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Computing Hilbert Functions using the Syzygy and LCM-lattice methods

Maria Barouti

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Maria Barouti

School of Mathematical Sciences

Rochester Institute of Technology

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Thesis Committee:

Major Professor	Dr. Manuel Lopez
	Dr. Anurag Agarwal
	Dr. James Marengo

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Abstract

The Hilbert function for any graded module $M = \bigoplus_{i \in \mathbb{N}} M_i$ over a field k is defined by

$$\mathrm{HF}(M, b) = \dim_k M_b,$$

where integer b indicates the graded component being considered.

One standard approach to computing the Hilbert function is to come up with a free-resolution for the graded module M and another is via a Hilbert power series which serves as a generating function. Using combinatorics and homological algebra we develop three alternative ways to generate the values of a Hilbert function when the graded module is a quotient ring over a field. Two of these approaches (which we've called the lcm-Lattice method and the Syzygy method) are conceptually combinatorial and work for any polynomial quotient ring over a field. The third approach, which we call the Hilbert function table method, also uses syzygies but the approach is better described in terms of homological algebra.

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Chapter 1

Preliminaries

In this work we address only polynomial rings and their quotient rings. Therefore all definitions pertaining a ring are meant to apply to *commutative* rings. As a consequence all our modules are two-sided modules and all our ideals are two-sided ideals. In fact we require more structure of our objects: we require that they be graded objects. this is made precise by the first two definitions.

Definition 1.0.1. *A graded ring R is a ring that has a direct sum decomposition into abelian additive groups $R = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} R_n = R_0 \bigoplus R_1 \bigoplus R_2 \bigoplus R_3 \bigoplus ...$ such that $R_s R_r \subseteq R_{s+r}$ for all $r, s \geq 0$.*

There is also the closely related concept of a graded modulo.

Definition 1.0.2. *A graded module M over any graded ring R is a module that can be written as a direct sum $M = \bigoplus_{i \in \mathbb{Z}^{\geq 0}} M_i$ satisfying $R_i M_j \subseteq M_{i+j}$ for all $i, j \geq 0$.*

Both concepts of a graded object are standard; see for example [**?**] pages 12 and 13.

An example of a graded ring and also of a graded module is the polynomial ring $k[x_1, x_2, ..., x_a]$ over a field k. The direct decomposition in this case is $R = k[x_1, x_2, ..., x_a] = \bigoplus_{b \in \mathbb{Z}^{\geq 0}} R_b$ where each $R_b = \text{span}_k\{\text{monomials of degree } b\}$. This means that each R_b is a k-vector

space. Moreover, since every ideal I of a ring R is an R-module one can easily prove the following result.

Lemma 1.0.1. *An ideal I is a graded ideal of a graded ring $R = \bigoplus_{n \in \mathbb{Z}^{\geq 0}} R_n$ if it can be written as a direct sum of ideals such that each summand corresponds to $I \cap R_n$ for some n.*

Definition 1.0.3. *An ideal I of $k[x_1, x_2, ..., x_a]$ is homogeneous if and only if every homogeneous component of every polynomial $p(\bar{\mathrm{x}})$ is in I, where $\bar{\mathrm{x}}$ denotes an a-tuple $(x_1, x_2, ...x_a)$.* (See for example [2] page 299.)

Here are some easy-to-prove facts relevant to the present discussion about monomial ideals.

- A monomial ideal in $k[x_1, x_2, ..., x_a]$ is, by definition, one generated by monomials (see [2] page 318). Therefore it is a homogeneous ideal since every monomial is a homogeneous polynomial. A monomial ideal is also a graded ideal because $k[x_1, x_2, ..., x_a]$ is a graded ring hence we may apply Lemma 1.0.1.

- Given a monomial ideal I in the polynomial ring R then R/I is a graded module because it has the following direct sum decomposition

$$R/I = \bigoplus_{b \in \mathbb{Z}^{\geq 0}} (R_b + I)/I, \text{ where } b \text{ is the grading.}$$

Observe that every summand is also a module over the base ring.

Our object of study is the graded modules R/I, where R is a polynomial ring in finitely many variables over a field k and I is a finitely generated monomial ideal in R. In this setting we have that for each $b \geq 0$ the summand $(R_b + I)/I$ is indeed a vector space since it is a module over a field.

Furthermore, since the number of variables is finite each such summand is a finite dimensional vector space. This brings up a natural question: Given a summand with grading

b what is its dimension as a vector space over the base field k? This is in fact how the Hilbert function for the graded module is defined.

Let us illustrate this definition by considering $R = k[x_1, x_2, x_3, x_4]$ and $I = \langle x_2^4, x_1x_4, x_3^2 \rangle$. Then $R/I = \bigoplus_{i=0}^{\infty} R_i$ where

$$R_i = \{\text{all polynomials equivalence classes in } R/I \text{ with representatives of degree i}\}.$$

Such R_i are no longer rings on their own but they are k-vector spaces. The dimension of these vectors spaces are, $\dim R_0 = 1$, $\dim R_1 = 4$, $\dim R_2 = 8$, $\dim R_3 = 12$, $\dim R_4 = 15$, and for all $i \geq 5$ $\dim R_i = 16$.

In general, for any graded module $M = \bigoplus_{i \in \mathbb{N}} M_i$ we define the Hilbert function of M as $\mathrm{HF}(M, b) = \dim_k M_b$. In particular, a basic result facilitating our computations is the "rank-nullity" theorem which says that the rank plus the nullity of a linear transformation $T : V \longrightarrow W$ is equal to the dimension of V.

Theorem 1.0.1. $\mathrm{HF}(R, b) = \mathrm{HF}(R/I, b) + \mathrm{HF}(I, b)$, *where* $R = k[x_1, x_2, ..., x_a]$ *and* I *is a monomial ideal.*

Proof. Let $T : V \longrightarrow W$ be a linear transformation; then using the inclusion map i of the kernel *into* V we get a sequence of linear transformations:

$$0 \longrightarrow \mathrm{ker(T)} \xrightarrow{i} V \xrightarrow{T} \mathrm{coker(T)} \longrightarrow 0.$$

If T is onto then $\mathrm{coker(T)} \cong V/\mathrm{ker(T)}$. Then

$$\mathrm{dim(ker(T))} - \mathrm{dim(V)} + \mathrm{dim(coker(T))} = 0.$$

$\square$

Definition 1.0.4. *An exact sequence of modules is a sequence either finite or infinite of modules and homomorphisms between them such that the image of one homomorphism equals the kernel of the next homomorphism.* (For a reference see [2] page 378.)

An example of an exact sequence is the sequence in the next lemma (see [5] page 98) and the free resolution used in the Hilbert Syzygy theorem below (see [3] page 3). We shall refer to the exact sequence in the next lemma as *the short exact sequence.* Bookkeeping often requires a shift in the grading. If $M = \bigoplus_{i=0}^{\infty} M_i$ is a finitely generated $\mathbb{Z}^{\geq 0}$-graded module over R, then we denote $M(-d)$ to be the regrading of M obtained by a shift of the grading of M. In this case, the graded component M_i of M becomes M_{i+d} grading component of $M(-d)$.

Lemma 1.0.2. *Let M be a graded R-module. If $x_n \in R_d$ with $\deg(\mathrm{x_n}) = d$, then there is a degree preserving exact sequence*

$$0 \to (0 : x_n)(-d) \to M(-d) \xrightarrow{x_n} M \xrightarrow{\phi} M/x_nM \to 0,$$

where $\phi(m) = m + x_nM$ and $(0 : x_n) = \{m \in M | x_nm = 0\}$.

The draw back of this sequence is that not all objects are necessarily free R-modules. Free R-modules are R-modules that are isomorphic to a direct sum of copies of R. The traditional approach (see [1]) to computing the Hilbert function of a finitely graded R-module M (of which our quotient polynomial rings are examples) is based in the following theorem.

Theorem 1.0.2. (Hilbert Syzygy Theorem)

Any finitely generated graded R-module M has a finite graded free resolution

$$0 \to P_n \xrightarrow{\phi_n} P_{n-1} \to ...\to P_1 \xrightarrow{\phi_1} P_0.$$

This exact sequence can also be written as

$$0 \to P_n \xrightarrow{\phi_n} P_{n-1} \to ...\to P_1 \xrightarrow{\phi_1} P_0 \to M \to 0.$$

Since each P_i is a free R-module for $0 \leq i \leq n$. If R is a graded ring the above exact sequence is in fact an exact sequence of graded free modules and graded homomorphisms and each term in the free resolution is of the form $P_i = R_{1_i}(-d_{1_i}) \oplus R_{2_i}(-d_{2_i}) \oplus \cdots \oplus R_{l_i}(-d_{l_i})$ then applying the Theorem 1.0.1 in an inductive argument one obtains the following method for computing $HF(M,t)$

$$\mathrm{HF}(M,t) = \sum_{i=0}^{n} (-1)^i \left(\mathrm{HF}(R_{1_i}(-d_{1_i}),t) + \mathrm{HF}(R_{2_i}(-d_{2_i}),t) + \cdots + \mathrm{HF}(R_{l_i}(-d_{l_i}),t) \right).$$

Another standard approach to computing the Hilbert function is via the Hilbert series.

Definition 1.0.5. *Let $R = \bigoplus R_n$ be a graded ring. The Hilbert series of R is defined to be the generating function*

$$\mathrm{HS}(R,t) = \sum_{n=0}^{\infty} \mathrm{HF}(R,n)t^n.$$

Similarly, if I is a homogeneous ideal of R, then the Hilbert series of I is the formal power series

$$\mathrm{HS}(I,t) = \sum_{n=0}^{\infty} \mathrm{HF}(I,n)t^n.$$

Convergence is not an issue since we are working with formal power series.

For the Hilbert series we have a counterpart to our result derived from the "rank-nullity" theorem.

Theorem 1.0.3. *Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring and $I = \bigoplus_{n \geq 0} I_n$ be a graded ideal. Then*

$$\mathrm{HS}(R/I,t) = \mathrm{HS}(R,t) - \mathrm{HS}(I,t).$$

Proof. Theorem 1.0.1 implies that $\mathrm{HF}(R/I,n) = \mathrm{HF}(R,n) - \mathrm{HF}(I,n)$ and by summing over all values of n the theorem follows. $\square$

In other words, for computing the dimension of R_n/I_n, we count the number of monomials in R_n and we subtract the number of monomials spanning I_n; this is because the mono-

mials spanning R_n is a basis for R_n as a vector space over k and similarly the monomials spanning I_n form a basis for I_n as a vector space over k.

