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A NEAR TIME OPTIMAL
ADAPTIVE CONTROL ALGORITHM
FOR SECOND ORDER SYSTEMS

BY

Carlton J. Warren

A Thesis Submitted

in

Partial Fulfillment

of the

Requirements for the Degree of

MASTER OF SCIENCE

in

Electrical Engineering

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DEPARTMENT OF ELECTRICAL ENGINEERING

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ABSTRACT

During the past few years, many non-linear and/or adaptive control algorithms have been developed for industrial processes. Many have been rather complex schemes either requiring or specifically developed for an on line digital computer. As an alternative to such systems, at least on low order plants, the development of a near time optimal, adaptive control algorithm is proposed. This rule must encompass a significantly large group of the systems to be encountered and yet be simple enough for hardware implementation as a single loop controller.

The author's attention is confined primarily to systems whose transfer functions may be approximated by

$$\frac{K}{S^2+AS+B}, \text{ and whose inputs (setpoints) and disturbances}$$

are essentially step functions. System parameter variations are considered slow relative to the frequency of disturbance inputs or setpoint changes. The desired, or optimal, closed loop response for these systems is assumed to be the fastest possible response to a step input, with no overshoot. This time optimal deadbeat response is unique for a system with given dynamics and fixed accelerating and braking power sources. An algorithm

is developed which provides total response time within a few percent of the time optimal deadbeat response on any system within the general classification. Additionally, the algorithm is adaptive and need not be tuned or adjusted in any way at startup or to compensate for system age or drift.

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INTRODUCTION

In many processes the most desirable closed loop control is that which provides the fastest possible response to a step change in input or disturbance, with little or no overshoot. Fastest possible in this case refers to the theoretical limitation imposed by the system dynamics and the available drive or power sources, rather than the characteristics of any control device. Response to dynamic inputs other than step functions is not a concern as setpoints* remain fixed in normal operation and are changed only infrequently, from one fixed operating point to another.

The desired response discussed above may be obtained in general by applying maximum input to the system up to the time when further application makes overshoot unavoidable and then applying the one input sequence which will bring the system to rest at the new operating point without overshoot. Any other input sequence will result in overshoot. In a first order system, full power may be applied until the system reaches the new operating point, followed by the input required to maintain that operating point. A second order system may be driven

*Setpoints are controller inputs corresponding to desired system outputs.

at full power to the state from which continuous application of minimum power or maximum reverse power is required to bring the system to rest at the new operating point. This transient* must be followed by the input required to maintain the new operating point. Higher order systems require a unique sequence of full forward-full reverse power applications to follow this 'fastest' trajectory, again followed by the input required to maintain the new operating point. In every case this minimum transient from one state to another is uniquely determined by the system transfer function, the maximum and minimum inputs available and the initial and final states. Such a transient is usually called time optimal and will be referred to as such throughout this work.

While the time optimal response may be the most desirable, it cannot be obtained with standard 3-mode** control. Such linear control may be defined to provide a time optimal response between two specific operating points

*"Transient" will denote the sequence of events occurring between any starting state and the stable state to which the system is driven.

**Proportional plus integral plus derivative control.
 Controller output = $AE + B \int_0^t E dt + C \frac{dE}{dt}$, where E is error.

in certain systems with fixed parameters*, but this control will not provide an optimal response between any other points. Control will also deteriorate with any change in system parameters. In a system with time varying parameters, 3-mode controllers must be adjusted for stability under the worst of the varying conditions. Response under more favorable conditions must therefore suffer to guarantee stability in the worst case. .

A review of the literature^{1,2} has indicated that in many cases the control of low order single loop systems (particularly adaptive and/or time optimal control) has been accomplished using algorithms of such complexity that a digital computer is required. Use of these techniques seems a tremendous waste of computing power and money, unless many such loops are to be controlled by one computer. Even multi-loop computer control suffers from a reliability standpoint, since all loops are dependent on a single critical element. As an alternative to such systems, at least on low order plants, the development of a near time optimal adaptive control algorithm is proposed. This rule will encompass a significantly large group of the

*That is, a "linear" controller's proportional and derivative gains may both be set very high, to cause controller saturation, and with a ratio such that the controller output switches polarity at the proper time.

systems to be encountered and yet be simple enough for hardware implementation as a single loop controller. Such an algorithm will allow the production of a controller competitive in cost and similar in mechanical configuration to present 3-mode analog controllers but providing some of the more desirable control characteristics of the computer based systems discussed. Since the largest application for this device would be the industrial process control field, attention shall be confined to that area.

The main elements of a typical industrial control loop are shown in figure 1. The plant is of course the boiler, motor or other device or process being controlled. V1 is the temperature, pressure, position, speed, or similar output parameter being controlled. R1 is the controlling or drive power for the plant. This may be fuel flow, electrical power, steam pressure or any number of other quantities. In every case however, there is some limitation on the maximum and minimum drive available. Perhaps a maximum fuel flow rate, an electrical voltage or current equal to that of the main supply, or a maximum safe input to the system is the limitation. The minimum fuel flow rate is zero, but the concept of a negative or reverse input (and maximum limit) is inherent in electrical and other types of system drive. The

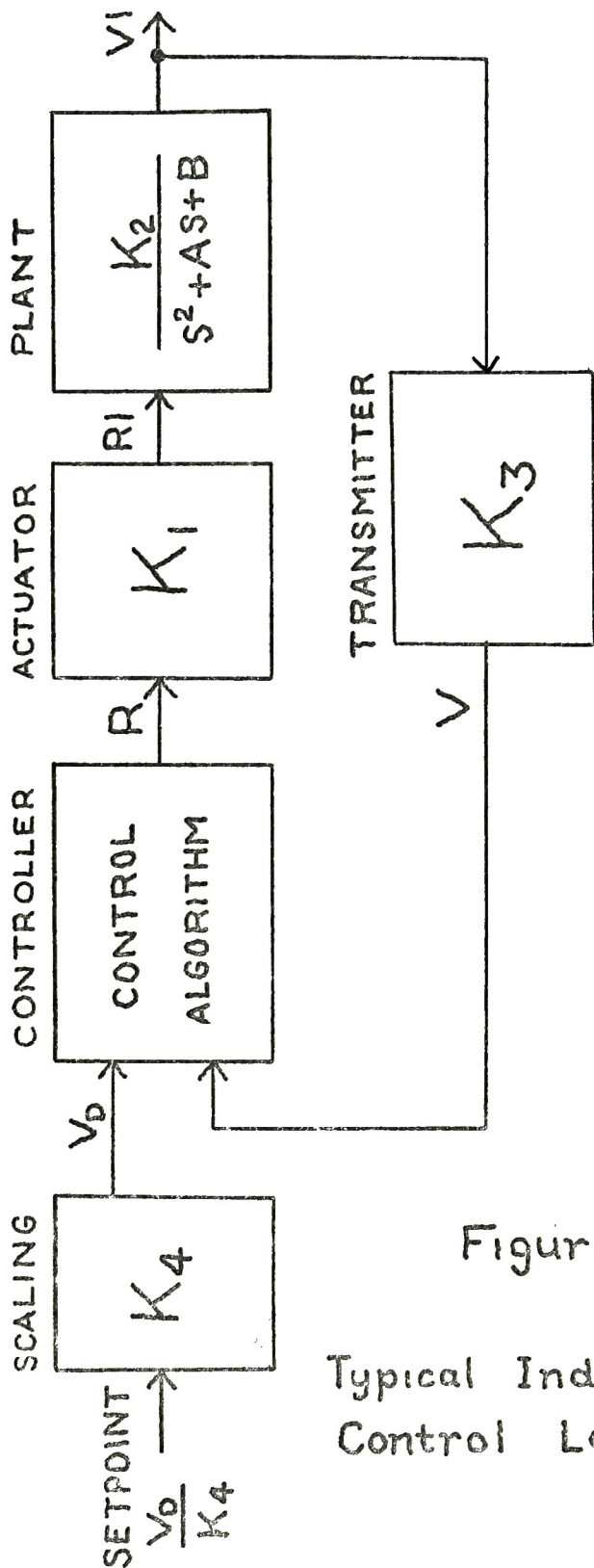


Figure 1

Typical Industrial
Control Loop

system actuator controls the drive power anywhere in the minimum to maximum range, in response to its own control input R . Generally, the actuator cannot drive the system past its input limits, and in fact may set those limits. A motor driven valve for instance, cannot be driven beyond fully open or fully closed. In most cases the dynamic characteristics of the actuator are not significant when compared to those of the plant, and its transfer function is indicated as simply the gain K_1 . Similarly, the transmitter may almost always be represented as a gain K_3 . Its function is to monitor the controlled variable and produce an output signal V proportional to the controlled variable and suitable for use by the controller. A thermocouple transmitter for instance, might be used to measure a process temperature and produce a proportional voltage or current signal, in a standard range such as 4 to 20 milliamps. The setpoint is usually the position of a control knob on a standard analog controller, but could be a remote electrical or pneumatic signal. In any case it represents the desired output V_1 . The scaling block converts the setpoint to a proportional signal V_D for use by the controller. In an electronic controller with a control knob as a setpoint, the scaling would be simply an internal potentiometer. The controller

itself computes the error $E = V_D - V$, and produces a corrective output R (the actuator input) according to some algorithm. This element may be the computer discussed, a specially built "hard wired" computer (analog or digital), or a standard 3-mode controller. The 3-mode controller uses the algorithm

$$R = AE + B \int_0^t E dt + C \frac{dE}{dt}, \quad .$$

where A , B , and C are manual adjustments used for "tuning" the controller to a particular process. It should be recognized that since the analog controller is a basically linear device, its output must spend considerable time in its linear range in driving the system to a new operating point. Since a time optimal response requires the controller output to be at one or the other of the saturation limits throughout the transient, linear control cannot produce a time optimal response*. Two- or three-mode control parameters are often adjusted for the fastest response (defined as time to the first peak, or to attain some error criterion) possible while remaining within some tolerable level of overshoot.

*With the exception of a first order system, where a very high proportional gain may be used to provide a near time optimal response.

This often involves temporary saturation in the system or in the controller. When this is the case, the control system performance will differ for different sized input steps and different operating points. Finally, since the linear controller's parameters must be adjusted for best control of a particular system, time varying system parameters present obvious problems.

As indicated, this work is concerned primarily with those systems for which the normal inputs are step functions and for which the desired output response is time optimal. On the premise that a large number of the described systems may be approximated as second order with no zeros, the work is further limited to

the systems $\frac{V(s)}{R(s)} = \frac{K_2}{s^2 + As + B}$, where A or B or both

may be zero. First order systems will also be considered as a limiting case. From this point on, the "system" referred to will include the plant, the actuator, and the transmitter, so that the systems of interest of

$$\frac{V(s)}{R(s)} = \frac{K_1 K_2 K_3}{s^2 + As + B} = \frac{K}{s^2 + As + B} .$$

Figure 2a shows the time optimal response, as defined previously, for a typical second order system. The system is first accelerated toward the new operating point by maximum input R , and then braked, by maximum reverse R , until output V comes to rest at the new operating point. R then becomes fixed at the value (not necessarily zero of course) required to maintain that output. Note that the time optimal response is by definition a deadbeat response. That is, the input (R) and output (V) transients have a finite number of cycles, or beats. The response of figure 2b might find equal or greater acceptance in certain applications. In this case the optimal or desired response is that which will meet and maintain a given error criterion in the least amount of time. This criterion may be met in slightly less time than that afforded by the figure 2a response, since the overshoot and oscillatory "tail" of the 2b response are within the allowable error. It should be recognized that the algorithm resulting from the present work, although intended to produce (a close approximation to) the time optimal deadbeat response, could produce the response of figure 2a with only slight modification.

The control concept to be developed involves a high gain limiting amplifier, whose limits represent the

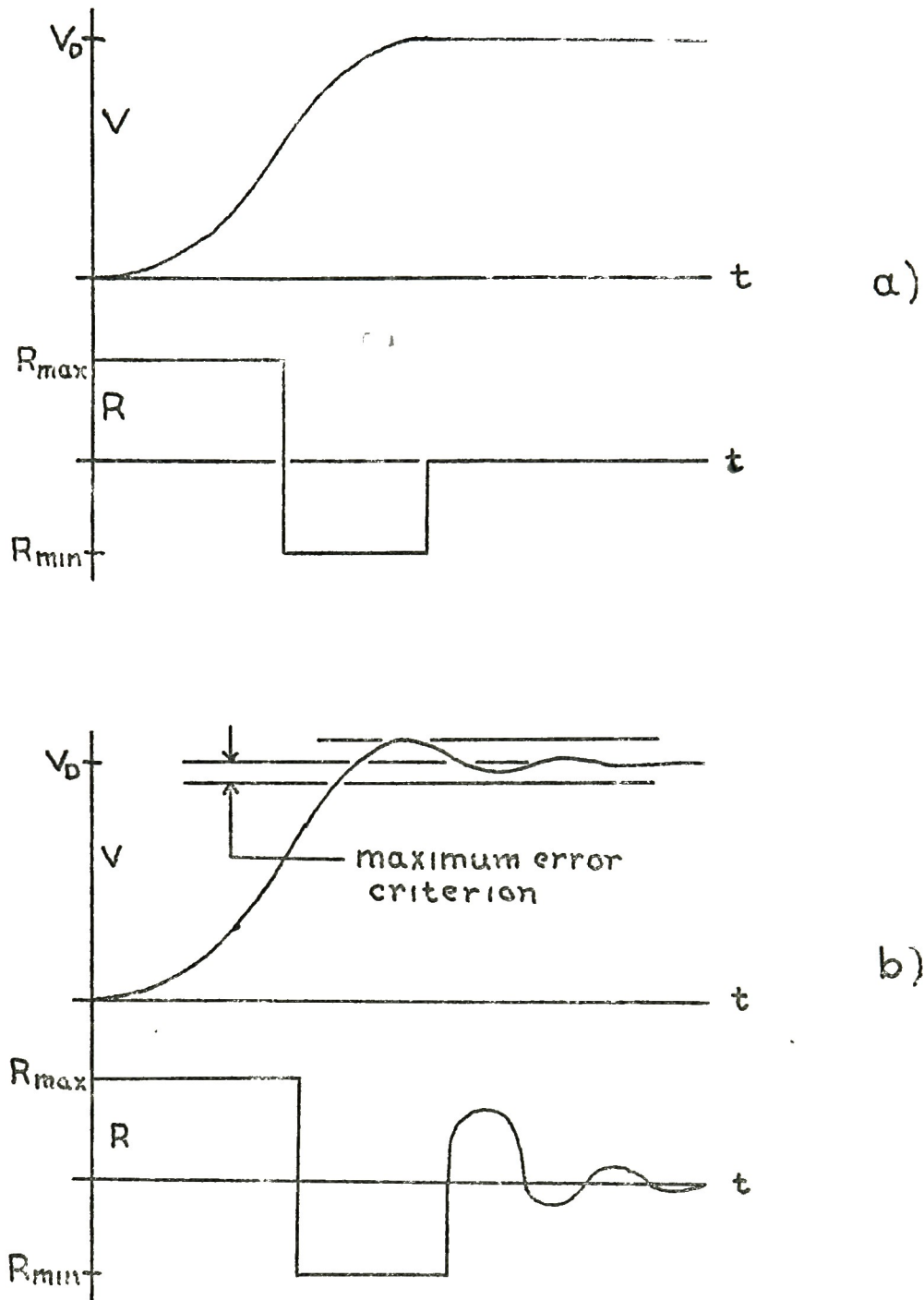


Figure 2 Optimal Responses
 a) deadbeat response b) minimum time to maximum error criterion

input saturation limits of the controlled system.

The amplifier, whose output is the controller

output, is driven by a modified error signal. This

error signal consists of the (scaled) setpoint V_D

minus a predicted final system output V_F . The

prediction is based on measurements of the system's

current and, to a lesser degree, past states.

It represents the final output if maximum braking

input* were applied from the present state until the

"velocity" $\frac{dV}{dt}$ is equal to zero.

Several members of the general family of systems will

be investigated as models from which to define the

rule for prediction. An analog simulation will be

performed with $\frac{V(S)}{R(S)} = \frac{K}{S^2}$ as the model. Results

of the simulation will be analyzed and appropriate

modifications and additions made to the rule in

order to obtain acceptable performance on all systems

of the general form. Suggestions will be made for

*Braking input is that input which drives dV/dt to zero. Accelerating input increases the magnitude of dV/dt .

extension of the rule to include higher order systems and systems with lead terms, and several potentially valuable simplified forms of the rule will be discussed.

The final form of the algorithm will use the predicted

output $V_F = V + \frac{\dot{V}|\dot{V}|}{2|R_b K - BV|}$, where K and B (system

parameters) are defined (approximated) in terms of \ddot{V}^* at transition times (where the limiting amplifier switches from full power into its linear region or vice-versa, as in figure 2) and R_b is available braking input (one of the amplifier limits). It will provide a very close approximation to the time optimal deadbeat response on all systems of the general form

$$\frac{V(S)}{R(S)} = \frac{K}{S^2 + AS + B},$$

where K may have any value and A or B or both may be zero (or any other value). The storage and arithmetic capability required suggest the use of the "microcomputer" concept for implementation. Evidence will be presented to show that this is not inconsistent with the desire for a cost competitive, single loop, "hardware" controller.

$$*\dot{V} = dV/dt, \quad \ddot{V} = d^2V/dt^2$$

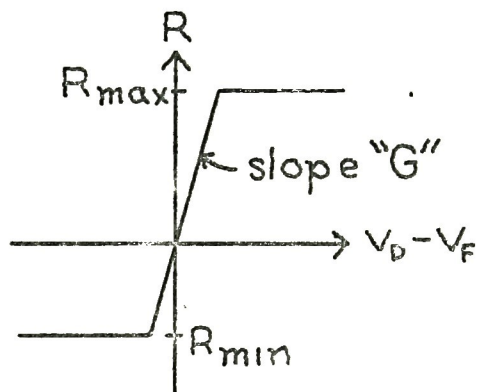
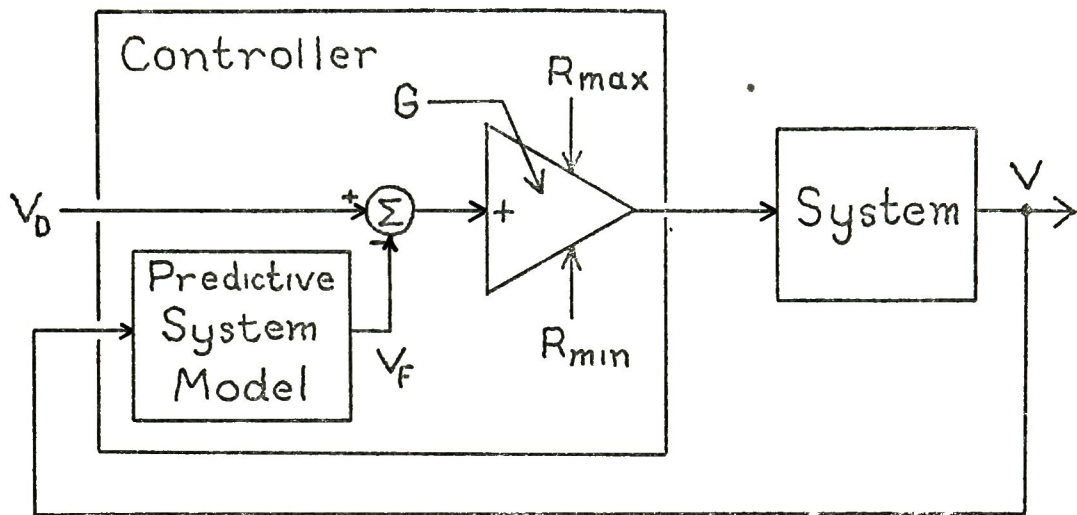
PROJECT DEFINITION

The project's objective has been defined as the development of a control algorithm which will provide adaptive time optimal control of the systems

$$\frac{V(S)}{R(S)} = \frac{K}{S^2 + AS + B} \cdot$$

To produce the time optimal response, the controller must provide an appropriate sequence of two output levels corresponding to system input saturation. Therefore it may be said to operate in a switched mode during the transient. However, the system output must be maintained once the transient is past and the controller must therefore be able to produce any output between the two saturation limits. A controller configuration meeting these requirements is shown in figure 3. The controller consists of a limiting amplifier and a mathematical model used for predicting the future value of the system output V . The precise function of the model is to predict the final value of V should maximum braking input be applied to the system from the current instant of time until \dot{V} were equal to zero. This projected final value is designated V_F . The limiting amplifier is driven by the modified error signal $V_D - V_F$.

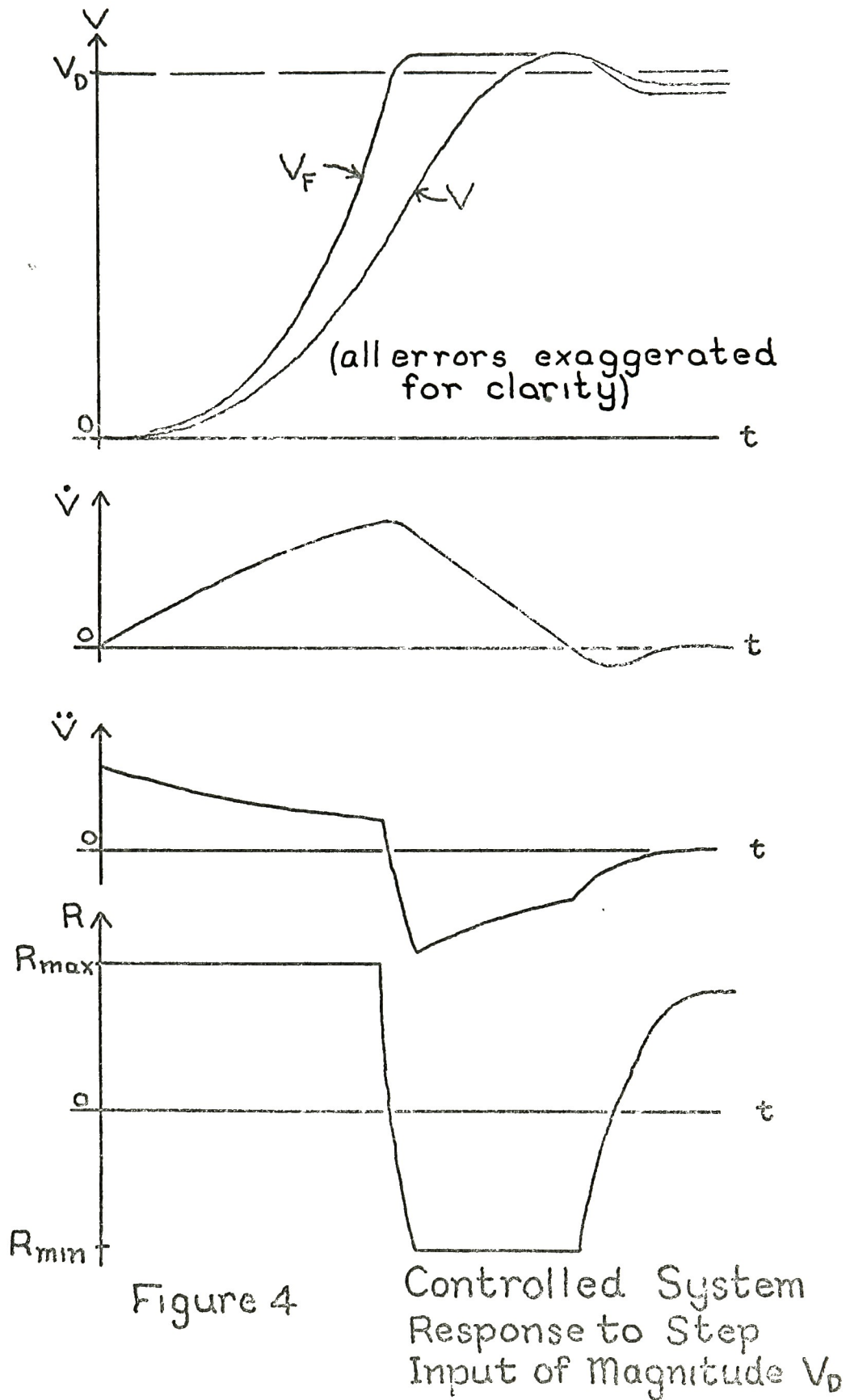
Figure 3
Proposed Controller Configuration



Limiting amplifier
transfer characteristic

It has a very high gain G so that a small error will drive the amplifier to one of its saturation limits. These limits are adjusted to equal the saturation inputs of the system, or to other desired limits within the bounds of input saturation.

Figure 4 illustrates the controlled systems response to an input (V_D) step function. The limiting amplifier is held in saturation at R_{\max} until its driving signal $V_D - V_F$ approaches zero. At this point R crosses rapidly through its linear zone to saturation at R_{\min} . V_F is by definition the value of V at which \dot{V} becomes zero if R_b is applied until $V = V_F$. V_F therefore remains constant after R reaches saturation at R_{\min} , since R_{\min} in this case is R_b . V_F is slightly greater than V_D at this point because of the finite gain G of the limiting amplifier. The finite gain causes R to leave saturation at R_{\max} before $V_F = V_D$ and requires $V_F > V_D$ to achieve saturation at R_{\min} . As a result the system is accelerated to a velocity \dot{V} just short of the theoretical limit before switching, and full braking input (R_{\min}) is not applied until the small overshoot is unavoidable. When $\dot{V} = 0$, $V_F = V$, and as \dot{V} becomes negative, R_{\max} becomes



R_b .* Since $V_D - V_F$ was just large enough to maintain saturation when $\dot{V} = 0$, R now leaves saturation and remains in its linear region. Again because of the gain G , the output comes to rest at a slight error. All three of the errors noted are exaggerated for clarity in figure 4. They become insignificant with a sufficiently large value of G , the price being an increase in "noise" on the controller output R at equilibrium. This potential problem could be dealt with by switching to a less responsive control mode when near equilibrium.

In order to provide adaptive control, the predictive equation used in the controller must provide valid results with any of the systems in the general classification. It was felt that because of the basic similarity in second order responses to a step input, a single model system, appropriately scaled, might be used to predict the future output of the controlled system. Only for the case where the controlled system is identical to the model system except for scale, would perfect results be achieved. However, the model would provide increasingly correct information as the setpoint was neared, so prospects for rapid damping should be good. Scaling would be obtained

*Since \dot{V} is being increased in magnitude by R_{min} .

on a continuous basis, or as required, by the controller itself, as opposed to the externally programmed parameters (proportional, integral and derivative gains) of conventional controllers.

