 & e^{-\lambda c_1} \frac{(\lambda c_1)^\nu}{\nu!} & 0 & 0 & \dots 0 \\ 0 & 0 & e^{-\lambda c} \frac{(\lambda c)^\nu}{\nu!} & 0 & \dots 0 \\ 0 & 0 & 0 & e^{-\lambda c} \frac{(\lambda c)^\nu}{\nu!} & \dots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{-\lambda c} \frac{(\lambda c)^\nu}{\nu!} & 0 & 0 & 0 & \dots 0 \end{bmatrix} \quad (17)$$

$$B_\nu = \begin{bmatrix} 0 & e^{-\lambda(c_1+c_2)} \frac{(\lambda(c_1+c_0))^\nu}{\nu!} & 0 & 0 & \dots 0 \\ 0 & 0 & e^{-\lambda c} \frac{(\lambda c)^\nu}{\nu!} & 0 & \dots 0 \\ 0 & 0 & 0 & e^{-\lambda c} \frac{(\lambda c)^\nu}{\nu!} & \dots 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{-\lambda c} \frac{(\lambda c)^\nu}{\nu!} & 0 & 0 & 0 & \dots 0 \end{bmatrix} \quad (18)$$

$A_0$  is the matrix that describes the probability of the system's transition from one level when the system is not idle to the next lower level;  $A_1$  describes the probability of the system staying in the same non-idle level and  $A_j, j \geq 2$ , is the matrix of probabilities describing the transition of the system  $(j-1)$  levels up when the system is not idle. The matrices  $B_\nu$  have similar interpretations but for an empty system. Since the matrix  $A$  is doubly stochastic, its invariant vector is given by  $\underline{\pi} = \frac{1}{m} \underline{e}$ , where  $\underline{e}$  is a unit column vector. In view of the special structure of the matrices  $A_\nu$ , the vector  $\underline{\beta}$  in equation (7), becomes

$$\underline{\beta} = \sum_{k=1}^{\infty} k A_k \underline{e} = [\lambda c_1, \lambda c, \dots, \lambda c]^T \quad (19)$$

so that the stability condition in equation (8) simplifies to  $\frac{\lambda c_1 + (m-1)\lambda c}{m} < 1$ .

## 6. Numerical Examples

The algorithms described in the earlier section were implemented in Maple for different choices of input parameter values of  $m$  (number of services required for a special service time),  $c$  (normal service time),  $c_1$  ( $m$ th service time), and  $c_0$  (initial service time).

Values of these parameters are chosen for illustrative purposes. Also, the choice of values for the parameters  $c$ ,  $c_1$ , and  $c_0$  was determined by their interdependence in satisfying the stability condition (1). The arrival rate was set, without loss of generality, to 1. The value of  $m$  was set to 5 so that a special service is rendered for every fifth customer. The iterations in computing the matrix  $G$  were stopped when successive iterates differed by less than  $10^{-5}$ .

Below, we provide the results of our numerical examples describing different situations where the initial, normal and  $m$ th service times vary and study their impact on two basic performance measures -- the average queue length and the conditional average queue length. The scenarios have two of the service times equal to one another and the third is allowed to vary to isolate the impact of a particular service time. The goal of these numerical examples is to understand the interaction among these system parameters in determining the system performance.

Example 1:

We first set the normal service time to be less than the initial service which is equal to the  $m$ th service time. In figure (1) we can see that the average queue length grows, as it should, as the normal service time increases. But the interesting result of this scenario is how the average queue length starts out large, for the larger size  $c_1$  and  $c_0$  and continues to increase at a faster rate than the small and medium sized  $c_1$  and  $c_0$ . Even though the normal service time is smaller than the other two service times, a high level of normal service time induces sufficient backlog in the system to make the average queue size grow more rapidly, especially when the other two service times are large.



