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Global character of the solutions of rational difference

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The Global Character of the Solutions of

$$X_{n+1} = \frac{AX_{n-l}}{1+(2-\alpha)X_n + \alpha X_{n-l}}, n = 0,1,2,\ldots$$

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Abstract

It is our goal to investigate the global character of the positive solutions of the following rational difference equation:

$$X_{n+1} = \frac{AX_{n-l}}{1 + (2-\alpha)X_n + \alpha X_{n-l}}, \quad n = 0, 1, 2, \ldots,$$

where $A > 0$, $0 < \alpha < 2$, $l = 1, 2, 3\ldots$ and the initial conditions are positive real numbers. We will examine the global stability, periodic nature, boundedness and the monotonicity character of the positive solutions.

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1 Introduction

We will investigate the global character of the positive solutions of the difference equation:

$$X_{n+1} = \frac{AX_{n-l}}{1 + X_n + X_{n-l}} \quad , \quad n = 0, 1, 2, \dots, \tag{1}$$

where $A > 0$, $l = 1, 2, 3\dots$ and the initial conditions are positive real numbers.

Our goal is to investigate the global stability, periodic nature, boundedness and the monotonicity of the positive solutions of Eq.(1).

In particular, we will examine the following cases:

(i) The case where $l = 1$.

(ii) The case where $l = 2k + 1; k = 0, 1, 2, \dots$.

(iii) The case where $l = 2k; k = 1, 2, 3, \dots$.

In addition, we will analyze the global behavior of the positive solutions of the following difference equation:

$$X_{n+1} = \frac{AX_{n-1}}{1 + (2-\alpha)X_n + \alpha X_{n-1}} \quad , \quad n = 0, 1, 2, \dots,$$

where $A > 0$, and $0 < \alpha < 1$ and $1 < \alpha < 2$.

Furthermore, it is our goal to compare the similarities and differences in global behavior of the solutions in all the situations.

2 Global Character of Solutions of $X_{n+1} = \frac{AX_{n-1}}{1+X_n+X_{n-1}}$

It is our goal in this section to study the global character of the positive solutions of the following difference equation:

$$X_{n+1} = \frac{AX_{n-1}}{1 + X_n + X_{n-1}}, \quad n = 0, 1, 2, \ldots, \tag{2}$$

where the given parameter $A > 0$, and the initial conditions X_{-1}, X_0 are positive real numbers. Observe that Eq.(2) is a second order non-linear difference equation and it is also a special case of Eq.(1) where $l = 1$.

Other nonlinear second order rational difference equations were investigated in [2], [3], [5] - [9]. The study of these equations is quite challenging and rewarding and still in its infancy.

It is believed that nonlinear rational difference equations are of great importance in their own right and further results about such equations offer prototypes towards the development of the basic theory of the global behavior of nonlinear difference equations.

We determine the equilibrium points of Eq.(2) by setting

$$\bar{X} = \frac{A\bar{X}}{1 + \bar{X} + \bar{X}} = \frac{A\bar{X}}{1 + 2\bar{X}}.$$

Clearly $\bar{X} = 0$, and $1 + 2\bar{X} = A$ are the two solutions of the above equilibrium equation. So the two equilibrium points of Eq.(2) are $\bar{X}_1 = 0$ and $\bar{X}_2 = \frac{A-1}{2}$ when $A > 1$.

In this section we will prove the following properties of Eq.(2):

(i) If $A \leq 1$, then every solution of Eq.(2) converges to 0.

(ii) If $A > 1$, then Eq.(2) has solutions with minimal period 2 and every positive solution of Eq.(2) converges to a period 2 cycle or to the fixed point $\frac{A-1}{2}$.

2.1 The case $A \leq 1$

In this section we will assume that $A \leq 1$. Then $\bar{X}_1 = 0$ is the only equilibrium of Eq.(2). Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). We will show that

$$\lim_{n \to \infty} X_n = 0.$$

First we establish two useful Lemmas.

Lemma 2.1 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose that $A < 1$. Then*

$$\lim_{n \to \infty} X_n = 0.$$

Proof : Note that

$$X_1 = \frac{AX_{-1}}{1 + X_0 + X_{-1}} < AX_{-1} ,$$

$$X_3 = \frac{AX_1}{1 + X_2 + X_1} < AX_1 < A(AX_{-1}) = A^2 X_{-1} ,$$

$$X_5 = \frac{AX_3}{1 + X_4 + X_3} < AX_3 < A^2(AX_{-1}) = A^3 X_{-1} ,$$

$$\vdots$$

So we see that for all $n \geq 0$,

$$X_{2n+1} \ < \ A^{n+1} X_{-1} \ .$$

Hence

$$0 \ \leq \ \lim_{n \to \infty} X_{2n+1} \leq \lim_{n \to \infty} A^{n+1} X_{-1} \ = \ 0 \ .$$

Thus

$$\lim_{n \to \infty} X_{2n+1} = 0 \ . \tag{3}$$

Similarly we show that

$$\lim_{n \to \infty} X_{2n} = 0 \ . \tag{4}$$

Therefore the result follows via (3) and (4).

□

Lemma 2.2 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose $A = 1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : Notice that

$$X_1 = \frac{X_{-1}}{1 + X_0 + X_{-1}} < X_{-1} \ ,$$

$$X_3 = \frac{X_1}{1 + X_2 + X_1} < X_1 \ ,$$

$$\vdots$$

Therefore, there exists $L_1 \geq 0$ such that

$$\lim_{n\to\infty} X_{2n+1} = L_1.$$

In addition, observe that

$$X_2 = \frac{X_0}{1 + X_1 + X_0} < X_0 \ ,$$

$$X_4 = \frac{X_2}{1 + X_3 + X_2} < X_2 \ ,$$

$$\vdots$$

Therefore, there exists $L_2 \geq 0$ such that

$$\lim_{n\to\infty} X_{2n+2} = L_2.$$

It suffices to show that

$$L_1 = L_2 = 0.$$

Observe that via Eq.(2) we get

$$L_2 = \frac{L_2}{1 + L_1 + L_2},$$

from which we see that

$$L_1 + L_2 = 0.$$

Thus the result follows.

□

The following Theorem shows that $\bar{X}_1 = 0$ is a global attractor when $A \leq 1$.

Theorem 2.1 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose that $A \leq 1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : The proof follows from Lemma 2.1 and Lemma 2.2. □

The following examples will graphically illustrate the convergence of solutions of Eq.(2) to zero. The first example will show the convergence to zero when $A = .5 < 1$, $X_{-1} = 1$ and $X_0 = 2.7$.

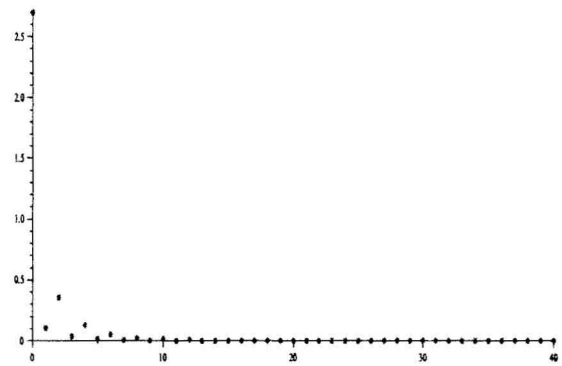

Figure 1: Convergence of solutions of Eq.(2) to zero when $A = .5 < 1$.

The next example will show the convergence to zero when $A = 1$, $X_{-1} = .7$ and $X_0 = 2$.

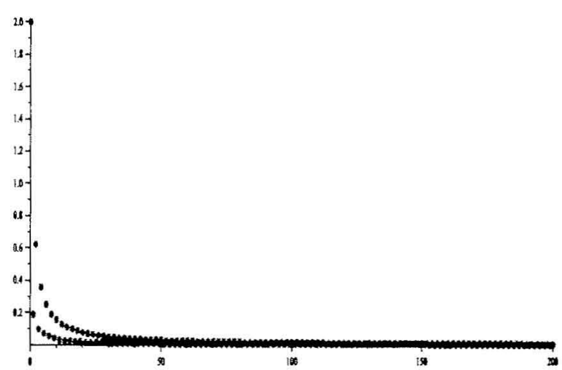

Figure 2: Convergence of solutions of Eq.(2) to zero when $A = 1$.

2.2 The case $A > 1$

In this section we will assume that $A > 1$ and prove that every positive solution of Eq.(2) is either periodic with period 2 or converges to a period 2 cycle. In addition, observe that when $A > 1$, we have a positive equilibrium point $\bar{X} = \frac{A-1}{2}$. The following Lemma will be used to prove that every positive solution of Eq.(2) converges to a period 2 cycle.

Lemma 2.3 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2) and suppose that $A > 1$. Then the following statements are true:*

(i) If $X_{-1} + X_0 < A - 1$, then

$$X_n + X_{n+1} < A - 1 \quad \text{for all} \quad n \geq -1 \ .$$

(ii) If $X_{-1} + X_0 = A - 1$, then

$$X_n + X_{n+1} = A - 1 \quad \text{for all} \quad n \geq -1 \ .$$

(iii) If $X_{-1} + X_0 > A - 1$, then

$$X_n + X_{n+1} > A - 1 \quad \text{for all} \quad n \geq -1 \ .$$

Proof : We will prove (i); (ii) and (iii) are similar and will be omitted. Suppose that

$$X_{-1} + X_0 < A - 1.$$

It suffices to show that

$$X_0 + X_1 < A - 1. \tag{5}$$

To prove (5), we want to know if

$$X_0 + X_1 = X_0 + \frac{AX_{-1}}{1 + X_0 + X_{-1}} \stackrel{?}{<} A - 1,$$

and this gives us

$$X_0 + X_0^2 + X_0X_{-1} + AX_{-1} \stackrel{?}{<} (A-1)(1 + X_0 + X_{-1}) = A + AX_0 + AX_{-1} - 1 - X_0 - X_{-1}.$$

Thus we get

$$X_0 + X_0^2 + X_0X_{-1} + X_{-1} \stackrel{?}{<} A + AX_0 - 1 - X_0.$$

Therefore we want to know if

$$X_0(1 + X_0) + X_{-1}(1 + X_0) \stackrel{?}{<} A(1 + X_0) - (1 + X_0),$$

that gives us

$$X_0 + X_{-1} \stackrel{?}{<} A - 1,$$

which is true. The result then follows by induction.