To build on this result we need the following notation for the Hilbert function of a module M shifted by degree d

$$\mathrm{HF}\{M(-d)\} := \mathrm{HF}(M, t-d).$$

Lemma 1.0.3. *A principal ideal has the Hilbert function of a polynomial ring shifted by the degree of the generator. If $I = \langle p \rangle$, where p is a monomial of degree n in $k[\bar{\mathrm{x}}]$ and $\bar{\mathrm{x}}$ represents the a-tuple $(x_1, x_2, x_3, ..., x_a)$ then*

$$\mathrm{HF}(I, t) = \mathrm{HF}\{k[\bar{\mathrm{x}}](-n)\}.$$

Proof. By definition $\mathrm{HF(I,t)}$ is the dimension of the vector space spanned by all polynomials in I of uniform degree t. A basis for such a vector space can be chosen to be all monomials in I of degree t. These are of the form $f \cdot p$, where f is monomial of $\deg(f) = t - \deg(p)$ so there are as many such monomials as there are monomials of degree $t - n$ in $k[\bar{\mathrm{x}}](-n)$. □

Before working through our first example it would be helpful to refer the following corollary to our last lemma.

Corollary 1.0.1. *For a principal ideal $I = \langle p \rangle$ we have that*

$$\mathrm{HF}(R/I, t) = \mathrm{HF}(R, t) - \mathrm{HF}(R(-\deg(p)), t)$$

Proof. Apply the above lemma to the Theorem 1.0.1 □

Example 1.0.1. *Here is an example of the use of the above corollary to find the Hilbert function of $M = k[x, y, z]/\langle x^5 \rangle$*

Let $R = k[x, y, z]$. Then the Hilbert function of the module M is given by the following formula

$$\mathrm{HF}(M,t) = \mathrm{HF}(R,t) - \mathrm{HF}(R(-\deg(x^5)),t) = \mathrm{HF}(R,t) - \mathrm{HF}(R(-5),t).$$

Therefore,

HF$\{R\}$	$-$HF$\{R\}(-5)\}$	HF$\{M\}$
1	0	**1**
3	0	**3**
6	0	**6**
10	0	**10**
15	0	**15**
21	-1	**20**
28	-3	**25**
36	-6	**30**
45	-10	**35**
55	-15	**40**
..	..	..
..	..	..

Regardless of our approach to the Hilbert function of polynomial quotient rings, it is clear that computing the Hilbert function of rings of the form $k[x_1, x_2, ..., x_a]$ is essential. That will be our first task.

Chapter 2

Hilbert Function tables, motivating applications and examples

We study Hilbert functions by placing them into families. The simplest such family will be the Hilbert functions corresponding to the indexed set $\{k[x_1, x_2, \ldots, x_a] \, : \, a \geq 1\}$. Then we generalize the idea of the Pascal table to construct the Hilbert Function tables. To motivate this generalization we use the Stanley–Reisner ring of a complex which we gradually build in a form that is analogous to the way the corresponding Hilbert Function table would be generated. Finally one must address the difficulties of generating a row of the Hilbert Function table which involves the introduction of one or more monomials in the ideal being used for the quotient ring corresponding to that row. We illustrate the difficulties at the end of this chapter and develop a different method of solving this problem in each of the next two chapters.

2.1 Pascal Table and more general Hilbert Function Tables

Consider the indexed set $\{k[x_1, x_2, \ldots, x_a] : a \geq 1\}$ of polynomial rings. We use the index value a to determine the row and the degree b of the monomials being counted to determine the column in the table below.

HF of $k[x_1]$	1	1	1	1	1	1	1	1	...
HF of $k[x_1, x_2]$	1	2	3	4	5	6	7	8	...
HF of $k[x_1, x_2, x_3]$	1	3	6	10	15	21	28	36	...
HF of $k[x_1, x_2, x_3, x_4]$	1	4	10	20	35	56	84	120	...
HF of $k[x_1, x_2, x_3, x_4, x_5]$	1	5	15	35	70	126	210	330	...
HF of $k[x_1, x_2, x_3, x_4, x_5, x_6]$	1	6	21	56	126	252	462	792	...
HF of $k[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$	1	7	28	84	210	462	924	1716	...
HF of $k[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$	1	8	36	120	330	792	1716	3432	...
.	.	.	.	.	.	.	.	.	...

The reader would have undoubtedly noticed that the number patterns displayed in the above table are those of the Pascal triangle. For this reason we refer to the above table as the Pascal table. These numerical patterns lead us to the following proposition.

Proposition 2.1.1. *$F(a, b) = F(a-1, b) + F(a, b-1)$, where $F(a, b)$ denotes the number of monomials of degree b in $k[x_1, x_2, ..., x_a]$.*

Proof. Let S be the set of monomials in $k[x_1, x_2, ..., x_a]$ of degree b. Then we can write S as the union of the set S_1 of monomials of degree b in the variables $x_1, x_2, ..., x_{a-1}$ and a set S_2 disjoint from S_1. Observe $|S_1| = F(a-1, b)$. Now we consider any element of S_2, such an element has a factor x_a. So if $p(\bar{x}) \in S_2$ then there is a unique $\hat{p}(\bar{x})$ such that $p(\bar{x}) = \hat{p}(\bar{x}) \cdot x_a$ and $\deg(\hat{\mathrm{p}}(\bar{\mathrm{x}})) = \mathrm{b} - 1$. On the other hand, if $\hat{q}(x) \in k[x_1, x_2, ..., x_a]$ and has degree $b-1$ then $(\hat{q}(x)) \cdot x_a \in S_2$. Therefore there is a bijection from the set of monomials of degree $b-1$ in $k[x_1, x_2, ..., x_a]$ to the set S_2. Consequently, $|S| = F(a-1, b) + F(a, b-1)$. $\square$

Now we prove by induction that each element of the table is given by the following proposition. Please be aware that the row count starts with 1 but the column count starts with zero. This is because the row count matches the number of variables used and the column count corresponds to the constant degree of the set of monomials being counted.

Proposition 2.1.2. $F(a,b) = \frac{(a-1+b)!}{(a-1)!b!}$, *where* $F(a,b)$, $a \geq 1, b \geq 0$, *denotes the entry that lies in the* a^{th} *row and the* b^{th} *column of the above table.*

Proof. We have that $F(1,b)$ is the number of monomials of degree b in a single variable. Since x_1^b is the only monomial in $K[x_1]$ of degree b then $F(1,b) = 1$ for all $b \geq 0$. Also $F(a,0) = 1$ for all $a \geq 1$ because in the ring $k[x_1, x_2, ..., x_a]$ there is only one monomial of degree zero which is $x_1^0 \cdot x_2^0 \cdot .... \cdot x_a^0$.

Inductive Step

Suppose $a > 1$and $b > 0$ then given that

$$F(a-1,b) = \frac{(a-1+b-1)!}{(a-2)!b!}$$

and

$$F(a,b-1) = \frac{(a-1+b-1)!}{(a-1)!(b-1)!}.$$

Then using the previous proposition, we have

$$\begin{aligned} F(a,b) &= F(a-1,b) + F(a,b-1) \\ &= \frac{(a-1+b-1)!}{(a-2)!b!} + \frac{(a-1+b-1)!}{(a-1)!(b-1)!} \\ &= \frac{(a-1+b)\cdot(a-2+b)!}{(a-1)!b!} \\ &= \frac{(a-1+b)!}{(a-1)!b!}. \end{aligned}$$

□

Both meanings assigned to $F(a,b)$ are equivalent. Thus, for example, we can say that by choosing $a=2$, we regard $F(2,b)$ as the value in the 2^{nd} row and b^{th} column of the table or the number of monomials of degree b that can be written with two distinct variables. Observe also that the above proposition together with Corollary 1.0.1 gives a concrete formula for the Hilbert function of a principal ideal, because now we can write for $R = k[\bar{\mathrm{x}}]$ and $p \in R$,

$$\mathrm{HF}(R/I,b) = \mathrm{F}(a,b) - \mathrm{F}(a,b-\deg(p)) = \frac{(a-1+b)!}{(a-1)!b!} - \frac{(a-1+b-\deg(p))!}{(a-1)!(b-\deg(p))!}. \qquad (2.1)$$

Proposition 2.1.1 is also valid for generating some rows of more general families of Hilbert functions. We can prove it using either a counting argument or some homological algebra machinery. We prefer the latter since our solution strategy is to use algebra to avoid delicate counting procedures.

Proposition 2.1.1 allows for an inductive construction of other expressions for computing values Hilbert function table. Let us illustrate this by expressing $F(a,b)$ in terms of the ascending factorial $[a]^n = a\cdot(a+1)\cdot(a+2)\cdot\ldots\ldots\cdot(a+n-1)$ with the convention $[a]^0 = 1$.

Proposition 2.1.3. *The Hilbert function $F(a,b)$ defined as above it can be computed by either one of the following formulas*

$$F(a,b) = \sum_{i=0}^{a-1} \frac{1}{i!}[b]^i \qquad \textit{or} \qquad F(a,b) = \sum_{j=0}^{b} \frac{1}{j!}[a-1]^j.$$

Proof. To prove the first formula we observe that $F(1,b) = 1$ for all $b \geq 0$ and this is precisely $F(1,b) = \sum_{i=0}^{1-1} \frac{1}{i!}[b]^i$.

We do induction on the first parameter of $F(a,b)$ namely $a \geq 2$. Suppose

$$F(a-1,b) = \sum_{i=0}^{a-2} \frac{1}{i!}[b]^i.$$

Now we use the result that

$$F(a,b) = \begin{cases} F(a-1,b)+0, & \text{for } b=0 \\ F(a-1,b)+F(a,b-1), & \text{for } b>0 \end{cases}$$

Observe that $F(a-1,0)=1$, for all $a \geq 1$.

Therefore

$$\begin{aligned} F(a,b) &= F(a-1,b)+F(a,b-1) \\ &= \sum_{i=0}^{a-2} \frac{1}{i!}[b]^i + \frac{(a-1+b-1)!}{(a-1)!(b-1)!} \\ &= \sum_{i=0}^{a-2} \frac{1}{i!}[b]^i + \frac{1}{(a-1)!} \cdot (b \cdot (b+1) \cdot .... \cdot (b+a-2)) \\ &= \sum_{i=0}^{a-2} \frac{1}{i!}[b]^i + \frac{1}{(a-1)!} \cdot [b]^{(a-1)} \\ &= \sum_{i=0}^{a-1} \frac{1}{i!}[b]^i. \end{aligned}$$

The second formula follows immediately from the first formula since the left hand side is invariant when variables $a-1$ is interchanged with b. Therefore we have that

$$F(a,b) = \sum_{j=0}^{b} \frac{1}{j!}[a-1]^j.$$

$\square$

As an example take the graded module $k[x_1, x_2, x_3]$ then

$F(3,b) = [b]^0 + \frac{1}{1!}[b]^1 + \frac{1}{2!}[b]^2 = 1 + b + \frac{1}{2}(b^2+b)$, where $b = 0, 1, 2, ...$

Now we proceed to create a more robust version to computes the Hilbert function of a quotient ring by introducing the meaning of the Hilbert function table.