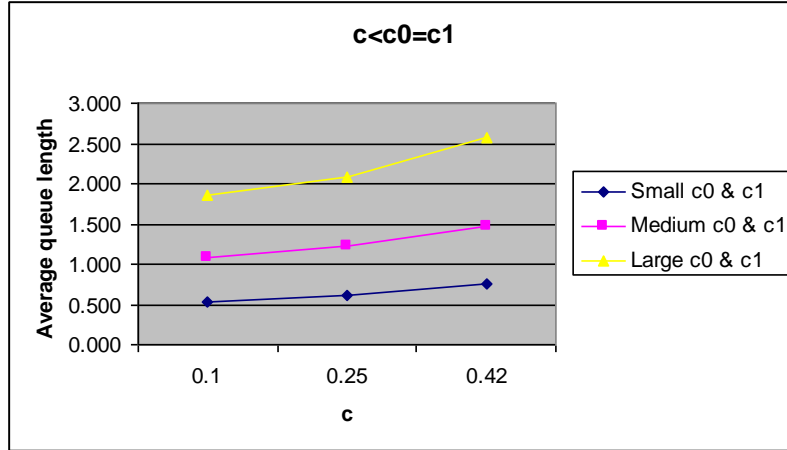


Figure 1

Example 2:

Next, we are looking at a system in which the normal service time is greater than the initial and  $m^{\text{th}}$  service times. Figure (2) shows similar results as Figure (1) but the average queue length grows even faster as the normal service time grows. In this case we see that the average queue length starts out smaller than in Figure (1) but we also see how it increases at a faster rate. This happens because the initial and  $m^{\text{th}}$  service times are smaller than the normal service times. The relatively small service times for the first customer and the  $m^{\text{th}}$  customer keep the queue sizes in check even though the normal service time is higher than that in example 1. So, we gather that the special service times seem to exert a larger impact on the average queue length under these conditions.

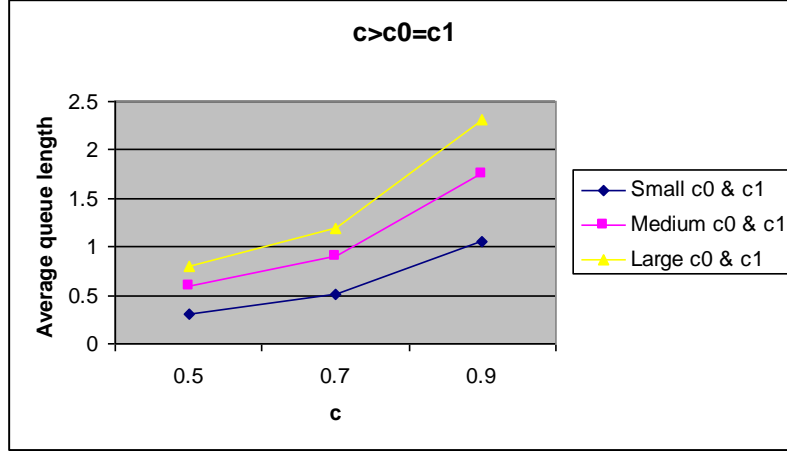


Figure 2

Example 3:

In this example, we set the instance where the  $m^{\text{th}}$  service time is less than the normal and initial service times. This graph in Figure (3) shows that the average queue length grows at a slow pace as the  $m$ th service time increases. This case has an interesting situation, for large  $c$  and  $c_0$  the graph starts out with a much higher average queue length and grows faster than the small and medium  $c$  and  $c_0$ . This is occurring because the normal service time is large. This case shows us that the  $m$ th service time does not have a big effect on the average queue length when the initial and normal service times are larger. The special service time is over shadowed by the long service times of succeeding customers.

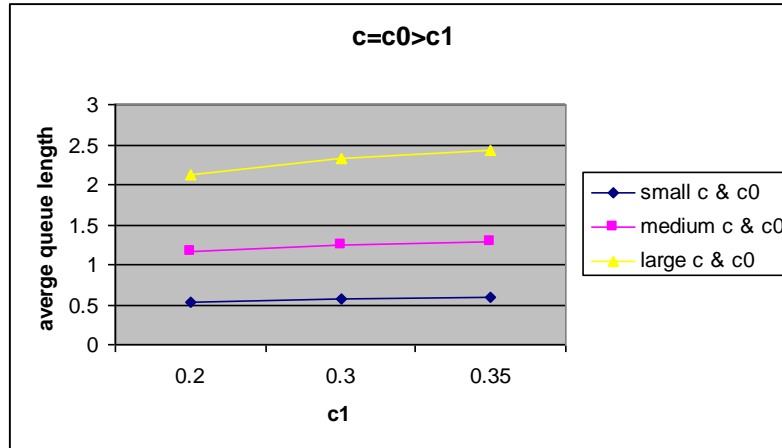


Figure 3

Example 4:

The next scenario describes the case when the  $m^{\text{th}}$  service time is greater than the normal and initial service times, we can see the results in Figure (4). The figure shows us that the average queue length starts out small and increases fast as the  $m^{\text{th}}$  service time increases. We have an interesting case in which the queue that builds up during the large  $m^{\text{th}}$  service time is compounded by the relatively large regular and initial service periods. We note that for relatively smaller regular and initial service times, the impact of a larger  $m^{\text{th}}$  period is mild.