□

Theorem 2.2 *Suppose that $A > 1$. Then every positive solution of Eq.(2) is either periodic with period 2 or converges to a period 2 cycle.*

Proof : Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Via Lemma 2.3 we will consider the following three cases:

<u>**Case I:**</u> Suppose that $X_{-1} + X_0 < A - 1$, then we know that

$$X_n + X_{n+1} < A - 1 \quad \text{for all} \quad n \geq -1 \, .$$

Observe that by iterations and inequalities we get

$$X_1 = \frac{AX_{-1}}{1 + (X_0 + X_{-1})} > \frac{AX_{-1}}{1 + (A - 1)} = X_{-1} \, ,$$

$$X_3 = \frac{AX_1}{1 + (X_2 + X_1)} > \frac{AX_1}{1 + (A - 1)} = X_1 \, ,$$

$$\vdots$$

Also note that

$$X_2 = \frac{AX_0}{1 + (X_1 + X_0)} > \frac{AX_0}{1 + (A - 1)} = X_0 \, ,$$

$$X_4 = \frac{AX_2}{1 + (X_3 + X_2)} > \frac{AX_2}{1 + (A - 1)} = X_2 \, ,$$

$$\vdots$$

Thus there exist $L_1 \leq A - 1$ and $L_2 \leq A - 1$ such that

$$\lim_{n \to \infty} X_{2n+1} = L_1 \text{ and } \lim_{n \to \infty} X_{2n+2} = L_2 \, .$$

By taking the limit of Eq.(2) we see that $\{X_n\}_{n=-1}^{\infty}$ actually converges to a period 2 cycle.

<u>**Case II:**</u> Suppose that $X_{-1} + X_0 = A - 1$, then we know that

$$X_n + X_{n+1} = A - 1 \quad \text{for all} \quad n \geq -1 \, .$$

Observe that by iterations and equalities we get

$$X_1 = \frac{AX_{-1}}{1 + (X_0 + X_{-1})} = \frac{AX_{-1}}{1 + (A - 1)} = X_{-1} \ ,$$

$$X_3 = \frac{AX_1}{1 + (X_2 + X_1)} = \frac{AX_1}{1 + (A - 1)} = X_1 \ ,$$

$$\vdots$$

Also note that

$$X_2 = \frac{AX_0}{1 + (X_1 + X_0)} = \frac{AX_0}{1 + (A - 1)} = X_0 \ ,$$

$$X_4 = \frac{AX_2}{1 + (X_3 + X_2)} = \frac{AX_2}{1 + (A - 1)} = X_2 \ ,$$

$$\vdots$$

Thus there exist L_1 and L_2 such that

$$\lim_{n \to \infty} X_{2n+1} = L_1 = X_{-1} \text{ and } \lim_{n \to \infty} X_{2n+2} = L_2 = X_0 \ .$$

This is clearly a period 2 cycle of Eq.(2).

Note: In this case every solution of Eq.(2) whose initial conditions satisfy $X_0 + X_{-1} = A - 1$ is periodic with period 2.

Case III: Suppose that $X_{-1} + X_0 > A - 1$, then we know that

$$X_n + X_{n+1} > A - 1 \quad \text{for all} \quad n \geq -1 \ .$$

Observe that by iterations and inequalities we get

$$X_1 = \frac{AX_{-1}}{1 + (X_0 + X_{-1})} < \frac{AX_{-1}}{1 + (A - 1)} = X_{-1} \ ,$$

$$X_3 = \frac{AX_1}{1 + (X_2 + X_1)} < \frac{AX_1}{1 + (A - 1)} = X_1 \ ,$$

$$\vdots$$

Also note that

$$X_2 = \frac{AX_0}{1 + (X_1 + X_0)} < \frac{AX_0}{1 + (A - 1)} = X_0 \ ,$$

$$X_4 = \frac{AX_2}{1 + (X_3 + X_2)} < \frac{AX_2}{1 + (A - 1)} = X_2 \ ,$$

$$\vdots$$

Thus there exist L_1 and L_2 such that

$$\lim_{n \to \infty} X_{2n+1} = L_1 \text{ and } \lim_{n \to \infty} X_{2n+2} = L_2 \ .$$

By taking the limit of Eq.(2) we see that $\{X_n\}_{n=-1}^{\infty}$ actually converges to a period 2 cycle. Hence the result follows.

$\square$

Remark: From cases I, II, III in Theorem 2.2 based on the Stable Manifold Theorem, there exist non-trivial solutions $\{X_n\}_{n=-1}^{\infty}$ of Eq.(2) such that

$$\lim_{n \to \infty} X_{2n+1} = L_1 = \frac{A-1}{2} = L_2 = \lim_{n \to \infty} X_{2n+2}.$$

The following examples will graphically illustrate the existence of a period 2 cycle and convergence of solutions of Eq.(2) to a period 2 cycle. The first example will show the existence of solution with minimal period 2 when $A = 3$, $X_{-1} = .5$ and $X_0 = 1.5$.

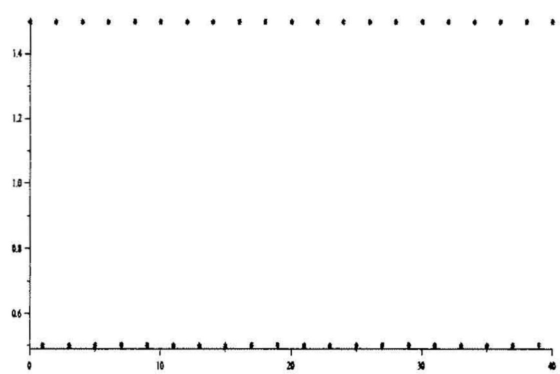

Figure 3: Existence of solutions of Eq.(2) with minimal period 2 when $A = 3$.

The next example will show the convergence to a period 2 cycle when $A = 3$, $X_{-1} = 1$ and $X_0 = 1.7$; $X_{-1} + X_0 > A - 1 = 2$.

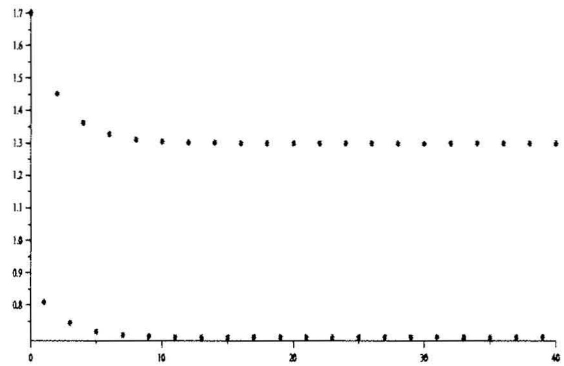

Figure 4: Convergence of solutions of Eq.(2) to a period 2 cycle when $A = 3$.

The next example will show the convergence to a period 2 cycle when $A = 3$, $X_{-1} = .4$ and $X_0 = 1$; $X_{-1} + X_0 < A - 1 = 2$.

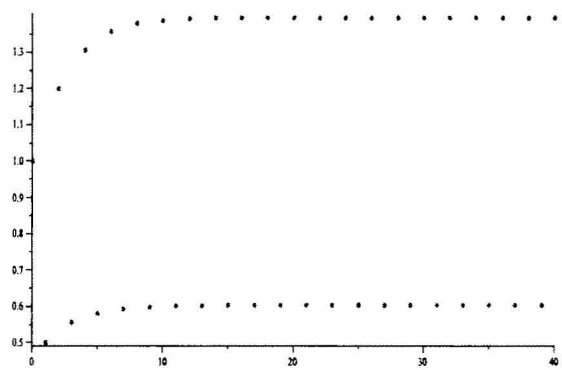

Figure 5: Convergence of solutions of Eq.(2) to a period 2 cycle when $A = 3$.

3 Global Character of Solutions of $X_{n+1} = \frac{AX_{n-(2k+1)}}{1+X_n+X_{n-(2k+1)}}$

It is our goal in this section to study the global character of the positive solution of the following difference equation:

$$X_{n+1} = \frac{AX_{n-(2k+1)}}{1 + X_n + X_{n-(2k+1)}} \quad , \quad n = 0, 1, 2, \ldots, \tag{6}$$

where $k = 0, 1, 2, \ldots$ and the initial conditions $X_{-(2k+1)}, X_{-2k}, \ldots, X_{-1}, X_0$ are positive real numbers. Observe that in this section, Eq.(6) is Eq.(2) delayed by an odd order and is a special case of Eq.(1) where $l = 2k + 1$.

In this section, we will prove the following properties of Eq.(6):

(i) If $A \leq 1$, then every solution of Eq.(6) converges to 0.

(ii) If $A > 1$, then Eq.(6) has solutions with minimal period 2.

In addition, notice that the global character of solutions of Eq.(6) is identically the same as the global character of solutions of Eq.(2).

3.1 Introduction and Preliminaries

First we will state the theorem below that will be used to analyze local and global stability character of the zero equilibrium point.

Theorem A (Clark's Theorem)

Consider the following linear difference equation:

$$y_{n+1} + p_1 y_n + p_2 y_{n-1} + \ldots + p_r y_{n-(r-1)} = 0 \quad , \quad n = 0, 1, 2, \ldots \ , \tag{7}$$

where $p_r \neq 0$ *and* $r = 2, 3, 4 \ldots$. *Also consider the characteristic polynomial*

$$\lambda^r + p_1 \lambda^{r-1} + p_2 \lambda^{r-2} + \ldots + p_{r-1}\lambda + p_r = 0. \tag{8}$$

Then the following statements are true:

(i) *The equilibrium point* $\bar{y} = 0$ *of Eq.(7) is a* <u>*sink*</u> *if* $|p_1| + |p_2| + \ldots + |p_r| < 1$.

(ii) *The equilibrium point* $\bar{y} = 0$ *of Eq.(7) is a* <u>*saddle point equilibrium*</u> *if Eq.(8) has at least one root with modulus bigger than one and all the other roots with modulus less than one.*

(iii) *The equilibrium point* $\bar{y} = 0$ *of Eq.(7) is a* <u>*repeller*</u> *if all roots of Eq.(8) have modulus bigger than one.*

3.2 Local Stability of the Zero Equilibrium Point

We determine the equilibrium points by setting

$$\bar{X} = \frac{A\bar{X}}{1+\bar{X}+\bar{X}} = \frac{A\bar{X}}{1+2\bar{X}}.$$

Clearly $\bar{X} = 0$, and $1+2\bar{X} = A$ are the two solutions of the above equilibrium equation. So the two equilibrium points of Eq.(6) are $\bar{X_1} = 0$ and $\bar{X_2} = \frac{A-1}{2}$ when $A > 1$.