Definition 2.1.1. *A Hilbert function table associated to a quotient ring* $k[x_1, x_2, ..., x_d]/I$*, where* I *is a monomial ideal in* $k[x_1, x_2, ..., x_d]$ *is an array whose entry indexed by* (a, b) *is the value of* $\mathrm{HF}(k[x_1, x_2, ..., x_a]/I_a, b)$ *, where* I_a *is the ideal generated by the generators of* I *that involve only the set of variables* $\{x_1, x_2, ..., x_a\}$*.*

As a result of the above definition, the Pascal table is a Hilbert function table for graded modules of the form $k[\bar{\mathrm{x}}]$ where $\bar{\mathrm{x}} = (x_1, x_2, ..., x_a)$ and a takes the values $1, 2, 3, 4, ...$

We can also observe that if $a \geq d$ then $I_a = I$. Moreover, the order of the variables $x_1, x_2, ..., x_d$ will affect the Hilbert function table. In fact, two different Hilbert function tables for the same quotient ring need not have the same rows for $1 \leq a < d$. This is because altering the order of $x_1, x_2, ..., x_d$ will alter the sequence of ideals $I_1, I_2, ...I_{d-1}$. However, two Hilbert function tables for $k[x_1, x_2, ..., x_a]/I$ will agree in rows d and higher because $I_a = I$ for $a \geq d$. After the d^{th} row, every new variable does not introduce a new monomial in the ideal. Therefore, producing the rows after the d^{th} row is a straightforward application of the following result.

Theorem 2.1.1. *Given* $\mathrm{HF}(j, b)$ *the Hilbert Function of* $k[x_1, x_2, ..., x_d, x_{d+1}, ..., x_{d+j}]/I$*, with* $j > 0$*, where* I *is a monomial ideal of the fixed set of variables* $\{x_1, ..., x_d\}$ *for* $b \geq 0$ *we have that* $\mathrm{HF}(j, b) = \mathrm{HF}(j-1, b) + \mathrm{HF}(j, b-1)$*.*

Proof. For $j \geq 1$, let $M_j = k[x_1, x_2, ...x_d, x_{d+1}, \ldots, x_{d+j}]/I$ and let $z = x_{d+j}$. We use the short exact sequence

$$0 \longrightarrow (0 : z)(-1) \xrightarrow{\text{incl}} M(-1) \longrightarrow M \longrightarrow M/zM \longrightarrow 0 \tag{2.2}$$

found in [5]. In this short exact sequence let the term $M = M_j$. Applying what are

commonly known as the second and third isomorphism theorems,

$$
\begin{aligned}
zM_j &= z\,(k[x_1, x_2, \ldots\ldots x_d, x_{d+1}, \ldots, x_{d+j}]/I) \\
&\cong z\,(k[x_1, x_2, \ldots\ldots x_d, x_{d+1}, \ldots, x_{d+j}]/(I \cap \langle z \rangle)) \\
&\cong (z\,k[x_1, x_2, \ldots\ldots x_d, x_{d+1}, \ldots, x_{d+j}] + I)/I
\end{aligned}
$$

therefore,

$$
\begin{aligned}
M_j/zM_j &= (k[x_1, x_2, \ldots\ldots x_d, x_{d+1}, \ldots, x_{d+j}]/I)/\,(z\,k[x_1, x_2, \ldots\ldots x_d, x_{d+1}, \ldots, x_{d+j}] + I/I) \\
&\cong M_{j-1} = k[x_1, x_2, \ldots\ldots x_d, x_{d+1}, \ldots, x_{d+j}]/I.
\end{aligned}
$$

Since $z \notin I$ the only element $x \in M_j$ such that $zx = 0$ is $x = 0$. In other words the annihilator of multiplication by z is zero. This implies the short exact sequence, $0 \longrightarrow (0 : z)(-1) \xrightarrow{\text{incl}} M_j(-1) \longrightarrow M_j \longrightarrow M_j/zM_j \longrightarrow 0$ and the corresponding alternating sum $\mathrm{HF}\{M_J/zM_j\} - \mathrm{HF}\{M_j\} + \mathrm{HF}\{M_j(-1)\} = 0$. □

2.2 Motivating example: the Stanley-Reisner Ring

The Stanley-Reisner ring is a polynomial quotient ring assigned to a finite simplicial complex. First we must bring to the attention of the reader what is meant by a *finite simplicial complex.*

Definition 2.2.1. *A finite simplicial complex Δ consists of a finite set V of vertices and a collection Δ of subsets of V called faces such that*

- *(i) If $u \in V$, then $u \in \Delta$.*
- *(ii) If $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.*

Note: The empty set is a face of every simplex.

Let Δ be a simplicial complex and let F be a face of Δ. Define the dimensions of F and Δ by dimF $= |F| - 1$ and dim$\Delta = \sup\{\text{dimF}|F \in \Delta\}$ respectively. A face of dimension q is called a q-face or a q-simplex. Associate a distinct variable x_i to each distinct vertex in the set V. If F is a face of Δ then the product of all corresponding x_i is a square-free monomial associated with F. This is due to the fact that at most one q–face can exist for a given $(q+1)$–set of vertices. The Stanley-Reisner ring can be written in following form:

$$K[x_1, x_2, ..., x_n]/I,$$

where I is an ideal of square free monomials ideal in the variables $x_1, x_2, ..., x_n$ corresponding to the non-face of Δ. For convenience let us denote the Stanley-Reisner ring associated with Δ by $k[\Delta]$.

By definition a simplicial complex Δ is a set theoretic construct but it is often the case we work with its geometric realization. That is associate with Δ a topological space that is a subspace of $\mathbb{R}^{\dim\Delta}$ and it is a union of simplices corresponding to the faces of Δ. Since Δ can be written as a disjoint union of its i-dimensional components $\Delta = \bigcup_{i=0}^{\dim\Delta} \Delta_i$ consequently the Stanley Reisner ring of Δ admits a direct sum decomposition

$$k[\Delta] = \bigoplus_{i=0}^{\dim\Delta} k[\Delta_i]$$

whose summands $k[\Delta_i]$ are vector spaces with a basis of monomials (not necessarily square-free) supported on the i-dimensional faces of Δ.

Example 2.2.1. *We illustrate how construction of the complex with Stanley–Reisner ring*

$$M = k[x, \hat{x}, y, z, w]/\langle x\hat{x}, yzw\rangle.$$

mirrors the generating of the corresponding Hilbert Function table by adding one variable at a time and including all relevant monomials in the ideal used in the quotient.

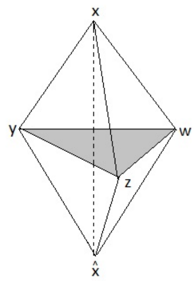

Figure 2.1: 4-vertices, 3-edge, 0-faces

Start with the complex C_0 corresponding to the point x we have the polynomial ring $k[x]$. Bringing the next variable $\hat{x}$ we have a new complex C_1 corresponding to the points $x, \hat{x}$. So we have $k[x, \hat{x}]/\langle x\hat{x}\rangle$. When the next variable y shows up we have the complex C_2 corresponding to the points $x, \hat{x}, y$ and the edges xy and $\hat{x}y$. By the same way, when z shows up we have the complex C_3 corresponding to the points $x, \hat{x}, y, z$, the edges $xy, xz, yz, y\hat{x}, z\hat{x}$ and the faces xyz and $y\hat{x}z$. To generate the table below we invoke Theorem 2.1.1

HF$\{k[x]\}$	1	1	1	1	1	1	1	1	...
HF$\{k[x, \hat{x}]/\langle x\hat{x}\rangle\}$	1	2	2	2	2	2	2	2	...
HF$\{k[x, \hat{x}, y]/\langle x\hat{x}\rangle\}$	1	3	5	7	9	11	13	15	...
HF$\{k[x, \hat{x}, y, z]/\langle x\hat{x}\rangle\}$	1	4	9	16	25	36	49	64	...

Let $M_1 = k[x, \hat{x}, y, z]/\langle x\hat{x}\rangle$.

Then by using the short exact sequence (s.e.s.) for M we have

0	$\longrightarrow$	$b_3 j$	$\xrightarrow{\text{inclusion}}$	$b_2 j$	$\xrightarrow{\text{multiply by } w}$	$b_1 j$	$\longrightarrow$	$b_0 j$	$\longrightarrow$	0
0	$\longrightarrow$	$(0:w)_M(-1)$	$\xrightarrow{\text{inclusion}}$	$M(-1)$	$\xrightarrow{\text{multiply by } w}$	M	$\longrightarrow$	$M/wM \cong M_1$	$\longrightarrow$	0
0		0		0		**1**		1		0
0		0		1		**5**		4		0
0		0		5		**14**		9		0
0		1		14		**29**		16		0
0		4		29		**50**		25		0
0		9		50		**77**		36		0
0		16		77		**110**		49		0
0		25		110		**149**		64		0
...		...		...		...		...		...
...		...		...		...		...		...

The justification for the values in the left most column is based on the annihilator

$$(0:w) = \{q \in M : qw = 0 \in M\}$$

associated with the map which is multiplication by w. A basis for the b-graded component of the module $(0:w)$ is the following set:

$$\begin{aligned}
B &= \{\text{nonzero } p \in M_2 \text{ of degree } b : yz|p\} \\
&= \{(yz)(r) : \text{nonzero } r \in M_1 \text{ with degree } b-2 \text{ and } (x\hat{x}) \nmid yzr\} \\
&= \{(yz)(r) : \text{nonzero } r \in M_1 \text{with degree } b-2\}.
\end{aligned}$$

Thus $\mathrm{HF}\{(0:w)\} = |B| = \mathrm{HF}\{M_1(-2)\}$. Having accounted for all annihilator elements and using the fact that $b_2 j = b_1 j + b_3 j - b_0 j$ we find the Hilbert function for M.

Example 2.2.2. *We are looking for the Hilbert function on the module*

$$M = k[x, y, z]/\langle x^2yz^3, x^3z, y^2z^2\rangle.$$

By rearranging the variables in our example we have that $M = k[y, z, x]/\langle y^2z^2, x^2yz^3, x^3z\rangle$ and based on Theorem 2.1.1 we have:

HF$\{k[y]\}$	1	1	1	1	1	1	1	1	...
HF$\{k[y, z]/\langle y^2z^2\rangle\}$ = HF$\{M_1\}$	1	2	3	4	4	4	4	4	...