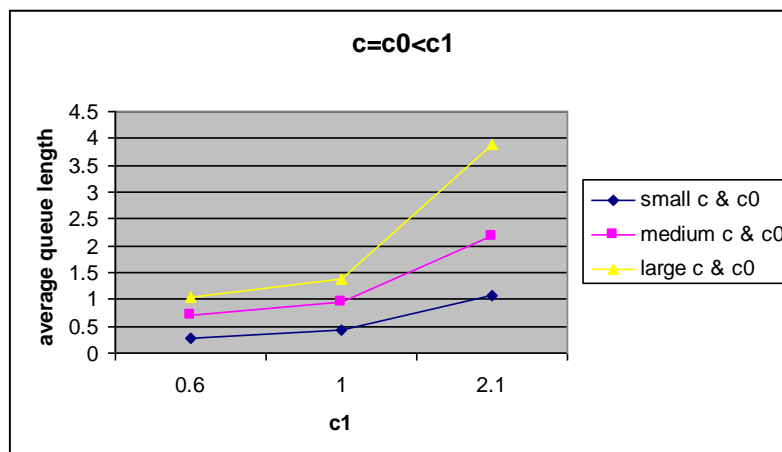


Figure 4

Example 5:

Here, the initial service time is less than the other two service times, and we can see that the average queue length grows at a slow rate as the initial service time grows. It is interesting to note the relatively large average queue length even for small values of  $c_0$ , when  $c_1$  and  $c$  are large. Customers who arrive to an empty system get served quickly. However, with the large service times of regular and special customers, the time between empty systems is relatively large and the queue size builds up during this interval. This is the cause of the larger average queue sizes.

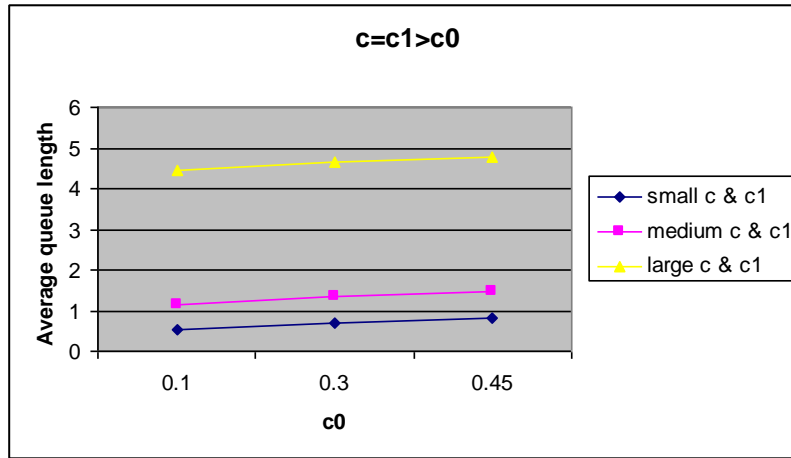


Figure 5

Example 6:

The last case, where the initial service time is greater than the  $m^{\text{th}}$  and normal service times is illustrated in Figure (6). As we look at this graph we can see that the results are what we should expect, the average queue length is increasing as the initial service time is increasing. The interesting case in this example is for the small  $c$  and  $c_1$ , the average queue length stays consistent no matter how big the initial service time is. This is occurring due to the fast services of the  $m^{\text{th}}$  and normal customers, but when the  $m^{\text{th}}$  and normal

service times become larger, the initial service time has more of an effect on the average queue length.