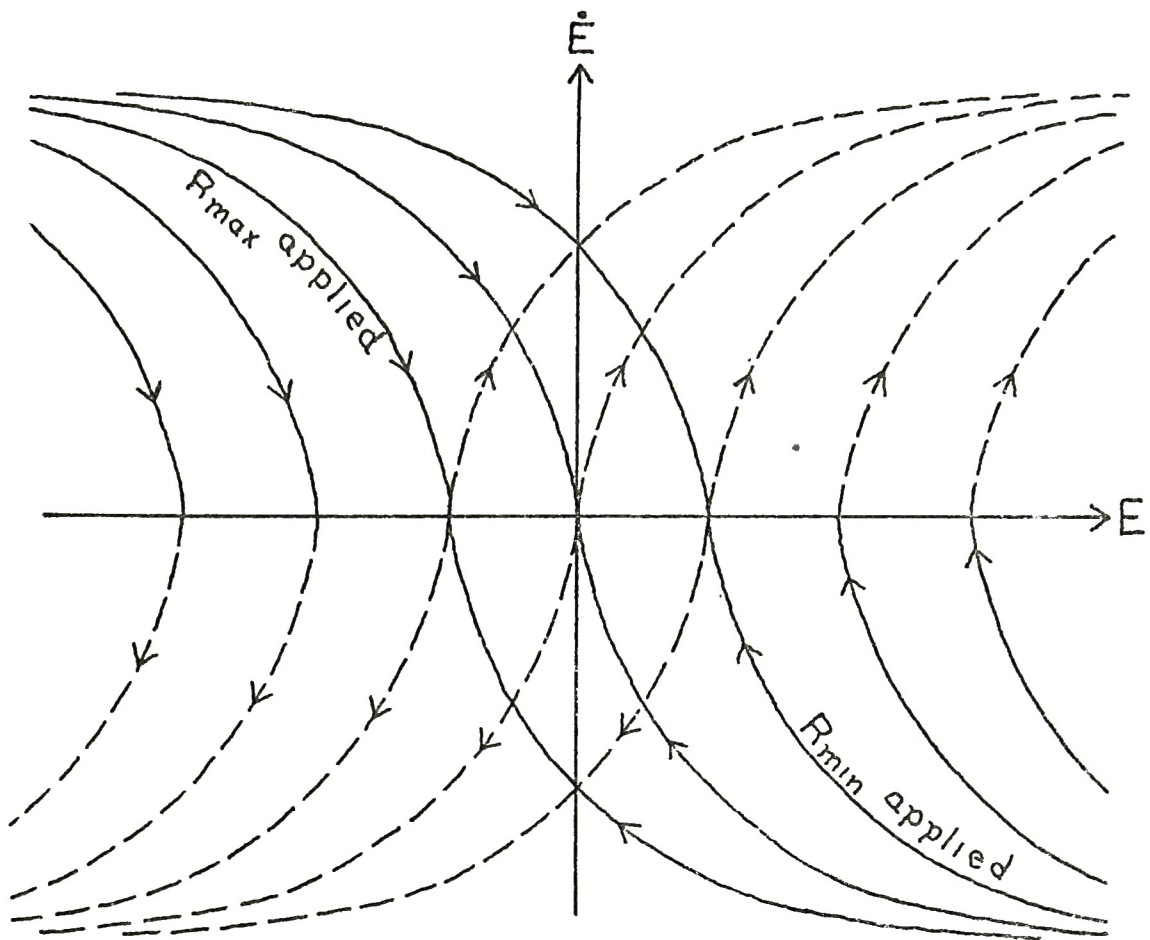
For a given system, the exact predictive equation may be developed by first obtaining the equation for the systems response to an input step function. This may be differentiated and set equal to zero. The resulting equation can be solved for the time (as measured from the application of the step) required to force \dot{V} to zero. This time, hereafter called T_F , will be in terms of the system parameters, the magnitude of the input step, and \dot{V} at the time of application of the step. T_F may then be substituted into the original equation for V , to obtain an equation for V_F in terms of the system parameters, the magnitude of the input step, and the system state (V and \dot{V}) at the time of application of the step. The input step function referred to is of course the application of maximum braking input to the system when $V_F = V_D$. This procedure is illustrated for several systems in Appendices 1-4.

A plot of error $E = V_D - V$ versus \dot{E} is referred to as a phase plane plot. Since $\dot{E} = -\dot{V}$, such a plot is easily obtained from the equation for V_F , given a particular V_F , and

either R_{\max} or R_{\min} . For a typical second order system, several such plots are shown in figure 5. Note that application of R_{\max} drives \dot{E} negative (\dot{V} positive) while application of R_{\min} drives \dot{E} positive (\dot{V} negative). The intersection of each trajectory with the E axis is $V_D - V_F$. The dotted portion represents the trajectory the system would follow if the same input were still applied after $\dot{E} = 0$. Any system transient obtained by driving the system at R_{\min} or R_{\max} must follow a trajectory in one of the two families. A time optimal trajectory between any two equilibrium states (where $\dot{E} = 0$) or from any state to an equilibrium state must follow a unique path consisting of one segment from each of the two families.* Phase plane trajectories are thus useful for the visualization of time optimal concepts.

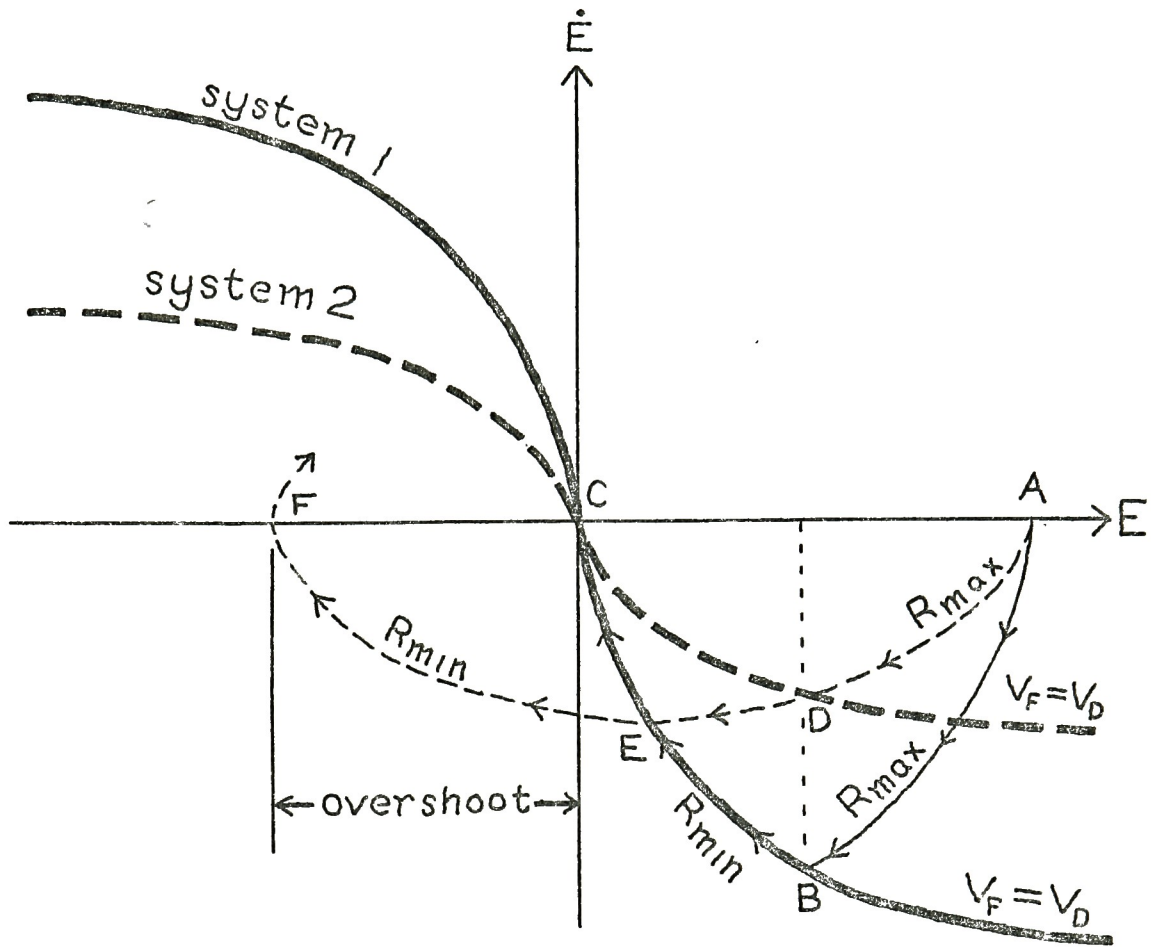
The heavy solid line on figure 6 is the solution for $V_F = V_D$, for a particular V_D and a particular system; system 1. Assume that system 1 were driven from rest at state A to equilibrium at state C (zero error, zero rate of change of error) in a time optimal manner.

*An exception is the trivial case where both points lie on the same segment.



$$E = V_D - V \quad \dot{E} = -\dot{V}$$

Figure 5
Phase Plane Plots for Typical
Second Order System



→→→→→
A B C

controlled response of system 1

→---→---→---
A E F

controlled response of system 2

---→---→---
A D C

optimal response for system 2

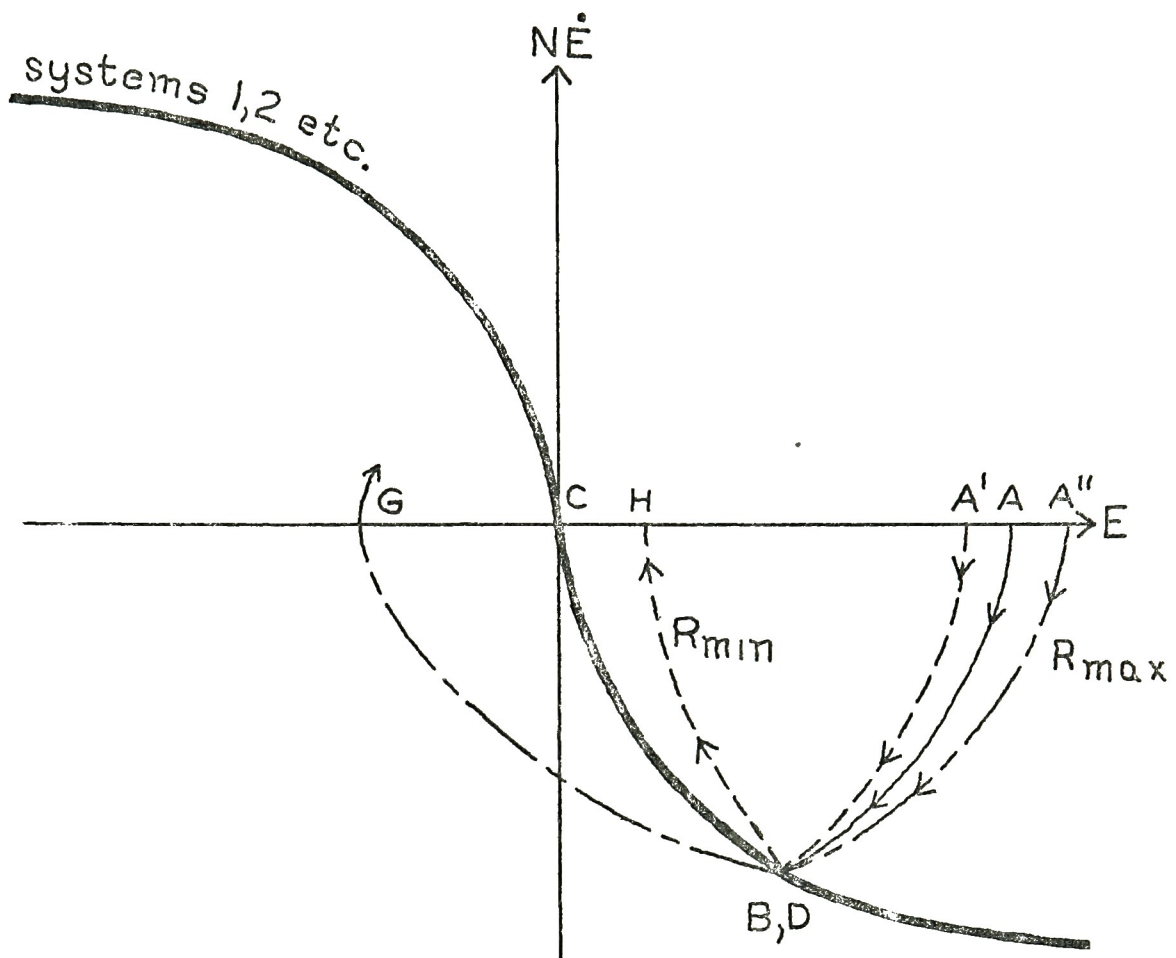
Figure 6

Controlled System Response, with Fixed V_F
used in Controller of Figure 3

(V_F used is that derived for system 1 above)

It would be forced along the solid line by R_{\max} until it reached state B, the time optimal switch point. At this instant R_{\min} would be applied and the system state would follow the heavy line to the origin. If this equation* for V_F were used in the controller of figure 3 time optimal control would be realized on system 1. Now consider system 2 (of figure 6). This system differs from system 1 only in time scale and/or gain, however, if the above control were applied to system 2, switching would occur at state E resulting in the overshoot shown. The optimal switch point is of course D. On the other hand, if V_F as used in the controller is written in general terms rather than the specific gains, time constants, etc. of system 1, and if the controller is able to measure these parameters either continuously or at reasonable intervals, then V_F becomes adaptive. It will be shown that sufficient information for this process is contained in V , \dot{V} , and \ddot{V} . The addition of such scaling allows the "normalization" of the phase plane plot, so that all systems differing from system 1 only in gain or time scale will follow the same normalized trajectory. The vertical axis in the normalized phase plane will be designated NE, as in figure 7.

*That is, V_F for system 1, as derived by the procedure discussed previously.



- systems 1,2 etc.
 - - - - - } system 3, more stable than 1,2
 - - - - - }
 - . - . - . system 4, less stable than 1,2

Figure 7
Controlled System Response, with Adaptive V_F used in Controller of Figure 3

(V_F derived in general terms for systems 1,2,etc.)

With the adaptive form of V_F , the controller would provide time optimal response when driving any system of the same form as the model, system 1. For instance, if system 1

were $\frac{V(S)}{R(S)} = \frac{4}{S^2+7S}$, then time optimal control would

be provided on all systems of the form $\frac{V(S)}{R(S)} = \frac{K}{S^2+AS}$.

As mentioned earlier, it is felt that use of a V_F derived from a single model system might provide acceptable performance on all systems of interest in this study, as well as near ideal performance on those differing only in scale. In addition to the trajectory for model system 1 and all similar systems, figure 7 also shows trajectories for systems 3 and 4. System inputs are switched from R_{\max} to R_{\min} at state B, as dictated by V_F . For want of a better term, system 3 shall be described as more stable than the model system, and system 4 as less stable than the model. For instance, in relation to the above example

of systems 1 and 2, system 4 might be $\frac{V(S)}{R(S)} = \frac{2}{S^2}$,

or $\frac{V(S)}{R(S)} = \frac{K}{S^2}$. The less stable system would overshoot

the zero error point before coming to rest ($\dot{E} = 0$) while

the more stable system would never reach zero error. Actually, the controller of figure 3 would force the more stable system to the origin along the $V_F = V_D$ line, or model system trajectory, by simply applying less than maximum braking input to the system after reaching state B. However, since R_{\min} is maximum braking input, the controller can do nothing to reduce the overshoot of system 4. Figure 8a shows the path the controlled system 4 would take to the origin if V_F were not adaptive, while figure 8b illustrates the increased damping provided by the adaptive V_F . The adaptive V_F is increasingly close to the ideal for system 4 as the controlled system nears the origin. Briefly, this occurs because for relatively small* time intervals the difference in the step responses of various second order systems are primarily differences of scale. This is illustrated in figure 9.

With the control approach established, the problem becomes one of choosing the model system and performing the mathematical manipulations required to use it in the manner described. It is apparant from the phase plane considerations that performance will be quite sensitive

*Small relative to system speed.

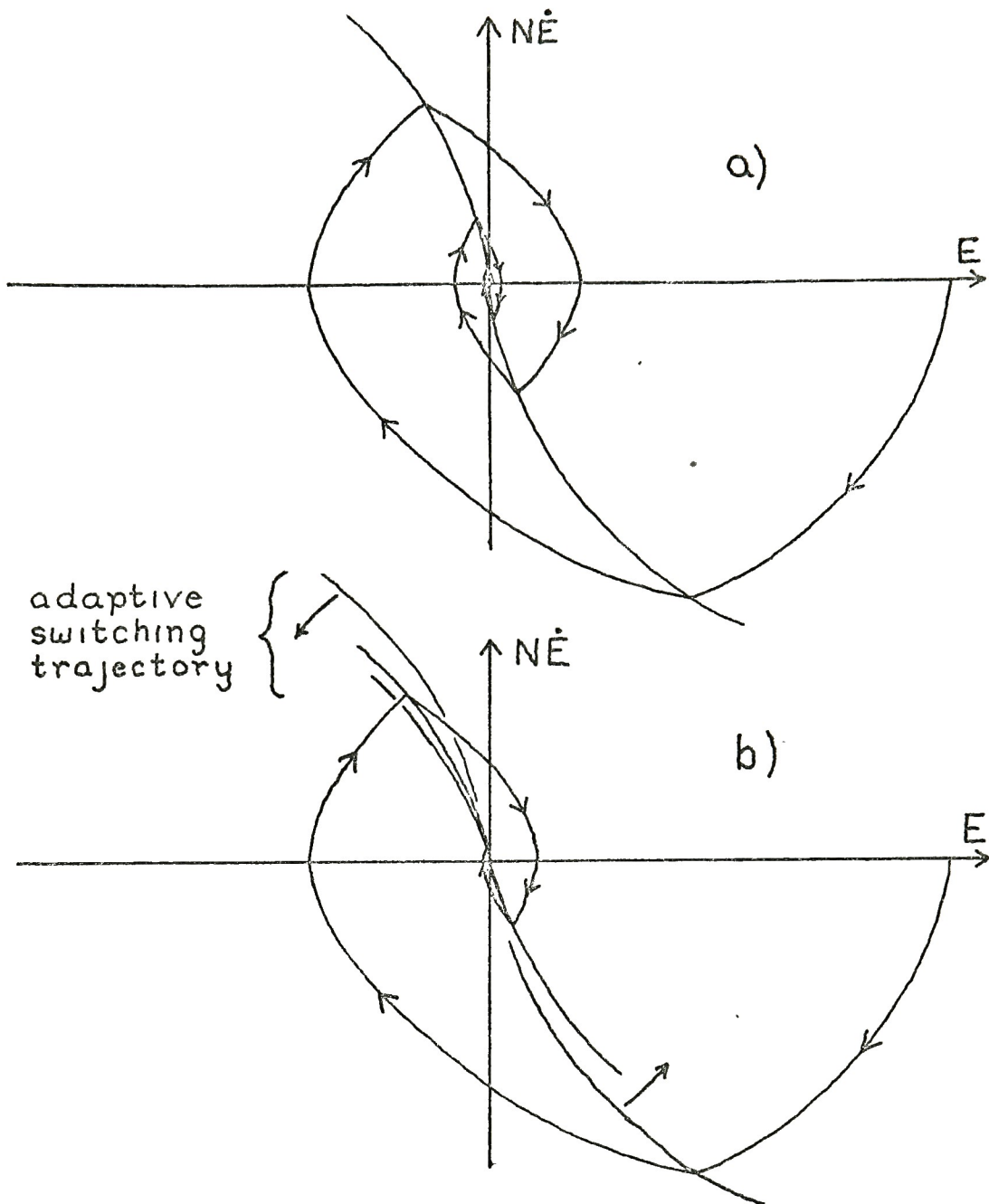


Figure 8

Controlled Response of System 4

a) fixed V_F b) adaptive V_F

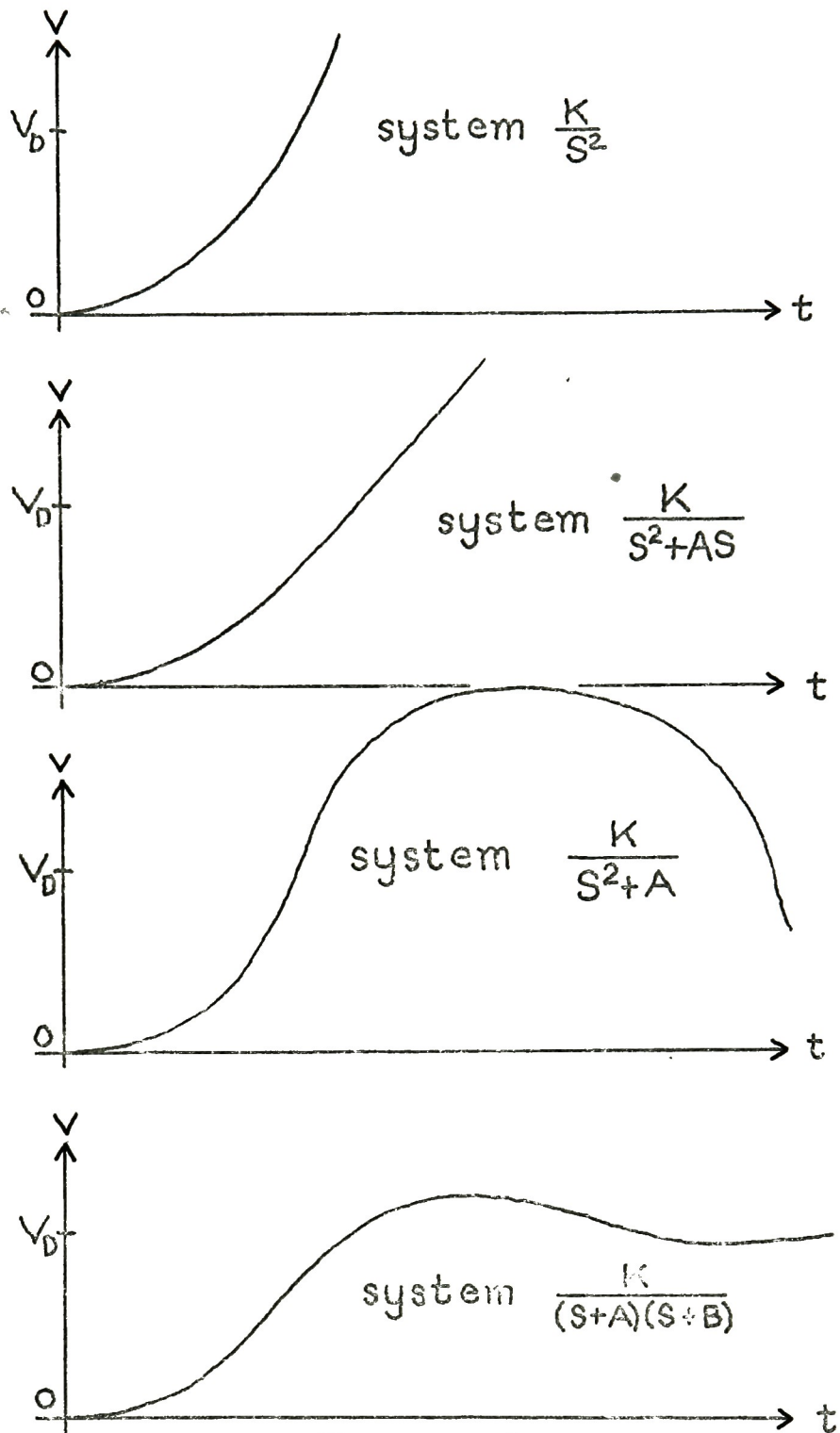


Figure 9

Response of Various Second Order Systems to Input Step Function of Magnitude V_D

to the model used. Once the model is chosen and the equations of the algorithm determined, the model must be evaluated both for performance in the controller and ease of implementation. Modifications and compromises will be necessary to achieve a practical result.

INITIAL DEFINITION OF THE ALGORITHM

Four basic transfer functions break out of the general

form $\frac{K}{S^2+AS+B}$. These are $\frac{K}{S^2}$, $\frac{K}{S^2+B}$, $\frac{K}{S^2+AS}$

and $\frac{K}{(S+A)(S+B)}$. Each of these was considered, at least

briefly, as the model system for the algorithm. The

system $\frac{K}{(S+A)(S+B)}$ was considered first, as it was

felt that this might be the most commonly encountered system. Its equations quickly became unwieldy and work was halted. Further study disclosed the fact that considerable work had been done in time optimal control of relay servomechanisms^{2,3} using the simple servo

transfer function $\frac{K}{S(S+A)}$. This, as well as its relatively

simple equations, made it a likely candidate.

As discussed in the previous section, our approach to the control problem will be to measure the state of the controlled system, assume that the model is in that state, and apply to the controlled system the input which would

drive the model system to equilibrium along time optimal trajectories. This was illustrated previously on the phase plane of figures 5-8 and is now illustrated on the time axes of figure 10. With reference to figure 10a, R_{\max} (maximum accelerating input) is applied to the system until it reaches a state from which the model would require continuous application of R_{\min} (maximum braking) to bring it to rest ($\dot{V} = 0$) at $V = V_D$. The system trajectory is shown dotted up to this point. The solid line, for both V and R , indicates the time optimal* model trajectory which coincides with the system trajectory at the switch point. From the switch point on then, the solid line is the trajectory predicted for the controlled system by the model. V_F has become equal to V_D . As V_F becomes slightly greater than V_D , the controller output R (see figure 3) is suddenly driven negative. However, it drives only as far negative as is necessary to keep $V_F = V_D + \frac{R}{G}$, where G is the controller amplifier gain.

Since this may be accomplished with less than maximum braking, the system follows the model trajectory (for V and \dot{V} , not for R) to equilibrium. Note that the total controlled system transient is optimal up to the switch point, since the system is driven by R_{\max} , and suboptimal

*Time optimal for the model or systems of similar form.