Therefore, based on the short exact sequence we have

0	$\longrightarrow$	$(0 : x)(-1)$	$\longrightarrow M(-1) \longrightarrow$	M	$\longrightarrow$	M_1	$\longrightarrow 0$
0		0	0	**1**		$1 = \{1\}$	0
0		0	1	**3**		$2 = \{y, z\}$	0
0		0	3	**6**		$3 = \{y^2, yz, z^2\}$	0
0		0	6	**10**		$4 = \{y^3, y^2z, yz^2, z^3\}$	0
0		$1 = \{x^2z\}$	10	**12**		$4 = \{y^4, y^3z, yz^3, z^4\}$	0
0		$2 = \{yx^2z, z^2x^2\}$	12	**13**		$4 = \{y^5, yz^4, y^4z, z^5\}$	0
0		$4 = \{1, y^2x^2z, yzx^2z, z^2x^2z\}$	13	**14**		4	0
...		...	...	...		...	...
...		...	...	...		...	...

In order to figure out the Hilbert function of the annihilator module we need to find all the non zero elements in M. Those elements should be either multiple of xyz^3 or x^2z. Therefore, we cannot have a factor of y and a factor of y^2z. In other words, there are no elements in M_1 that create x^2yz^3 and x^3z. However, there are elements in M_1 that create y^2z^2. By this way and using the fact that the alternating sum is zero we create the above table. In this example we can observe that the draw back is that computing the Hilbert function of the annihilator ideal would require counting.With the next examples we illustrate basic approaches to avoid counting.

2.3 Examples

In this section we use the basic results earlier in this chapter to find the Hilbert function of key examples that provide the motivation for the techniques we develop in chapters 3 and 4. To be systematic we find it convenient to group the examples based on the number of monomials generating the ideal used to produce the quotient ring.

2.3.1 The ideal used to produce the quotient polynomial ring is a principal ideal

Consider $M = k[x_1, x_2, \ldots, x_a]/\langle u\rangle$ where $\deg u = d$. Using Equation (2.1) we obtain the following:

$$\mathrm{HF}(M,b) = \begin{cases} F(a,b), & \text{for } 0 \leq b \leq d-1 \\ F(a,b) - F(a,b-d), & \text{for } b \geq d \end{cases} \tag{2.3}$$

This approach combined with the result in Proposition 2.1.2 immediately yields

$$\mathrm{HF}(M,b) = \begin{cases} F(a,b) = \displaystyle\sum_{j=0}^{b} \frac{1}{j!}[a-1]^j, & \text{for } 0 \leq b \leq d-1 \\ F(a,b) - F(a,b-d) = \displaystyle\sum_{j=b-(d-1)}^{b} \frac{1}{j!}[a-1]^j, & \text{for } b \geq d \end{cases}$$

and this in turn can be encoded as matrix multiplication using an infinite matrix and infinite column vectors corresponding to the right-hand side of the above equation.

$$\begin{pmatrix} \frac{1}{0!} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{0!} & \frac{1}{1!} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{0!} & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \dots & \frac{1}{(d-1)!} & 0 & 0 & \dots \\ 0 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & \dots & \frac{1}{(d-1)!} & \frac{1}{d!} & 0 & \dots \\ 0 & 0 & \frac{1}{2!} & \frac{1}{3!} & \dots & \frac{1}{(d-1)!} & \frac{1}{d!} & \frac{1}{(d+1)!} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \cdot \begin{pmatrix} [a-1]^0 \\ [a-1]^1 \\ [a-1]^2 \\ [a-1]^3 \\ [a-1]^4 \\ [a-1]^5 \\ [a-1]^6 \\ \dots \\ \dots \end{pmatrix} = \begin{pmatrix} \mathrm{HF}(\mathbf{M},\mathbf{0}) \\ \mathrm{HF}(\mathbf{M},\mathbf{1}) \\ \mathrm{HF}(\mathbf{M},\mathbf{2}) \\ \vdots \\ \mathrm{HF}(\mathbf{M},\mathbf{d-1}) \\ \mathrm{HF}(\mathbf{M},\mathbf{d}) \\ \mathrm{HF}(\mathbf{M},\mathbf{d+1}) \\ \dots. \\ \dots. \end{pmatrix}.$$

In what follows we concentrate our efforts in finding ways to compute the Hilbert function of a polynomial ring as finite sums and differences of the Pascal table row corresponding to the number of variables in in our polynomial ring. In each such case one can do as above and use Proposition 2.1.2 to produce a matrix multiplication approach similar to the above. We'll leave this for the reader to try using the methods in chapter four as a starting point. Here are two examples to illustrate the above computations more concretely.

Example 2.3.1. *We are looking for the Hilbert function of the module*

$$M = k[x, y, z]/\langle xy^2 \rangle.$$

Equation (2.3) indicates the following recurrence relation for this quotient ring

$$\mathrm{HF}(M, b) = \begin{cases} F(a, b), & \text{for } 0 \le b \le 2 \\ F(a, b) - F(a, b-3), & \text{for } b \ge 3 \end{cases}$$

where the coefficients of $F(a, b)$ and $F(a, b-3)$ denote the first entry of the Pascal triangle.

Therefore, the Hilbert function of the module $M = k[x,y,z]/\langle xy^2\rangle$ is expressed by the following sequence of numbers

$$\mathrm{HF}(M,b)\text{: } \mathbf{1} \quad \mathbf{3} \quad \mathbf{6} \quad \mathbf{9} \quad \mathbf{12} \quad \mathbf{15} \quad \mathbf{18} \quad \mathbf{...}$$

Finally, we can rewrite the second part of the above function as

$$F(a,b) - F(a,b-3) = \sum_{j=b-2}^{b} \frac{1}{j!}[a]^j, \text{ for } b \geq 3.$$

An alternative way to express the Hilbert function of R is given by the following way

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & \frac{1}{1!} & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & \frac{1}{1!} & \frac{1}{2!} & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{1!} & \frac{1}{2!} & \frac{1}{3!} & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & 0 & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} & 0 & \dots \\ 0 & 0 & 0 & 0 & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \cdot \begin{pmatrix} 2^{(0)} \\ 2^{(1)} \\ 2^{(2)} \\ 2^{(3)} \\ 2^{(4)} \\ 2^{(5)} \\ 2^{(6)} \\ \dots \\ \dots \end{pmatrix} = \begin{pmatrix} \mathbf{1} \\ \mathbf{3} \\ \mathbf{6} \\ \mathbf{9} \\ \mathbf{12} \\ \mathbf{15} \\ \mathbf{18} \\ \dots \\ \dots \end{pmatrix}.$$

2.3.2 The ideal used in the quotient polynomial ring consists of two monomials

Suppose $M = k[\bar{x}]/\langle u,v\rangle$ with $\deg(u) = d_u$ and $\deg(v) = d_v$. As before the key to computing the $\mathrm{HF}(M,b)$ is finding a way to account exactly once for the monomials of degree b belonging to the ideal $\langle u,v\rangle$. There are three possibilities which the reader may visualize as the Venn Diagram of two overlaying regions: one corresponding to $\langle u\rangle$ and the other to the corresponding $\langle v\rangle$. In fact $q \in \langle u\rangle$ and $q \in \langle v\rangle$ iff $q \in \langle \mathrm{lcm}(u,v)\rangle$. To see this

observe $u \mid q$ and $v \mid q \Leftrightarrow \operatorname{lcm}(u,v) \mid q$. Therefore with the use of the inclusion-exclusion principle we have

$$\operatorname{HF}(\langle u, v\rangle, b) = \operatorname{HF}(\langle u\rangle, b) + \operatorname{HF}(\langle v\rangle, b) - \operatorname{HF}(\operatorname{lcm}(u,v), b).$$

Since every ideal on the right-hand side is a principal ideal we apply lemma 1.0.3 and the "rank–nullity" reasoning from Chapter 1 to get after setting $d_{\mathrm{lcm}} = \deg(\operatorname{lcm}(u,v))$, $d_{\min} = \min(d_u, d_v)$ and $d_{\max} = \max(d_u, d_v)$,

$$\operatorname{HF}(M,b) = \begin{cases} F(a,b), & \text{for } 0 \leq b < d_{\min} \\ F(a,b) - F(a, b - d_{\min}) & \text{for } d_{\min} \leq b < d_{\max} \\ F(a,b) - F(a,b-d_u) - F(a,b-d_v) & \text{for } d_{\max} \leq b < d_{\mathrm{lcm}} \\ F(a,b) - F(a,b-d_u) - F(a,b-d_v) + F(a,b-d_{\mathrm{lcm}}) & \text{for } b \geq d_{\mathrm{lcm}} \end{cases}$$

Example 2.3.2. *We are looking for the Hilbert function of the module*

$$M = k[x,y,z]/\langle x^2y, xz^2\rangle.$$

Equation (2.3) indicates the following recurrence relation for this quotient ring

$$\operatorname{HF}(M,b) = \begin{cases} F(a,b), & \text{for } 0 \leq b \leq 2 \\ F(a,b) - 2F(a,b-3) + F(a,b-5), & \text{for } b \geq 3 \end{cases}$$

Therefore, the Hilbert function of the module $M = k[x,y,z]/\langle x^2y, xz^2\rangle$ is expressed by the following sequence of numbers

$$\operatorname{HF}(M,b): \quad \mathbf{1} \quad \mathbf{3} \quad \mathbf{6} \quad \mathbf{8} \quad \mathbf{9} \quad \mathbf{10} \quad \mathbf{11} \quad \mathbf{\ldots}$$

In the next chapter we make full use of the *Principle of Inclusion and Exclusion* to develop what we will call the *lcm–lattice method* to handle any monomial ideal with a finite number of monomials. Before moving to the next chapter that let us take advantage

of this example to illustrate an alternative which accounts for the monomials of degree b in the ideal only once. In other words, the principle of inclusion-exclusion is a sequence of corrections for alternating over-counts and under-counts which corresponds to regions of the Venn diagram where two, three, four, etc... sets overlaps. Our goal here is to partition the union of all sets in the Venn diagram into disjoint sets as to avoid alternating inclusions with exclusions. This is accomplished by ordering our sets $E_1, E_2, E_3, \ldots$ then letting $F_1 = E_1, F_2 = E_2 \setminus F_1, F_3 = E_3 \setminus (F_1 \cup F_2), \ldots$. This is an approach conceptually similar to the Gramm-Schmidt process in linear algebra.