The next lemma will describe the local stability character of the zero equilibrium point of Eq.(6). It will apply the results from Theorem A.

Lemma 3.4 *The following statements are true:*

(i) *The equilibrium point $\bar{X_1} = 0$ of Eq.(6) is locally asymptotically stable if $A < 1$.*

(ii) *The equilibrium point $\bar{X_1} = 0$ of Eq.(6) is an unstable repeller if $A > 1$.*

Proof : The Linearized Equation of Eq.(6) about $\bar{X_1} = 0$ is the following difference equation:

$$y_{n+1} = \frac{\partial f}{\partial x}(\bar{X_1}, \bar{X_1})y_n + \frac{\partial f}{\partial y}(\bar{X_1}, \bar{X_1})y_{n-(2k+1)} \quad , \quad n = 0, 1, 2, \dots .$$

Now let

$$f(x,y) = \frac{Ay}{1+x+y} = Ay(1+x+y)^{-1}.$$

The partial derivatives of $f(x,y)$ with respect to x and y are:

$$f_x(x,y) = -Ay(1+x+y)^{-2} = \frac{-Ay}{(1+x+y)^2} < 0 \ ,$$

$$f_y(x,y) = \frac{A(1+x+y) - Ay}{(1+x+y)^2} = \frac{A+Ax}{(1+x+y)^2} > 0 \ .$$

Also we see that

$$f_x(\bar{X_1}, \bar{X_1}) = \frac{-A(0)}{(1+0+0)^2} = 0 \ ,$$

$$f_y(\bar{X_1}, \bar{X_1}) = \frac{A+A(0)}{(1+0+0)^2} = A \ .$$

So the Linearized Equation of Eq.(6) about $\bar{X}_1 = 0$ is:

$$y_{n+1} - Ay_{n-(2k+1)} = 0 \text{ for } \ n = 0, 1, 2, \dots \ .$$

Therefore it follows via Theorem A that the following statements are true:

(i) The equilibrium point $\bar{X} = 0$ is locally asymptotically stable if $A < 1$.

(ii) The equilibrium point $\bar{X} = 0$ is an unstable repeller if $A > 1$.

□

3.3 Attracting Intervals of Eq.(6)

In this section, we establish the fact that every positive solution of Eq.(6) is eventually attracted to the interval $(0, A)$.

Theorem 3.3 *Let* $\{X_n\}_{n=-2k-1}^{\infty}$ *be a positive solution of Eq.(6). Then*

$$X_n < A \ \textit{for all} \ \ n \geq 1.$$

Proof : Observe that by computation and inequalities we get

$$X_1 = \frac{AX_{-2k-1}}{1 + X_0 + X_{-2k-1}} < \frac{AX_{-2k-1}}{X_{-2k-1}} = A \ ,$$

$$X_2 = \frac{AX_{-2k}}{1 + X_1 + X_{-2k}} < \frac{AX_{-2k}}{X_{-2k}} = A \ ,$$

$$X_3 = \frac{AX_{-2k+1}}{1 + X_2 + X_{-2k+1}} < \frac{AX_{-2k+1}}{X_{-2k+1}} = A \ .$$

$$\vdots$$

Hence the result follows by induction.

□

3.4 Global Stability of the Zero Equilibrium Point

It is our goal in this section to show that every positive solution of Eq.(6) converges to zero when $A \leq 1$. First we establish two useful Lemmas.

Lemma 3.5 *Let $\{X_n\}_{n=-2k-1}^{\infty}$ be a positive solution of Eq.(6). Suppose that $A < 1$. Then*

$$\lim_{n \to \infty} X_n = 0.$$

Proof : As in Lemma 2.1 by iteration and inequalities we get

$$X_1 = \frac{AX_{-2k-1}}{1 + X_0 + X_{-2k-1}} < AX_{-2k-1} \ ,$$

$$X_{1+(2k+2)} = \frac{AX_1}{1 + X_{2k+2} + X_1} < AX_1 < A(AX_{-2k-1}) = A^2 X_{-2k-1} \ ,$$

$$X_{1+(2k+2)2} = \frac{AX_{1+(2k+2)}}{1 + X_{(2k+2)2} + X_{1+(2k+2)}} < AX_{1+(2k+2)} < A(A^2 X_{-2k-1}) = A^3 X_{-2k-1} \ ,$$

$$\vdots$$

It follows by induction that for all $n \geq 0$,

$$X_{1+(2k+2)n} \ < \ A^{n+1} X_{-2k-1} \ .$$

Hence we see that

$$\lim_{n \to \infty} X_{1+(2k+2)n} \ = \ 0 \ .$$

We continue this process, and as in Lemma 2.1 we get

$$\lim_{n \to \infty} X_n = 0 \ .$$

□

Lemma 3.6 *Let $\{X_n\}_{n=-2k-1}^{\infty}$ be a positive solution of Eq.(6). Suppose that $A = 1$. Then*

$$\lim_{n \to \infty} X_n = 0.$$

Proof : Since $A = 1$, then we get

$$X_{n+1} = \frac{AX_{n-(2k+1)}}{1 + X_n + X_{n-(2k+1)}} = \frac{X_{n-(2k+1)}}{1 + X_n + X_{n-(2k+1)}} \quad , \quad n = 0, 1, 2, \ldots$$

Via Theorem 3.3 we get

$$X_n < 1 \quad \text{for all} \quad n \geq 1.$$

By computation and inequalities we get:

$$X_1 = \frac{X_{-2k-1}}{1 + X_0 + X_{-2k-1}} < X_{-2k-1} \ ,$$

$$X_2 = \frac{X_{-2k}}{1 + X_1 + X_{-2k}} < X_{-2k} \ ,$$

$$X_3 = \frac{X_{-2k+1}}{1 + X_2 + X_{-2k+1}} < X_{-2k+1} \ ,$$

$$\vdots$$

There exist $L_1, L_2, \ldots, L_{2k+2} \geq 0$ such that

$$\lim_{n\to\infty} X_{(2k+2)n+1} = L_1 \ ,$$

$$\lim_{n\to\infty} X_{(2k+2)n+2} = L_2 \ ,$$

$$\lim_{n\to\infty} X_{(2k+2)n+3} = L_3 \ ,$$

$$\lim_{n\to\infty} X_{(2k+2)n+4} = L_4 \ ,$$

$$\vdots$$

$$\lim_{n\to\infty} X_{(2k+2)n+(2k+2)} = L_{2k+2} \ .$$

Via Theorem 3.3, we see that

$$L_1, L_2, L_3, \ldots, L_{2k+2} \leq 1.$$

It suffices to show that

$$L_1 = L_2 = L_3 = \ldots = L_{2k+2} = 0. \tag{9}$$

Observe that via Eq.(6), it follows that

$$L_{2k+2} = \frac{L_{2k+2}}{1 + L_{2k+1} + L_{2k+2}},$$

which gives us

$$L_{2k+1} + L_{2k+2} = 0.$$

Thus it follows that

$$L_{2k+1} = L_{2k+2} = 0.$$

Similarly we see that

$$L_{2k} = L_{2k+1} = 0.$$

We continue this process from which (9) follows.

□

The following Theorem shows that $\bar{X}_1 = 0$ is a global attractor when $A \leq 1$.

Theorem 3.4 *Let* $\{X_n\}_{n=-2k-1}^{\infty}$ *be a positive solution of Eq.(6). Suppose that* $A \leq 1$. *Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : The proof follows from Lemma 3.5 and Lemma 3.6.

□

The following examples will graphically illustrate the convergence of the solutions of Eq.(6) to zero. The first example will show the convergence to zero when $k = 3$, $A = .5 < 1$, $X_{-3} = 4$, $X_{-2} = 3$, $X_{-1} = 2$ and $X_0 = 1$.

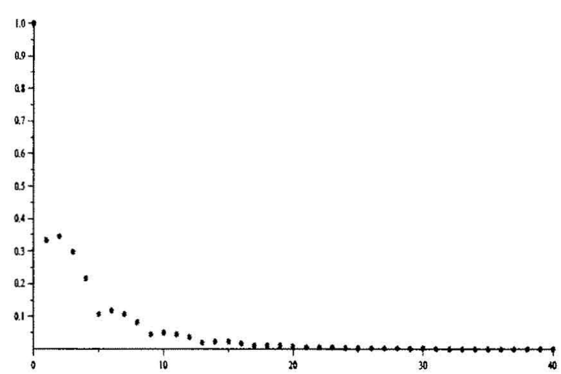

Figure 6: Convergence of solutions of Eq.(6) to zero when $k = 3$ and $A = .5 < 1$.

The next example will show the convergence to zero when $k = 5, A = 1, X_{-5} = 2, X_{-4} = 1.8, X_{-3} = 1.7, X_{-2} = 1.5, X_{-1} = 1.3$, and $X_0 = 1$.

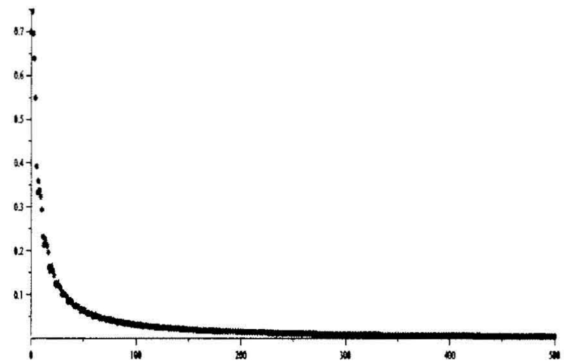

Figure 7: Convergence of solutions of Eq.(6) to zero when $k = 5$ and $A = 1$.

3.5 Existence of Solutions with Minimal Period 2

In this section we will investigate the periodic character of the solutions of Eq.(6).

The following Lemma gives necessary and sufficient conditions for the existence of solutions with minimal period 2.