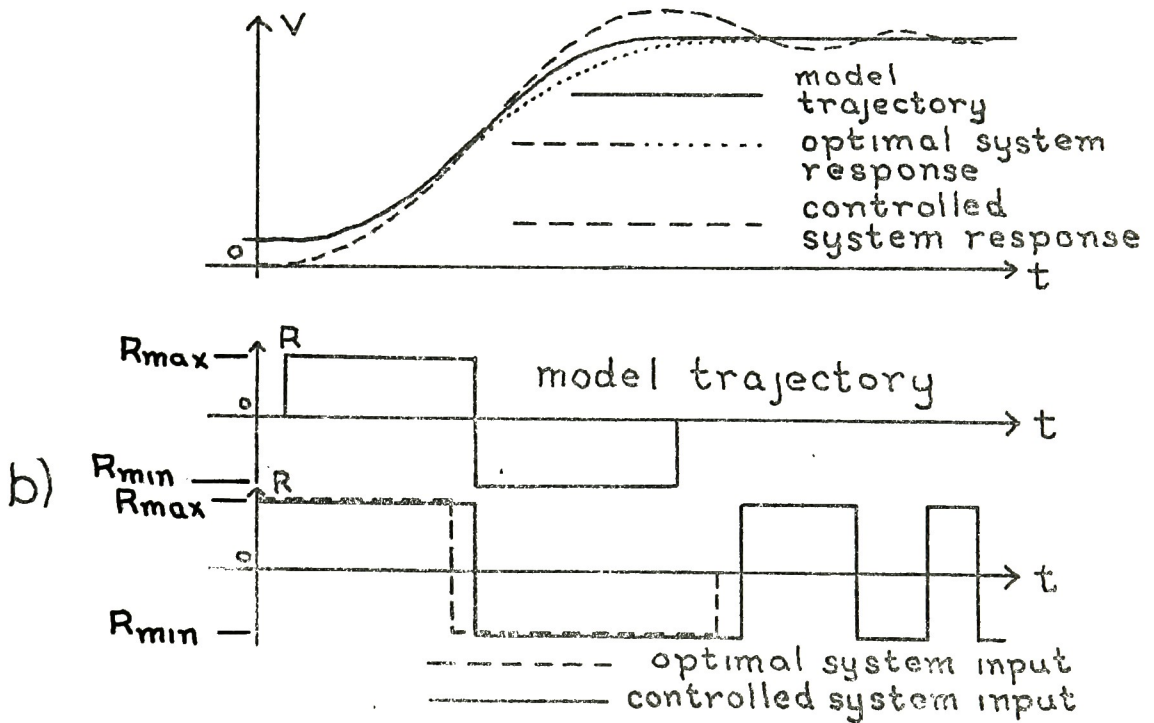
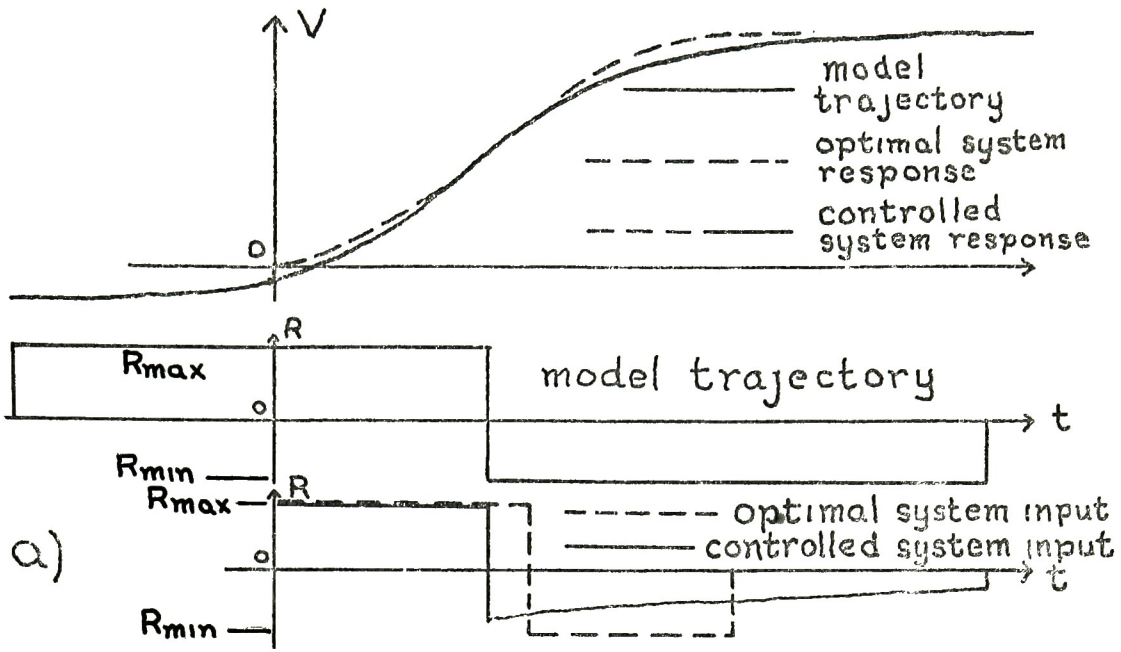


Figure 10
 Controlled System Response, with Controller
 of Figure 3 and Controlled System
 a) more stable than model, b) less stable than model

thereafter. The continuation of the actual time optimal response for the system is shown dotted. Figure 10b shows the controlled system response when the system is less stable than the model. Here the switch point predicted by $V_F = V_D$ is too late and the overshoot cannot be avoided. This response might be entirely satisfactory however, if the first peak were limited to a specified maximum, and could even be considered optimal by the definition of figure 2b.

In order to achieve the above result with $\frac{K}{S(S+A)}$ as a model, the mathematical manipulations of appendix 1 were carried out. First, using standard Laplace transform techniques, the system response to a step input R/S ,

$$V(t) = V_0 + \frac{\dot{V}_0}{A} - \frac{RK}{A^2} + \frac{RKt}{A} + \left[\frac{RK}{A^2} - \frac{\dot{V}_0}{A} \right] e^{-At}, \text{ was}$$

derived. $V(t)$ was differentiated and \dot{V} was set equal to zero. The resulting equation,

$$\dot{V} = \frac{RK}{A} + \left(\dot{V}_0 - \frac{RK}{A} \right) e^{-At} = 0 \text{ at } t = T_F, \quad R = R_b, \text{ was solved}$$

for $T_F = \frac{1}{A} \ln \left(1 - \frac{A \dot{V}}{R_b K} \right)$. Substituting T_F in the equation

for $V(t)$, with present V and \dot{V} being initial conditions,

led to the equation for the predicted final output;

$$V_F = V + \frac{\dot{V}}{A} + \frac{R_b K}{A^2} \ln \left[1 - \frac{A \dot{V}}{R_b K} \right] .$$

The solution is not complete at this point, since the unknowns K and A must be expressed in terms of the current state. V , \dot{V} and \ddot{V} provide enough equations, but the solution is complex. A better approach might be to use

$$A = \frac{-\ddot{V}}{\dot{V}} .$$

Since one would not expect rapid changes

(relative to system time constants) in the effective value of A , these derivatives could be heavily filtered to reduce the associated noise problems. By substituting

$$\ddot{V} \text{ into } \dot{V}, \text{ one finds that } K = \frac{1}{R} (A \dot{V} + \ddot{V}) .$$

As previously

defined, R is the current system input, and R_b is whichever of the saturation limits (R_{\max} or R_{\min}) will drive \dot{V} toward zero. K and A have now been defined in terms of the current system state. V_F is therefore adaptive and totally dependant on the state of the controlled system.

At this point an algorithm has been defined, and a theoretical evaluation of its properties may proceed.

The control was applied mathematically to the system

$\frac{20}{(S+2)(S+1)}$ with the result shown in figure 11. The

excellent performance with the system tested was an encouraging result and considerable thought was given to methods of implementing this control. These will not be discussed because, mainly due to the difficulty in obtaining the model parameters without the third

derivative, the use of the $\frac{K}{S(S+A)}$ model was dropped.

Several things of later importance were brought out during this phase. Since assumptions had to be made about the available braking input, it was required that it be equal to the accelerating input or that the ratio of the two be included in the algorithm. Also it was recognized that some form of automatic (but not necessarily continuous) magnitude scaling of the controlled system derivatives would be necessary to keep them within the useful range of physical devices, and several techniques were investigated. A basic difference was noted between systems with and without free integrations in that the effective ratio of accelerating to braking force in the latter case is affected strongly by the system output,

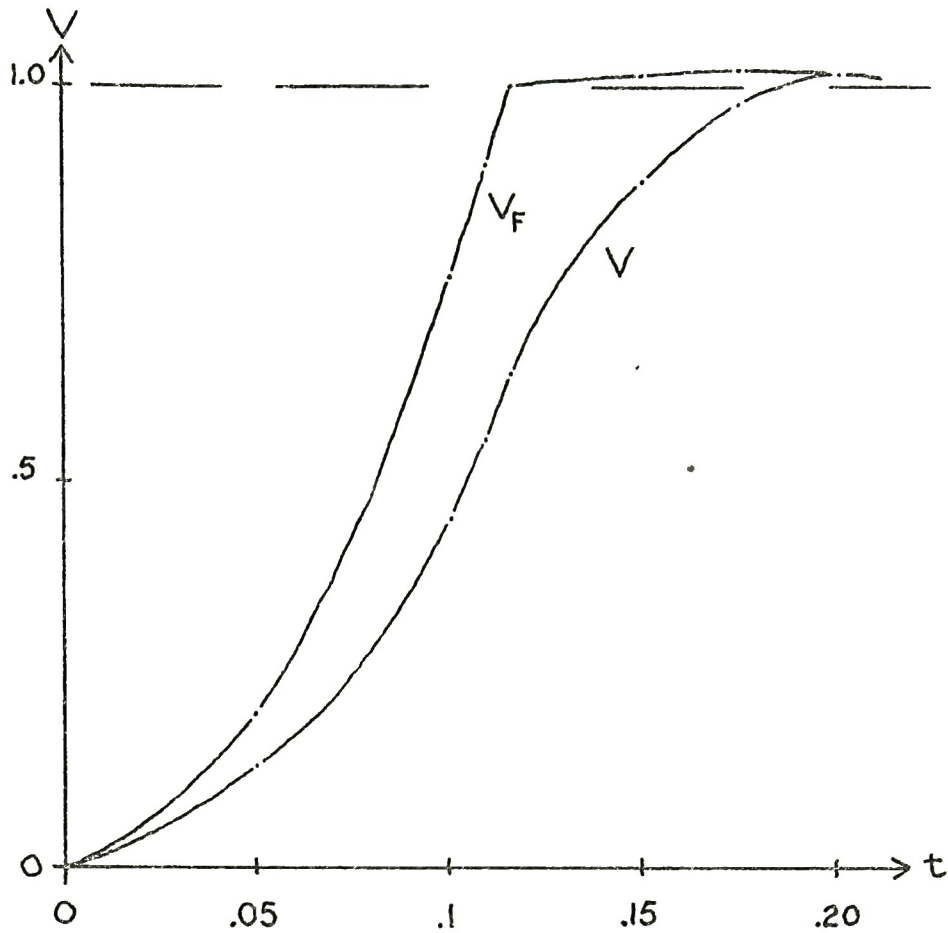


Figure 11

Controlled Response of System $\frac{20}{(s+2)(s+1)}$
to Input Step of Magnitude 1.0

In Controller of Figure 3,

$$V_F = V + \frac{\dot{V}}{A} + \frac{R_b K}{A^2} \ln\left(1 - \frac{A\dot{V}}{R_b K}\right), \text{ where}$$

$$A = \frac{-\ddot{V}}{\dot{V}}, \text{ and } K = \frac{1}{R} \left(\dot{V} + \frac{\ddot{V}}{A} \right).$$

$$\text{Also, } R_{\max} = 5, R_{\min} = -5$$

unless a considerable overdrive* is available. Finally, a number of interesting control approaches for subclasses of the general system classification considered here resulted from the work.

Following the work with $\frac{K}{S(S+A)}$, the general objectives of

the project and the chosen approach to those ends were again considered. There was some concern at this point because of the small overshoot of the two time constant system of figure 11, and it seemed likely that one of the most unstable systems in the general classification might be the best choice from the standpoint of guaranteed

stability. $\frac{K}{S^2}$ was the logical choice, as its equations

are much simpler than $\frac{KB}{S^2+B}$. Appendix 2 shows the

derivation of V_F for this model in a manner similar

to that discussed previously. This time, $V_F = V - \frac{\dot{v}^2}{2R_b K}$

*The term overdrive indicates more input power is available than is required to maintain maximum desired system output.

is in terms of V , \dot{V} , and only the single model parameter K .

K may be replaced by $\frac{\ddot{V}}{R}$, so that $V_F = V - \frac{\dot{V}^2}{\frac{R_b \ddot{V}}{R}}$, which if $\frac{R_b}{R} = 1^*$ is equivalent to $V_F = V + \frac{\dot{V}|\dot{V}|}{2|\ddot{V}|}$. In this case

\ddot{V} may be filtered, as it represents a relatively slowly changing constant. As a check on the derivation of V_F , the control was applied mathematically to the system

$\frac{1}{S(S+1)}$, with the result shown in figure 12a. Note that

final equilibrium (a deadbeat response is obtained) is reached at a time only 22% greater than the time optimal response. This information was obtained by calculating V_F points until the switch point (where $V_F = V_D$) was found and then calculating the value of R immediately after the switch, using $V_F = V_D$. Then assuming R stayed constant, V , \dot{V} , and \ddot{V} were calculated (and plotted on figure 12a) for $t = T_S = .1$. ** Again using $V_F = V_D$, a new value of R was calculated and, assuming R stayed constant at this value, V , \dot{V} , and \ddot{V} were calculated at $t = T_S + .2$. This process

*This is not a restriction, but simplifies the following work.

** T_S indicates the switch point.

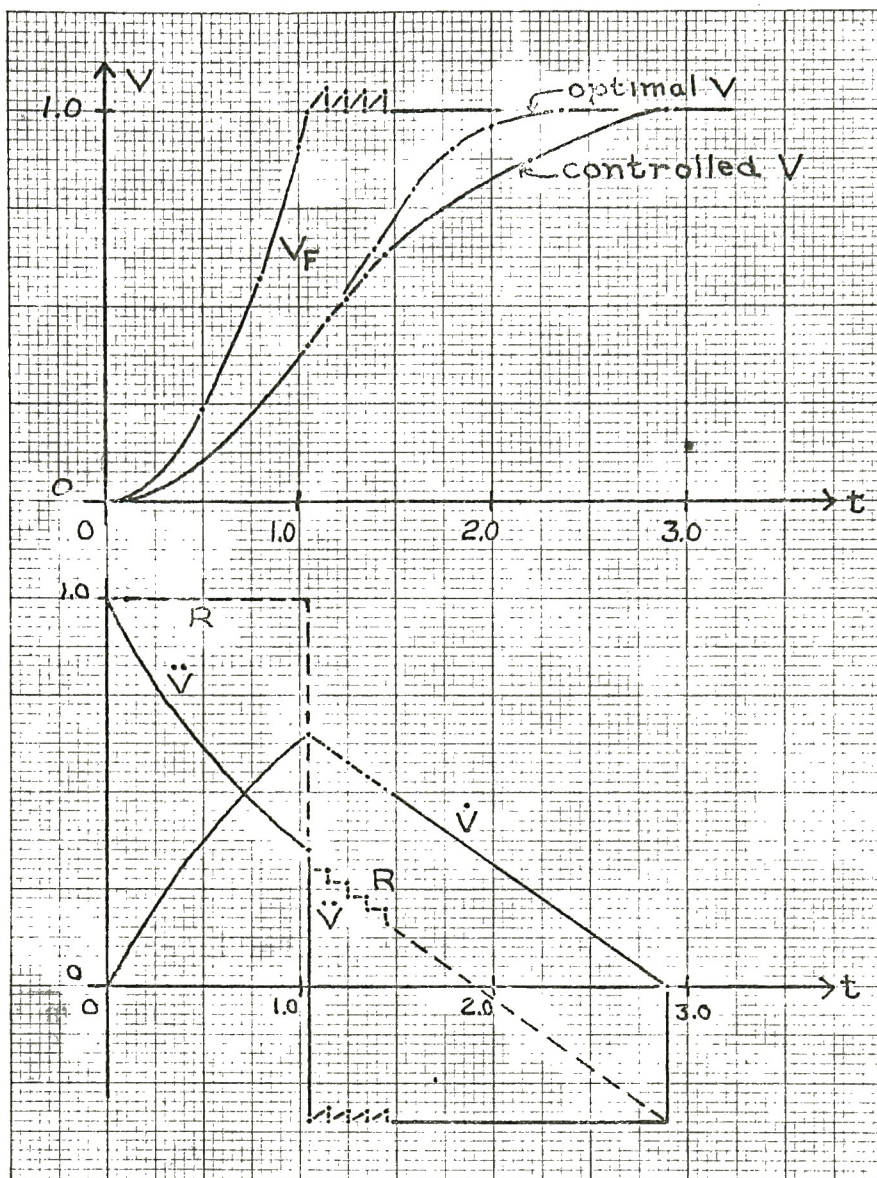


Figure 12 a
Controlled Response of System $\frac{1}{s(s+1)}$
to Input Step of Magnitude 1.0

In Controller of Figure 3,

$$V_F = V + \frac{\dot{V}|\dot{V}|}{2|\ddot{V}|}$$

$$R_{\max} = 1.0, \quad R_{\min} = -1.0$$

Figure 12b

Sample Calculations for Figure 12a

$$V = V_0 + \dot{V}_0 t - R + Rt + (R - \dot{V}_0) e^{-t}$$

$$\dot{V} = R - (R - \dot{V}_0) e^{-t}$$

$$\ddot{V} = (R - \dot{V}_0) e^{-t}$$

@ $t = 1.0$,

$$V = -1 + t + e^{-t} = .368$$

$$\dot{V} = 1 - e^{-t} = .632$$

$$\ddot{V} = e^{-t} = .368$$

$$V_F = V + \frac{\dot{V}^2}{2\ddot{V}} = .911$$

@ $t = 1.05$,

$$V = .4$$

$$\dot{V} = .65$$

$$\ddot{V} = .35$$

$V_F = 1.0$ and $t = 1.05 = T_S$ is switchpoint.

After switch,

$$V_F = V - \frac{\dot{V}^2}{2\ddot{V}} = 1$$

$$\ddot{V} = \frac{\dot{V}^2}{2(V-1)} = -.35$$

$$R = \dot{V}_0 + \ddot{V} = .3$$

Now, @ $t = T_S + .1$, assuming $R = .3$,

$$V = .4 + .65 - .3 + .3(.1) + (.3 - .65) e^{-.1} = .463$$

$$\dot{V} = .3 - (.3 - .65) e^{-.1} = .617$$

$$\ddot{V} = (.3 - .65) e^{-.1} = .317$$

But if $V_F = 1$,

$$\ddot{V} = \frac{\dot{V}^2}{2(V-1)} = \frac{.617^2}{2(-.539)} = -.353$$

$$R = \dot{V}_0 + \ddot{V} = .617 - .353 = .264$$

Now @ $t = T_S + .2$, assuming $R = .264$,

$$V = .463 + .617 - .264 + .264(.1) + (.264 - .617)e^{-.1} = .522$$

$$\dot{V} = .264 - (.264 - .617)e^{-.1} = .584$$

$$\ddot{V} = (.264 - .617)e^{-.1} = -.353$$

But if $V_F = 1$,

$$\ddot{V} = \frac{\dot{V}^2}{2(V-1)} = \frac{.584^2}{2(.522-1)} = -.356$$

$$R = \dot{V}_0 + \ddot{V} = .584 - .356 = .228$$

was repeated until the trajectories became obvious, (sample calculations are shown in figure 12b), and then checked mathematically. That is, the integral of the constant \ddot{V} shown (in figure 12a), with the initial condition $V(T_S)$ is the ramp \dot{V} shown, and similarly, its integral, with initial condition $V(T_S)$ is the V shown. Also of course, $V_F = V_D$ is valid from T_S on. The particular model trajectory on which the controlled system is driven is that of

$\frac{.35}{s^2}$, whose apparent point of origin is at $V = -.2$ and

$t = -.8$. It was later noted that this information could have been obtained from the solution of the

differential equation $V_F = 1 = \frac{\ddot{V}^2}{.35}$, rather than the

incremental approach taken.

This algorithm showed enough promise, in terms of simplicity and ease of implementation, to justify an analog simulation to determine its performance over the entire class of systems. A simulation, either digital or analog, was the logical next step because manual calculation of response curves for a large number of systems was impractical.

ANALOG SIMULATION WITH MODEL $\frac{K}{S^2}$

The circuit of figure 13 was first used for a crude simulation of the controller. No attempt at extreme accuracy was made in this breadboard because one objective of the simulation was to demonstrate that the controller was not unduly sensitive to its own parameters. Also, mainly qualitative results were desired. Ten percent tolerances were used throughout and zero offsets were ignored. R_{\min} and R_{\max} , the controller output limits, were set equal in magnitude for convenience. A large amount of filtering (R1, R2) was added to the differentiators for noise suppression. Most of the noise was line frequency pickup due to hasty design and poor breadboard layout and was sufficient to completely saturate the differentiator outputs without filtering. While irritating at this point in time, it probably provided a useful foretaste of practical noise levels.

Figure 14 illustrates the circuit used for system simulation with the controller breadboard of Figure 13. Time scaling was chosen to provide computer time transients of approximately 45 seconds duration. All simulation results (figure 15, 16, 19-26) were recorded by tracing a crt display, and transient duration was chosen to facilitate

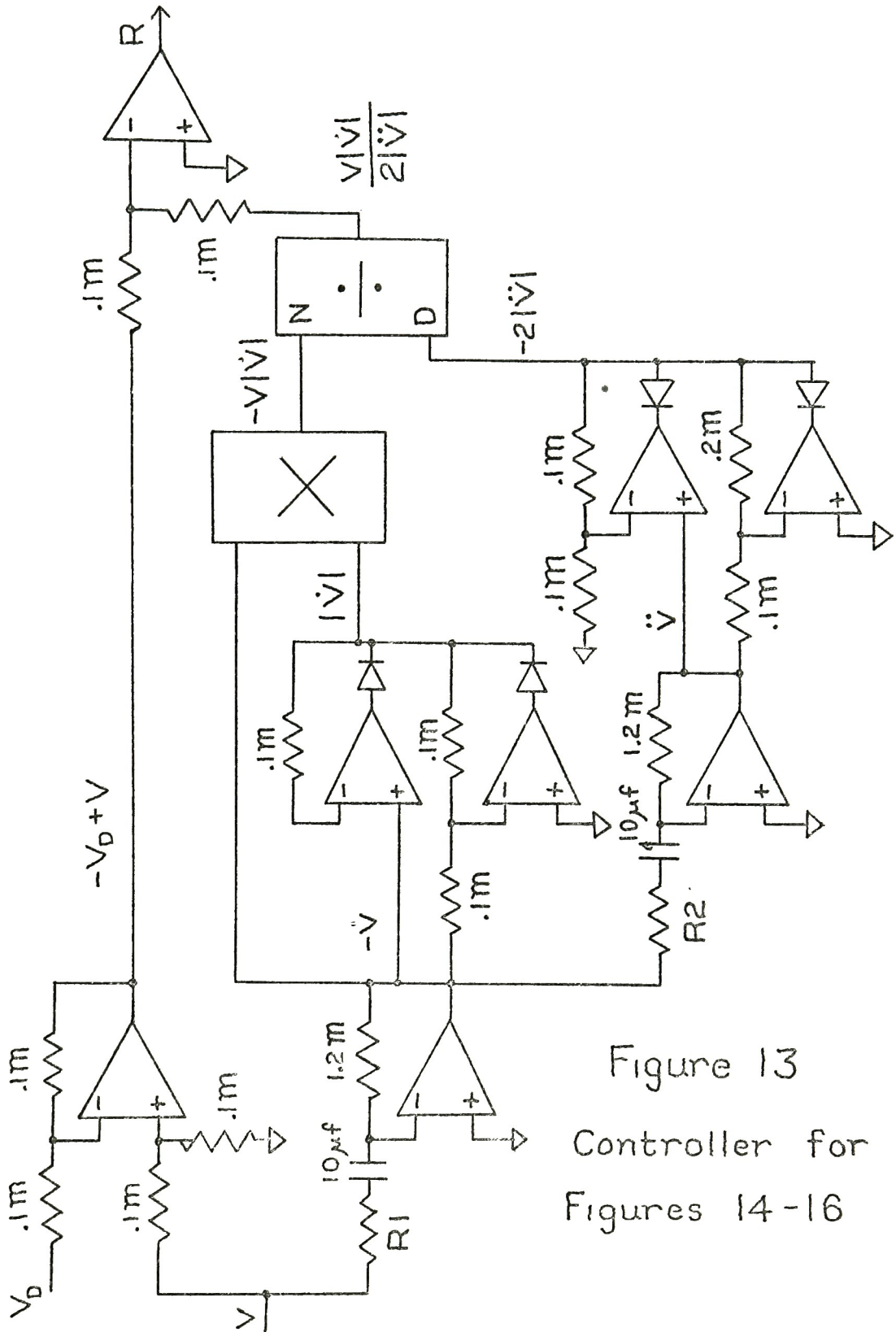
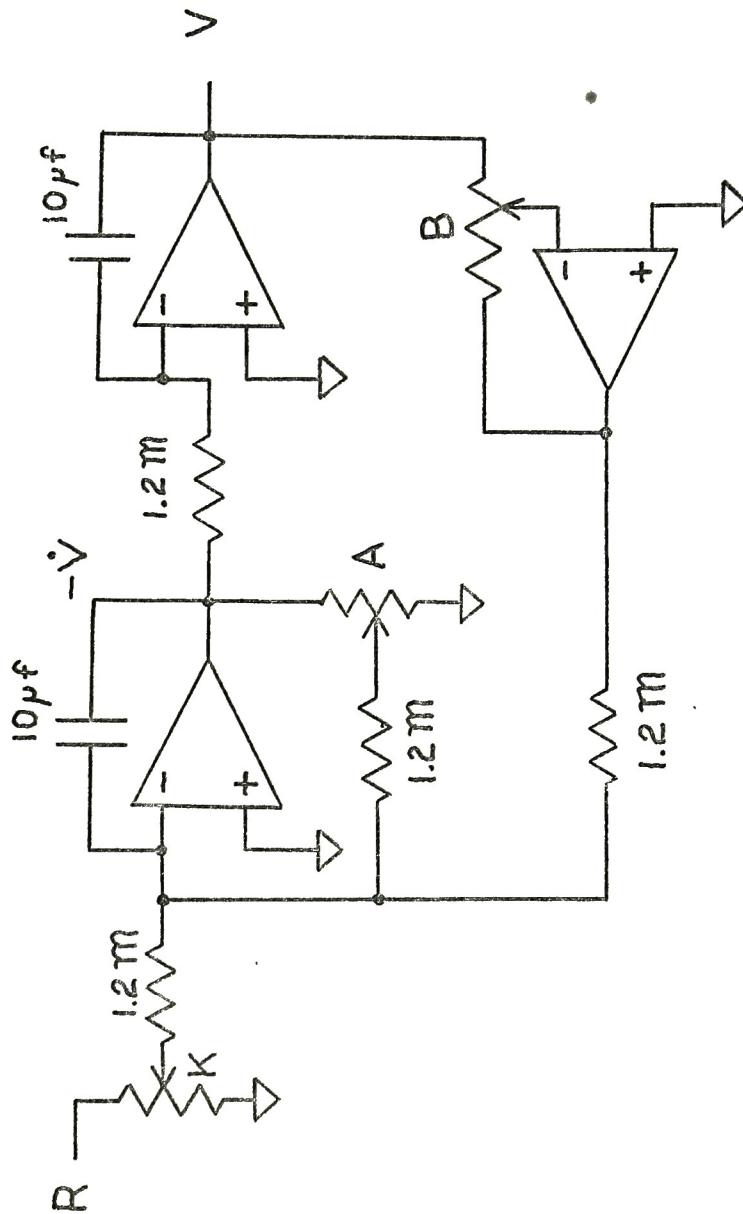


Figure 13
Controller for
Figures 14-16

Figure 14

Simulation of $\frac{K}{s^2 + As + B}$
for Figures 13, 15, 16



this technique. The first system simulated was $\frac{.5}{s^2}$, since if the controller didn't perform properly with this system it was either improperly constructed or the algorithm was incorrectly defined. ($\frac{K}{s^2}$ was the model for the algorithm.) The results for a positive input step from 0 to .8 are shown in figure 15, and for a negative step from .8 to 0 in figure 16. While not perfect, the results were as close as would be expected considering the accuracy of the test. The radical behavior of $\frac{\dot{v}|v|}{2|\ddot{v}|}$ near the switch point is caused by the filtering of v . That is, after the switch point the measured $|\ddot{v}|$ drops to zero before settling to its correct value. The slight overshoots are caused by the imbalance in positive and negative inputs and small gain errors in the various amplifiers. Other systems were investigated briefly with this configuration, but the thought occurred that the troublesome differentiators need not have been used, as the derivatives could have been made available from system simulation. The system simulation was accordingly changed to that of figure 17, and the controller reduced to that of figure 18. This was done only for ease of simulation and does not imply that derivatives will be

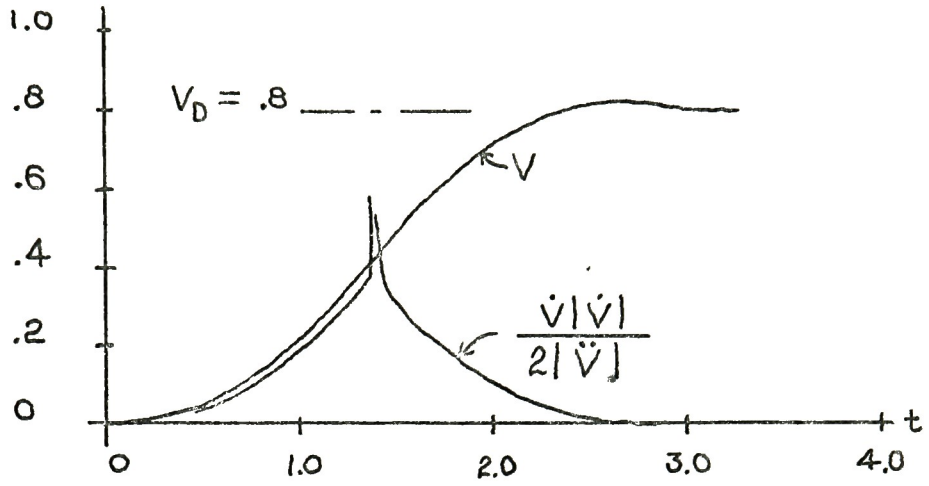


Figure 15

Results of Analog
Simulation of $\frac{.5}{S^2}$.