Let $u = x^2y$ and $v = xz^2$. Let also $E_1 = \langle u \rangle$ and $E_2 = \langle v \rangle$ then $F_1 = E_1$ and $F_2 = \{\text{all monomials which are multiple of } v \text{ but not of } u\}$. Since E_1 and E_2 are graded modules then F_1 and F_2 will be graded sets. For example, for degree 4, E_1 and E_2 are disjoint so no monomials of degree 4 needs to be excludes for F_2. However, for degree 5, for example $uz^2 = vxy$; in this case we want to count x^2yz^2 as a multiple of u (i.e. belonging to F_1) but prevent it being counted as a multiple of v. So we want to disallow in this case the factor xy from being multiplied by v. Observe

$$\frac{\mathrm{lcm}(x^2y, xz^2)}{xz^2} = \frac{x^2yz^2}{xz^2} = xy$$

which is known as a syzygy. How this would work with example 2.3.2 illustrated up to degree 6 in the following table,where

$$G(a, b) = \mathrm{HF}\{M\} = \mathrm{HF}\{k[x, y, z]\} - |F_1| - |F_2|.$$

Degree k	0	1	2	3	4	5	6	...
HF{k[x,y,z]}	1	3	6	10	15	21	28	...
F_1				u	$ux, uy, uz,$	$ux^2, uy^2, uz^2,$ uxy, uyz, uxz	$ux^3, uy^3, uz^3, xyzu,$ $ux^2y, uxy^2, ux^2z,$ uy^2z, uxz^2, uyz^2	... ...
F_2				v	vx, vy, vz	$vx^2, vy^2, vz^2,$ xzv, yzv	$vx^3, vy^3, vz^3,$ $vx^2z, vy^2z, vxz^2, vyz^2$	... ...
$\lvert F_1\rvert$				1	3	6	10	...
$\lvert F_2\rvert$				1	3	5	7	...
G(a,b)	1	3	6	8	9	10	11	...

Chapter 3

LCM–Lattice Method

As discussed at the end of the previous the challenge remains to find the Hilbert function of a monomial ideal with more than one monomial generator. Our first approach, which we develop in this chapter, uses the well known *Principle of Inclusion and Exclusion* (which the reader will find in the standard reference [4]). First we need the following:

Proposition 3.0.1. $\langle u\rangle \cap \langle v\rangle = \langle \mathrm{lcm}(u,v)\rangle$

Proof. $p \in \langle u\rangle \cap \langle v\rangle \Leftrightarrow u \mid p$ and $v \mid p \Leftrightarrow \mathrm{lcm}(u,v) \mid p$ □

Corollary 3.0.1. $\langle p_1\rangle \cap \langle p_2\rangle \cap \langle p_3\rangle \cap ... \cap \langle p_r\rangle = \langle \mathrm{lcm}(p_1, p_2, p_3, ..., p_r)\rangle$

Proof. (By Induction)

- The above proposition is the above case.
- Suppose $\bigcap_{i=1}^{r-1} \langle p_i\rangle = \langle \mathrm{lcm}(p_1, p_2, p_3, ..., p_{r-1})\rangle$ then

  $\bigcap_{i=1}^{r} \langle p_i\rangle = \bigcap_{i=1}^{r-1} \langle p_i\rangle \cap \langle p_r\rangle = \langle \mathrm{lcm}(\mathrm{lcm}(p_1, p_2, p_3, ..., p_{r-1}), p_r)\rangle = \langle \mathrm{lcm}(p_1, p_2, p_3, ..., p_r)\rangle.$

□

Further use of enclusion-exclusion; this time with n monomials we get

$$\begin{aligned}
\mathrm{HF}(\langle p_1, p_2, p_3, ..., p_n \rangle, b) &= |\{\text{monomials of degree } b \text{ in } \langle p_1, p_2, p_3, ..., p_n \rangle\}| \\
&= |\{\text{monomials of degree } b \text{ in } \langle p_1 \rangle \text{ OR } \langle p_2 \rangle \text{ OR... OR } \langle p_n \rangle\}| \\
&= \sum_{1 \le j_1 \le n} |\langle p_{j_1} \rangle| - \sum_{1 \le j_1 < j_2 \le n} |\langle \mathrm{lcm}(p_{j_1}, p_{j_2}) \rangle| \\
&\quad + \sum_{1 \le j_1 < j_2 < j_3 \le n} \langle (p_{j_1}, p_{j_2}, p_{j_3}) \rangle| \\
&\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\quad + (-1)^{r-1} \sum_{1 \le j_1 < j_2 < \ldots < j_r \le n} |\langle \mathrm{lcm}(p_{j_1}, p_{j_2}, ..., p_{j_r}) \rangle| \\
&\quad + (-1)^{n-1} |\langle \mathrm{lcm}(p_1, p_2, ..., p_n) \rangle| \\
&= \sum_{r=1}^{n} \left((-1)^{r-1} \sum_{1 \le j_1 < j_2 < \ldots < j_r \le n} |\langle \mathrm{lcm}(p_{j_1}, p_{j_2}, ..., p_{j_r}) \rangle| \right).
\end{aligned}$$

Assigning $d_{j_1 j_2 j_3 \ldots j_r} = \deg(\mathrm{lcm}(p_1, p_2, p_3, ..., p_r))$, where $1 \le r \le n$ and $1 \le j_1 < j_2 < j_3 < ... < j_r \le n$. To facilitate expressing the Hilbert function let's expand $F(a, b) = 0$ if $b < 0$.

Then $\mathrm{HF}(\langle p_1, p_2, p_3, ..., p_n \rangle, b) = \sum_{r=1}^{n} \left((-1)^{n-1} \sum_{1 \le j_1 < j_2 < \ldots < j_r \le n} F(a, b - d_{j_1 j_2 j_3 \ldots j_r}) \right)$

and

$$\mathrm{HF}(k[\bar{x}]/\langle p_1, p_2, p_3, ..., p_r \rangle, b) = F(a, b) - \sum_{r=1}^{n} \left((-1)^{n-1} \sum_{1 \le j_1 < j_2 < \ldots < j_r \le n} F(a, b - d_{j_1 j_2 j_3 \ldots j_r}) \right).$$

The argument starting at the top of this page proves the validity of the method we now describe. The starting point of building up the lcm–lattice is what we call layer 1. Layer 1 is a row containing all the monomials of the given ideal. Finding the lcm of all the pairs we create the 2$^{\text{nd}}$ layer. Next we find the lcm of all the triples in layer 1 and we call this layer 3. Following the same pattern we create as many layers as there are monomials in

the given ideal. The last layer will contain the lcm of all the monomials given in the ideal. If the ideal I contains n monomials then the number of monomials in the lcm-lattice in layers 1, 2, 3, ..., n will be $\binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \ldots, \binom{n}{n}$ correspondingly. These values are those found in the n^{th} row of the Pascal triangle to the right and including $\binom{n}{1}$.

The following examples give a nice view of the above description.

Example 3.0.3. *Finding the Hilbert function of the module $M = R/\langle x^2, y^3\rangle$ where $R = k[x, y, z]$.*

In the case that we have two monomials in the ideal the lcm lattice is simple. Start by building up the lcm lattice. Layer 1 is called the row that has all the monomials of the ideal. Afterwards, we take the lcm of the two monomials and we have the following

$$\begin{array}{cccl} x^2 & & y^3 & \text{layer 1} \\ & x^2y^3 & & \text{layer 2} \end{array}$$

According now to the above lcm lattice, we are left with a lattice of monomials on which we use inclusion - exclusion at each row to produce the alternating sum that computed the Hilbert function

HF$\{R\}$	layer 1(-)		layer 2(+)	HF$\{M\}$
1	0	0	0	**1**
3	0	0	0	**3**
6	-1	0	0	**5**
10	-3	-1	0	**6**
15	-6	-3	0	**6**
21	-10	-6	1	**6**
28	-15	-10	3	**6**
36	-21	-15	6	**6**
45	-28	-21	10	**6**
55	-36	-28	15	**6**
..	..	..	..	..
..	..	..	..	..

We apply now our lcm–Lattice method to quotient rings where the monomial ideal consists of three monomials.

Example 3.0.4. *Given the quotient ring* $M = k[x, y, z]/\langle xz, yz, x^2y\rangle$*, find its Hilbert function.*

Start by building up the lcm lattice.

$$\begin{array}{cccl} \mathbf{x^2y} & xz & yz & \text{layer 1} \\ \mathbf{x^2yz} & x^2yz & \mathbf{xyz} & \text{layer 2} \\ & \mathbf{x^2yz} & & \text{layer 3} \end{array}$$

So we are left with the above lattice of monomials on which we use inclusion - exclusion at each row to produce the alternating sum that computed the Hilbert function. Let $R = k[x, y, z]$. Since we observe that there are monomials of the same degree in adjacent rows of the lcm-lattice lattice, we exclude these pairs of monomials from the alternating sum in our table. We do that cancellation because the contribution of such a pair to

the alternating sum is zero. Therefore, the monomials that are cancelled are displayed in bold-faced. By this way we have the following

HF$\{R\}$	layer 1(-)	layer 2(+)	HF$\{M\}$
1	0	0	**1**
3	0	0	**3**
6	-2	0	**4**
10	-6	0	**4**
15	-12	1	**4**
21	-20	3	**4**
28	-30	6	**4**
36	-42	10	**4**
45	-56	15	**4**
55	-72	21	**4**
..	..	..	..
..	..	..	..

Example 3.0.5. *Compute the Hilbert function of the module*

$M = k[x, y, z]/\langle x^2y^3z, xz^3, xy^4z, x^2z^2\rangle$. *As before, for the sake of simplicity, we will let* $R = k[x, y, z]$.

Start by building up the lcm lattice and we have

$$\begin{array}{ccccccl}
 & x^2y^3z & xz^3 & xy^4z & x^2z^2 & & \text{layer 1} \\
\mathbf{x^2y^3z^3} & x^2y^4z & x^2y^3z^2 & xy^4z^3 & x^2z^3 & \mathbf{x^2y^4z^2} & \text{layer 2} \\
 & \mathbf{x^2y^4z^3} & \mathbf{x^2y^3z^3} & x^2y^4z^3 & \mathbf{x^2y^4z^2} & & \text{layer 3} \\
 & & & \mathbf{x^2y^4z^3} & & & \text{layer 4}
\end{array}$$

We typeset in bold face the monomials that are cancelled. By that way we have the following the table

HF$\{R\}$	layer 1		layer 2			layer 3	HF$\{M\}$
1	0	0	0	0	0	0	**1**
3	0	0	0	0	0	0	**3**
6	0	0	0	0	0	0	**6**
10	0	0	0	0	0	0	**10**
15	-2	0	0	0	0	0	**13**
21	-6	0	1	0	0	0	**16**
28	-12	-2	3	0	0	0	**17**
36	-20	-6	6	2	0	0	**18**
45	-30	-12	10	6	1	0	**20**
55	-42	-20	15	12	3	-1	**22**
..	..	..	..	..	..	..	..
..	..	..	..	..	..	..	..