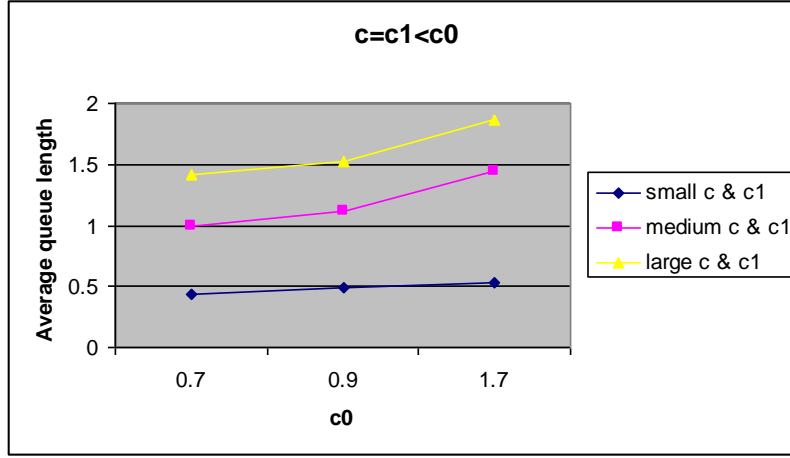


Figure 6

In the next set of examples we investigate the behavior of the conditional mean queue length as the service time parameters change. The conditional mean queue length is the average number of customers in the queue in each state.

Example 7:

The two figures below represent two queues with the same traffic intensity for varying values of the normal service time ( $c$ ), the  $m^{\text{th}}$  service time ( $c_1$ ) and the initial service time ( $c_0$ ). In Figure (7), the initial service time is larger than the normal service time and the  $m^{\text{th}}$  service time.

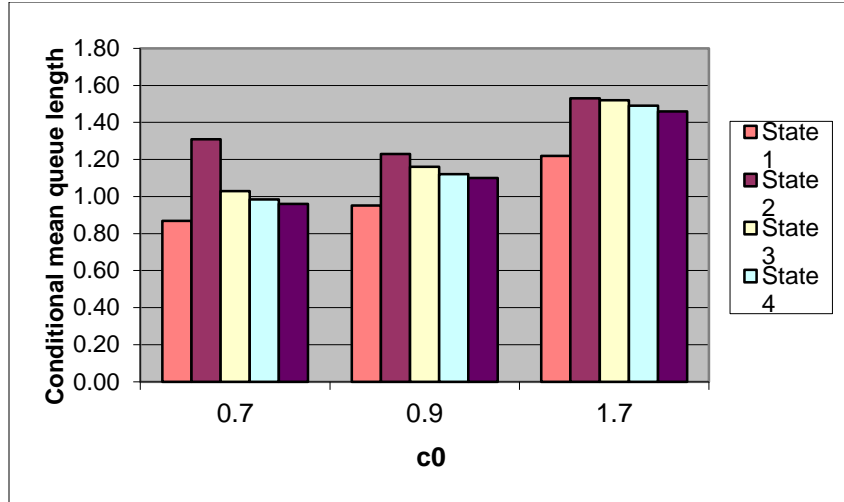


Figure 7

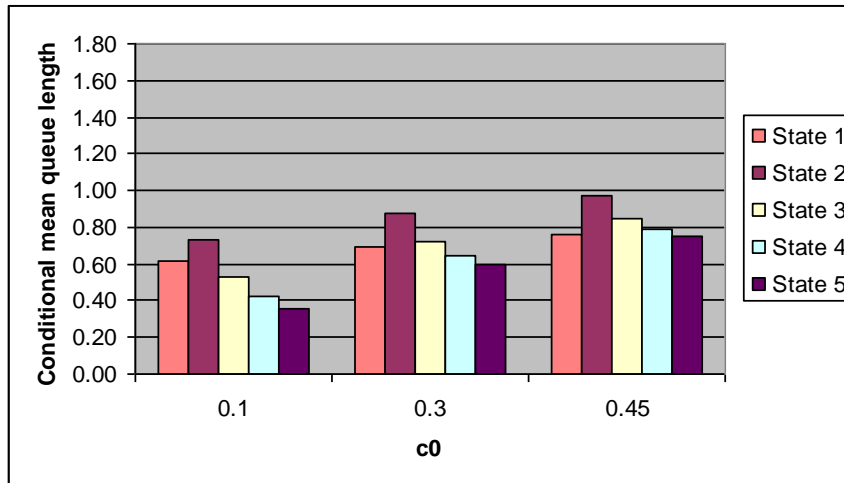


Figure 8

and in Figure (8) the initial service time is smaller than the normal and  $m$ th service times.