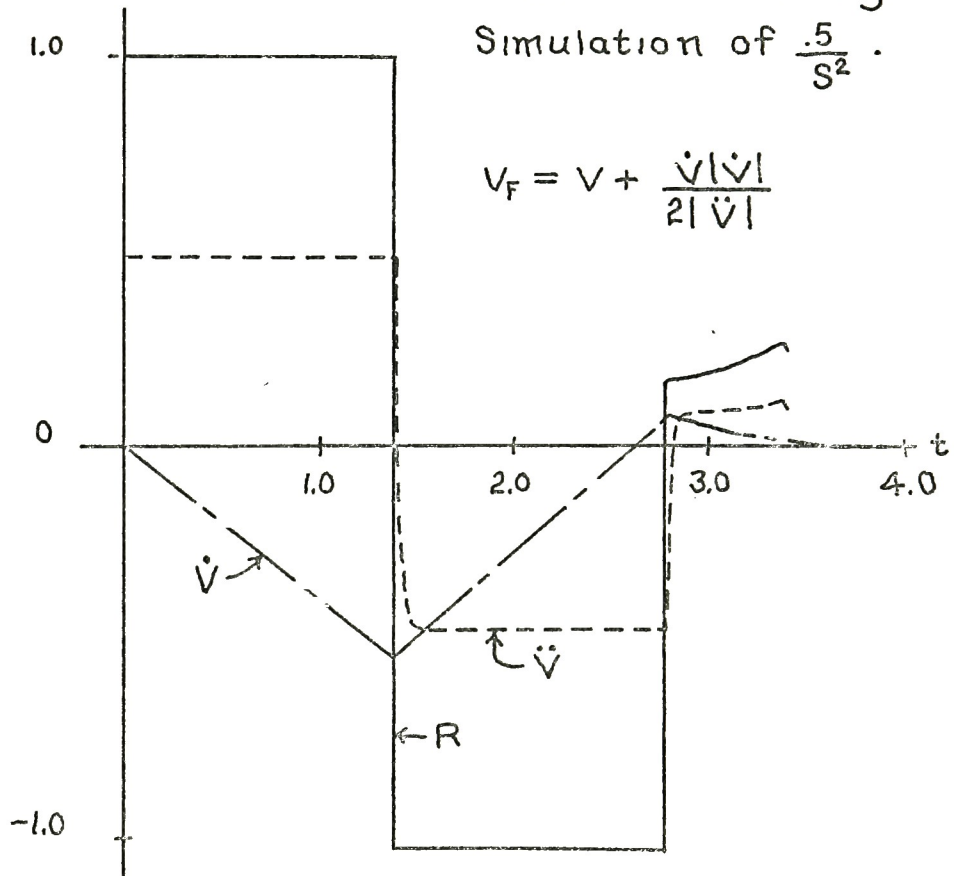
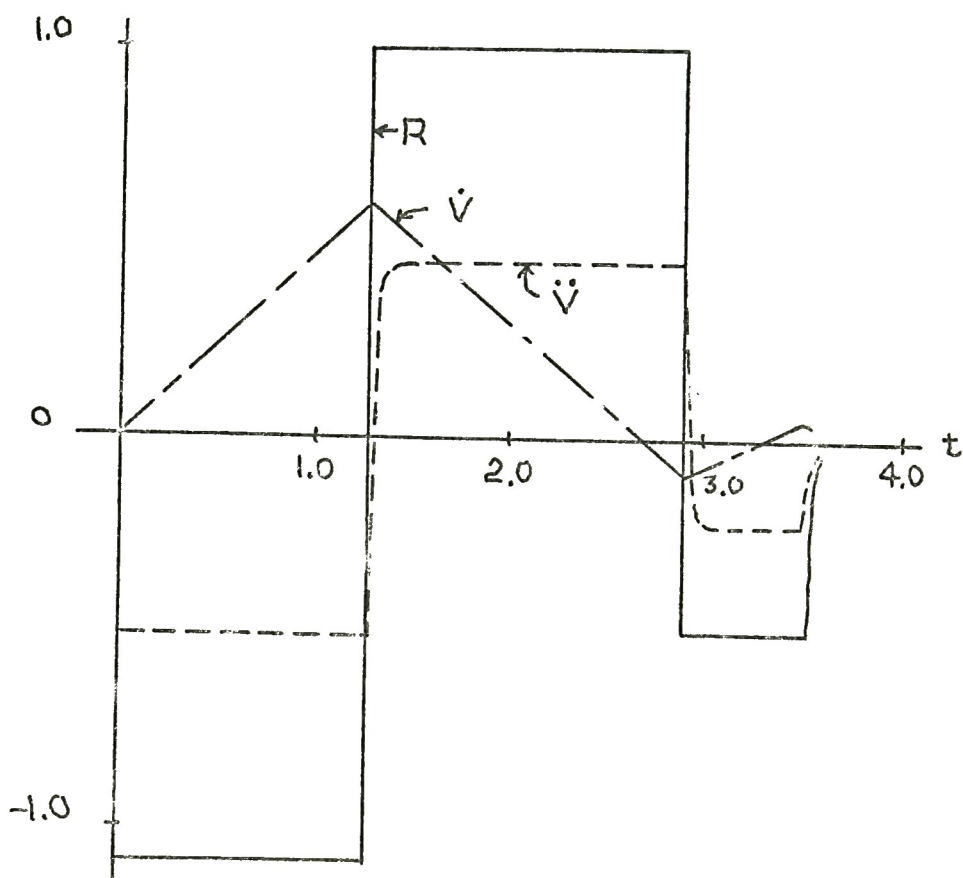
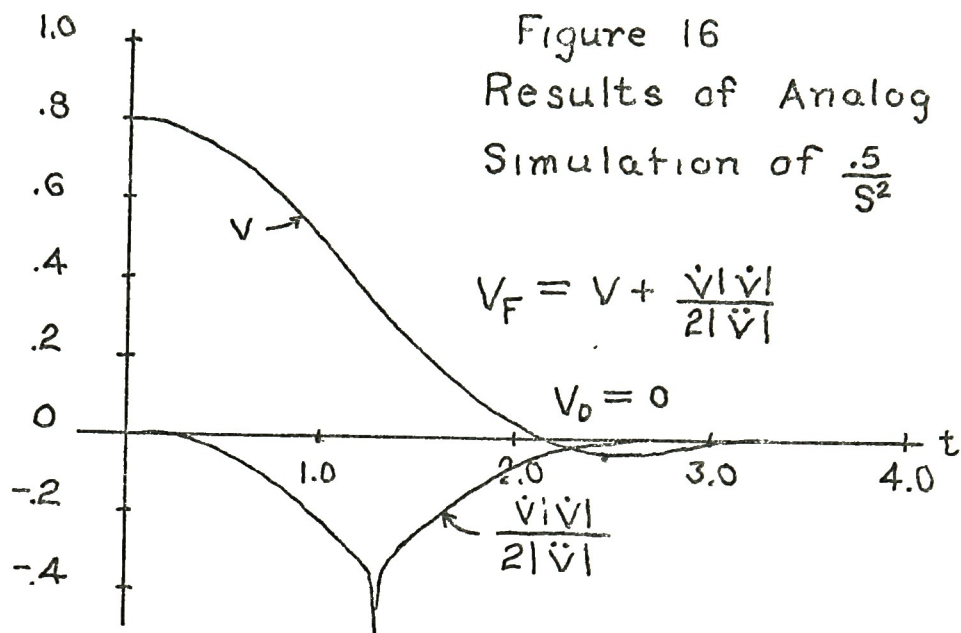
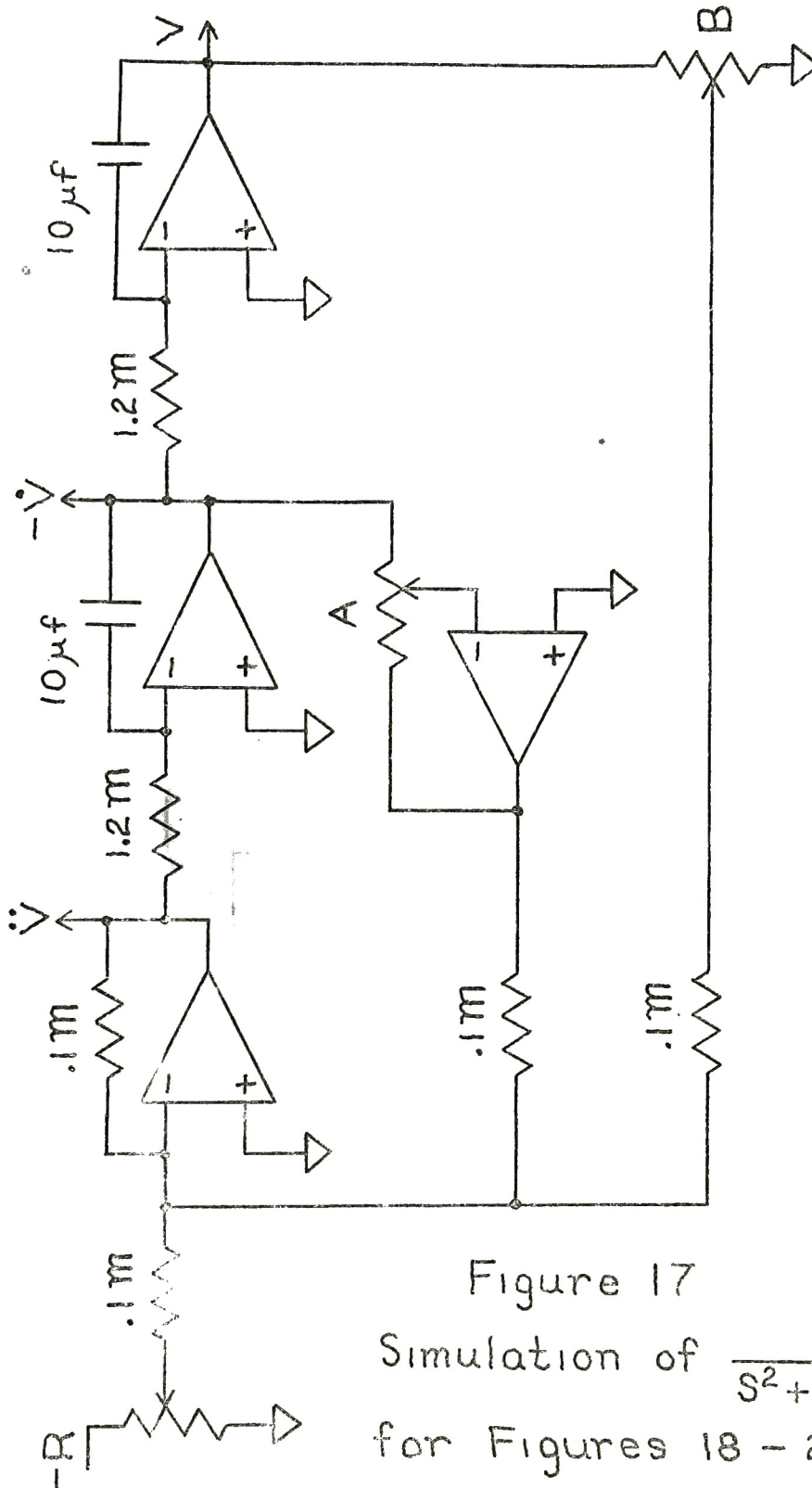


Figure 16
Results of Analog
Simulation of $\frac{.5}{S^2}$





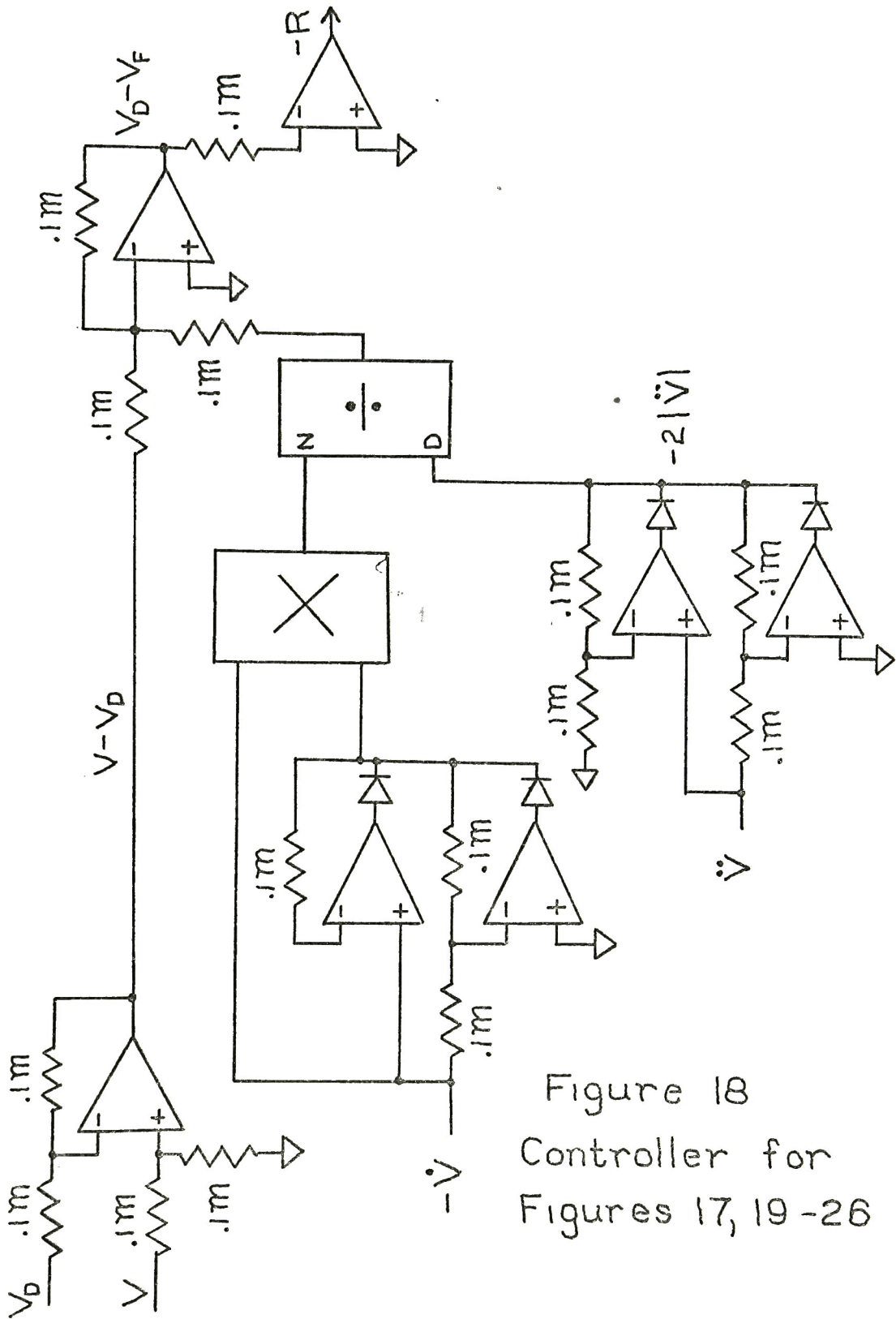


Figure 18
 Controller for
 Figures 17, 19-26

available from an actual system. The system $\frac{.5}{S^2}$ was

again tested and results were slightly better than above, mainly because supply voltages were adjusted to equalize positive and negative R^* , and zero adjustments were made on the multiplier and divider. Aberrations at the switch point caused by \ddot{V} filtering were gone.

$|\ddot{V}|$ still must drop to zero after the switch point, since \ddot{V} reverses polarity, but without filtering this transient is very fast.

Two members of the $\frac{K}{S(S+A)}$ family, figures 19 and 20, were

tested next, with results as predicted by the prior analytical techniques (see figure 12). Negative going (from X to 0) input steps produced responses identical to those produced by positive going input steps. Following

this, results for $\frac{.4}{(S+.6)(S+.6)}$, $\frac{.4}{S^2+.4}$, and $\frac{.477}{S+.477}$

were obtained, as shown in figure 21 through 26.

* $R_{\min} = -R_{\max}$ was chosen for convenience and the controller breadboard assumes this to be true in calculating V_F . A practical version would use the actual ratio.

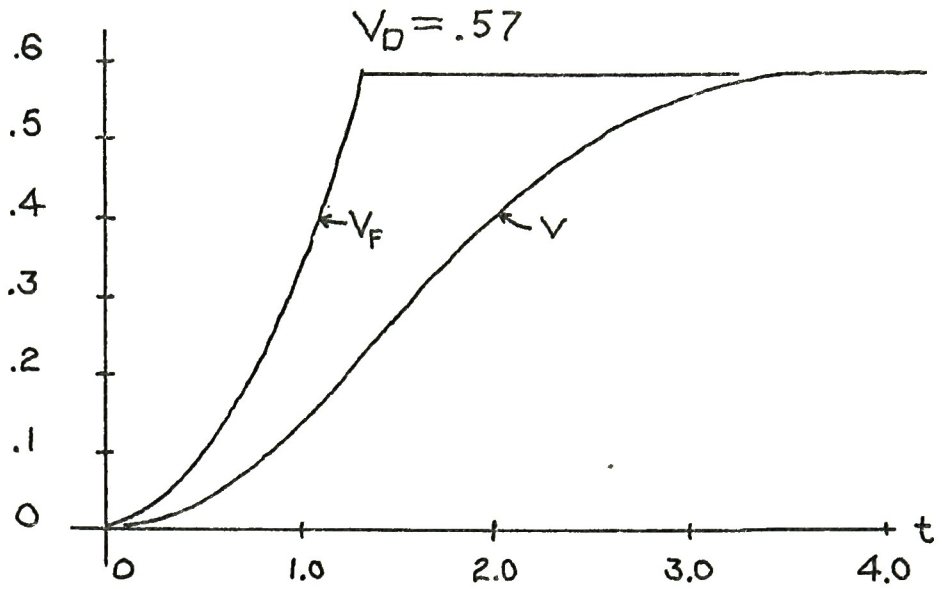
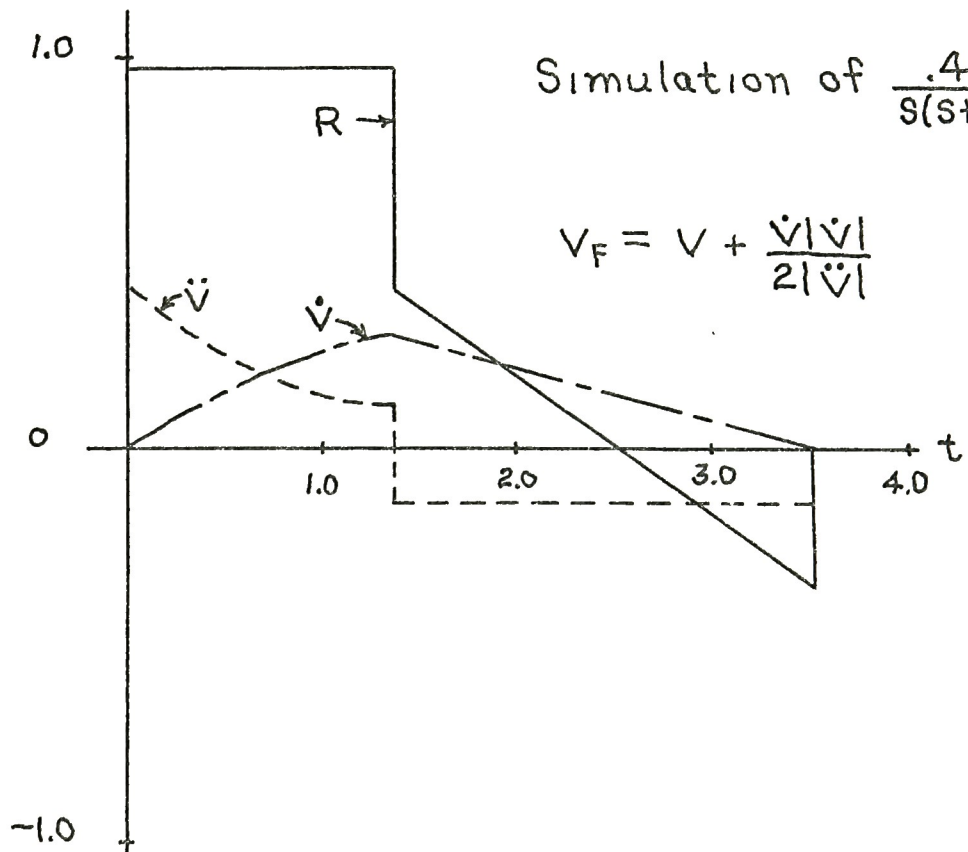