Lemma 3.7 *Eq.(6) has positive solutions with minimal period 2 if and only if $A > 1$.*

Proof : Via Theorem 3.4 we showed that when $A \leq 1$ then

$$\lim_{n\to\infty} X_n = 0.$$

Therefore, it suffices to consider the case where $A > 1$. Now we will suppose that $X_{-1} \neq X_0$ and set

$$X_{-2k-1} = X_{-2k+1} = \ldots = X_{-3} = X_{-1}$$

and

$$X_{-2k} = X_{-2k+2} = \ldots = X_{-2} = X_0.$$

So we see that:

$$X_1 = \frac{AX_{-(2k+1)}}{1 + X_0 + X_{-(2k+1)}} = \frac{AX_{-1}}{1 + X_0 + X_{-1}} = X_{-1}\ , \text{ and}$$

$$X_2 = \frac{AX_{-2k}}{1 + X_1 + X_{-2k}} = \frac{AX_0}{1 + X_{-1} + X_0} = X_0\ .$$

By solving for A we get:

$$1 + X_{-1} + X_0 \ = \ A \ = \ 1 + X_0 + X_{-1}\ .$$

Therefore,

$$X_0 = (A-1) - X_{-1} \geq 0,$$

and

$$X_{-1} = (A-1) - X_0 \geq 0.$$

Hence

$$X_{-1} \leq A - 1 \text{ and } X_0 \leq A - 1.$$

Therefore Eq.(6) has positive solutions with minimal period 2 if and only if

$$A > 1.$$

In addition, the solutions with minimal period 2 appear in the following pattern:

$$X_{-1},\ X_0 = (A-1) - X_{-1},\ X_{-1},\ X_0 = (A-1) - X_{-1},\ X_{-1} \ldots\ .$$

Furthermore, notice that the following conditions hold:

$$0 < X_{-2k-1} = X_{-2k+1} = \ldots = X_{-3} = X_{-1} < A - 1$$

and

$$0 < X_{-2k} = X_{-2k+2} = \ldots = X_{-2} = X_0 = (A - 1) - X_{-1} < A - 1.$$

□

Extensive numerical computations suggest the following conjecture:

Conjecture 1 *Every positive solution of Eq. (6) converges to a period 2 cycle when* $A > 1$.

The following examples will graphically illustrate the existence of a period 2 cycle and convergence of solutions of Eq.(6) to a period 2 cycle. The first example will show the existence of a solution with minimal period 2 when $k = 3$, $A = 3$, $X_{-3} = .5$, $X_{-2} = 1.5$, $X_{-1} = .5$ and $X_0 = 1.5$; $X_{-1} + X_0 = A - 1 = 2$.

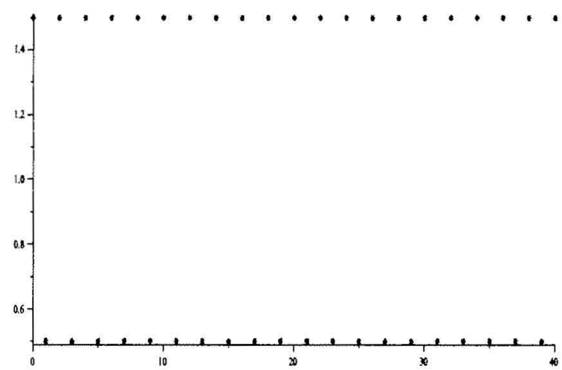

Figure 8: Existence of solutions of Eq.(6) with minimal period 2 when $k = 3$ and $A = 3$.

The next example will show the convergence to a period 2 cycle when $k = 3$, $A = 3$, $X_{-3} = 1$, $X_{-2} = 1.7$, $X_{-1} = 1$ and $X_0 = 1.7$; $X_{-1} + X_0 > A - 1 = 2$.

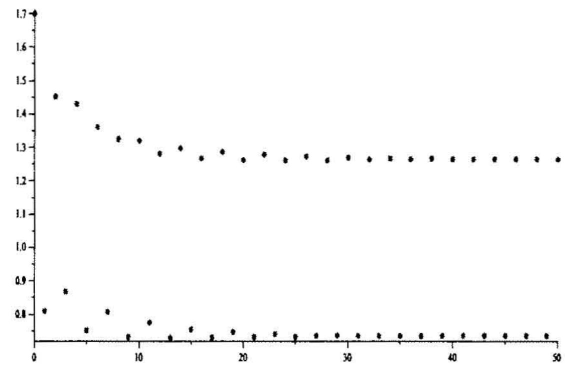

Figure 9: Convergence of solutions of Eq.(6) to a period 2 cycle when $k = 3$ and $A = 3$.

The next example will show the convergence to a period 2 cycle when $k = 3$, $A = 3$, $X_{-3} = .4$, $X_{-2} = 1$, $X_{-1} = .4$ and $X_0 = 1$; $X_{-1} + X_0 < A - 1 = 2$.

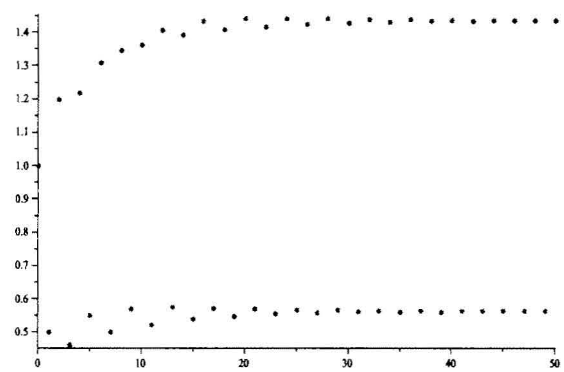

Figure 10: Convergence of solutions of Eq.(6) to a period 2 cycle when $k = 3$ and $A = 3$.

4 Global Character of Solutions of $X_{n+1} = \frac{AX_{n-2k}}{1+X_n+X_{n-2k}}$

It is our goal in this section to study the global character of the positive solutions of the following difference equation:

$$X_{n+1} = \frac{AX_{n-2k}}{1 + X_n + X_{n-2k}} \quad , \quad n = 0, 1, 2, \ldots, \tag{10}$$

where $k = 1, 2, 3, \ldots$ and the initial conditions $X_{-2k}, X_{-(2k-1)}, \ldots, X_{-1}, X_0$ are positive real numbers. Observe that in this section, Eq.(10) is Eq.(2) delayed by an even order, and is a special case of Eq.(1) where $l = 2k$.

In this section we will prove the following properties of Eq.(10):

(i) If $A \leq 1$, then every solution of Eq.(10) converges to 0.

(ii) If $A > 1$, then Eq.(10) has no solutions with minimal period 2.

4.1 Local Stability of the Zero Equilibrium Point

We determine the equilibrium points by setting

$$\bar{X} = \frac{A\bar{X}}{1 + \bar{X} + \bar{X}} = \frac{A\bar{X}}{1 + 2\bar{X}}.$$

Clearly $\bar{X} = 0$, and $1 + 2\bar{X} = A$ are the two solutions of the above equilibrium equation. So the two equilibrium points of Eq.(10) are $\bar{X}_1 = 0$ and $\bar{X}_2 = \frac{A-1}{2}$ when $A > 1$.

The next lemma will describe the stability character of the zero equilibrium point of Eq.(10).

Lemma 4.8 *The following statements are true:*

(i) The equilibrium point $\bar{X}_1 = 0$ of Eq.(10) is locally asymptotically stable if $A < 1$.

(ii) The equilibrium point $\bar{X}_1 = 0$ of Eq.(10) is an unstable repeller if $A > 1$.

Proof : The Linearized Equation of Eq.(10) about $\bar{X}_1 = 0$ is the following difference equation:

$$y_{n+1} = \frac{\partial f}{\partial x}(\bar{X}_1, \bar{X}_1)y_n + \frac{\partial f}{\partial y}(\bar{X}_1, \bar{X}_1)y_{n-2k} \quad , \quad n = 0, 1, 2, \dots .$$

Now let

$$f(x, y) = \frac{Ay}{1 + x + y} = Ay(1 + x + y)^{-1}.$$

The partial derivatives of $f(x, y)$ with respect to x and y are:

$$f_x(x, y) = -Ay(1 + x + y)^{-2} = \frac{-Ay}{(1 + x + y)^2} < 0 \ ,$$

$$f_y(x, y) = \frac{A(1 + x + y) - Ay}{(1 + x + y)^2} = \frac{A + Ax}{(1 + x + y)^2} > 0 \ .$$

Also we see that

$$f_x(\bar{X}_1, \bar{X}_1) = \frac{-A(0)}{(1 + 0 + 0)^2} = 0 \ ,$$

$$f_y(\bar{X}_1, \bar{X}_1) = \frac{A + A(0)}{(1 + 0 + 0)^2} = A \ .$$

So the Linearized Equation of Eq.(10) about $\bar{X}_1 = 0$ is:

$$y_{n+1} - Ay_{n-2k} = 0 \text{ for } n = 0, 1, 2, \dots .$$

Therefore it follows via Theorem A that the following statements are true:

(i) The equilibrium point $\bar{X} = 0$ is <u>locally asymptotically stable</u> if $A < 1$.

(ii) The equilibrium point $\bar{X} = 0$ is an <u>unstable repeller</u> if $A > 1$.

□

4.2 Attracting Intervals of Eq.(10)

In this section, we establish the fact that every positive solution of Eq.(10) is eventually attracted to the interval $(0, A)$.

Theorem 4.5 *Let* $\{X_n\}_{n=-2k}^{\infty}$ *be a positive solution of Eq.(10). Then*

$$X_n < A \text{ for all } n \geq 1.$$

Proof : Observe that by computation and inequalities we get

$$X_1 = \frac{AX_{-2k}}{1 + X_0 + X_{-2k}} < \frac{AX_{-2k}}{X_{-2k}} = A\ ,$$

$$X_2 = \frac{AX_{-2k+1}}{1 + X_1 + X_{-2k+1}} < \frac{AX_{-2k+1}}{X_{-2k+1}} = A\ ,$$

$$X_3 = \frac{AX_{-2k+2}}{1 + X_2 + X_{-2k+2}} < \frac{AX_{-2k+2}}{X_{-2k+2}} = A\ ,$$

$$\vdots$$

Hence the result follows by induction.

□

4.3 Global Stability of the Zero Equilibrium Point

It is our goal in this section to show that every positive solution of Eq.(10) converges to zero when $A \leq 1$. First we establish two useful Lemmas.