In Chapter 4 we develop an alternative approach based on the Syzygy of pairs of monomials.

Chapter 4

The Syzygy Method

In this chapter we extend the second approach to the example 2.3.2 to handle ideals with finitely many monomials as generators. When implemented as a recursive algorithm this method will break down a Hilbert function computation into a sum–difference expression of Hilbert functions all of which involve a principal ideals. The computation is finished by invoking Corollary 2.3. Unlike the lcm–method, the principal ideals used will be generated by always taking syzygys of pairs of monomials (we never consider three or more of the given monomials in a computational step). The key recursive step is given by the following theorem.

Theorem 4.0.1. (Syzygy method) *Let $M = k[\bar{\mathrm{x}}]/I$, where $I = \langle p_1, p_2, p_3, ..., p_r \rangle$, a monomial ideal generated by $p_1, p_2, p_3, ..., p_r \in k[\bar{\mathrm{x}}]$. Then, using the notation $d_j = \deg(p_j)$ and $m_{ij} = \frac{\mathrm{lcm}(p_i,p_j)}{p_j} \in k[\bar{\mathrm{x}}]$ with $i < j$, we have*

$$\mathrm{HF}(M,t) = F(a,t) - F(a,t-d_1) - \sum_{j=2}^{r} \mathrm{HF}(k[\bar{\mathrm{x}}]/\langle m_{1j}, m_{2j}, m_{3j}, ..., m_{(j-1)j} \rangle, t - d_j).$$

Proof. (By Induction)

- Base case $r = 1$ then this hold by the corollary 1.0.1.

- Suppose $r > 1$ and

$$\begin{aligned}\mathrm{HF}(k[\bar{\mathrm{x}}]/\langle p_1, p_2, p_3, ..., p_{r-1}\rangle, t) &= F(a,t) - F(a, t-d_1) \\ &\quad -\sum_{j=2}^{r-1} \mathrm{HF}(k[\bar{\mathrm{x}}]/\langle m_{1j}, m_{2j}, m_{3j}, ..., m_{(j-1)j}\rangle, t-d_j).\end{aligned}$$

- We show that

$$\begin{aligned}\mathrm{HF}(k[\bar{\mathrm{x}}]/\langle p_1, p_2, p_3, ..., p_r\rangle, t) &= \mathrm{HF}(k[\bar{\mathrm{x}}]/\langle p_1, p_2, p_3, ..., p_{r-1}\rangle, t) \\ &\quad - \mathrm{HF}(k[\bar{\mathrm{x}}]/\langle m_{1r}, m_{2r}, m_{3r}, ..., m_{(r-1)r}\rangle, t-d_r).\end{aligned}$$

A monomial $q \in k[\bar{\mathrm{x}}]$, of degree t represent a nonzero element in $k[\bar{\mathrm{x}}]/\langle p_1, p_2, p_3, ..., p_{r-1}\rangle$ and is zero in $k[\bar{\mathrm{x}}]/\langle p_1, p_2, p_3, ..., p_r\rangle$ if and only if $p_i \nmid q$ for all $1 \leq i < r$ and $p_r | q$. If we call the set of all such monomials $\Gamma(t)$ then we have that

$$\mathrm{HF}(k[\bar{\mathrm{x}}]/\langle p_1, p_2, p_3, ..., p_r\rangle, t) = \mathrm{HF}(k[\bar{\mathrm{x}}]/\langle p_1, p_2, p_3, ..., p_{r-1}\rangle, t) - |\Gamma(t)|.$$

A monomial $q \in k[\bar{\mathrm{x}}]$ satisfies $q \in \Gamma(t) \Leftrightarrow q = a \cdot p_r$ where a is a monomial in $k[\bar{\mathrm{x}}]$ of degree $t - d_r$ and $p_i \nmid a \cdot p_r$ for all $1 \leq i < r$. This is equivalent to $m_{ir} \nmid a$ for all $1 \leq i < r$.

Since a is a monomial we have that,

$$\begin{aligned}& a \notin \langle m_{ir}\rangle, \text{ for all } 1 \leq i \leq r-1 \\ &\quad \Leftrightarrow a \notin \langle m_{1r}, m_{2r}, m_{3r}, ..., m_{(r-1)r}\rangle \\ &\quad \Leftrightarrow a \in k[\bar{\mathrm{x}}]/\langle m_{1r}, m_{2r}, m_{3r}, ..., m_{(r-1)r}\rangle.\end{aligned}$$

Finally, to finish the proof and establish that

$$|\Gamma(t)| = \mathrm{HF}(k[\bar{\mathrm{x}}]/\langle m_{1r}, m_{2r}, m_{3r}, ..., m_{(r-1)r}\rangle, t-d_r)$$

we only need to observe that a is uniquely determined by $q \in \Gamma(t)$ and every $a \in k[\bar{\mathrm{x}}]/\langle m_{1r}, m_{2r}, m_{3r}, ..., m_{(r-1)r}\rangle$ uniquely determines a monomial q.

$\square$

Example 4.0.6. *Both the lcm-lattice-method and the Syzygy method produce similar formulas for computing the Hilbert function. We apply the Syzygy method to establish that the lcm-lattice method holds for a monomial ideal with three monomials. The reader should observe that this will confirm of that result without the use of inclusion-exclusion.*

Consider I generated by three (not necessarily distinct) monomials p_1, p_2, p_3 with degrees d_1, d_2, d_3 respectively. We need to show that

$$\begin{aligned} HF\{k[\bar{\mathrm{x}}]/\langle p_1, p_2, p_3\rangle\} &= F(a,t) - F(a, t-\deg(p_1)) \\ &\quad -F(a, t-\deg(p_2)) - F(a, t-\deg(p_3)) \\ &\quad +F(a, t-\deg(\mathrm{lcm}(p_1,p_2))) + F(a, t-\deg(\mathrm{lcm}(p_2,p_3))) \\ &\quad +F(a, t-\deg(\mathrm{lcm}(p_1,p_3))) - F(a, t-\deg(\mathrm{lcm}(p_1,p_2,p_3))). \end{aligned}$$

By the syzygy method we obtain the following equality which we call the syzygy equality

$$\begin{aligned} HF\{k[\bar{\mathrm{x}}]/\langle p_1, p_2, p_3\rangle\} &= F(a,t) - F(a,t-d_1) - HF\{k[\bar{\mathrm{x}}]/\langle m_{12}\rangle(-d_2)\} \\ &\quad -HF\{k[\bar{\mathrm{x}}]/\langle m_{13}, m_{23}\rangle(-d_3)\} \end{aligned}$$

Applying the syzygy method to the third and fourth summands on the right hand side we have

$$\begin{aligned} HF\{k[\bar{\mathrm{x}}]/\langle m_{12}\rangle(-d_2)\} &= F(a, t-d_2) - F(a, t-d_2-\deg(m_{12})) \\ &= F(a, t-d_2) - F(a, t-\deg(\mathrm{lcm}(p_1,p_2))) \end{aligned}$$

and

$$\begin{aligned} HF\{k[\bar{\mathrm{x}}]/\langle m_{12}, m_{23}\rangle(-d_3)\} &= F(a, t-d_3) - F(a, t-d_3-\deg(m_{13})) \\ &\quad -HF\left\{k[\bar{\mathrm{x}}]/\left\langle \mathrm{lcm}(m_{13}, m_{23})m_{23}^{-1}\right\rangle(-d_3-\deg(m_{23}))\right\}. \end{aligned}$$

and

$HF\{k[\bar{\mathrm{x}}]/\langle \frac{\mathrm{lcm}(m_{13},m_{23})}{m_{23}}\rangle(-d_3-\deg(m_{23}))\} =$

$$\begin{aligned} &= F(a, t-d_3-\deg(m_{23}) - F\left(a, t-d_3-\deg(m_{23}) - \deg\left(\frac{\mathrm{lcm}(m_{13}, m_{23}}{m_{23}}\right)\right) \\ &= F(a, t-\deg(\mathrm{lcm}(p_2, p_3))) - F(a, t-d_3-\deg\mathrm{lcm}(m_{13}, m_{23})) \\ &= F(a, t-\deg(\mathrm{lcm}(p_2, p_3))) - F(a, t-(d_3+\deg\mathrm{lcm}(m_{13}, m_{23})) \\ &= F(a, t-\deg(\mathrm{lcm}(p_2, p_3))) - F\left(a, t-(\deg(p_3)+\deg\left(\mathrm{lcm}\left(\frac{\mathrm{lcm}(p_1, p_3)}{p_3}, \frac{\mathrm{lcm}(p_2, p_3)}{p_3}\right)\right)\right) \\ &= F(a, t-\deg(\mathrm{lcm}(p_2, p_3))) - F\left(a, t-\left(\deg(p_3)+\deg\left(\frac{\mathrm{lcm}(p_1, p_2, p_3)}{p_3}\right)\right)\right) \\ &= F(a, t-\deg(\mathrm{lcm}(p_2, p_3))) - F(a, t-\deg(\mathrm{lcm}(p_1, p_2, p_3)))). \end{aligned}$$

Back substituting the iterated results of Syzygy method into the Syzygy equation we get the same alternating sum produced by lcm-method.

Thus providing the lcm-lattice method valid.

The following examples are based on the Syzygy method.