Figure 19
Results of Analog
Simulation of $\frac{.4}{s(s+1)}$



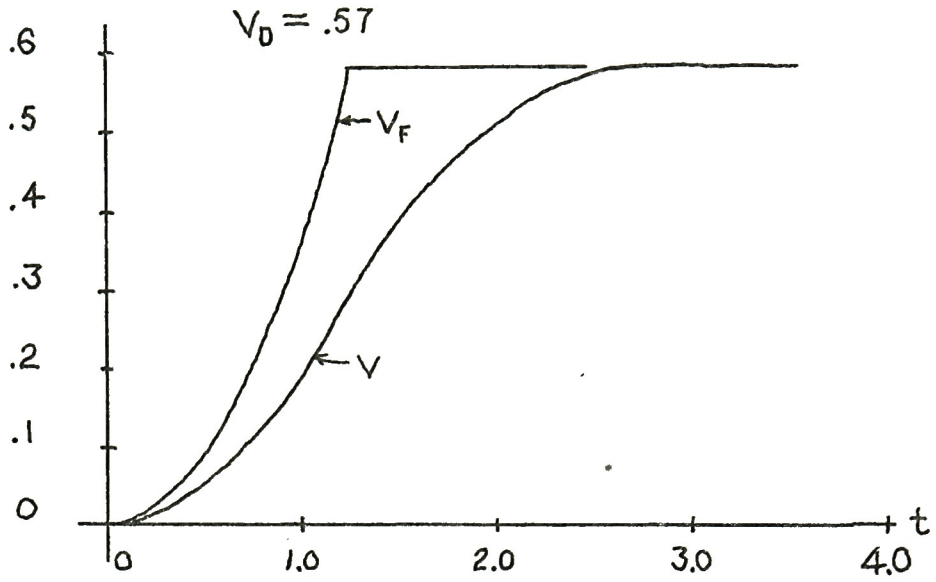
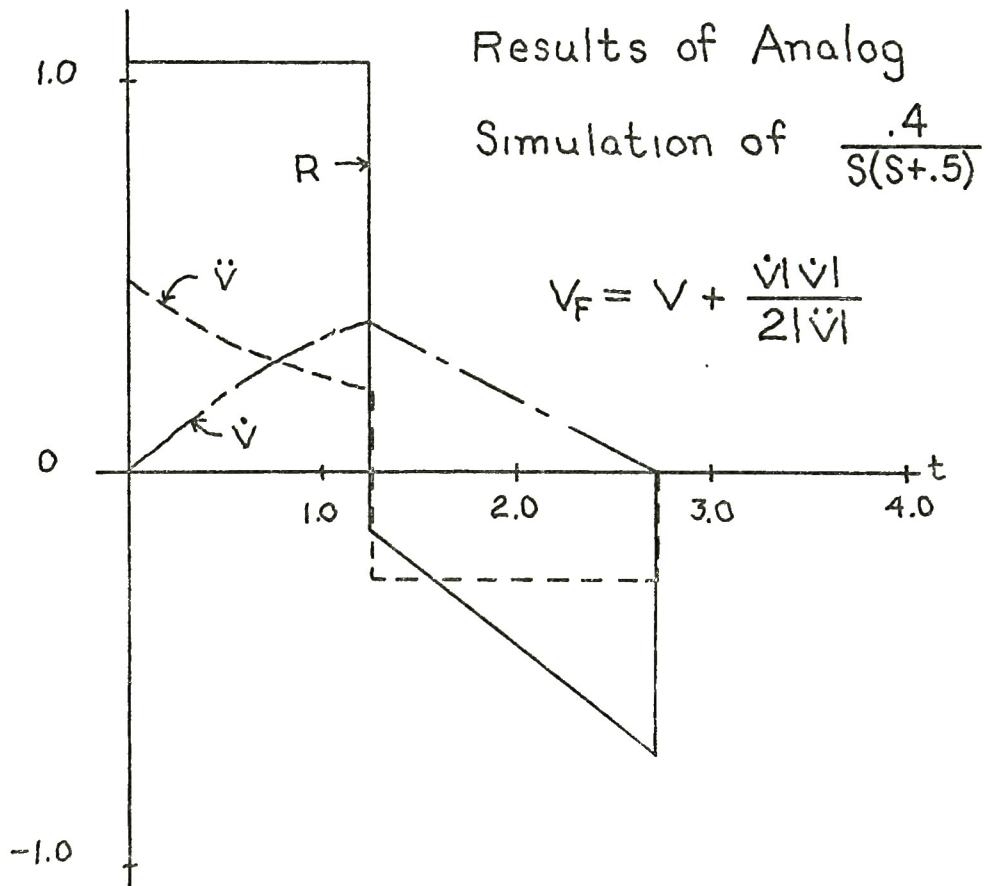
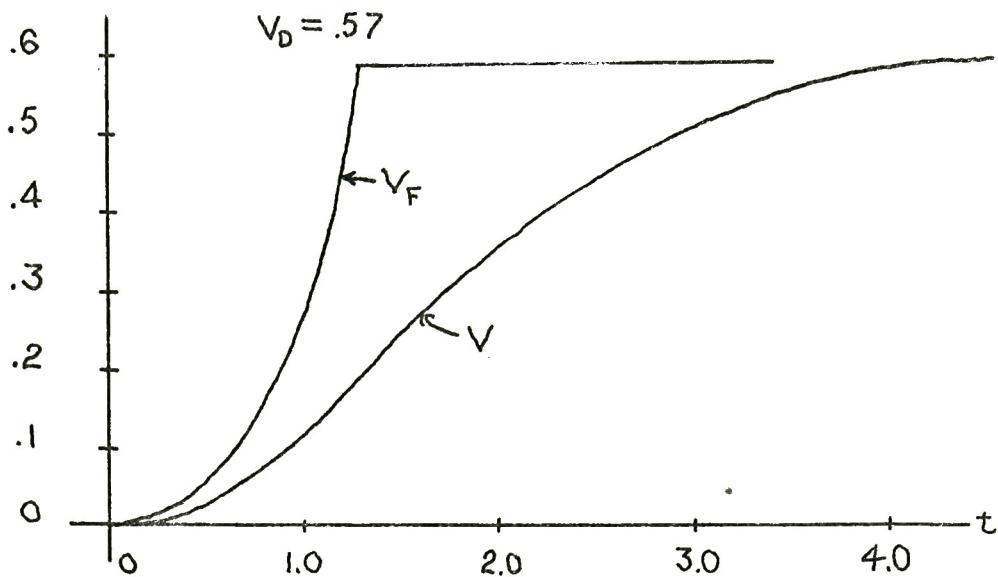


Figure 20





$$V_F = V + \frac{\dot{V}|\dot{V}|}{2|\ddot{V}|}$$

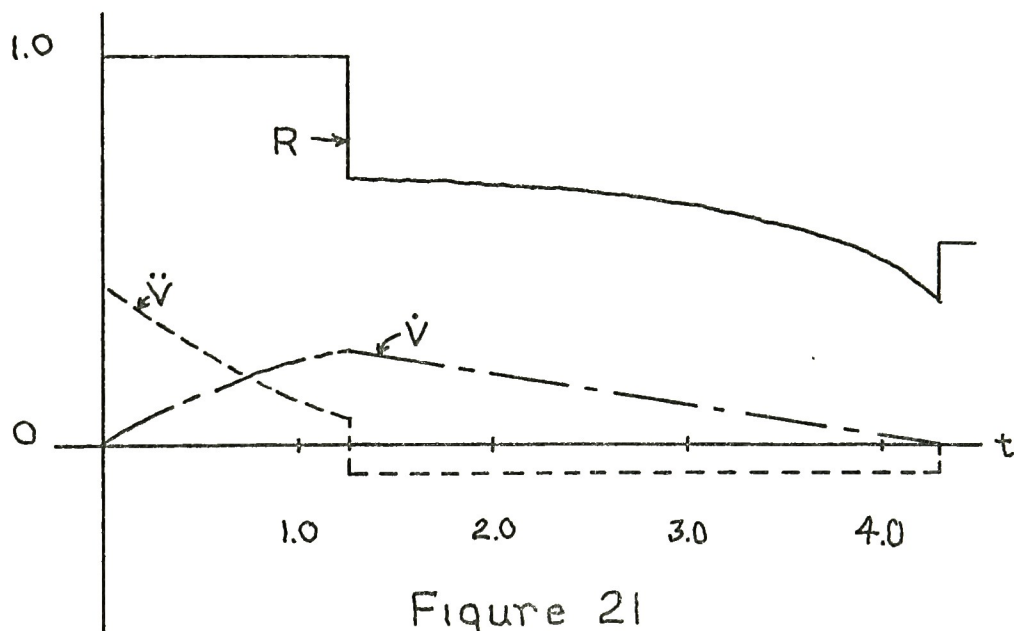


Figure 21
 Results of Analog
 Simulation of $\frac{.4}{(s+.6)(s+.6)}$

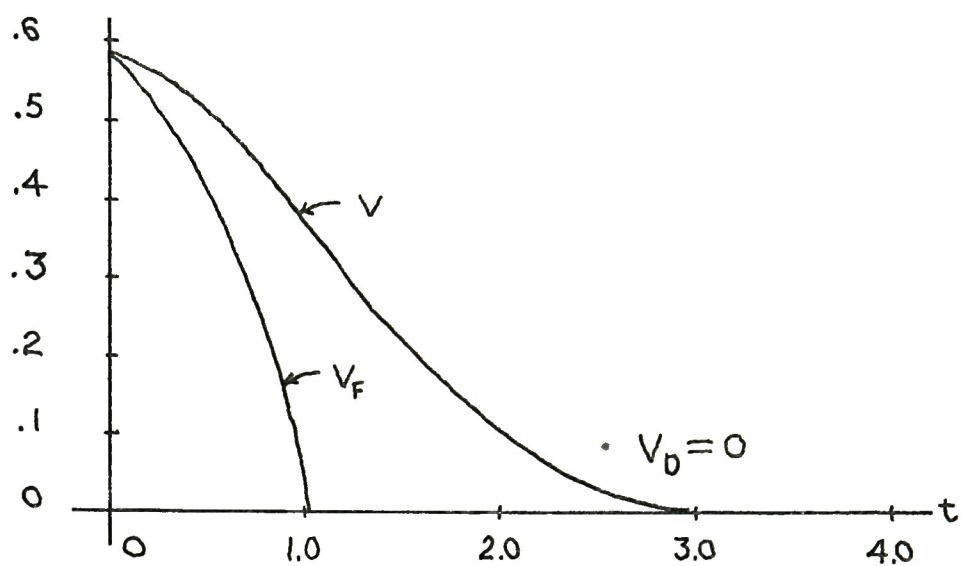
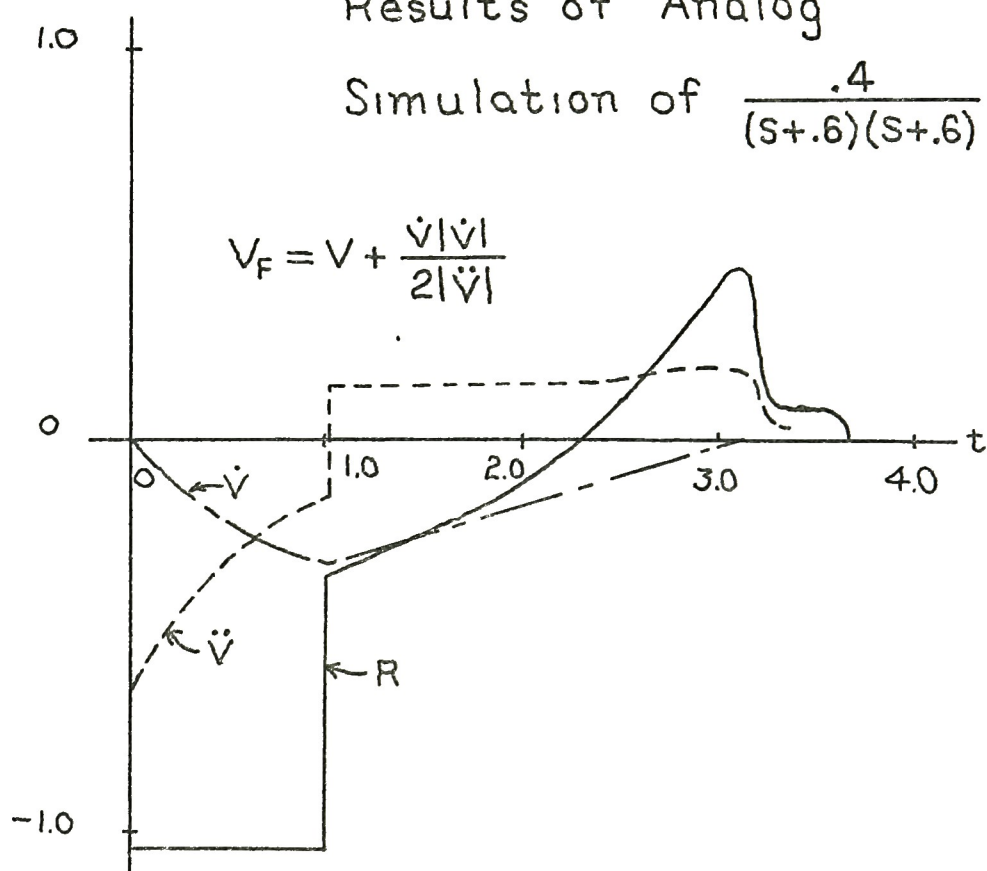


Figure 22

Results of Analog

Simulation of $\frac{.4}{(s+.6)(s+.6)}$ 

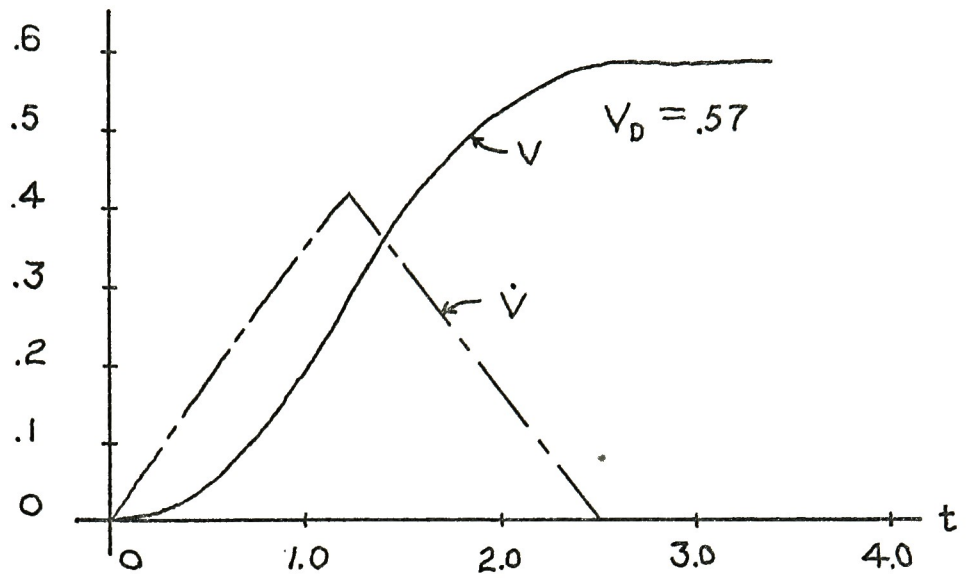
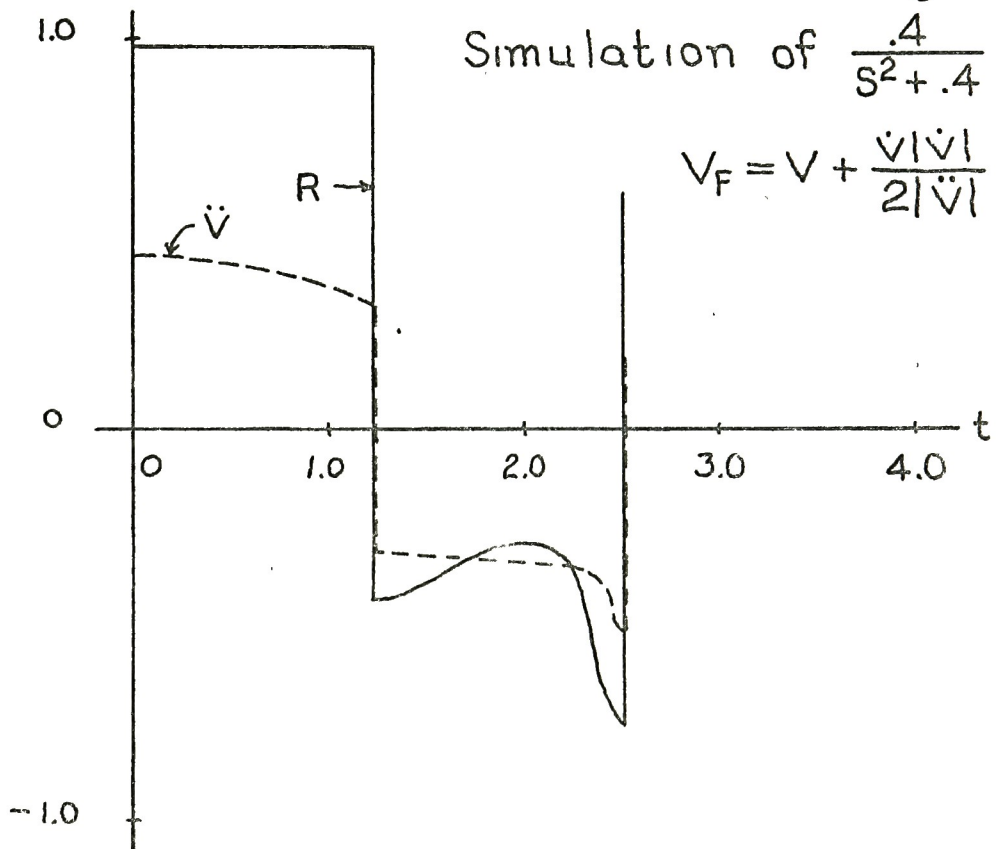
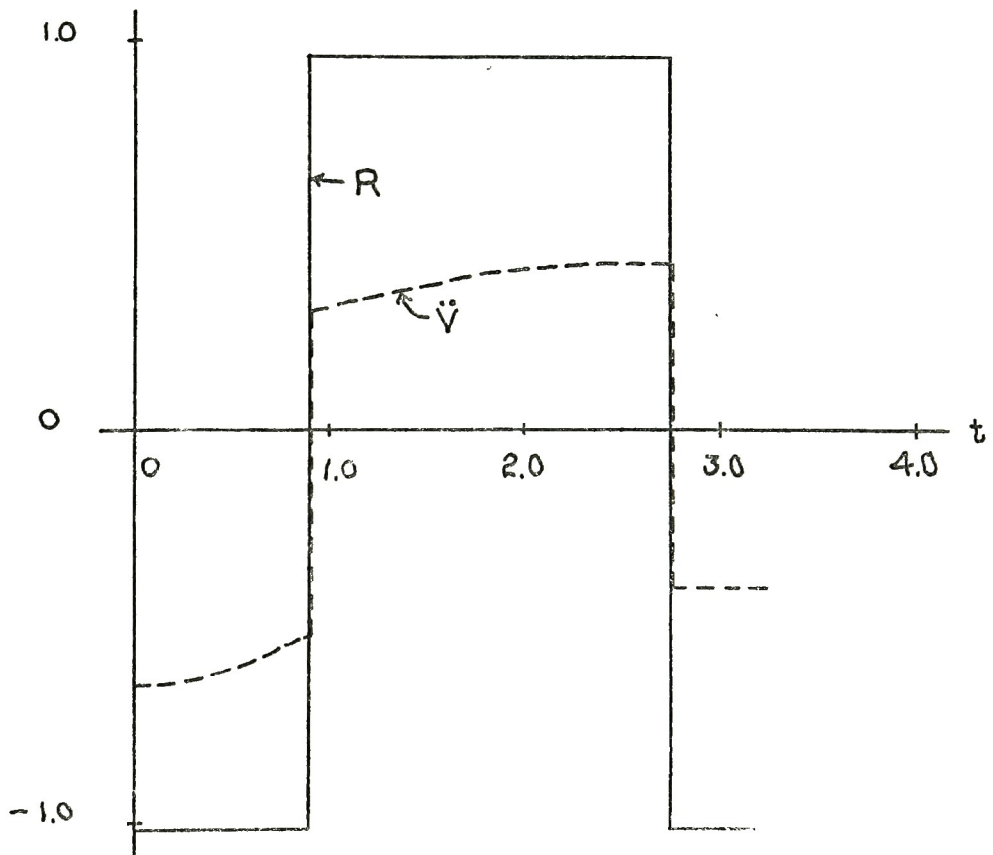
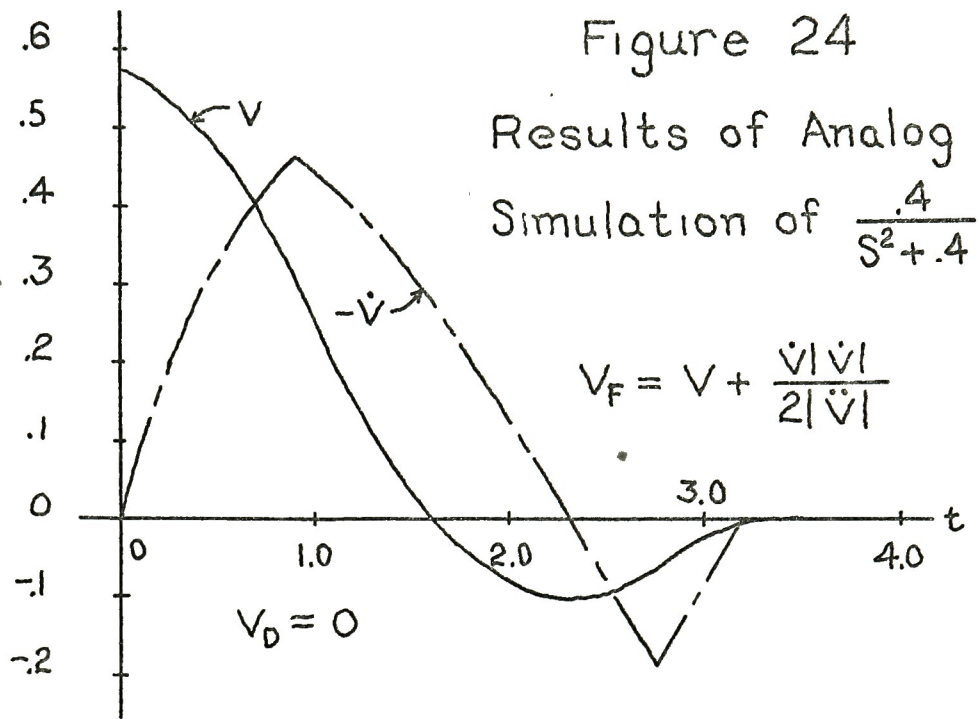


Figure 23

Results of Analog
Simulation of $\frac{.4}{s^2 + .4}$





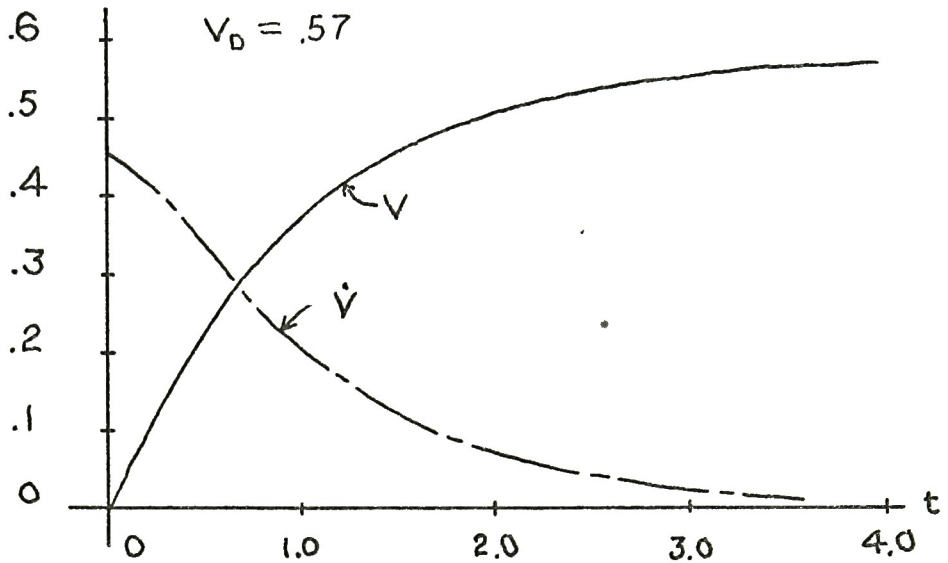
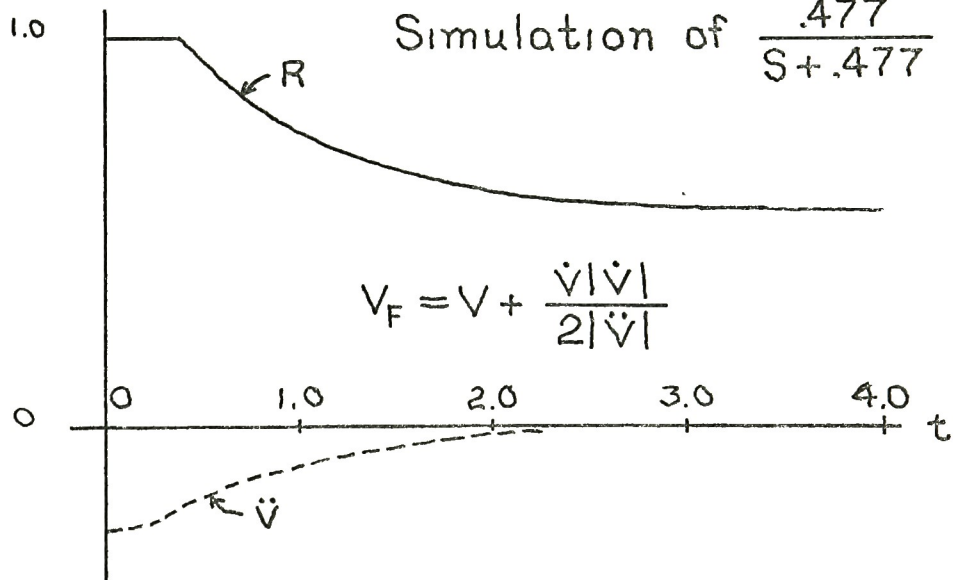


Figure 25

Results of Analog

Simulation of $\frac{.477}{S+.477}$ 

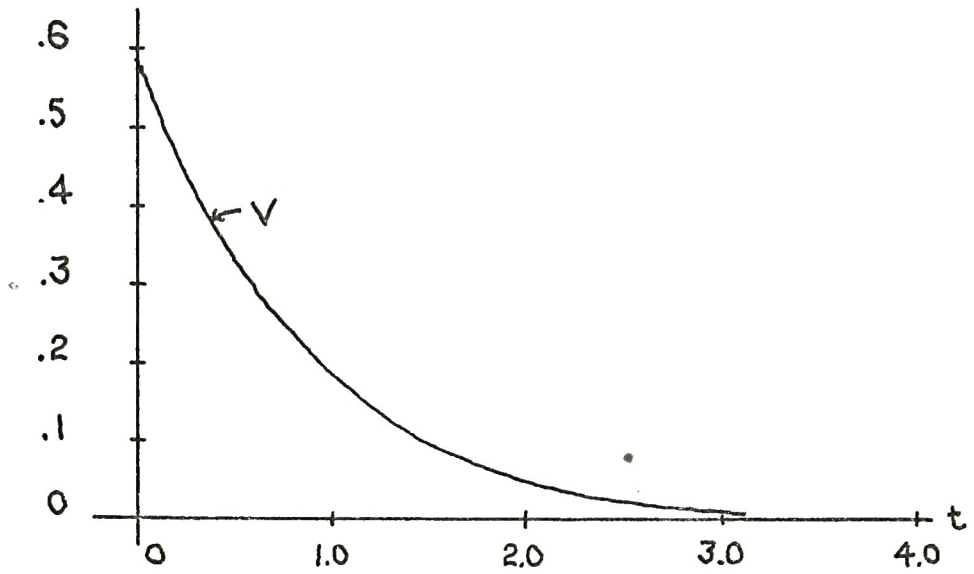
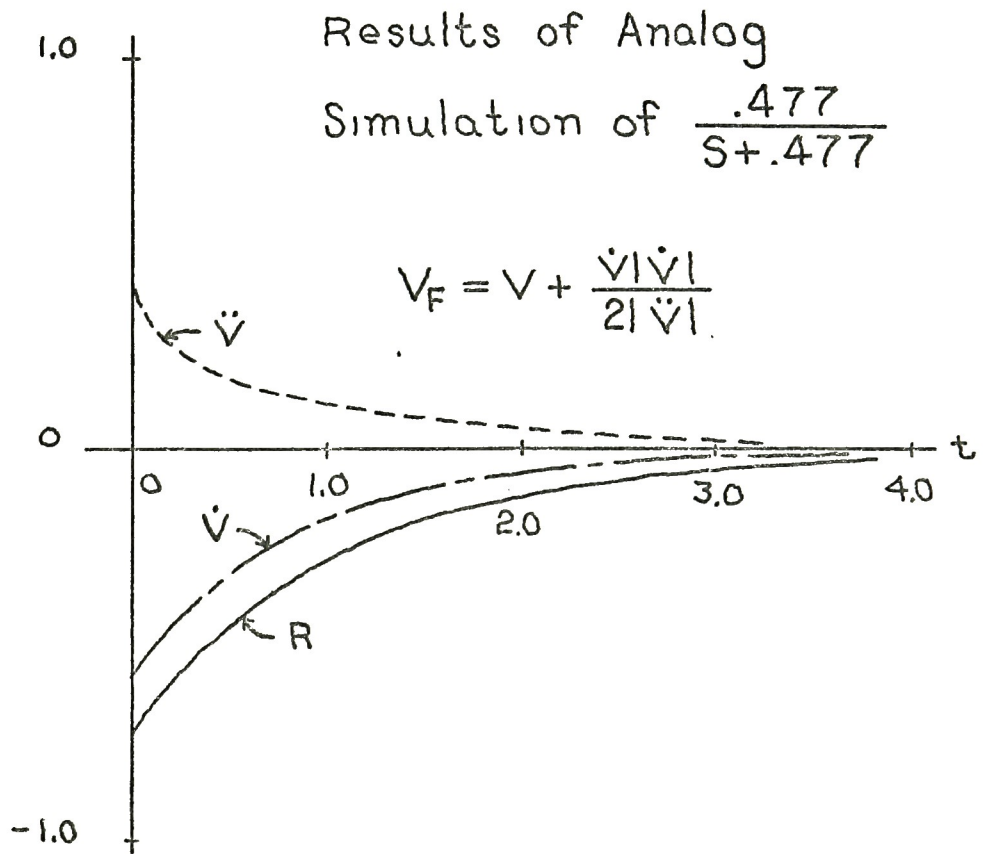


Figure 26



It is evident from the results of the simulation that the algorithm does not possess all of those fine qualities desired of it. Figures 19 and 20 indicate that excellent performance can be expected if the controlled system is

near the same type as the model $\left(\frac{K}{S^2}\right)$. For instance

$\frac{K}{S(S+A)}$ transients approach optimal as A becomes small. In

a servo application this would mean load friction was small in comparison with load inertia. With a given amount of friction, the greater the inertia, the more optimal the control. Unfortunately, as relative friction increases the switch point comes increasingly early and control becomes sluggish* (Figure 19 is more sluggish than figure 20).