Lemma 4.9 *Let $\{X_n\}_{n=-2k}^{\infty}$ be a positive solution of Eq.(10). Suppose that $A < 1$. Then*

$$\lim_{n \to \infty} X_n = 0.$$

Proof : As in Lemma 2.1 we iterate and get

$$X_1 = \frac{AX_{-2k}}{1 + X_0 + X_{-2k}} < AX_{-2k} \ ,$$

$$X_{1+(2k+1)} = \frac{AX_1}{1 + X_{2k+1} + X_1} < AX_1 < A(AX_{-2k}) = A^2 X_{-2k} \ ,$$

$$X_{1+(2k+1)2} = \frac{AX_{1+(2k+1)}}{1 + X_{(2k+1)2} + X_{1+(2k+1)}} < AX_{1+(2k+1)} < A(A^2 X_{-2k}) = A^3 X_{-2k} \ ,$$

$$\vdots$$

It follows by induction that for all $n \geq 0$,

$$X_{1+(2k+1)n} \ < \ A^{n+1} X_{-2k} \ .$$

Hence we see that

$$\lim_{n \to \infty} X_{1+(2k+1)n} \ = \ 0 \ .$$

Similarly as in Lemma 2.1, we get

$$\lim_{n \to \infty} X_n = 0 \ .$$

□

Lemma 4.10 *Let $\{X_n\}_{n=-2k}^{\infty}$ be a positive solution of Eq.(10). Suppose that $A = 1$. Then*

$$\lim_{n \to \infty} X_n = 0.$$

Proof : Since $A = 1$, then we get

$$X_{n+1} = \frac{AX_{n-2k}}{1 + X_n + X_{n-2k}} = \frac{X_{n-2k}}{1 + X_n + X_{n-2k}} \quad , \quad n = 0, 1, 2, \ldots$$

Via Theorem 4.5 we get

$$X_n < 1 \quad \text{for all} \quad n \geq 1.$$

By computation and inequalities we get:

$$X_1 = \frac{X_{-2k}}{1 + X_0 + X_{-2k}} < X_{-2k} ,$$

$$X_2 = \frac{X_{-2k+1}}{1 + X_1 + X_{-2k+1}} < X_{-2k+1} ,$$

$$X_3 = \frac{X_{-2k+2}}{1 + X_2 + X_{-2k+2}} < X_{-2k+2} ,$$

$$\vdots$$

There exist $L_1, L_2, \ldots, L_{2k+1} \geq 0$ such that

$$\lim_{n \to \infty} X_{(2k+1)n+1} = L_1 ,$$

$$\lim_{n \to \infty} X_{(2k+1)n+2} = L_2 ,$$

$$\lim_{n \to \infty} X_{(2k+1)n+3} = L_3 ,$$

$$\lim_{n \to \infty} X_{(2k+1)n+4} = L_4 ,$$

$$\vdots$$

$$\lim_{n \to \infty} X_{(2k+1)n+(2k+1)} = L_{2k+1} .$$

Via Theorem 4.5, we see that

$$L_1, L_2, L_3, \ldots, L_{2k+1} \leq 1.$$

It suffices to show that

$$L_1 = L_2 = L_3 = \ldots = L_{2k+1} = 0. \tag{11}$$

Observe that via Eq.(10), it follows that

$$L_{2k+1} = \frac{L_{2k+1}}{1 + L_{2k} + L_{2k+1}},$$

which gives us

$$L_{2k} + L_{2k+1} = 0.$$

Thus it follows that

$$L_{2k} = L_{2k+1} = 0.$$

Similarly we see that

$$L_{2k-1} = L_{2k} = 0.$$

We continue this process from which (11) follows.

□

The following Theorem shows that $\bar{X}_1 = 0$ is a global attractor when $A \leq 1$.

Theorem 4.6 *Let $\{X_n\}_{n=-2k}^{\infty}$ be a positive solution of Eq.(10). Suppose that $A \leq 1$. Then*

$$\lim_{n \to \infty} X_n = 0.$$

Proof : The proof follows from Lemma 4.9 and Lemma 4.10 and will be omitted.

□

The following examples will graphically illustrate the convergence of the solutions of Eq.(10) to zero. The first example will show the convergence to zero when $k = 4$, $A = .5 < 1$, $X_{-4} = 1.5$, $X_{-3} = .9$, $X_{-2} = .7$, $X_{-1} = .5$ and $X_0 = .3$.

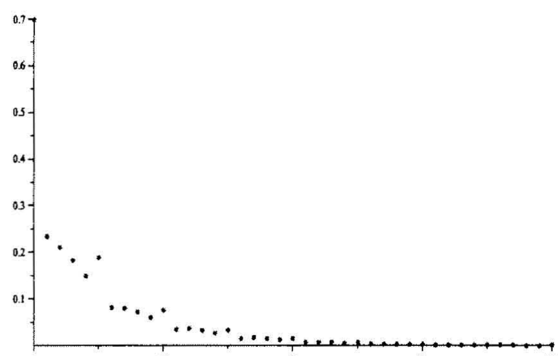

Figure 11: Convergence of solutions of Eq.(10) to zero when $k = 4$ and $A = .5 < 1$.

The next example will show the convergence to zero when $k = 6$, $A = 1$, $X_{-6} = 5$, $X_{-5} = 4$, $X_{-4} = 3$, $X_{-3} = 2$, $X_{-2} = 1.7$, $X_{-1} = 1$ and $X_0 = .7$.

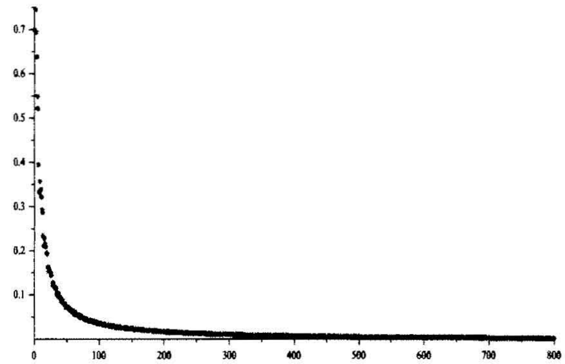

Figure 12: Convergence of solutions of Eq.(10) to zero when $k = 6$ and $A = 1$.

4.4 No Solutions with Minimal Period 2

The following Lemma shows that Eq.(10) has no solutions with minimal period 2.

Lemma 4.11 *Eq.(10) has no solutions with minimal period 2.*

Proof : Let $\{X_n\}_{n=-2k}^{\infty}$ be a positive solution of Eq.(10). Notice that when $A \leq 1$, then we know that

$$\lim_{n\to\infty} X_n = 0.$$

Therefore, it suffices to consider the case where $A > 1$. For the sake of contradiction, we will assume that Eq.(10) has a solution with minimal period 2. Suppose that $X_{-1} \neq X_0$. Then it follows that

$$X_0 = X_{-2} = X_{-4} = \ldots = X_{-2k}$$

and

$$X_{-1} = X_{-3} = X_{-5} = \ldots = X_{-2k+1}.$$

Then by iteration we get the following two relations:

$$X_1 = \frac{AX_{-2k}}{1 + X_0 + X_{-2k}} = \frac{AX_0}{1 + X_0 + X_0} = \frac{AX_0}{1 + 2X_0} = X_{-1}\ , \text{ and}$$

$$X_2 = \frac{AX_{-2k+1}}{1 + X_1 + X_{-2k+1}} = \frac{AX_{-1}}{1 + X_{-1} + X_{-1}} = \frac{AX_{-1}}{1 + 2X_{-1}} = X_0\ .$$

Then we get

$$AX_0 = X_{-1} + 2X_{-1}X_0$$

and

$$AX_{-1} = X_0 + 2X_{-1}X_0.$$

Hence we see that

$$2X_{-1}X_0 = AX_0 - X_{-1} = AX_{-1} - X_0,$$

from which it follows that

$$AX_0 - AX_{-1} = X_{-1} - X_0. \tag{12}$$

Therefore, via (12) we get $A = -1$, which is a contradiction.

□

Extensive numerical computations suggest the following conjecture:

Conjecture 2 *Every positive solution of Eq. (10) converges to the positive equilibrium point $\frac{A-1}{2}$ when $A > 1$.*

The following examples will graphically illustrate the convergence of the solutions of Eq.(10) to $\frac{A-1}{2}$. The first example will show the convergence to $\frac{A-1}{2}$ when $k = 4$, $A = 3$, $X_{-4} = .9$ $X_{-3} = .7$, $X_{-2} = .5$, $X_{-1} = .3$ and $X_0 = .1$.

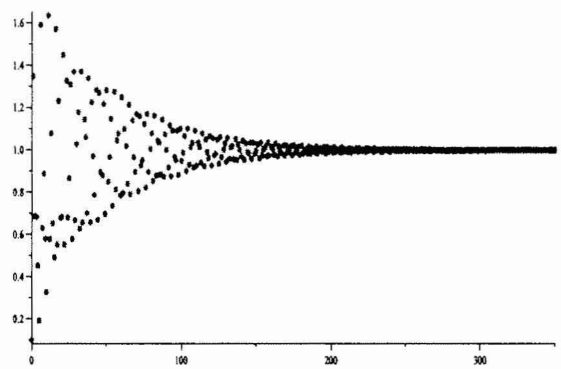

Figure 13: Convergence of solutions of Eq.(10) to $\frac{A-1}{2} = 1$ when $k = 4$ and $A = 3$.

The next example will show the convergence to $\frac{A-1}{2}$ when $k = 6$, $A = 3$, $X_{-6} = 1.5$, $X_{-5} = 1.3$, $X_{-4} = .9$, $X_{-3} = .7$, $X_{-2} = .5$, $X_{-1} = .3$ and $X_0 = .1$.

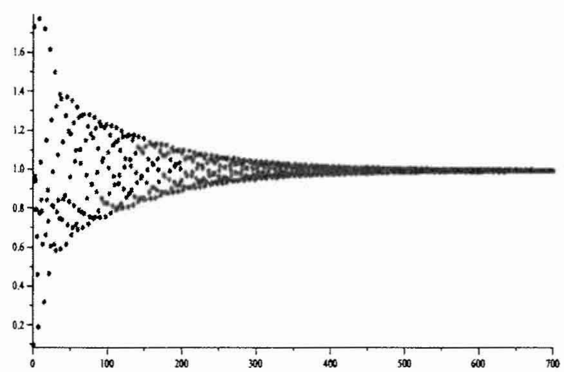

Figure 14: Convergence of solutions of Eq.(10) to $\frac{A-1}{2} = 1$ when $k = 6$ and $A = 3$.

5 Global Character of Solutions of $X_{n+1} = \frac{AX_{n-1}}{1+(2-\alpha)X_n+\alpha X_{n-1}}$

It is our goal in this section to study the global character of the solutions of the following difference equation:

$$X_{n+1} = \frac{AX_{n-1}}{1+(2-\alpha)X_n+\alpha X_{n-1}} \quad , \quad n = 0, 1, 2, \ldots, \tag{13}$$

where $A > 0$ and $\alpha \in [0, 2]$.