We can see that the two figures have similar behavior but Figure (7), which applies for a system with larger values of  $c_0$ , has much bigger the conditional mean queue lengths.

This shows us that the values of  $c_0$ ,  $c$ ,  $c_1$  and their relative sizes dictate the relative growth of the queue sizes at various states and the traffic intensity will be a poor descriptor of the qualitative behavior of such systems.

Example 8:

We consider two systems. In each system, the normal and initial service times are held at the same level. In the first system, the  $m^{\text{th}}$  service time is larger than these and in the second system, it is smaller. The traffic intensities in the two systems although not equal are very close.

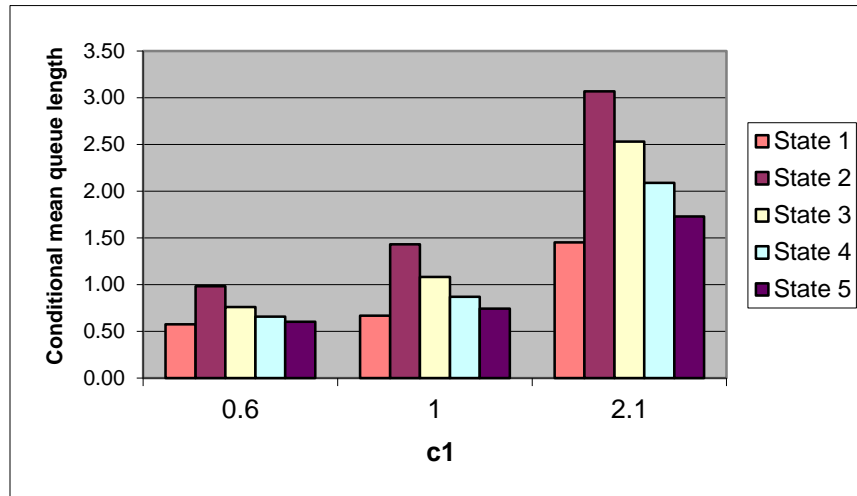


Figure 9

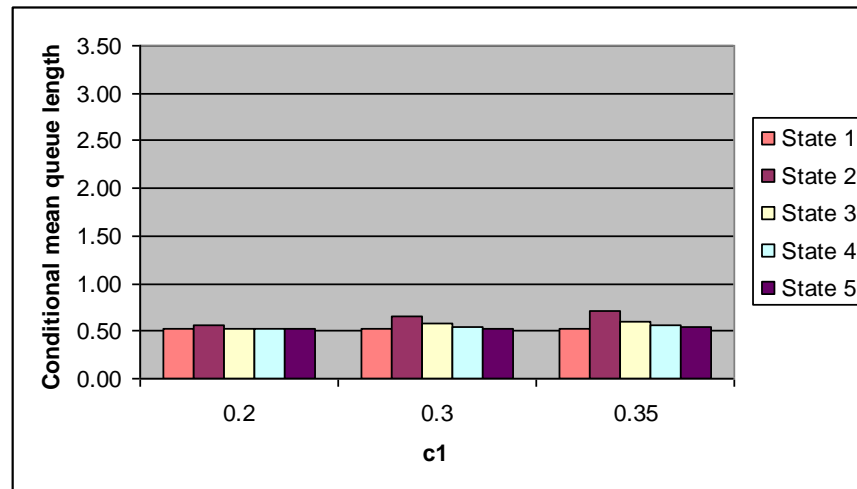


Figure 10

We note that with the smaller values of the  $m^{\text{th}}$  service times, the conditional mean queue length stays about the same across these values as can be seen in Figure (10). However, for larger values of the special service times, as this value increases, there is a substantial difference in how the conditional mean queue length changes. The increase in the average size of the queue when different types of customers are at service is evident from Figure (9) indicating a strong influence of the special service time on the conditional mean queue length.

## 7. Conclusion

In this project we studied a variation of the M/G/1 system with three different service times: normal, initial and maintenance service times. Using the results from matrix analytic methods, we implemented algorithms to compute performance measures to study the behavior of the system as its parameters are changed and used Maple to implement the algorithms. The results explain the complex nature of the dependence of these parameters. The interplay among the three different service times and the threshold value and their impact on the performance measures was seen much more clearly than what a summary measure such as the traffic intensity might suggest.



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