All systems without a free integration will suffer from the above problem (that as the controlled systems inherent stability increases the control becomes less optimal) and from a variation in effective braking force to

*Qualitative terms such as sluggish, as used in this section refer to deviation from the optimal response for the system in question, not to inherent differences between that system and any other system.

accelerating force ratio with output level. The latter is very evident in figures 22 and 23. Notice the radically different responses, depending on direction, between the same two operating points, 0 and .57. Other results from the simulation showed very nearly optimal results when V stayed near the origin (say $0 \pm .2$) and horribly unbalanced response when V was near the extremes (± 1). Response to a step input change from .95 to .9 for instance, would be several hundred percent overshoot followed by a deadbeat response. This is perhaps the redeeming feature of the control; that even though the overshoot may be large, response is always complete after one half cycle of overshoot. In fact, the control might be entirely satisfactory on a system with sufficient overdrive. For a system with 70-80% overdrive, the worst possible (magnitude, not percent) overshoot would be that shown in figure 24. If the system equation included a damping term the maximum overshoot would be reduced or eliminated, as in figure 22. This would represent a fortuitous partial cancellation of the two problems discussed. Of course if damping were increased further, the sluggishness induced by the damping would override the overestimation (or compound the underestimation) of braking input caused by lack of a free integration and dominate the response. The limiting case would be

the single time constant, figures 25 and 26, where the response is not much faster than that obtained by applying an open loop step input equal to the final control value.

Finally, there are two problems associated with the use of \ddot{V} in the algorithm. To be practical the control must operate with a heavily filtered version of \ddot{V} , but as demonstrated by the simulation of figure 13 through 16, this causes errant behaviour of V_F near the switch point. While this has little effect on response to a large step, response to small input changes would be far from optimal. For a small step input the switch point would occur early because of the filter's underestimation of \ddot{V} and the resulting large V_F . At the switch point, V_F would show a drastic increase due to the inordinately long time the filtered \ddot{V} spends near zero in changing polarity. By the time it (V_F) recovers something close to its pre-switch point magnitude, (R will be at maximum braking until this time) \dot{V} will have decreased drastically and \ddot{V} required to keep V_F equal to the setpoint will be very small. Given the proper conditions, the controller would either remain in its linear region after this time, for an extremely sluggish deadbeat response, or swing back to the accelerating input. A little extension of this

reasoning (hardly mathematically sound, but satisfying nonetheless) shows an unavoidable limit cycle.* A second, related problem is that V_F will "blow up" at equilibrium as \ddot{V} goes to zero, a condition also indicating the possibility of a limit cycle. This was recognized before the simulation and not considered a problem as an alternate control mode could be entered near equilibrium. Limit cycles from .1% to 1% were observed on all systems.

In summary it should be noted that while the algorithm will

not provide satisfactory control for the entire $\frac{K}{S^2+AS+B}$

family considered, it does have useful application to subclasses. Excellent potential is noted for load insensitive time optimal servo drives, for instance, where the load is primarily inertial and the system may be

approximated as $\frac{K}{S(JS+F)}$. K may also be time varying

in this application and therefore the only requirement is for relatively $(\frac{F}{J})$ small load friction. Tachometer

*A limit cycle is a stable (limited amplitude) oscillation about the equilibrium point.

feedback, which could supply \dot{V} directly, is often used in servo drives and should make this algorithm even more attractive. Application to systems where B is greater than zero are less straightforward. The algorithm will work well if some minimum overdrive ($\frac{K}{B} > 1$) can be guaranteed. That is, for a given minimum overdrive, worst case performance is known. If this is satisfactory, then performance will always be better than satisfactory as system gain varies. Relative damping affects response in much the same way as friction in the servo example.

Again, if a maximum damping ($\frac{A}{B}$) can be guaranteed and performance is satisfactory with that degree of damping, then it will always be more than satisfactory as damping varies. There are really two criteria then for successful

application of the algorithm to $\frac{K}{S^2 + AS + B}$ with K, A, and

B finite and time varying. Sufficient input overdrive (greater than that required for maximum output) must

be available to guarantee a minimum $\frac{K}{B}$ consistent with

allowable overshoots (figure 24). Secondly, the maximum

system damping $\frac{A}{B}$ must be less than some limit acceptable

to the user in terms of how sluggish (or suboptimal) a response can be tolerated. (See figures 21 and 22 versus figures 23 and 24. The major difference is the damping term.) There are tradeoffs here. The greater the overdrive, the less significant the damping becomes. On the other hand, the greater the damping, the less the overdrive required to minimize overshoots. In general, the amount of overdrive and damping would not be available for change by the control engineer and the decision to apply this control would depend on the amount of variation in the system parameters (is adaptive control required?), and the worst case performance predicted for this algorithm. Also, in both examples mentioned, provision would have to be made to eliminate the limit cycle, providing it was not acceptable to the user. Fortunately, a system with time varying parameters is likely to include considerable overdrive.

PROPOSED MODIFICATIONS TO THE ALGORITHM

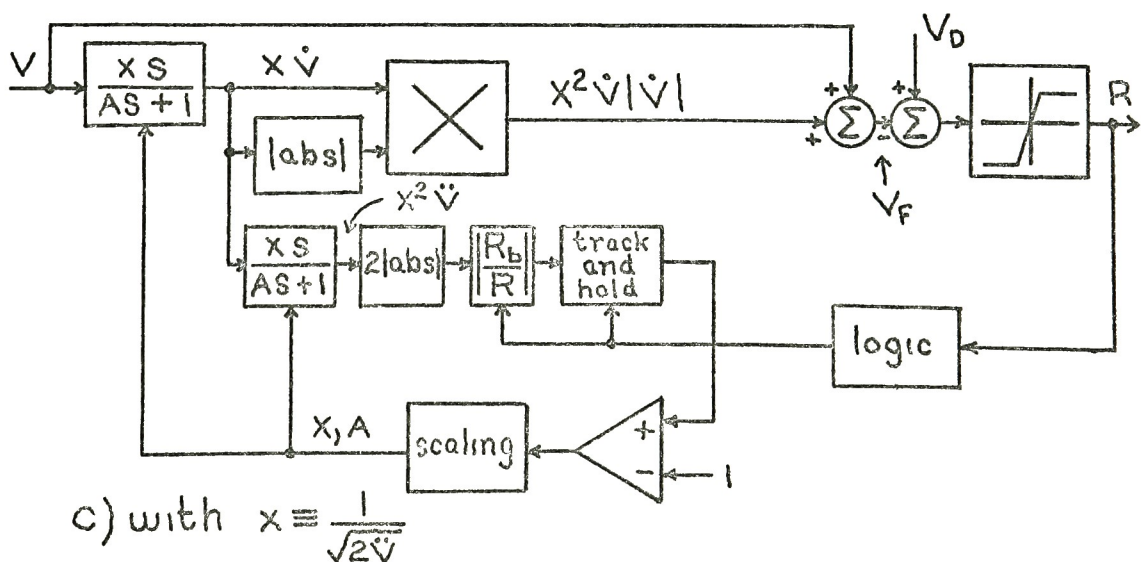
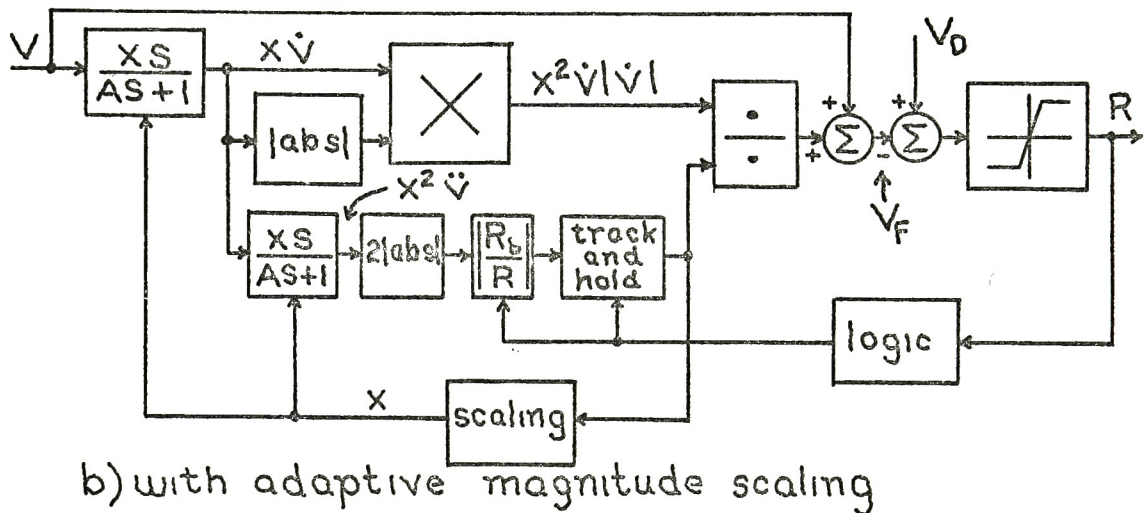
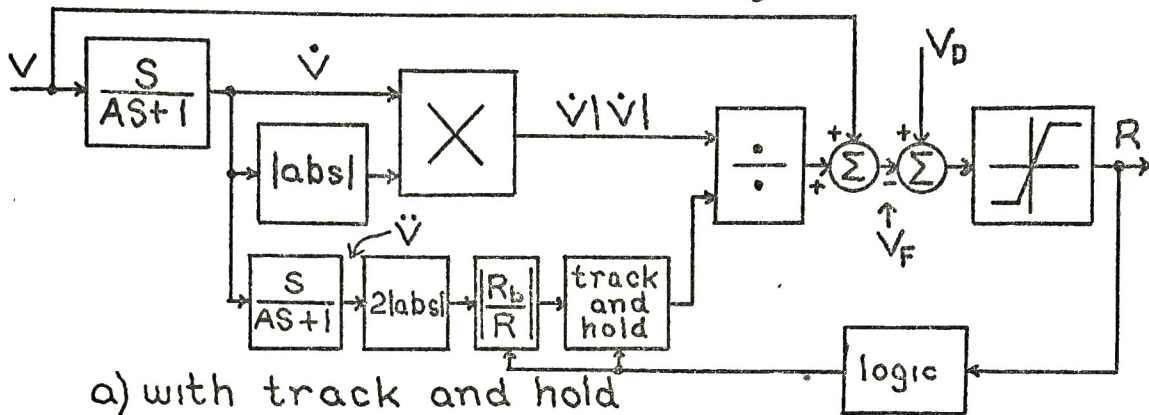
If the control is to be of any general use, the limit cycle mentioned must be eliminated. Two problems, the "blowup" of V_F as \ddot{V} goes to zero, and the small signal errors caused by \ddot{V} filtering, exist. The former may be eliminated by using the measured value of \ddot{V} only when R is R_{\max} or R_{\min} and storing the last value under that condition for use when R is between limits. The latter may be eliminated by requiring R to be at R_{\max} or R_{\min} for a period of time equal to the \ddot{V} filter risetime before control is transferred from the stored value to the measured value. A block diagram of the controller with the addition of this "track and hold" device is shown in figure 27a. Note that this controller uses the

$$\text{algorithm } V_F = V + \frac{\dot{V}|\dot{V}|}{2\left|\frac{R_b\ddot{V}}{R}\right|}. \quad \text{For convenience, } \frac{R_b}{R} \text{ was}$$

assumed equal to 1 in the previous section.

As mentioned earlier, in order to implement the algorithm with available physical components some form of magnitude scaling must be done on the derivatives to keep them within the useful range of the devices used. The derivatives used must also be filtered in accordance with

Figure 27
Controller Block Diagrams



system bandwidth in order to separate useful information from the expected noise. The magnitude of \ddot{V} itself may be used as a measure of system bandwidth, and filtering may be adjusted accordingly. The larger the magnitude of \ddot{V} , the higher (frequency) the filter breakpoint should be. Both derivatives can be scaled by the factor X in order to keep them in some acceptable range, with no

effect on the calculation of V_F , since
$$\frac{X\dot{V} |X\dot{V}|}{2X \left| \frac{R_b}{R} X\ddot{V} \right|} = \frac{\dot{V} |\dot{V}|}{2 \left| \frac{R_b}{R} \ddot{V} \right|} .$$

This scaling may be continuous or discontinuous. These additions are shown in figure 27b.

Note that, if the scaling is continuous, and implemented by

forcing $X^2 \left| \frac{R_b}{R} \ddot{V} \right|$ to equal one, the divider may be eliminated

from the block diagram as in figure 27c. We now have a controller wherein a filtered, sometimes sampled version of \ddot{V} is used to determine a scale factor for use in a proportional plus derivative type control. True, the derivative control is nonlinear ($R = G(V_D - V - X^2 \dot{V} |\dot{V}|)$, limited at R_{\max} and R_{\min}) but it may be more palatable to some because of successful industry experience with derivative compensation in standard controllers.

The scaling and storage discussed above are merely suggestions for practical implementation of the algorithm. Performance will be as detailed in the previous section and as shown qualitatively in figure 28. Attention will now be given to modifications which will extend the usefulness of the algorithm. These will be considered in order of increasing complication, as any large increase in complexity defeats the stated purpose of the algorithm. It is intended that these modified forms of the algorithm be applied with scaling, filtering, and storage techniques similar to those above.

The first alteration considered was from $V_F = V + \frac{\dot{v}|\dot{v}|}{2\left|\frac{R_b}{R}\ddot{v}\right|}$

to $V_F = V + \frac{\dot{v}|\dot{v}|}{2\left|\frac{R_b}{R}\ddot{v}_0\right|}$. Physically, this change is

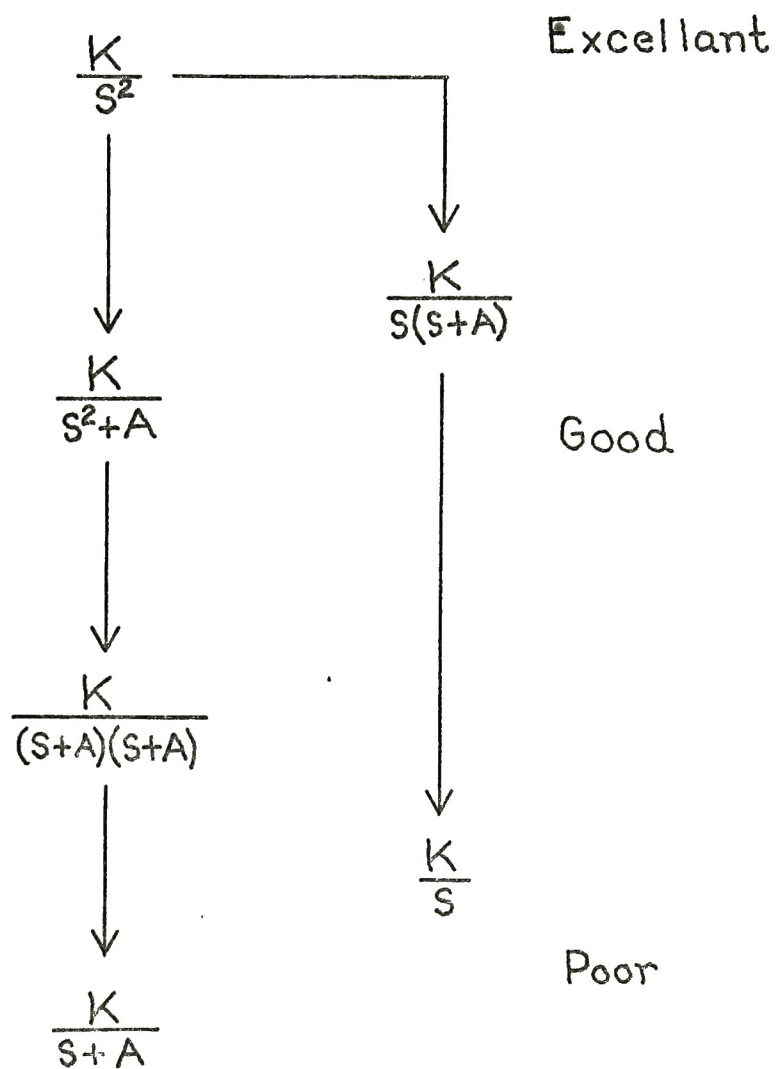
relatively minor if the track and hold approach of figure 27c has been adapted, since it can store a good approximation to \ddot{v}^* . Assuming a perfect determination of \ddot{v}_0 , V_F may be written analytically, as for the (servo)

system $\frac{K}{S(S+A)}$ in figure 29. Note that as A becomes large,

* \ddot{v}_0 is \ddot{v} immediately after the last switch point, where R_{\max} or R_{\min} was applied.

Figure 28

Performance of $V_F = V + \frac{\dot{V}|V|}{2|V|}$



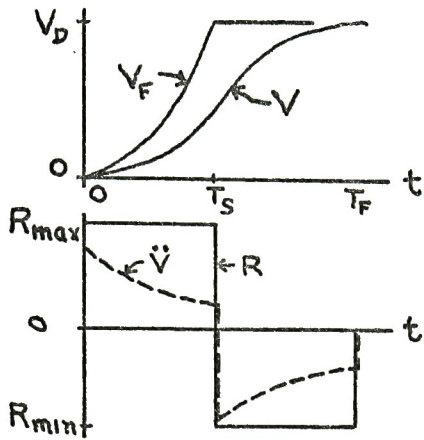


Figure 29

Derivation of V_F for System $\frac{K}{s(s+A)}$ when $V_F = V - \frac{V^2}{2 \frac{R_b}{R} \ddot{V}_0}$ from $t = 0$ to $t = T_s$,

$$V = -\frac{R_{\max} K}{A^2} + \frac{R_{\max} K t}{A} + \frac{R_{\max} K}{A^2} e^{-At}$$

$$\dot{V} = \frac{R_{\max} K}{A} - \frac{R_{\max} K}{A} e^{-At}$$

$$\ddot{V} = R_{\max} K e^{-At}$$

$$\ddot{V}_0 = R_{\max} K$$

$$V_F = V + \frac{\dot{V}|\dot{V}|}{2 \frac{R_b}{R} \ddot{V}_0} = V - \frac{\dot{V}^2}{2 \frac{R_b}{R} \ddot{V}_0}$$

$$V_F = -\frac{R_{\max} K}{A^2} + \frac{R_{\max} K t}{A} + \frac{R_{\max} K}{A^2} e^{-At} - \frac{\left(\frac{R_{\max} K}{A} - \frac{R_{\max} K}{A} e^{-At}\right)^2}{2 \frac{R_{\min}}{R_{\max}} R_{\max} K}$$

$$V_F = -\frac{R_{\max} K}{A^2} \left(1 + \frac{R_{\max}}{2 R_{\min}}\right) + \frac{R_{\max} K t}{A} + \left(1 + \frac{R_{\max}}{R_{\min}}\right) \frac{R_{\max} K}{A^2} e^{-At} + \frac{R_{\max}^2 K}{2 R_{\min} A^2} e^{-2At}$$

and,

$$\lim_{A \rightarrow \infty} V_F = \lim_{A \rightarrow \infty} V = \frac{R_{\max} K t}{A}$$

or as the system approaches $\frac{K}{AS}$, V_F approaches V , which is the optimal control for $\frac{K}{AS}$. This is in distinct contrast to the control provided by the original V_F , a control which became extremely sluggish as A became large.

Performance between the $\frac{K}{S^2}$ and $\frac{K}{AS}$ limits appears to be

only slightly suboptimal. Appendix 5 demonstrates

mathematically the application of this V_F to $\frac{.4}{S(S+.1)}$,

including the effects of an arbitrary amount of filtering on \ddot{V} , which makes the result slightly less optimal than predicted by the general V_F above. Note that this system, which had been somewhat slow with the previous V_F (figure 19), obtains a deadbeat response in elapsed time only 3% greater than optimal (figure 30). These results were corroborated by analog simulation. A similar analysis of

several other systems could generate a curve of $\frac{T_{\text{elapsed}}}{T_{\text{optimal}}}$

versus $\frac{K}{A}$, which could be used to determine the response

for any $\frac{K}{A}$. In view of the performance expected, this is

probably not necessary. $V_F = V + \frac{\dot{V}|\dot{V}|}{2\left|\frac{R_b}{R}\ddot{V}_0\right|}$ has been shown

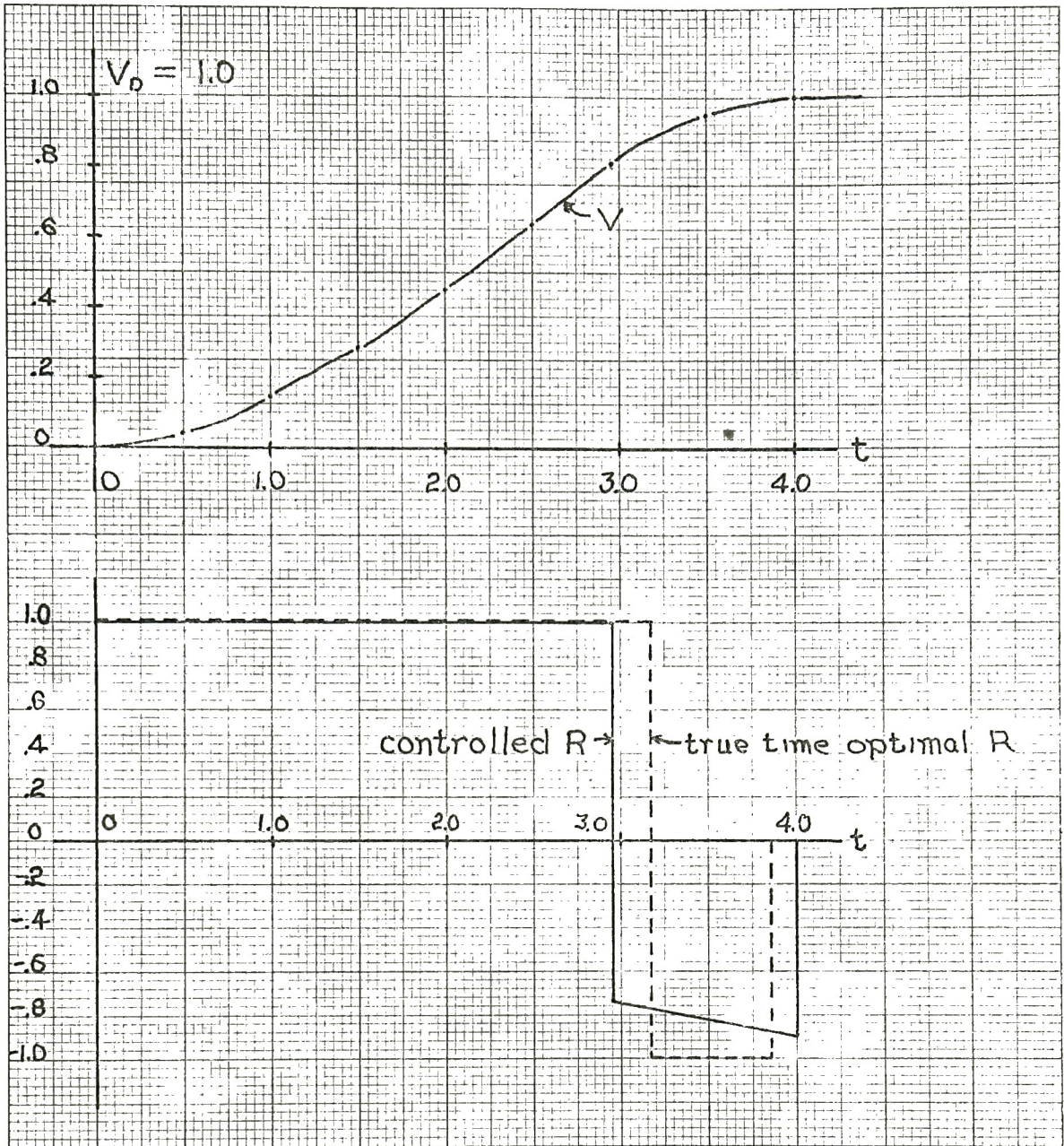


Figure 30

Response of Controlled System $\frac{.4}{s(s+1)}$,
 when $V_f = V + \frac{\dot{V}|\dot{V}|}{2|\ddot{V}_0|}$, and
 $\ddot{V}_0 \approx \frac{\dot{V}(\tau) - \dot{V}_0}{T}$, $T = .08 / \ddot{V}_0$

to be almost perfectly adaptive and time optimal for all

systems in the $\frac{K}{S^2} \rightarrow \frac{K}{S(S+A)} \rightarrow \frac{K}{AS}$ subclass, however,

it has the disadvantage of not adapting as rapidly to

parameter changes as the previous $V_F = V = \frac{\dot{V}|\dot{V}|}{2\left|\frac{R_b}{R}\ddot{V}\right|}$.