5.1 Introduction and Preliminaries

The following properties of Eq.(13) were proved in [3] and [6]:

(i) If $\alpha = 0$, then the following statements are true:

- Every positive solution of Eq.(13) converges to 0 if $A < 1$.
- Every positive solution of Eq.(13) converges to a period 2 cycle if $A = 1$.
- Eq.(13) has unbounded solutions if $A > 1$.

The following examples will graphically illustrate the convergence of the solutions of Eq.(13) to zero if $A < 1$, the convergence of the solutions of Eq.(13) to a period 2 cycle if $A = 1$, and the existence of unbounded solutions of Eq.(13) if $A > 1$. The first example will show the convergence to zero when $A = .5 < 1$, $\alpha = 0$, $X_{-1} = 1$ and $X_0 = .7$.

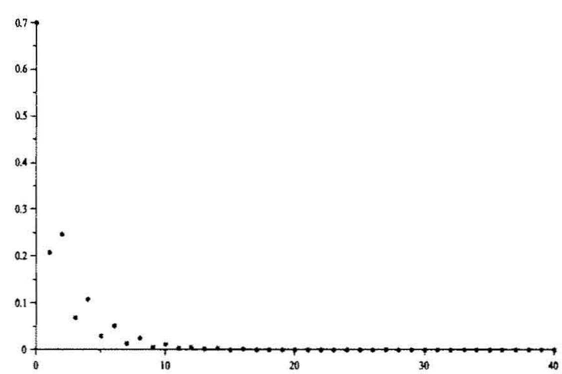

Figure 15: Convergence of solutions of Eq.(13) to zero when $\alpha = 0$ and $A = .5 < 1$.

The next example will show the existence of a period 2 cycle when $A = 1$, $\alpha = 0$, $X_{-1} = 1$ and $X_0 = 0$.

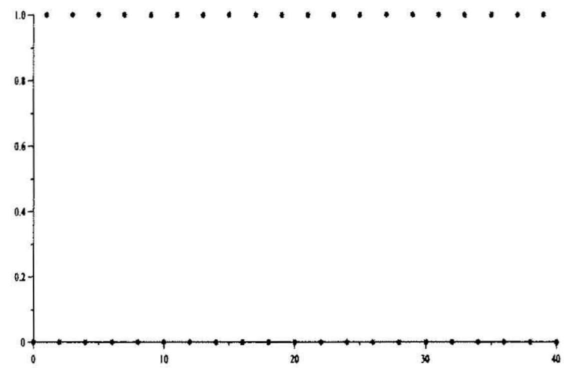

Figure 16: Existence of a period 2 solutions of Eq.(13) when $\alpha = 0$ and $A = 1$.

The next example will show the convergence to a period 2 cycle when $A = 1$, $\alpha = 0$, $X_{-1} = 1$ and $X_0 = 1.5$.

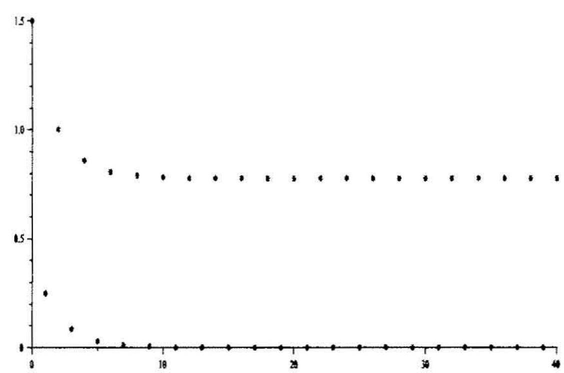

Figure 17: Convergence of solutions of Eq.(13) to a period 2 cycle when $\alpha = 0$ and $A = 1$.

The next example will show the existence of unbounded solutions when $A = 1.3$, $\alpha = 0$, $X_{-1} = 1$ and $X_0 = 2.7$.

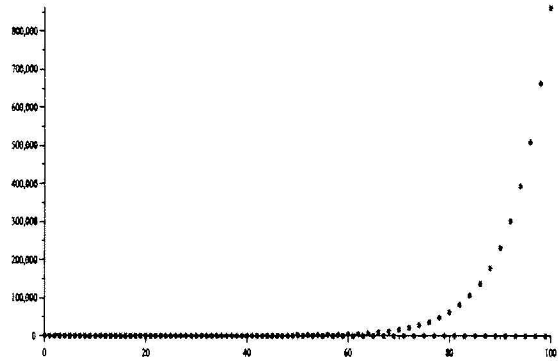

Figure 18: Existence of unbounded solutions of Eq.(13) when $\alpha = 0$ and $A = 1.3$.

(ii) If $\alpha = 2$, then the following statements are true:

- Every positive solution of Eq.(13) converges to 0 if $A \leq 1$.
- Every positive solution of Eq.(13) converges to $\frac{A-1}{2}$ if $A > 1$.

The following examples will graphically illustrate the convergence of the solutions of Eq.(13) to zero if $A \leq 1$, the convergence of the solutions of Eq.(13) to $\frac{A-1}{2}$ if $A > 1$. The first example will show the convergence to zero when $A = .5 < 1$, $\alpha = 2$, $X_{-1} = 1$ and $X_0 = .7$.

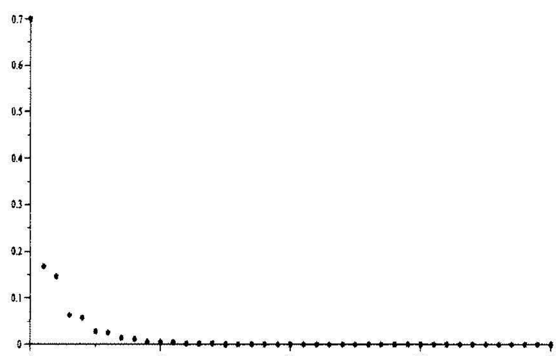

Figure 19: Convergence of solutions of Eq.(13) to zero when $\alpha = 2$ and $A = .5 < 1$.

The next example will show the convergence to zero when $A = 1$, $\alpha = 2$, $X_{-1} = 1$ and $X_0 = .7$.

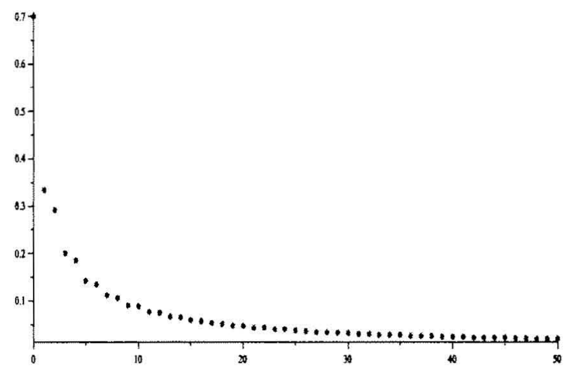

Figure 20: Convergence of solutions of Eq.(13) to zero when $\alpha = 2$ and $A = 1$.

The next example will show the convergence to $\frac{A-1}{2}$ when $A = 3$, $\alpha = 2$, $X_{-1} = .3$ and $X_0 = .7$.

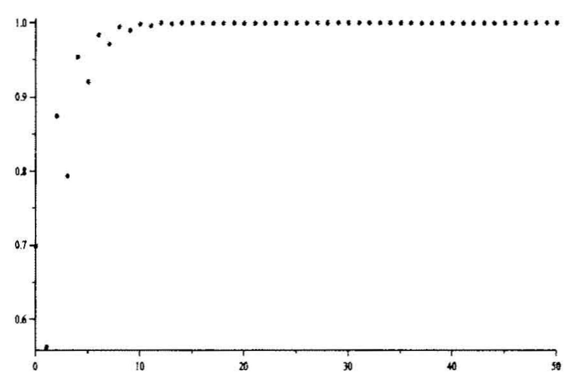

Figure 21: Convergence of solutions of Eq.(13) to $\frac{A-1}{2} = 1$ when $\alpha = 2$ and $A = 3$.

(iii) If $\alpha = 1$, then the results were proved in section 2.

Therefore, we will need to consider the cases where

$$0 < \alpha < 1 \qquad \text{and} \qquad 1 < \alpha < 2.$$

We determine the equilibrium points by setting

$$\bar{X} = \frac{A\bar{X}}{1+(2-\alpha)\bar{X}+\alpha\bar{X}} = \frac{A\bar{X}}{1+2\bar{X}}.$$

Clearly $\bar{X}=0$, and $1+2\bar{X}=A$ are the two solutions of the above equilibrium equation. So the two equilibrium points of Eq.(13) are $\bar{X}_1=0$ and $\bar{X}_2=\frac{A-1}{2}$ when $A>1$.

5.2 The case $A \le 1$

In this section we will assume that $A \le 1$. Then $\bar{X}_1 = 0$ is the only equilibrium of Eq.(13). Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(13). We will show that

$$\lim_{n\to\infty} X_n = 0.$$

First we establish two useful Lemmas.

Lemma 5.12 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(13). Suppose that $A<1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : Note that

$$X_1 = \frac{AX_{-1}}{1+(2-\alpha)X_0+\alpha X_{-1}} < AX_{-1}\ ,$$

$$X_3 = \frac{AX_1}{1+(2-\alpha)X_2+\alpha X_1} < AX_1 < A(AX_{-1}) = A^2X_{-1}\ ,$$

$$X_5 = \frac{AX_3}{1+(2-\alpha)X_4+\alpha X_3} < AX_3 < A^2(AX_{-1}) = A^3X_{-1}\ ,$$

$$\vdots$$

So we see that for all $n \ge 0$,

$$X_{2n+1} \ < \ A^{n+1}X_{2n-1}\ .$$

Hence

$$0 \ \le \ \lim_{n\to\infty} X_{2n+1} \le \lim_{n\to\infty} A^{n+1}X_{-1} \ = \ 0\ .$$

Thus

$$\lim_{n\to\infty} X_{2n+1} = 0\ . \tag{14}$$

Similarly we show that

$$\lim_{n\to\infty} X_{2n} = 0\ . \tag{15}$$

Therefore the result follows via (14) and (15). □

Lemma 5.13 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(13). Suppose $A = 1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : Notice that

$$X_1 = \frac{X_{-1}}{1 + (2-\alpha)X_0 + \alpha X_{-1}} < X_{-1} \ ,$$

$$X_3 = \frac{X_1}{1 + (2-\alpha)X_2 + \alpha X_1} < X_1 \ ,$$

$$\vdots$$

Therefore, there exists $L_1 \geq 0$ such that

$$\lim_{n\to\infty} X_{2n+1} = L_1.$$

In addition, observe that

$$X_2 = \frac{X_0}{1 + (2-\alpha)X_1 + \alpha X_0} < X_0 \ ,$$

$$X_4 = \frac{X_2}{1 + (2-\alpha)X_3 + \alpha X_2} < X_2 \ ,$$

$$\vdots$$

Therefore, there exists $L_2 \geq 0$ such that

$$\lim_{n\to\infty} X_{2n+2} = L_2.$$

It suffices to show that

$$L_1 = L_2 = 0. \tag{16}$$

Observe that via Eq.(13) we get

$$L_2 = \frac{L_2}{1 + (2-\alpha)L_1 + \alpha L_2},$$

from which we see that

$$(2-\alpha)L_1 + \alpha L_2 = 0.$$

Thus (16) follows as $0 < \alpha < 2$ and $\alpha \neq 1$.