Example 4.0.7. *Find the Hilbert function of $M = R/\langle x^2, y^3\rangle$, where $R = k[x, y, z]$.*

We will only need the syzygy $m_{12} = \frac{\mathrm{lcm}(x^2,y^3)}{y^3} = \frac{x^2y^3}{y^3} = x^2$. Computing the Hilbert function in this case requires only one use Theorem 4.0.1, which yields the following:

$$\begin{aligned} \mathrm{HF}\{M\} &= \mathrm{HF}\{R\} - \mathrm{HF}\{R(-\deg(x^2))\} - \mathrm{HF}\{R/\langle m_{12}\rangle(-\deg(y^3))\} \\ &= \mathrm{HF}\{R\} - \mathrm{HF}\{R(-2)\} - \mathrm{HF}\{R/\langle x^2\rangle(-3)\}. \end{aligned} \quad (4.1)$$

Based on the Corollary 1.0.3 we see that the last term in 4.1 it is shifted by 3, so we have

$$\begin{aligned} \mathrm{HF}\{R/\langle x^2\rangle(-3)\} &= \mathrm{HF}\{R(-3)\} - \mathrm{HF}\{R(-\deg(x^2))(-3)\} \\ &= \mathrm{HF}\{R(-3)\} - \mathrm{HF}\{R(-5)\}. \end{aligned} \tag{4.2}$$

Therefore, by substituting 4.2 into 4.1, we have

$$\begin{aligned} \mathrm{HF}\{M\} &= \mathrm{HF}\{R\} - \mathrm{HF}\{R(-2)\} - [\mathrm{HF}\{R(-3)\} - \mathrm{HF}\{R(-5)\}] \\ &= \mathrm{HF}\{R\} - \mathrm{HF}\{R(-2)\} - \mathrm{HF}\{R(-3)\} + \mathrm{HF}\{R(-5)\}. \end{aligned}$$

The Hilbert function of M shows in the last column of the following row-generating table

HF$\{R\}$	$-$HF$\{R(-2)\}$	-HF{R(-3)}	HF{R(-5)}	HF$\{M\}$
1	0	0	0	**1**
3	0	0	0	**3**
6	-1	0	0	**5**
10	-3	-1	0	**6**
15	-6	-3	0	**6**
21	-10	-6	1	**6**
28	-15	-10	3	**6**
36	-21	-15	6	**6**
45	-28	-21	10	**6**
55	-36	-28	15	**6**
66	-45	-36	21	**6**
78	-55	-45	28	**6**
..	..	..	..	..
..	..	..	..	..

Now apply the Syzygy method to quotient rings whose monomial ideal consists of three monomials.

Example 4.0.8. *Finding the Hilbert function of the quotient ring* $M = k[x, y, z]/\langle xz, yz, x^2y\rangle$.

$$\begin{aligned} m_{12} &= \tfrac{\operatorname{lcm}(xz,yz)}{yz} = \tfrac{xyz}{yz} = x, \\ m_{13} &= \tfrac{\operatorname{lcm}(xz,x^2y)}{x^2y} = \tfrac{x^2yz}{x^2y} = z, \\ m_{23} &= \tfrac{\operatorname{lcm}(yz,x^2y)}{x^2y} = \tfrac{x^2yz}{x^2y} = z. \end{aligned}$$

Let $R = k[x, y, z]$.

Using now the syzygy method we can express the Hilbert function of M as follows

$$\begin{aligned} \mathrm{HF}\{M\} &= \mathrm{HF}\{R\} - \mathrm{HF}\{R(-\deg(xz))\} - \mathrm{HF}\{R/\langle m_{12}\rangle(-\deg(yz))\} \\ &\quad -\mathrm{HF}\{R/\langle m_{13}, m_{23}\rangle(-\deg(x^2y))\} \\ &= \mathrm{HF}\{R\} - \mathrm{HF}\{R(-2)\} - \mathrm{HF}\{R/\langle x\rangle(-2)\} - \mathrm{HF}\{R/\langle z\rangle(-3)\}. \quad (4.3) \end{aligned}$$

Observe that the last two terms of 4.3 are equal to the second row of the Pascal table just shifted since

$$\mathrm{HF}\{R/\langle x\rangle(-2)\} \cong \mathrm{HF}\{k[y, z](-2)\}$$

and

$$\mathrm{HF}\{R/\langle z\rangle(-3)\} \cong \mathrm{HF}\{k[x, y](-3)\}.$$

By that way, we have that the Hilbert function of M is shown in the last column of th row-generating table below.

HF$\{R\}$	$-$HF$\{R(-2)\}$	$-$HF$\{R/\langle x\rangle(-2)\}$	$-$HF$\{R/\langle z\rangle(-3)\}$	HF$\{M\}$
1	0	0	0	**1**
3	0	0	0	**3**
6	-1	-1	0	**4**
10	-3	-2	-1	**4**
15	-6	-3	-2	**4**
21	-10	-4	-3	**4**
28	-15	-5	-4	**4**
36	-21	-6	-5	**4**
45	-28	-7	-6	**4**
55	-36	-8	-7	**4**
66	-45	-9	-8	**4**
78	-55	-10	-9	**4**
..	..	..	..	..
..	..	..	..	..

Example 4.0.9. *Compute the Hilbert function of the module*

$$M = k[x, y, z]/\langle x^2z^2, xz^3, xy^4z, x^2y^3z\rangle.$$

As before let $R = k[x, y, z]$. By finding now the syzygies, we have

$$m_{12} = \tfrac{x^2z^3}{xz^3} = x,$$
$$m_{13} = \tfrac{x^2y^4z^2}{xy^4z} = xz,$$
$$m_{23} = \tfrac{xy^4z^3}{xy^4z} = z^2,$$
$$m_{14} = \tfrac{x^2y^3z^2}{x^2y^3z} = z,$$
$$m_{24} = \tfrac{x^2y^3z^3}{x^2y^3z} = z^2,$$
$$m_{34} = \tfrac{x^2y^4z}{x^2y^3z} = y.$$

Based on the syzygy method we have

$$\begin{aligned}
\mathrm{HF}\{M\} &= \mathrm{HF}\{R\} - \mathrm{HF}\{R(-\deg(x^2z^2))\} - \mathrm{HF}\{R/\langle x\rangle(-\deg(xz^3))\} \\
&\quad -\mathrm{HF}\{R/\langle xz, z^2\rangle(-\deg(xy^4z))\} - \mathrm{HF}\{R/\langle z, z^2, y\rangle(-\deg(x^2y^3z))\} \\
&= \mathrm{HF}\{R\} - \mathrm{HF}\{R(-4)\} - \mathrm{HF}\{R/\langle x\rangle(-4))\} - \mathrm{HF}\{R/\langle xz, z^2\rangle(-6)\} \\
&\quad -\mathrm{HF}\{R/\langle z, z^2, y\rangle(-6)\}. \qquad (4.4)
\end{aligned}$$

From 4.4 we can see that

$$\mathrm{HF}\{R/\langle x\rangle(-4)\} \cong \mathrm{HF}\{k[y,z](-4)\} \qquad (4.5)$$

and

$$\mathrm{HF}\{R/\langle z, z^2, y\rangle(-6)\} \cong \mathrm{HF}\{R/\langle y, z\rangle(-6)\} \cong \mathrm{HF}\{k[x](-6)\}. \qquad (4.6)$$

Therefore, 4.5 is given by the 2^{nd} row of the Pacal table shifted down by four and 4.6 is given by the 1^{st} row shifted down by six.

Moreover, in order to find Hilbert function of M we need to find the $\mathrm{HF}\{R/\langle xz, z^2\rangle(-6)\}$. Applying again the syzygy method to the fourth summand on the right hand side of 4.4 we have that

$$m_{12} = \tfrac{xz^2}{z^2} = x.$$

Observe that the shifting is equally distributed in all the terms as follows

$$\begin{aligned}
\mathrm{HF}\{R/\langle xz, z^2\rangle(-6)\} &= \mathrm{HF}\{R(-6)\} - \mathrm{HF}\{R(-\deg(xz))(-6)\} - \mathrm{HF}\{R/\langle x\rangle(-\deg(z^2))(-6)\} \\
&= \mathrm{HF}\{R(-6)\} - \mathrm{HF}\{R(-2)(-6)\} - \mathrm{HF}\{R/\langle x\rangle(-2)(-6)\} \\
&= \mathrm{HF}\{R(-6)\} - \mathrm{HF}\{R(-8)\} - \mathrm{HF}\{R/\langle x\rangle(-8)\} \\
&= \mathrm{HF}\{R(-6)\} - \mathrm{HF}\{R(-8)\} - \mathrm{HF}\{k[y,z](-8)\}
\end{aligned}$$

This way we have the following row-generating table

HF$\{R(-6)\}$	$-$HF$\{R(-8)\}$	$-$HF$\{k[y,z](-8)\}$	HF$\{R/\langle xz, z^2\rangle(-6)\}$
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0
1	0	0	1
3	0	0	3
6	-1	-1	4
10	-3	-2	5
15	-6	-3	6
..	..	..	..
..	..	..	..

Substituting now 4.5,4.6,4.7 into 4.4 we have

HF$\{R\}$	$-$HF$\{R(-4)\}$	$-$HF$\{k[y,z](-4)\}$	$-$HF$\{R/\langle xz, z^2\rangle(-6)\}$	$-$HF$\{k[x](-6)\}$	HF$\{M\}$
1	0	0	0	0	**1**
3	0	0	0	0	**3**
6	0	0	0	0	**6**
10	0	0	0	0	**10**
15	1	1	0	0	**13**
21	-3	-2	0	0	**16**
28	-6	-3	-1	-1	**17**
36	-10	-4	-3	-1	**18**
45	-15	-5	-4	-1	**20**
55	-21	-6	-5	-1	**22**
66	-28	-7	-6	-1	**24**
..	..	..	..	..	..
..	..	..	..	..	..

Finally, the Hilbert function of M is shown in the last column.

Chapter 5

Syzygy method via homological algebra

The short exact sequence that involves ϕ_{x_a} := multiplication by x_a (see [5] page 98) works well with the assemblage row-by-row of a Hilbert function table. That is because the key homomorphism in the short exact sequence is multiplication by a variable followed by natural projection. Consequently the last non-zero object of the short exact sequence is the cokernel of ϕ_{x_a}. This cokernel (we saw in chapter 2) turns out to be the quotient ring corresponding to the row in the Hilbert function table immediately preceding the introduction of the variable x_a. In other words, of the two Hilbert function sequences that the short exact sequence needs to generate the the Hilbert function of $k[x_1, x_2, ..., x_a]/I_a$, one of them (the right-most) is the Hilbert function of $k[x_1, x_2, ..., x_{a-1}]/I_{a-1}$. Therefore any remaining difficulty would be confined to finding the Hilbert function for the kernel of ϕ_{x_a}.

In this chapter we make use of the same set up as in chapter 2. Let $S = \{p_1, p_2, p_3, ..., p_r\}$, where $p_1, p_2, p_3, ..., p_r$ are monomials in the variables $x_1, x_2, ...x_d$. Extend this set of

variables to an infinite set of variables $x_1, x_2, ..., x_d, x_{d+1}, ...$. For integer value a let $S_a = \{p_i \in S : p_i \in k[x_1, x_2, ..., x_a]\}$. Re-index if necessary the set S such that

1. $S'_a \subset S_a$ if $a' \geq a$ and ...