Thus a designer faced with step changes in load might find the earlier control a better choice. A step parameter change might not be recognized by the later control until a switch point occurred, while the earlier control would take immediate corrective action.

Also of interest in Appendix 5 work is the method of approximating \ddot{V}_0 . This involves the relationship

$\ddot{V}_0 \approx \frac{\dot{V}(T) - \dot{V}_0}{T}$, where T is a time sufficient for the

required noise rejection but small in comparison with the typical transient times in the system. In the example of

Appendix 5, $T = \frac{.03}{\ddot{V}_0}$, where \ddot{V}_0 represents an average of

previously determined \ddot{V}_0 values, is used. This is

consistent with the filtering requirements (that the filter bandwidth vary with \ddot{V}) outlined previously. Of more particular interest, however, is the fact that any direct measurement of \ddot{V} is eliminated from the algorithm,

$$\text{since } V_F \text{ may now be written } V_F = V + \frac{\dot{V}|\dot{V}|T}{2\left|\frac{R_b}{R}(V(T)-V_0)\right|} .$$

T is a stored "constant" whose value is not critical, although it must roughly track changes in system dynamics. The magnitude scaling discussed before is still applicable of course. Control is transferred from the stored value of \ddot{V}_0 at T. During the second (braking) phase of a transient, the newly determined value will be essentially equal to the stored value, if it can be obtained with less than maximum braking.

The alteration discussed will not significantly improve

performance with $\frac{K}{S^2+A}$, however the statements made

concerning the original algorithm $(V_F = V + \frac{\ddot{V}|\ddot{V}|}{2\left|\frac{R_b}{R}\ddot{V}\right|})$

when applied to the $\frac{K}{S^2+B} \rightarrow \frac{K}{S^2+AS+B} \rightarrow \frac{K}{S}$ subclass may

now be modified. Performance on all systems in this

class will be essentially the same as that provided by

the original algorithm on $\frac{K}{S^2+B}$, rather than becoming

sluggish as damping increases. Overall performance

for $V_F = V + \frac{\dot{V}|\dot{V}|}{2\left|\frac{R_b}{R}\ddot{V}_0\right|}$ is shown qualitatively in figure 31.

The encouraging results of the first modification, which virtually eliminated control sensitivity to system damping, stimulated the search for a further modification which could negate the imbalance effects associated with finite zero frequency gains. In the original algorithm, $2|RK|$ was associated with the accelerating force. The available braking force was assumed equal to, or related by a constant factor to, the accelerating force.

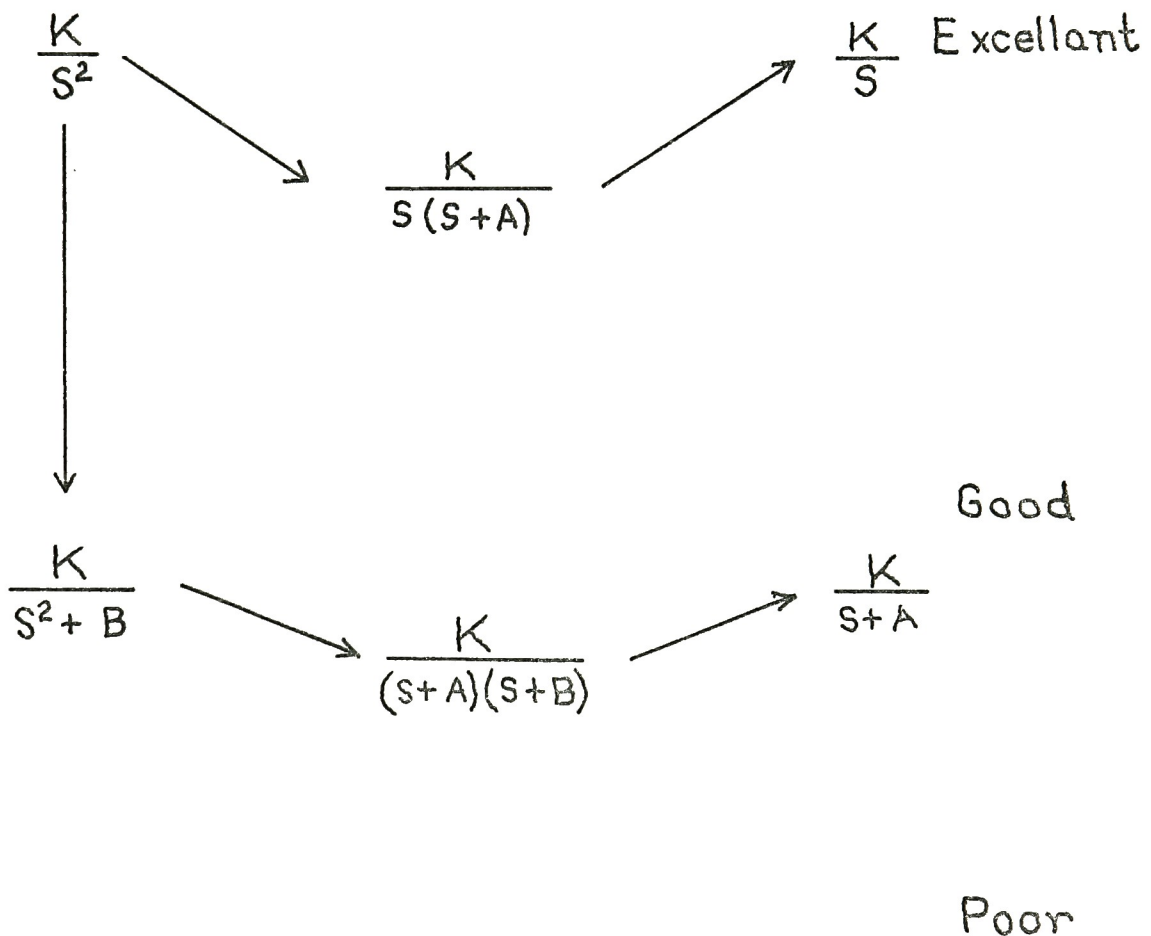
V , because of its relationship to RK in the model $\left(\frac{K}{S^2}\right)$, was used as a measure of available braking force. One notes however, that in the general second order system

$$\frac{K}{S^2+AS+B}, \quad \ddot{V}_0 = R_0K-BV_0 \quad (\text{see Appendices 1, 2, 3, 4})$$

and that therefore a more proper indicator of braking force might be $|R_0K-BV|$, where R_b is the braking input,

Figure 31

Performance of $V_F = V + \frac{\dot{v}|\dot{v}|}{2|\ddot{v}_0|}$



or that input which drives V toward zero. The advantages

of this concept ($V_F = V + \frac{\dot{V}|\dot{V}|}{2|R_b K - BV|}$) are obvious when

one considers its application to the system ($\frac{.4}{S^2+4}$) of

figures 23 and 24. \ddot{V}_0 of course equals RK when $V_0 = 0$, and the operation of the control under discussion in the vicinity of zero (relative to K/B) will thus be identical to that of the original algorithm. For small steps near zero, \ddot{V} will remain relatively constant at RK during the brief transient periods and

$$V_F \approx V - \frac{\dot{V}^2}{2R_b K} \approx V + \frac{\dot{V}|\dot{V}|}{2\left|\frac{R_b}{R}\ddot{V}\right|} \approx V + \frac{\dot{V}|\dot{V}|}{2\left|\frac{R_b}{R}\ddot{V}_0\right|} \approx V + \frac{\dot{V}|\dot{V}|}{2|R_b K - BV|} .$$

Figure 23 illustrates the underestimation of braking force as V is driven toward an extreme (toward K/B), with the result that braking is applied much too early for optimal response. If the algorithm had been based on

$V_F = V + \frac{\dot{V}|\dot{V}|}{2|R_b K - BV|}$, the switch point would have occurred

later, at the point where $R = R_{\min}$, (-1 in this case), would be required to keep V_F equal to the setpoint after the switch (the basic action of the algorithm).

The switch point will still occur slightly earlier than the time optimal switch point, since the effective average braking force available is something between $2|R_b K - BV|$ and $2|R_b K - BV_D|$, and is thus greater than $2|R_b K - BV|$.

This insures a deadbeat response. The same argument applies when the output is driven toward zero. Braking will be applied slightly before the time optimal switch point and the output may thus be driven to equilibrium along a deadbeat trajectory. Intuitively, it would seem that the control could be improved even further by replacing B with some function of B , V , and V_D accounting for the effective average braking force which, as mentioned, is slightly greater than $2|R_b K - BV|$.

In order to retain the adaptive nature of the control, the unknown system parameters K and B must be expressed in terms of measureable quantities, as RK was approximated by \ddot{V} in the original algorithm. This naturally results in more complication; more "memory" and computational ability required in the controller. Using the fact that $\ddot{V}_0 = R_0 K - BV$, both K and B can be expressed in terms of R, V , and \ddot{V} at (any) two different switch points (Appendix 6). These logically would be the values at a current switch point and the switch point immediately

preceding it. The values of K and B used in the control might also be averages of individual results obtained in this way. \ddot{V}_0 may be approximated from \dot{V} as discussed under the first modification. A block diagram of the controller is shown in figure 32.

Appendix 6 illustrates mathematically, the application

of $V_F = V + \frac{\dot{V}|\dot{V}|}{2|R_b K - BV|}$ to the system $\frac{.4}{S^2 + .4}$, with results

in figure 33. The transient from .58 to zero is of primary interest, because of the large overshoot in figure 24. This response (figure 33) is calculated exactly, both for time optimal switching and for control based on the algorithm. Note that final equilibrium is achieved at a time only 3% greater than optimal.

The effect of increased system damping on the control

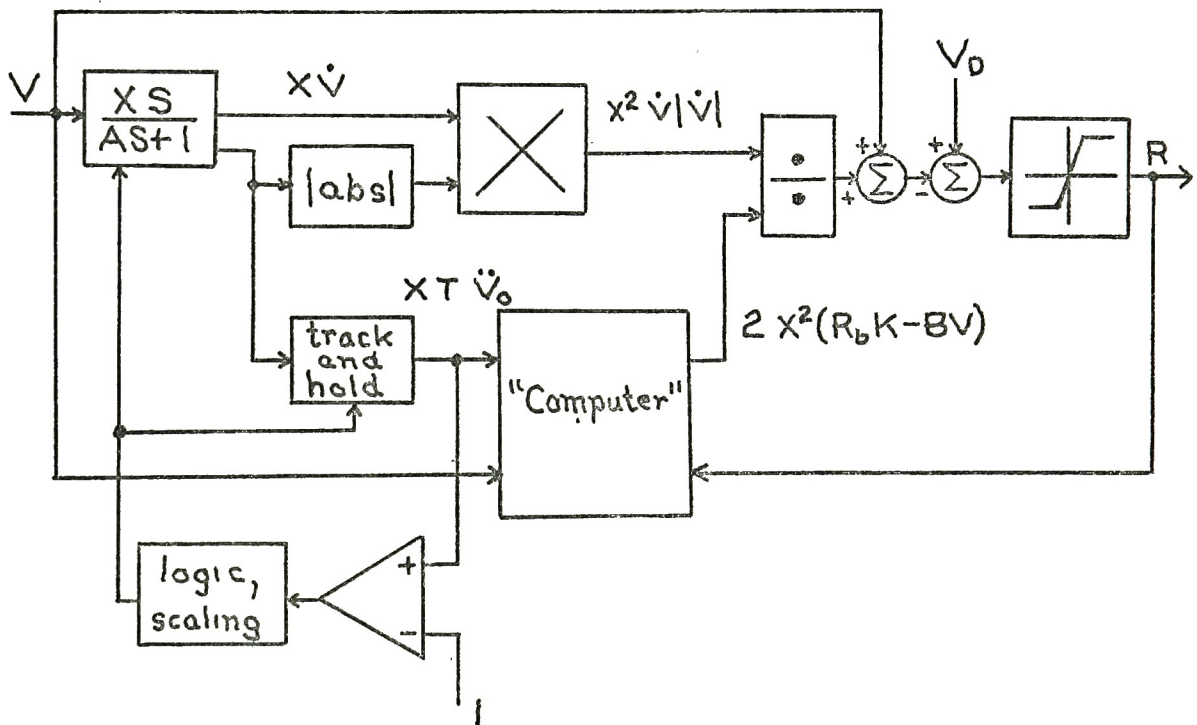
is expected to be small, as with $V_F = V + \frac{\dot{V}|\dot{V}|}{2\left|\frac{R_b}{R}\ddot{V}_0\right|}$,

the first modification. As system damping increases,

$\frac{K}{S^2 + AS + B}$ may be written as $\frac{K'CD}{(S+C)(S+D)}$, with C becoming

Figure 32
Controller Block Diagram

$$\text{for } V_F = V + \frac{\dot{V}|\dot{V}|}{2(R_b K - BV)}$$



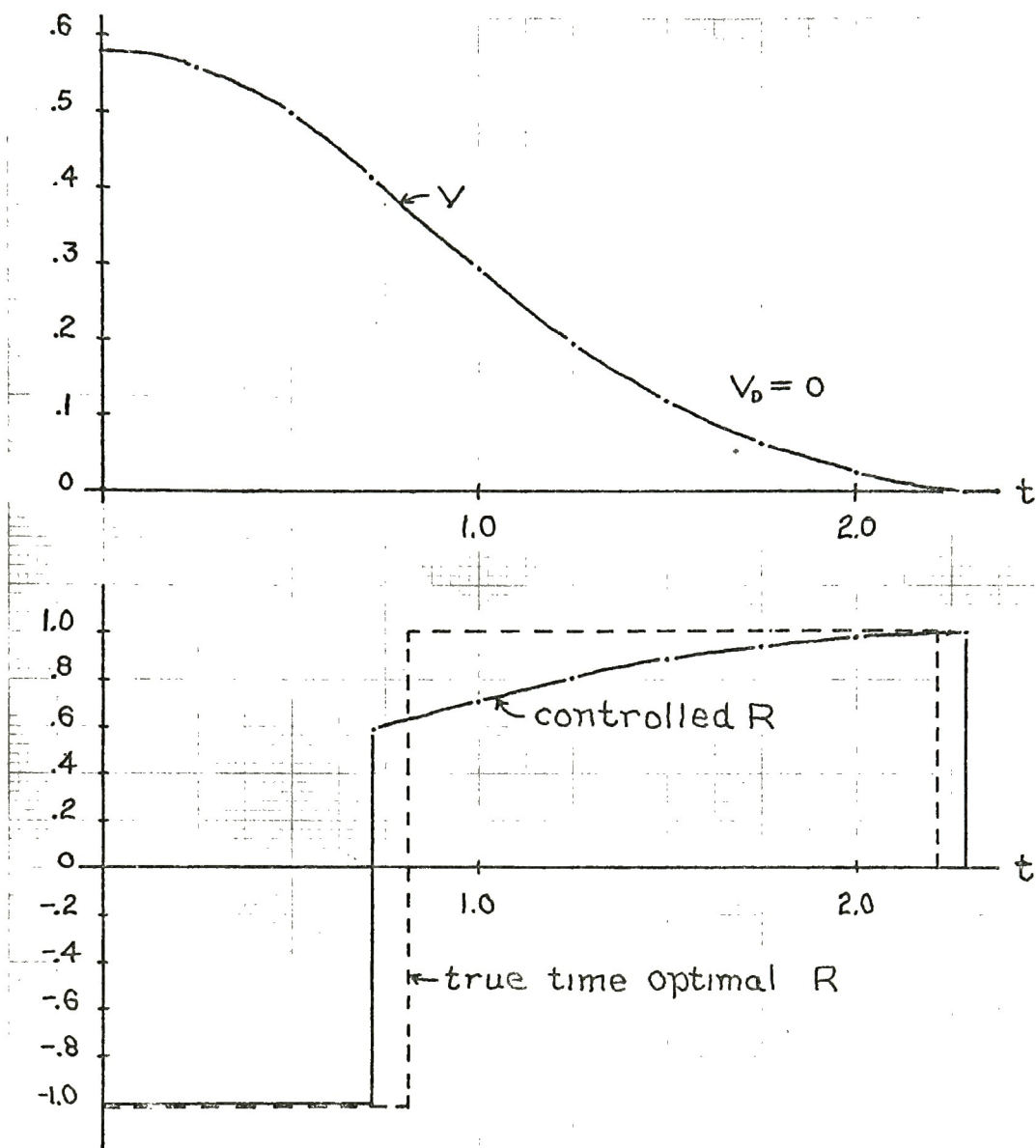


Figure 33

Response of Controlled System $\frac{.4}{s^2 + .4}$,
 when $V_F = V + \frac{\dot{V}|\dot{V}|}{2|R_b K - BV|}$, and
 K and B are defined as in Appendix 6

large with respect to D , and the limiting case being

$\frac{K'D}{S+D}$. Note that $K = K'CD$ and $B = CD$ both approach infinity

as C approaches infinity and therefore V_F approaches V ,

the optimal control for $\frac{K'D}{S+D}$. Appendix 6 also illustrates

the application of $V_F = V + \frac{\dot{v}|\dot{v}|}{2|R_b K - BV|}$ to $\frac{.4}{(S+.62)(S+.58)}$,

a system very near that of figures 21 and 22. Compare the sluggish response of figure 21 with that of figure 34. Time to equilibrium in this case is 9% over optimal.

This latest version of V_F , $V + \frac{\dot{v}|\dot{v}|}{2|R_b K - BV|}$, provides

satisfactory control on all systems in the original

classification ($V = \frac{RK}{S^2 + AS + B}$ where A , B and K take on

arbitrary values). It is somewhat more complicated than had been hoped for, but certainly within the realm of a single loop controller with hardware available today.

Speed of adaptation would be dependent on the techniques used for generation of K and B . Consideration should

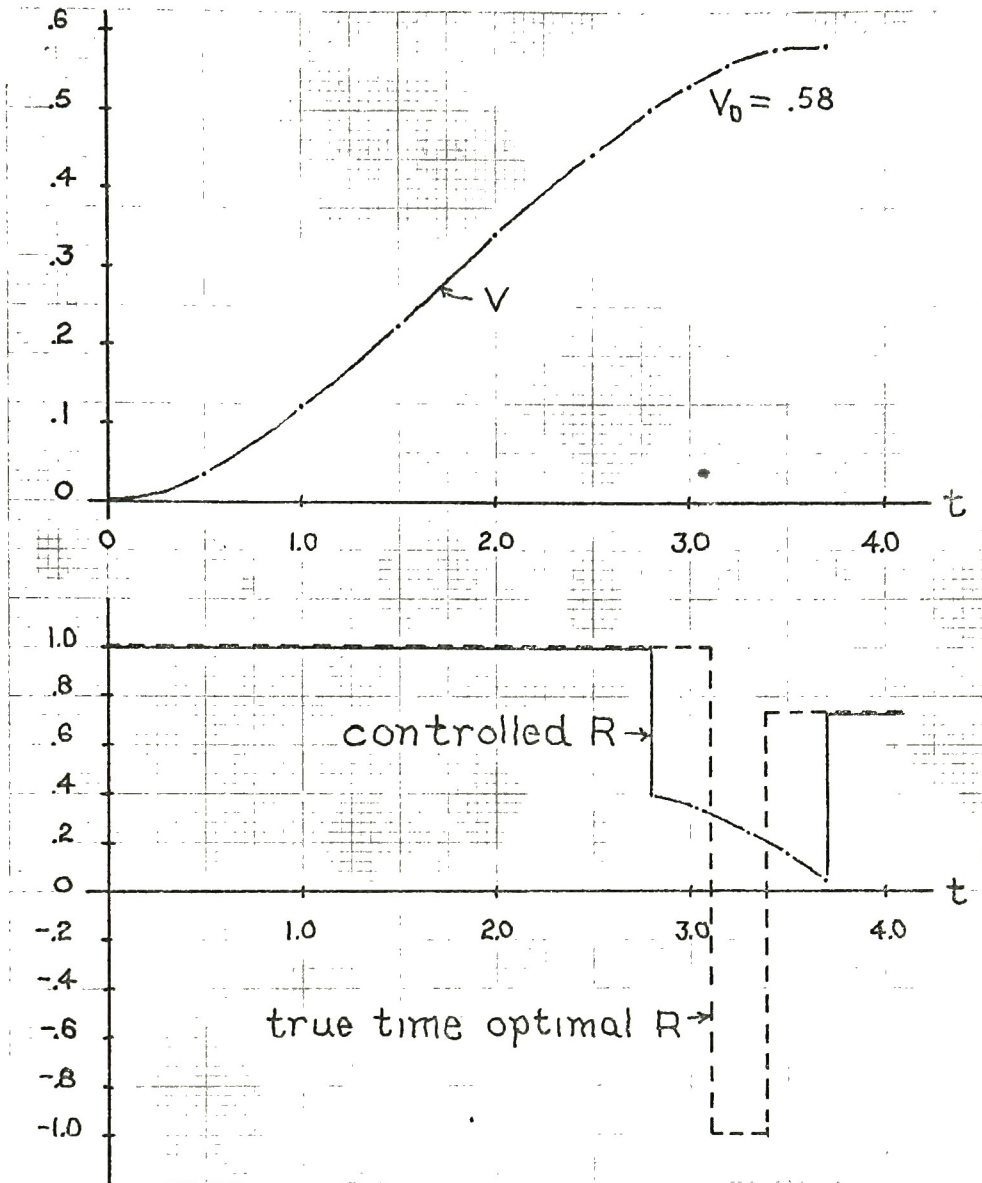


Figure 34

Response of Controlled System $\frac{.4}{(s+.62)(s+.58)}$,

when $V_F = V + \frac{\dot{V}|\dot{V}|}{2|R_b K - BV}$, and

K and B are defined as in Appendix 6

be given to providing initial values (when control goes on line) for K and B to minimize the possibility of a large overshoot occurring before the first determination of these parameters. If the K and B used are averages of several values, several large overshoots can occur before the controller "learns" the system. The solution might be initial values of $K = B = E$, where E is a suitably small number. The control would then progress from overdamped to optimal.

CONCLUSIONS

The main objective of this investigation has been met. That is, a control algorithm providing near time optimal,

adaptive control, to all systems of the form $\frac{K}{S^2+AS+B}$,

has been defined. Its application, at least within the scope of this study, has been limited to systems subjected primarily to step inputs, (as many, if not most, industrial processes are). With reference to figure 35, the constraints on the controlled system are as follows. First of course,

the system must be closely approximated by $\frac{K}{S^2+AS+B}$.

This expression includes the dynamic characteristics of the actuator, so that the actuator and transmitter as depicted here are strictly linear gain blocks. It is assumed that maximum actuator output does not result in saturation within the process, as long as the process is within its normal limits and that manual inputs to the controller limit maximum values of R to those values which result in maximum actuator outputs. Actuator saturation cannot be allowed, since the value of R is used in the algorithm. Maximum values for R, V (Transmitter gain), and V_D have been arbitrarily set at ± 1 , although this has no particular

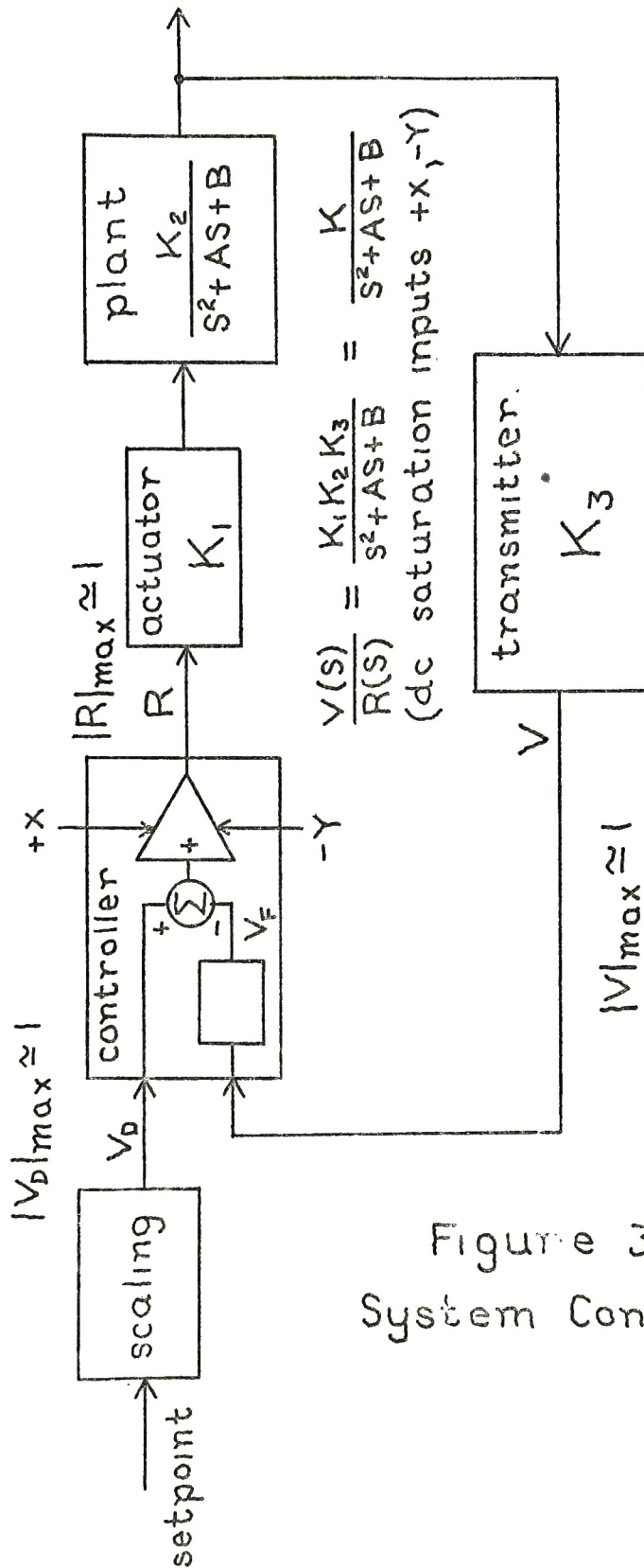


Figure 35
System Constraints

significance. Practically any system to which a standard 2 or 3 mode controller had been applied would meet the actuator-transmitter requirements with no modification.

The control algorithm itself is the modified error signal $V_D - V_F$, which represents the setpoint minus predicted final output, applied to a high gain limiting amplifier to produce controller output R. The amplifier limits are set in compliance with the actuator saturation constraint above.

Predicted final output V_F is equal to $V + \frac{\dot{V}|\dot{V}|}{2|R_b K - BV}$, where

R_b is the available braking input (controller output).

The adaptive feature of the control is in the automatic measurement and use of the system parameters K and B.