□

The following Theorem shows that $\bar{X}_1 = 0$ is a global attractor when $A \leq 1$.

Theorem 5.7 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(13). Suppose that $A \leq 1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : The proof follows from Lemma 5.12 and Lemma 5.13. □

The following examples will graphically illustrate the convergence of the solutions of Eq.(13) to zero. The first example will show the convergence to zero when $A = .5 < 1$, $\alpha = .5$, $X_{-1} = 1$ and $X_0 = 2.7$.

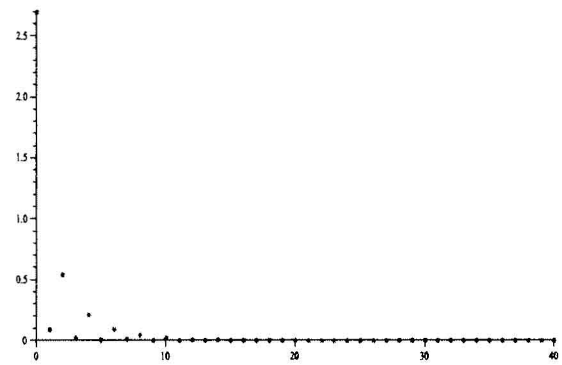

Figure 22: Convergence of solutions of Eq.(13) to zero when $\alpha = .5$ and $A = .5 < 1$.

The next example will show the convergence to zero when $A = .5 < 1$, $\alpha = 1.5$, $X_{-1} = 1$ and $X_0 = 2.7$.

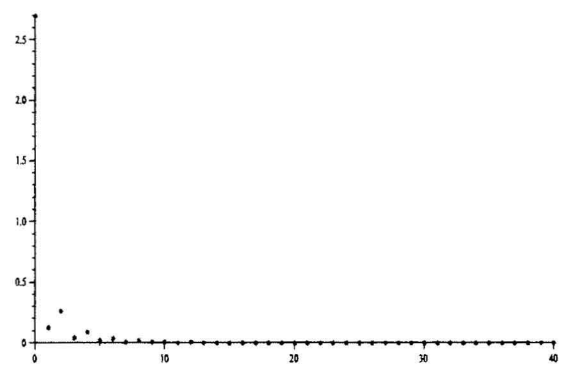

Figure 23: Convergence of solutions of Eq.(13) to zero when $\alpha = 1.5$ and $A = .5 < 1$.

The next example will show the convergence to zero when $A = 1$, $\alpha = .5$, $X_{-1} = .7$ and $X_0 = 2$.

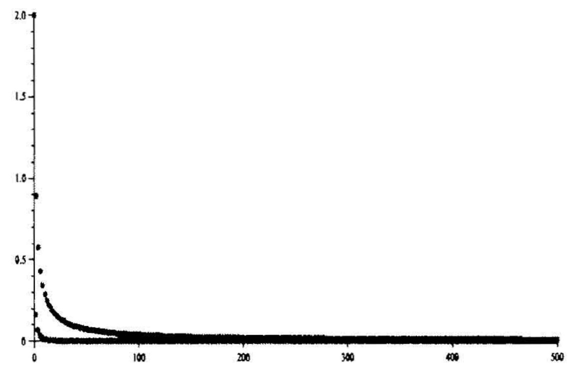

Figure 24: Convergence of solutions of Eq.(13) to zero when $\alpha = .5$ and $A = 1$.

The next example will show the convergence to zero when $A = 1$, $\alpha = 1.5$, $X_{-1} = .7$ and $X_0 = 2$.

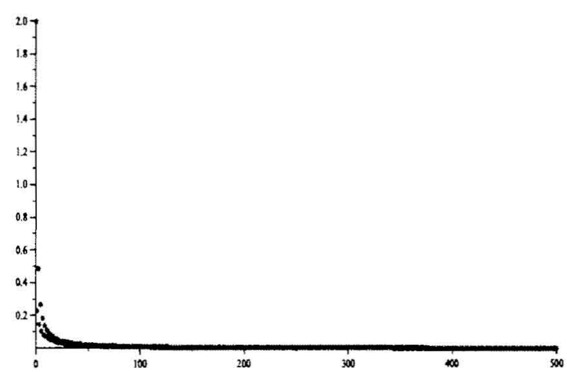

Figure 25: Convergence of solutions of Eq.(13) to zero when $\alpha = 1.5$ and $A = 1$.

5.3 The case $A > 1$

In this section we will assume that $A > 1$ and we will analyze the global behavior of the solutions of Eq.(13). In section 5.1 we showed that the two equilibrium points of Eq.(13) are

$$\bar{X}_1 = 0 \qquad \text{and} \qquad \bar{X}_2 = \frac{A-1}{2}.$$

It is our goal in this section to graphically illustrate that every positive solution of Eq.(13) converges to a period 2 cycle or to the positive equilibrium point.

The computer observations show the existence of the following period 2 cycle

$$X_{-1} = 0, \;\; X_0 = \frac{A-1}{\alpha}, \; \ldots \; .$$

The following examples will graphically illustrate the existence of a period 2 cycle of Eq.(13). The first example will show the existence of a period 2 cycle when $A = 3$, $\alpha = .5$, $X_{-1} = 0$ and $X_0 = 4$.

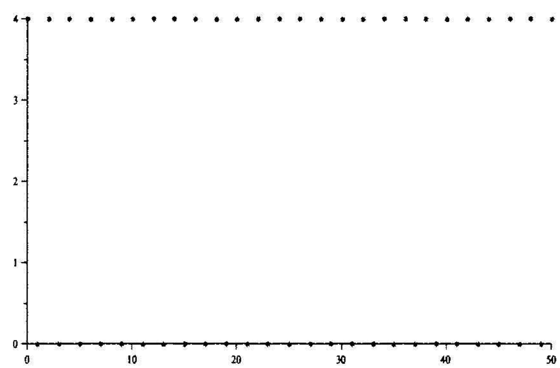

Figure 26: Existence of a period 2 solution of Eq.(13) when $\alpha = .5$ and $A = 3$.

The next example will show the existence of a period 2 cycle when $A = 7$, $\alpha = 1.5$, $X_{-1} = 0$ and $X_0 = 4$.

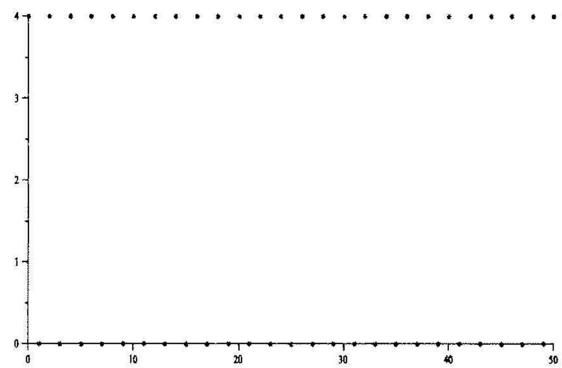

Figure 27: Existence of a period 2 solution of Eq.(13) when $\alpha = 1.5$ and $A = 7$.

5.3.1 The case $0 < \alpha < 1$

The following examples will graphically illustrate the convergence of solutions of Eq.(13) to a period 2 cycle. The first example will show the convergence to a period 2 cycle when $A = 2$, $\alpha = .7$, $X_{-1} = 1$ and $X_0 = 2.7$.

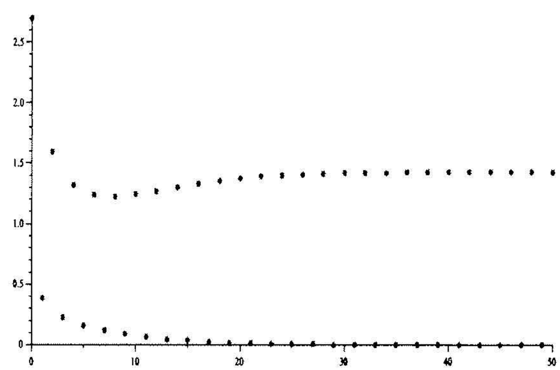

Figure 28: Convergence of solutions of Eq.(13) to a period 2 cycle when $\alpha = .7$ and $A = 2$.

The next example will show the convergence to a period 2 cycle when $A = 3$, $\alpha = .3$, $X_{-1} = 1$ and $X_0 = 2.7$.

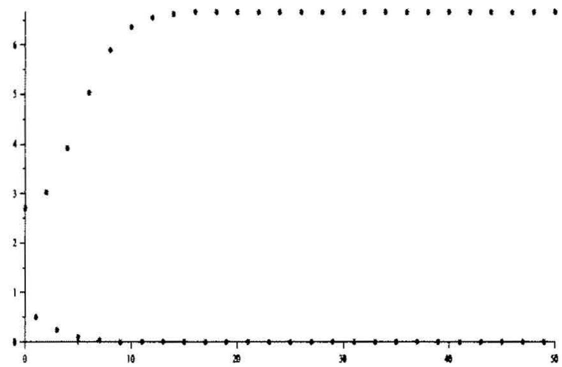

Figure 29: Convergence of solutions of Eq.(13) to a period 2 cycle when $\alpha = .3$ and $A = 3$.

The next example will show the convergence to a period 2 cycle when $A = 15$, $\alpha = .7$, $X_{-1} = 1$ and $X_0 = 2.7$.