2. For p_i, $p_j \in S_a$, $j > i$ only if the highest power of x_a dividing p_i also divides p_j.

The reader should observe that the first requirement of this re-indexing of the generators of I has the purpose of introducing the generators for the ideals I_a in consecutive order as the variables x_a are introduced one-by-one. The second criteria for the re-index ensures that, as the set S_{a-1} is enlarged to S_a, the new monomials are ordered in (non-strict) increasing order of the power of x_a. This second criteria is done to ensure that the variable x_a does not appear in the syzygies we might need compute as we generate the a^{th}-row of the Hilbert table. Also observe that if $S_a = \emptyset$ then set $I_a = 0$; otherwise set $I_a = \langle p_i \,|\, p_i \in S_a \rangle$. Let $M_a = k[x_1, x_2, ..., x_a]/I_a$. Construct an infinite array whose ath row is the sequence of Hilbert function values of M_a.

Consider the following *short exact sequence* where ϕ_{x_a} is multiplication by x_a, the module $M_a = k[x_1, x_2, \ldots, x_a]/I_a$, and $(0 : x_a)_{M_a} = \ker \phi_{x_a}$,

$$0 \to (0 : x_a)_{M_a}(-1) \to M_a(-1) \overset{\phi_{x_a}}{\to} M_a \to M_a/x_a M_a \to 0.$$

Set $S_0 = \emptyset$ and for $a \geq 1$, if $S_{a-1} \subsetneq S_a$ set

$$U_{x_a} = \{q_i = \frac{p_i}{x_a} \ : \ p_i \in S_a \setminus S_{a-1}\}.$$

If $S_{a-1} = S_a$ then set $U_{x_a} = \emptyset$.

Lemma 5.0.1. $(0 : x_a)_{M_a} = \langle q_i : q_i \in U_{x_a} \rangle_{M_a}$.

Proof. If $U_{x_a} = \emptyset \Leftrightarrow x_a \nmid p_i$ for all $p_i \in S_a \Leftrightarrow \forall g \in M_a, g \neq 0$ then $x_a g \neq 0$.

If $U_{x_a} \neq \emptyset$ the following equivalence holds:

$$\begin{aligned} x_a g = 0 \text{ in } M_a &\Leftrightarrow p_i \mid x_a g \text{ for some } p_i \in S_a \setminus S_{a-1} \\ &\Leftrightarrow q_i \mid g \text{ for some } q_i \in U_{x_a} \\ &\Leftrightarrow g \in \langle q_i \,:\, q_i \in U_{x_a} \rangle \end{aligned}$$

□

Remark 5.0.1. *Observe that if $U_{x_a} = \emptyset$ then from the above lemma follows that* $(0 : x_a)_{M_a} = 0$.

Using the same notation for syzygies as in the previous chapter, namely $m_{ij} = \frac{\text{lcm}(p_i, p_j)}{p_j}$ we now state the following lemma.

Lemma 5.0.2. *A non-zero monomial $g \in (0 : x_a)_{M_a}$ can be written as follows for one and only one $q_i \in U_{x_a}$,*

1. *$g = \alpha_1 q_1$ if $q_1 \in U_{x_a}$*
2. *$g = \alpha_j q_j$ if $q_j \in U_{x_a}$ and $m_{ij} \nmid \alpha_j$ for all $1 \geq i < j$*

and conversely any g satisfying one of the equations above, belongs to $(0 : x_a)_{M_a}$.

Proof. By the previous lemma all we are left to show is uniqueness.
Suppose $g \in (0 : x_a)_{M_a}$, let i be the smallest index such that $q_i \mid g$. Then for any $1 \geq i' < i$, g cannot be written as $g = \alpha_{i'} q_{i'}$.
If $i < j$ and $q_j \mid g$ then

$$\begin{aligned} \alpha_j = \frac{g}{q_j} \text{ but } q_i \mid g \text{ and } q_j \mid g &\Rightarrow \text{lcm}(q_i, q_j) \mid g \\ &\Leftrightarrow \frac{\text{lcm}(q_i, q_j)}{q_j} \mid \frac{g}{q_j} \Leftrightarrow m_{ij} \mid \alpha_j. \end{aligned}$$

Therefore α_j does not satisfy condition 2. □

Theorem 5.0.2. *With the notation of the two lemmas above, the Hilbert function of the annihilator of the homomorphism ϕ_{x_a} satisfies the following formula,*

$$\begin{aligned}\mathrm{HF}\{(0:x_a)_{M_a}\} =& \delta(a)\mathrm{HF}\{k[x_1, x_2, \ldots, x_{a-1}](-\deg q_1)\} \\ &+ \sum_{1<j\in \textit{ Index Set } U_{x_a}} \mathrm{HF}\{k[x_1, x_2, \ldots, x_{a-1}]/\langle m_{1j}, m_{2j}, \ldots, m_{(j-1)j}\rangle(-\deg q_j)\}\end{aligned}$$

where $\delta(a) = 0$ for $q_1 \notin U_{x_a}$, and $\delta(a) = 1$ for $q_1 \in U_{x_a}$.

Proof. If $q_1 \in U_{x_a}$ and $q_1 \mid g$ then $\alpha_1 \in k[x_1, x_1, \ldots, x_{a-1}]/I_{a-1}$ and $\deg(a_1) = b - \deg(q_1)$. But since $U_{x_{a-1}} = \emptyset$, then $I_{a-1} = 0$ which gives us the summand with $\delta(a) = 1$.
If $q_1 \notin U_{x_a}$ then $\delta(a) = 0$ and the first summand is irrelevant.
Moreover, for all $g \in (0 : x_a)_{M_a}$ expressible as $g = \alpha_j q_j$ with $m_{ij} \nmid \alpha_j$, $1 \geq i < j$, then

$$\begin{aligned}\alpha_j &\in (k[x_1, x_2, \ldots, x_{a-1}]/I_{a-1})\,/\langle m_{1j}, m_{2j}, \ldots, m_{(j-1)j}\rangle \\ &\cong k[x_1, x_2, \ldots, x_{a-1}]/\langle m_{1j}, m_{2j}, \ldots, m_{(j-1)j}\rangle.\end{aligned}$$

The last isomorphism being due to the second and third isomorphism theorems. □

Remark 5.0.2. *Observe that if $U_{x_a} = \emptyset$ then the sum in the theorem is zero, i.e.* $\mathrm{HF}\left((0 : x_a)_{M_a}, b\right) = 0$ *for all $b \geq 0$.*

With the Hilbert function for the annihilator $(0 : x_a)_{M_a}$ and the Hilbert function for $M_a/x_aM_a \cong M_{a_1}$ in hand it is straightforward to implement the procedure outlined in chapter 2 to generate the Hilbert function of M_a. For that reason we only show in the next example how to write the Hilbert function of the annihilator in terms of the Hilbert function of simpler quotient rings.

Example 5.0.10. *Use the above theorem to write a sum equivalent to the non-trivial annihilator ideals $(0 : x_a)_{M_a}$, where $I = \langle y^6, x^3y^5, x^2y^2z^2, x^3z, x^2yz^3\rangle$ where the variables are ordered $y, x, z, w_1, w_2, \ldots$.*

Before embarking in the computations we consider if the set of generators of I needs re-indexing given the order we have chosen to introduce the variables (this order is quirky in that the variable y is introduced before the variable x – and was chosen to illustrate we are free to select the order in which the variables are introduced). The criteria that $S'_a \subset S_a$ if $a' \geq a$ is satisfied by the order in which the monomials generating I are listed. However the second criteria; namely, that for p_i, $p_j \in S_a$, $j > i$ only if the highest power of x_a dividing p_i also divides p_j, requires that the order of the monomials $x^2y^2z^2$ and X^3z be swapped. Observe that adjusting the indexing to satisfy the second criteria does not interfere with the first criteria. In other words, after swapping the third and fourth monomials we get $I = \langle y^6, x^3y^5, x^3z, x^2y^2z^2, x^2yz^3\rangle$ which satisfies both re-indexing criteria. The first annihilator is $(0 : y)_{M_y}$, where $M_y = k[y]/\langle y^6\rangle$.
In this case $U_y = \{y^5\}$ and

$$\mathrm{HF}\{(0 : y)_{M_y}\} = \mathrm{HF}\{k(-5)\}.$$

The second annihilator is $(0 : x)_{M_x}$ where $M_x = k[y, x]/\langle y^6, x^3y^5\rangle$. In this case $U_x = \{x^2y^5\}$ and there is only the syzygy $m_{12} = y$ to consider. Therefore,

$$\mathrm{HF}\{(0 : x)_{M_x}\} = \mathrm{HF}\{k[y]/\langle y\rangle(-7)\} = \mathrm{HF}\{k(-7)\}.$$

The third and last non-trivial annihilator is $(0 : z)_{M_z}$, where

$$M_z = k[y, x, z]/\langle y^6, x^3y^5, x^3z, x^2y^2z^2, x^2yz^3\rangle.$$

In this case $U_z = \{x^3, x^2y^2z, x^2yz^2\}$. The syzygies to consider are

$$
\begin{array}{llll}
m_{13} = y^6 & m_{23} = y^5 & & \\
m_{14} = y^4 & m_{24} = xy^3 & m_{34} = x & \\
m_{15} = y^5 & m_{25} = xy^4 & m_{35} = x & m_{45} = y.
\end{array}
$$

Therefore,

$$
\begin{aligned}
\mathrm{HF}\{(0:z)_{M_z}\} =& \mathrm{HF}\{k[y,x]/\langle y^6, y^5\rangle(-3)\} \\
&+ \mathrm{HF}\{k[y,x]/\langle y^4, xy^3, x\rangle(-5)\} \\
&+ \mathrm{HF}\{k[y,x]/\langle y^5, xy^4, x, y\rangle(-5)\} \\
=& \mathrm{HF}\{k[y,x]/\langle y^5\rangle(-3)\} \\
&+ \mathrm{HF}\{k[y]/\langle y^4\rangle(-5)\} + \mathrm{HF}\{k(-5)\}.
\end{aligned}
$$

As the reader can see this approach is quite close in spirit to the Syzygy method discussed in the previous chapter. The only significant difference is in the tools used to prove it. All the information about the Hilbert function was obtain from the syzygies. We must conclude that we have developed two different approaches to computing the Hilbert function of a quotient ring: the lcm-lattice method and the syzygy method.

Bibliography

[1] Margherita Barile and Eric W. Weisstein. *Hilbert Function.* http://mathworld.wolfram.com/HilbertFunction.html.

[2] Richard M. Foote David S. Dummit. *Abstract Algebra, 3rd Edition.* John Wiley and Sons, INC., 2003.

[3] D Eisenbud. *Commutative Algebra with a View toward Algebraic Geometry.* Springer-Verlag, 1995.

[4] R. M. Wilson J. H. van Lint. *A Course in Combinatorics.* Cambridge University Press, 1992.

[5] Rafael H. Villarreal. *MONOMIAL ALGEBRAS.* Marcel Dekker, Inc., 2001.