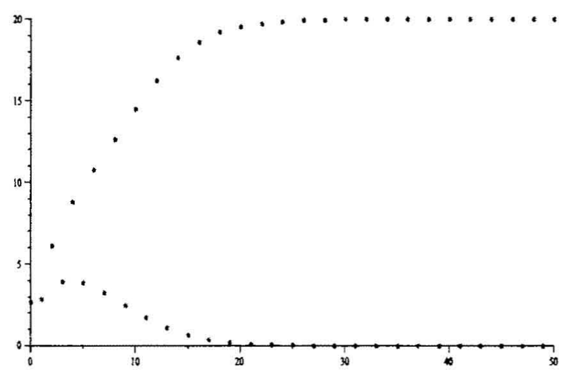

Figure 30: Convergence of solutions of Eq.(13) to a period 2 cycle when $\alpha = .7$ and $A = 15$.

5.3.2 The case $1 < \alpha < 2$

The following examples will graphically illustrate the convergence of solutions of Eq.(13) to the positive equilibrium. The first example will show the convergence to $\frac{A-1}{2}$ when $A = 2$, $\alpha = 1.3$, $X_{-1} = 1$ and $X_0 = 2.7$.

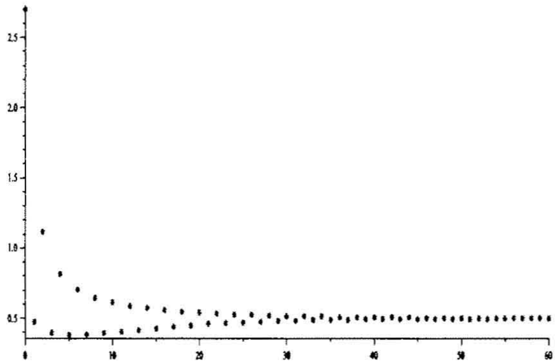

Figure 31: Convergence of solutions of Eq.(13) to $\frac{A-1}{2} = .5$ when $\alpha = 1.3$ and $A = 2$.

The next example will show the convergence to $\frac{A-1}{2}$ when $A = 5$, $\alpha = 1.3$, $X_{-1} = 1$ and $X_0 = 2.7$.

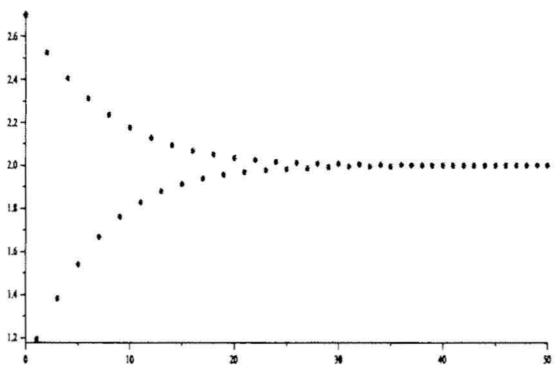

Figure 32: Convergence of solutions of Eq.(13) to $\frac{A-1}{2} = 2$ when $\alpha = 1.3$ and $A = 5$.

The next example will show the convergence to $\frac{A-1}{2}$ when $A = 5$, $\alpha = 1.3$, $X_{-1} = .5$ and $X_0 = 1.2$.

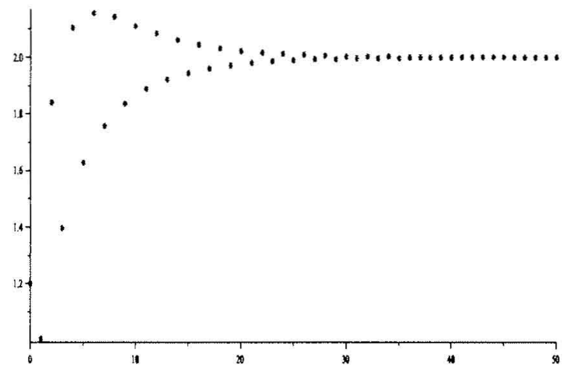

Figure 33: Convergence of solutions of Eq.(13) to $\frac{A-1}{2} = 2$ when $\alpha = 1.3$ and $A = 5$.

The next example will show the convergence to $\frac{A-1}{2}$ when $A = 10$, $\alpha = 1.7$, $X_{-1} = .5$ and $X_0 = 1.2$.

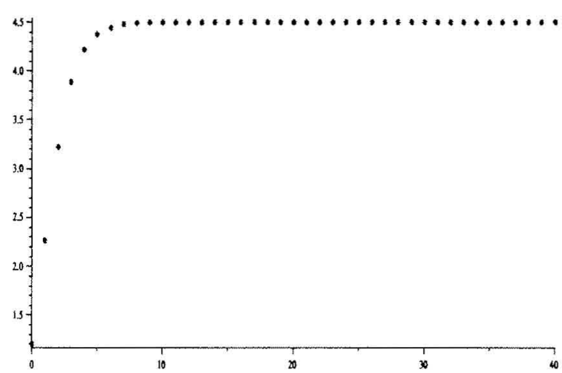

Figure 34: Convergence of solutions of Eq.(13) to $\frac{A-1}{2} = 4.5$ when $\alpha = 1.7$ and $A = 10$.

6 Conclusion and Future Work

It is our goal to continue the investigation of the global behavior of the positive solutions of Eq.(1). In particular, it is our goal to analyze the following difference equations:

(1)

$$X_{n+1} = \frac{AX_{n-1}}{1 + X_{n-s} + X_{n-1}} \quad , \quad n = 0, 1, 2, \ldots ,$$

where $s \geq 2$. We will investigate the boundedness nature, periodic nature, and monotonicity of solutions.

Furthermore note that computer observations suggest that the following properties are true:

(1.1) If s is even, then the following two statements are true:

(1.1.1) If $A \leq 1$, then every solution converges to 0.

(1.1.2) If $A > 1$, then every positive solution converges to a period 2 cycle.

(1.2) If s is odd, then the following two statements are true:

(1.2.1) If $A \leq 1$, then every solution converges to 0.

(1.2.2) If $A > 1$, then every positive solution converges to $\frac{A-1}{2}$.

(2)

$$X_{n+1} = \frac{A_n X_{n-1}}{1 + X_n + X_{n-1}} \quad , \quad n = 0, 1, 2, \ldots ,$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers. Moreover, we will examine how the even and odd order periods of $\{A_n\}$ affect the monotonic and periodic nature of the solutions.

7 Code (in Maple)

Second order equation

```
> with(linalg):
> number := 40:
> y[-1] := 1:
> y[0] := 2.7:
> for k from 0 to number do:
  y[k+1] := .5*y[k-1]/(1+y[k]+y[k-1]):
  od:
> ylist := [seq(y[k], k = 0 .. number)]:
> count := [seq(k, k = 0 .. number)]:
> ypts := zip((x, y) → [x, y], count, ylist):
> plot(ypts, style = point, symbol = solidcircle);
```

Fourth order equation

```
> with(linalg):
> number := 40:
> y[-3] := 4:
> y[-2] := 3:
> y[-1] := 2:
> y[0] := 1:
> for k from 0 to number do:
  y[k+1] := .5*y[k-1]/(1+y[k]+y[k-1]):
  od:
> ylist := [seq(y[k], k = 0 .. number)]:
> count := [seq(k, k = 0 .. number)]:
> ypts := zip((x, y) → [x, y], count, ylist):
> plot(ypts, style = point, symbol = solidcircle);
```

Fifth order equation

```
> with(linalg):
> number := 40:
> y[-4] := 1.5:
> y[-3] := .9:
> y[-2] := .7:
> y[-1] := .5:
> y[0] := .3:
> for k from 0 to number do:
  y[k+1] := .5*y[k-1]/(1+y[k]+y[k-1]):
  od:
> ylist := [seq(y[k], k = 0 .. number)]:
> count := [seq(k, k = 0 .. number)]:
> ypts := zip((x, y) → [x, y], count, ylist):
> plot(ypts, style = point, symbol = solidcircle);
```

Sixth order equation

```
> with(linalg):
> number := 40:
> y[-5] := 2:
> y[-4] := 1.8:
> y[-3] := 1.7:
> y[-2] := 1.5:
> y[-1] := 1.3:
> y[0] := 1:
> for k from 0 to number do:
  y[k+1] := y[k-1]/(1+y[k]+y[k-1]):
  od:
> ylist := [seq(y[k], k = 0 .. number)]:
> count := [seq(k, k = 0 .. number)]:
> ypts := zip((x, y) → [x, y], count, ylist):
> plot(ypts, style = point, symbol = solidcircle);
```

Seventh order equation

```
> with(linalg):
> number := 40:
> y[-6] := 5:
> y[-5] := 4:
> y[-4] := 3:
> y[-3] := 2:
> y[-2] := 1.7:
> y[-1] := 1:
> y[0] := .7:
> for k from 0 to number do:
  y[k+1] := y[k-1]/(1+y[k]+y[k-1]):
  od:
> ylist := [seq(y[k], k = 0 .. number)]:
> count := [seq(k, k = 0 .. number)]:
> ypts := zip((x, y) → [x, y], count, ylist):
> plot(ypts, style = point, symbol = solidcircle);
```

Second order equation with alpha (used in section 5)

```
> with(linalg):
> number := 40:
> a := 0:
> y[-1] := 1:
> y[0] := .7:
> for k from 0 to number do:
  y[k+1] := .5*y[k-1]/(1+(2-a)*y[k]+a*y[k-1]):
  od:
> ylist := [seq(y[k], k = 0 .. number)]:
> count := [seq(k, k = 0 .. number)]:
> ypts := zip((x, y) → [x, y], count, ylist):
> plot(ypts, style = point, symbol = solidcircle);
```

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[2] C.H. Gibbons, M.R.S. Kulenovic, and G. Ladas, *On the Recursive Sequence*

$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n},$$

Math. Sci. Res. Hot-Line 4 (2) (2000), 1-11.

[3] C.H. Gibbons, M.R.S. Kulenovic, and G. Ladas, *On the Recursive Sequence*

$$y_{n+1} = \frac{p + qy_n + ry_{n-1}}{1 + y_n},$$

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$$y_{n+1} = \frac{p + y_{n-1}}{qy_n + y_{n-1}},$$

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$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n + Cx_{n-1}},$$

J. Diff. Equa. Appl. 6(2000), 641-646.

[8] M.R.S. Kulenovic, G. Ladas and N.R. Prokup, *On the Recursive Sequence*

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{1 + x_n},$$

J. Diff. Equa. Appl. 6(2000), 563-576.

[9] M.R.S. Kulenovic, G. Ladas, and W.S. Sizer, *On the Recursive Sequence*

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\gamma x_n + \delta x_{n-1}}$$

Math. Sci. Res. Hot-Line 2 (5) (1998), 1-16.