Using the fact that $\ddot{V}_0 = R_0 K - BV_0$, K and B can be expressed in terms of R, V, and \ddot{V} at (any) two different switch points (where R makes a rapid transition from accelerating to braking, or vice-versa). These equations may be found in Appendix 6.

While stability problems should not be encountered with any system within the general classification, it could be advantageous to switch to a different control mode when within a given distance from the setpoint. One reason

could be noise triggering of the limiting amplifier, causing unnecessary "chattering" of the actuator . A slower but less noise sensitive control mode, entered only near equilibrium, could prevent this with no sacrifice in speed when changing operating points. A related matter is that while noise sensitivity increases with increased gain in the limiting amplifier, a system with low 'dc' gain will require high gain in the limiting amplifier to preserve low steady state error. Another possible reason for switching modes is that different performance characteristics might be desired at the operating point.

The algorithm is rather complex for analog implementation, especially since storage of the approximate system parameters is required, however the intended application is to single loop control, rather than multi-loop computer controlled systems. It is suggested that the controller could be built economically, in volume, with an analog to digital converter input, arithmetic unit, several storage locations, control logic, and digital to analog output converter, all of which is currently available in a handful of integrated circuit chips (consider the capabilities of the more advanced pocket calculators). If a second control mode were desired at equilibrium, it would be simply an addition to the control logic.

Regarding the extension of the algorithm to systems

with lead terms, $\frac{V(S)}{R(S)} = \frac{K_1 + K_2 S}{S^2 + AS + B}$, few applications

seem likely. However, should such extension be desirable, further work (not shown) has indicated that

$$V_{0+} = (R_{0+} - R_{0-})K_2 + \dot{V}_{0-}$$

and that the defined control algorithm can be used if V is modified to an equation of the form

$$V_F = V + \frac{[\dot{V} - (R_b - R)K_2 \dot{V}] [V - (R_b - R)K_2 \dot{V}]}{2 |R_b K_2 - BV|} \quad \text{where the}$$

derivative term exists only when of the same sign as V .

Regarding the possible extension to third order systems, further work (also not shown) has indicated a high probability of limit cycles or poorly damped responses. A third order algorithm (V_F) would probably be based on the term \ddot{V} rather than $\dot{V}|\dot{V}|$, however very little effort has been applied to the third order case.

Less sophisticated versions of the algorithm, both adaptive and fixed parameter, may find application to individual control problems. In applications where a near time

optimal control is desired, but adaptive control is not required, the algorithm could be used as defined, but without the automatic measurement of system parameters. These parameters (K and B) could be measured by any suitable method and included as fixed parameters of the controller. Since storage would no longer be required, an analog implementation would again be possible, at least in cases where derivative compensation in 3-mode control is possible now. In any case the controller would be considerably simpler in concept and hardware than the adaptive form. Its advantages over conventional controllers would be the time optimal performance and the knowledge that optimum performance was being obtained from the controller, a point frequently in doubt with the conventional units.

Simpler adaptive forms of the algorithm might also find

application in many areas. The form $V_F = V + \frac{\dot{V}|\dot{V}|}{2\left|\frac{R}{R}b\ddot{V}_0\right|}$

can perform nearly as well as the most complicated version of the algorithm, on systems with a free integration or with considerable overdrive (more input available than is required for maximum desired output). In its simplest application, storage of only one quantity (\ddot{V}_0) is required.

Complications might involve filtering or averaging of the \ddot{V}_0 term, filtering of \dot{V} , switched modes at equilibrium, etc. For systems requiring "instant" adaptation, one might

use the form $V_F = \frac{\dot{V}|\dot{V}|}{2\left|\frac{R}{R} \frac{\dot{V}}{V}\right|}$ which does not depend on

stored values and thus reacts immediately to changes in system parameters. The potential inadequacies of this V_F have been detailed, however, the appeal of the adaptation speed may override these.

If the controller was built as suggested, in digital form with analog inputs and outputs, arithmetic unit, storage and control logic, and in addition was modular in concept, any or all of the above versions of the algorithms could be implemented with modular additions or substitutions to the control logic and memory. Obviously many other algorithms could be implemented with the unit.

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Appendix 1

Development of V_F for model system

$$\frac{V(s)}{R(s)} = \frac{K}{s(s+A)}$$

System response to input $\frac{R}{s}$;

$$Vs^2 - V_0s - \dot{V}_0 + AVs - AV_0 = \frac{RK}{s}$$

$$V(s) = \frac{V_0s^2 + (AV_0 + \dot{V}_0)s + RK}{s^3 + As^2}$$

$$V(s) = \frac{B}{s} + \frac{C}{s^2} + \frac{D}{s+A}$$

$$V(s) = \frac{(B+D)s^2 + (AB+C)s + AC}{s^3 + As^2}$$

$$\therefore B+D = V_0, AC = RK, AB+C = AV_0 + \dot{V}_0$$

$$\text{and } C = \frac{RK}{A}$$

$$AB = AV_0 + \dot{V}_0 - \frac{RK}{A}$$

$$B = V_0 + \frac{\dot{V}_0}{A} - \frac{RK}{A^2}$$

$$D = V_0 - V_0 - \frac{\dot{V}_0}{A} + \frac{RK}{A^2} = \frac{RK}{A^2} - \frac{\dot{V}_0}{A}$$

$$\therefore V(t) = V = V_0 + \frac{\dot{V}_0}{A} - \frac{RK}{A^2} + \frac{RK}{A}t + \left(\frac{RK}{A^2} - \frac{\dot{V}_0}{A}\right)e^{-At}$$

Solution for V_F ;

$$\dot{V} = \frac{RK}{A} + \left(\dot{V}_0 - \frac{RK}{A}\right)e^{-At}$$

and when $R = R_b$, $t = T_F$,

$$\dot{V}(T_F) = \frac{R_bK}{A} + \left(\dot{V}_0 - \frac{R_bK}{A}\right)e^{-AT_F} = 0$$

$$e^{-AT_F} = \frac{1}{\left(1 - \frac{A\dot{V}_0}{R_b K}\right)}$$

$$T_F = \frac{1}{A} \ln\left(1 - \frac{A\dot{V}_0}{R_b K}\right)$$

$$V_F = V(T_F), \text{ when } V_0 = V(t), \dot{V}_0 = \dot{V}(t), R = R_b$$

$$V_F = V + \frac{\dot{V}}{A} - \frac{R_b K}{A^2} + \frac{R_b K}{A^2} \ln\left(1 - \frac{A\dot{V}}{R_b K}\right) + \frac{R_b K}{A^2}$$

$$V_F = V + \frac{\dot{V}}{A} + \frac{R_b K}{A^2} \ln\left(1 - \frac{A\dot{V}}{R_b K}\right)$$

Note that ;

$$\dot{V} = \frac{RK}{A} + \left(\dot{V}_0 - \frac{RK}{A}\right) e^{-At}$$

$$\ddot{V} = -A\left(\dot{V}_0 - \frac{RK}{A}\right) e^{-At}$$

$$\ddot{V} = A^2\left(\dot{V}_0 - \frac{RK}{A}\right) e^{-At}$$

and \therefore

$$\ddot{V} = \frac{\ddot{V}}{-A}$$

$$-A = \frac{-\ddot{V}}{\ddot{V}}$$

also ,

$$\dot{V} = \frac{RK}{A} - \frac{\ddot{V}}{A}$$

$$K = \frac{1}{R} (A\dot{V} + \ddot{V})$$

Appendix 2

Development of V_F for model system

$$\frac{V(s)}{R(s)} = \frac{K}{S^2}$$

System response to input $\frac{R}{S}$;

$$VS^2 - V_0S - \dot{V}_0 = \frac{RK}{S}$$

$$V = \frac{RK + \dot{V}_0S + V_0S^2}{S^3}$$

$$V = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{S^3} = \frac{AS^2 + BS + C}{S^3}$$

$$\therefore A = V_0, B = \dot{V}_0, C = RK$$

$$V(t) = V = V_0 + \dot{V}_0t + \frac{RK}{2}t^2$$

$$\dot{V} = \dot{V}_0 + Rkt$$

$$\ddot{V} = RK$$

Solution for V_F ;when $R = R_b$, $t = T_F$,

$$\dot{V}(T_F) = \dot{V}_0 + R_bKT_F = 0$$

$$T_F = \frac{-\dot{V}_0}{R_bK}$$

 $V_F = V(T_F)$, when $V_0 = V(t)$, $\dot{V}_0 = \dot{V}(t)$, $R = R_b$

$$V_F = V + \dot{V}T_F + \frac{R_bK}{2}T_F^2$$

$$V_F = V - \frac{\dot{V}^2}{R_bK} + \frac{R_bK\dot{V}^2}{2R_b^2K^2} = V - \frac{\dot{V}^2}{R_bK} + \frac{\dot{V}^2}{2R_bK}$$

$$V_F = V - \frac{\dot{V}^2}{2R_bK}$$

Appendix 3

Development of V_f for model system

$$\frac{V(s)}{R(s)} = \frac{K}{s^2 + B}$$

System response to input $\frac{R}{s}$;

$$Vs^2 - V_0s - \dot{V}_0 + BV = \frac{RK}{s}$$

$$V(s) = \frac{V_0s^2 + \dot{V}_0s + RK}{s^3 + Bs}$$

$$V(s) = \frac{A}{s} + \frac{C}{s + j\sqrt{B}} + \frac{D}{s - j\sqrt{B}}$$

$$V(s) = \frac{(A+C+D)s^2 + (j\sqrt{B}D - j\sqrt{B}C)s + AB}{s^3 + Bs}$$

$$\therefore AB = RK, j\sqrt{B}D - j\sqrt{B}C = \dot{V}_0, A+C+D = V_0$$

$$A = \frac{RK}{B}$$

$$C+D = V_0 - \frac{RK}{B} \quad \textcircled{1}$$

$$-C+D = \frac{\dot{V}_0}{j\sqrt{B}} \quad \textcircled{2}$$

$$2D = V_0 + \frac{\dot{V}_0}{j\sqrt{B}} - \frac{RK}{B} \quad \textcircled{1} + \textcircled{2}$$

$$\therefore D = \frac{V_0}{2} + \frac{\dot{V}_0}{2j\sqrt{B}} - \frac{RK}{2B}$$

$$C = D - \frac{\dot{V}_0}{j\sqrt{B}} = \frac{V_0}{2} - \frac{\dot{V}_0}{2j\sqrt{B}} - \frac{RK}{2B}$$

$$\therefore V(t) = \frac{RK}{B} + \frac{\dot{V}_0}{\sqrt{B}} \sin\sqrt{B}t + \left(V_0 - \frac{RK}{B}\right) \cos\sqrt{B}t$$

$$\dot{V} = \dot{V}_0 \cos\sqrt{B}t - \sqrt{B} \left(V_0 - \frac{RK}{B}\right) \sin\sqrt{B}t$$

$$\ddot{V} = -\sqrt{B} \dot{V}_0 \sin\sqrt{B}t - B \left(V_0 - \frac{RK}{B}\right) \cos\sqrt{B}t$$

Solution for V_F ;
 when $R = R_b$, $t = T_F$,

$$\dot{V}(T_F) = \dot{V}_0 \cos \sqrt{B} T_F - \sqrt{B} \left(V_0 - \frac{R_b K}{B} \right) \sin \sqrt{B} T_F = 0$$

$$\frac{\sin \sqrt{B} T_F}{\cos \sqrt{B} T_F} = \tan \sqrt{B} T_F = \frac{\dot{V}_0}{\sqrt{B} \left(V_0 - \frac{R_b K}{B} \right)}$$

$$T_F = \frac{1}{\sqrt{B}} \tan^{-1} \frac{\dot{V}_0}{\sqrt{B} \left(V_0 - \frac{R_b K}{B} \right)}$$

$V_F = V(T_F)$, when $V_0 = V(t)$, $\dot{V}_0 = \dot{V}(t)$, $R = R_b$

$$V_F = \frac{R_b K}{B} + \frac{\dot{V}}{\sqrt{B}} \sin \left(\tan^{-1} \frac{\dot{V}}{\sqrt{B} \left(V - \frac{R_b K}{B} \right)} \right) \\ + \left(V - \frac{R_b K}{B} \right) \cos \left(\tan^{-1} \frac{\dot{V}}{\sqrt{B} \left(V - \frac{R_b K}{B} \right)} \right)$$

Appendix 4

Development of V_F for model system

$$\frac{V(s)}{R(s)} = \frac{K}{(s+A)(s+B)}$$

System response to input $\frac{R}{s}$;

$$Vs^2 - V_0s - \dot{V}_0 + (A+B)Vs - (A+B)V_0 + ABV = \frac{RK}{s}$$

$$V(s) = \frac{V_0s^2 + (\dot{V}_0 + (A+B)V_0)s + RK}{s(s+A)(s+B)}$$

$$V(s) = \frac{X}{s} + \frac{Y}{s+A} + \frac{Z}{s+B}$$

$$V(s) = \frac{(X+Y+Z)s^2 + ((A+B)X + BY + AZ)s + ABX}{s(s+A)(s+B)}$$

$$\therefore X+Y+Z = V_0, (A+B)X + BY + AZ = \dot{V}_0 + (A+B)V_0, ABX = RK$$

$$X = \frac{RK}{AB}$$

$$Z = V_0 - X - Y = V_0 - \frac{RK}{AB} - Y$$

$$(A+B)\frac{RK}{AB} + BY + A(V_0 - \frac{RK}{AB} - Y) = \dot{V}_0 + (A+B)V_0$$

$$(B-A)Y = \dot{V}_0 + BV_0 - B\frac{RK}{AB}$$

$$\therefore Y = \frac{\dot{V}_0 + BV_0 - RK/A}{B-A}$$

$$Z = V_0 - \frac{RK}{AB} + \frac{\dot{V}_0 + BV_0 - RK/A}{A-B}$$

$$Z = \frac{\dot{V}_0 + AV_0 - RK/B}{A-B}$$

$$\therefore V(t) = \frac{RK}{AB} + \frac{(\dot{V}_0 + BV_0 - RK/A)}{B-A} e^{-At} + \frac{(\dot{V}_0 + AV_0 - RK/B)}{A-B} e^{-Bt}$$

$$V(t) = V = \frac{RK}{AB} + \dot{V}_0 \left(\frac{e^{-At}}{B-A} + \frac{e^{-Bt}}{A-B} \right) + \left(V_0 - \frac{RK}{AB} \right) \left(\frac{Be^{-At}}{B-A} + \frac{Ae^{-Bt}}{A-B} \right)$$

$$\dot{V} = \dot{V}_0 \left(\frac{-Ae^{-At}}{B-A} - \frac{Be^{-Bt}}{A-B} \right) + \left(V_0 - \frac{RK}{AB} \right) \left(\frac{-ABe^{-At}}{B-A} - \frac{ABe^{-Bt}}{A-B} \right)$$

$$\ddot{V} = \dot{V}_0 \left(\frac{A^2 e^{-At}}{B-A} + \frac{B^2 e^{-Bt}}{A-B} \right) + \left(V_0 - \frac{RK}{AB} \right) \left(\frac{A^2 Be^{-At}}{B-A} + \frac{AB^2 e^{-Bt}}{A-B} \right)$$

Solution for V_F ;

when $R = R_b$, $t = T_F$,

$$\dot{V}(T_F) = \dot{V}_0 \left(\frac{Ae^{-AT_F}}{A-B} + \frac{Be^{-BT_F}}{B-A} \right) + \left(V_0 - \frac{R_b K}{AB} \right) \left(\frac{ABe^{-AT_F}}{A-B} + \frac{ABe^{-BT_F}}{B-A} \right) = 0$$

$$0 = \left(\dot{V}_0 + A \left(V_0 - \frac{R_b K}{AB} \right) \right) \frac{B}{B-A} e^{(A-B)T_F} + \left(\dot{V}_0 + B \left(V_0 - \frac{R_b K}{AB} \right) \right) \frac{A}{A-B}$$

$$T_F = \frac{1}{A-B} \ln \frac{A\dot{V}_0 + ABV_0 - R_b K}{B\dot{V}_0 + ABV_0 - R_b K}$$

$V_F = V(T_F)$, when $V_0 = V(t)$, $\dot{V}_0 = \dot{V}(t)$, $R = R_b$

$$V_F = \frac{R_b K}{AB} + \frac{(\dot{V} + BV - R_b K/A)}{B-A} \frac{(A\dot{V} + ABV - R_b K)}{BV + ABV - R_b K} e^{\frac{-A}{A-B}} + \frac{(\dot{V} + AV - R_b K/B)}{A-B} \frac{(A\dot{V} + ABV - R_b K)}{B\dot{V} + ABV - R_b K} e^{\frac{-B}{A-B}}$$

$$V_F = \frac{R_b K}{AB} + \frac{(A\dot{V} + ABV - R_b K)^2}{A(B-A)(B\dot{V} + ABV - R_b K)} e^{\frac{-A}{A-B}} + \frac{A\dot{V} + ABV - R_b K}{B(A-B)} e^{\frac{-B}{A-B}}$$

Appendix 5

Application of $V_F = V + \frac{\dot{V}|\dot{V}|}{2|\ddot{V}_0|}$ to the system $\frac{V(s)}{R(s)} = \frac{.4}{s(s+1)}$

\ddot{V}_0 will be approximated as ;

$$\ddot{V}_0 \approx \frac{\dot{V}(T) - \dot{V}(0)}{T}, \text{ where } T = \frac{.08}{\bar{\ddot{V}}_0},$$

and $\bar{\ddot{V}}_0$ is an average of previously determined \ddot{V}_0 values.

Suppose input V_D steps from 0 to 1 ;

$$\bar{\ddot{V}}_0 \approx .4$$

$$\therefore T = \frac{.08}{.4} = .2$$

$$\dot{V}_0 = 0$$

$$\dot{V} = .4R - .4Re^{-t} \quad (\text{Appendix 1})$$

$$\therefore \ddot{V}_0 \approx \frac{.4 - .4e^{-.2}}{.2} = .363 \quad (R = R_{\max} = 1.0)$$

$$V_F = V + \frac{\dot{V}|\dot{V}|}{.726} = -.4 + .4t + .4e^{-t} + \frac{\dot{V}|\dot{V}|}{.726}$$

$$V_F = -.1798 + .4t - .0404e^{-t} + .2202e^{-2t}$$

$V_F = V_D = 1.0$ at $t = 2.96$, by trial and error.

Now, during braking phase,

$$R = R_{\min} = -1.0$$

$$V_F = 1 = V + \frac{\dot{V}^2}{.726}$$

$$\frac{dV_F}{dt} = 0 = \dot{V} + \frac{2\dot{V}\ddot{V}}{.726}$$

$$0 = 1 + \frac{2\ddot{V}}{.726}$$

$$\therefore \ddot{V} = -.363$$

$$\dot{V} = -.363t + \dot{V}_0 = .379 - .363t$$

$$V = .379t - .1815t^2 + V_0$$

$$V = .804 + .379t - .1815t^2$$

$$\text{and since } \ddot{V} + .4\dot{V} = .4R,$$

$$R = -.9075 + .4(.379 - .363t)$$

$$R = -.756 - .145t$$

$$\text{Now } \dot{V}(T_F) = .379 - .363T_F = 0$$

$$\therefore T_F = 1.043$$

$$\text{and the total transient time} = 2.96 + 1.043 = 4.003$$

However, for true time optimal switching,

$$V_F = V + \dot{V} - .4 \ln(1 + \dot{V}/.4), \text{ from Appendix 1.}$$

$$V_F = -.4 + .4t + .4e^{-t} + .4 - .4e^{-t} - .4 \ln(1 + 1 - e^{-t})$$

$$V_F = .4t - .4 \ln(2 - e^{-t})$$

$$V_F = 1.0 \text{ at } t = 3.18, \text{ by trial and error.}$$

Now, during braking phase,

$$V_0 = -.4 + .4(3.18) + .4e^{-3.18} = .889$$

$$\dot{V}_0 = .4 - .4e^{-3.18} = .383$$

$$\therefore V = .889 + (.383 + .4) - .4t + (-.4 - .383)e^{-t}, \text{ from Appendix 1.}$$

$$V = .783 - .4t - .783e^{-t}$$

$$\dot{V}(T_F) = -.4 + .783e^{-T_F} = 0$$

$$\therefore T_F = .672$$

$$\text{and the total optimal transient is } 3.18 + .672 = 3.852$$

Therefore, with $V_F = V + \frac{\dot{V}|\dot{V}|}{2|\ddot{V}_0|}$, and \ddot{V}_0 is approximated as $\frac{\dot{V}(T) - \dot{V}(0)}{T}$, where $T = \frac{.08}{\ddot{V}_0}$, the response of the controlled system $\frac{.4}{s(s+1)}$ to a unit step input is complete in time only 3.8% greater than optimal. All results are plotted in Figure 30.

Appendix 6

Application of $V_F = V + \frac{\dot{V}|\dot{V}|}{2|R_0K - BV|}$ to systems

$$\frac{V(S)}{R(S)} = \frac{K}{S^2 + AS + B}$$

For this general second order system,

$$\ddot{V}_0 = R_0K - BV_0 \quad (1)$$

Now define a previous \ddot{V}_0 as \ddot{V}_{-1} .

$$\text{Then } \ddot{V}_{-1} = R_{-1}K - BV_{-1} \quad (2)$$

$$\frac{V_0}{V_{-1}} \ddot{V}_{-1} = \frac{V_0}{V_{-1}} R_{-1}K - BV_0 \quad (3)$$

$$\ddot{V}_0 - \frac{V_0}{V_{-1}} \ddot{V}_{-1} = R_0K - \frac{V_0}{V_{-1}} R_{-1}K \quad (1) - (3)$$

$$\therefore K = \frac{V_{-1}\ddot{V}_0 - V_0\ddot{V}_{-1}}{V_{-1}R_0 - V_0R_{-1}}$$

similarly,

$$B = \frac{R_{-1}\ddot{V}_0 - R_0\ddot{V}_{-1}}{V_{-1}R_0 - V_0R_{-1}}$$

Now consider the system $\frac{.4}{s^2 + .4}$ with input V_D stepped from 0 to .58.

$$V = 1 - \cos .632t, \text{ from Appendix 3}$$

$$\dot{V} = .632 \sin .632t$$

$$\ddot{V} = .4 \cos .632t$$

$$\therefore V_F = 1 - \cos .632t + \frac{.4 \sin^2 .632t}{1.6 - .8 \cos .632t}$$

$V_F = V_D = .58$ at $t = 1.39$, by trial and error.

If \dot{V} were now driven to zero by application of

R_{\min} , final output V would be .562 at $T_F = .811$. The switchpoint chosen by V_F is therefore very close to optimal.

Now consider an input V_0 stepped from .58 to 0.

$$V = -1 + 1.58 \cos .632t, \text{ from Appendix 3.}$$

$$\dot{V} = -\sin .632t$$

$$\therefore V_F = -1 + 1.58 \cos .632t - \frac{\sin^2 .632t}{1.6 - 1.264 \cos .632t}$$

$$V_F = 0 \text{ at } t = .72, \text{ by trial and error.}$$

During the braking phase,

$$V_F = V - \frac{\dot{V}^2}{.8 - .8V} = 0$$

$$.8V - .8V^2 - \dot{V}^2 = 0 \quad (1)$$

$$.8\dot{V} - 1.6V\dot{V} - 2\dot{V}\ddot{V} = 0 \quad \left(\frac{d}{dt} (1)\right)$$

$$V = \frac{.8 - 2\ddot{V}}{1.6} = .5 - 1.25\ddot{V} \quad \omega = \sqrt{1/1.25} = .894$$

$$\therefore V = .5 + 1.25A \cos .894t + 1.25B \sin .894t$$

$$\dot{V} = -1.1175A \sin .894t + 1.1175B \cos .894t$$

$$\ddot{V} = -A \cos .894t - B \sin .894t$$

$$\text{but } \dot{V}_0 = -.439 = 1.1175B$$

$$\therefore B = -.3928$$

$$\text{and } V_0 = .419 = .5 + 1.25A$$

$$\therefore A = -.0648$$

$$\text{so } V = .5 - .081 \cos .894t - .491 \sin .894t$$

$$\dot{V} = .0724 \sin .894t - .439 \cos .894t$$

$$\ddot{V} = .0648 \cos .894t + .3928 \sin .894t$$

and since $.4R = \ddot{V} + .4V$,

$$R = 2.5 \ddot{V} + V$$

$$R = .5 + .081 \cos .894t + .491 \sin .894t$$

$$\dot{V}(T_F) = .0724 \sin .894T_F - .439 \cos .894T_F = 0$$

$$\frac{\sin .894T_F}{\cos .894T_F} = \tan .894T_F = \frac{.439}{.0724} = 6.07$$

$$.894T_F = 1.41$$

$$T_F = 1.58$$

$$\therefore \text{total transient time} = .72 + 1.58 = 2.30$$

However, for true time optimal switching,

$$V_F = 1 + \frac{\dot{V}}{.632} \sin \left(\tan^{-1} \frac{\dot{V}}{.632(V-1)} \right) + (V-1) \cos \left(\tan^{-1} \frac{\dot{V}}{.632(V-1)} \right)$$

from Appendix 3.

$$V_F = 0 \text{ at } t = .81, \text{ by trial and error.}$$

During braking phase,

$$V_0 = -1 + 1.58(.8719) = .377$$

$$\dot{V}_0 = -.4909$$

$$V = 1 - \frac{.4909}{.632} \sin .632t + (.377-1) \cos .632t$$

$$V = 1 - .777 \sin .632t - .623 \cos .632t$$

$$\dot{V} = -.4909 \cos .632t + .394 \sin .632t$$

$$\dot{V}(T_F) = -.4909 \cos .632T_F + .394 \sin .632T_F = 0$$

$$\frac{\sin .632T_F}{\cos .632T_F} = \tan .632T_F = \frac{.4909}{.394}$$

$$.632T_F = .895$$

$$T_F = 1.41$$

\therefore total optimal transient time is $.81 + 1.41 = 2.22$,
and the controlled system response is complete
in time 3.5% greater than optimal. These
results are plotted in Figure 33.

Now consider the system $\frac{.4}{(s+.62)(s+.58)}$, as input

V_D is stepped from 0 to .58.

$$V = 1 + 14.5 e^{-.62t} - 15.5 e^{-.58t}, \text{ from Appendix 4.}$$

$$\dot{V} = -9 e^{-.62t} + 9 e^{-.58t}$$

$$V_F = V + \frac{\dot{V} \dot{V}}{.8 - .72V}$$

$$V_F = .58 \text{ at } t = 2.8, \text{ by trial and error.}$$

During the braking phase,

$$V_F = .58 = V + \frac{\dot{V}^2}{.8 - .72V}$$

$$\dot{V}^2 + 1.2176 V - .72 V^2 = .464$$

$$2 \dot{V} \ddot{V} + 1.2176 \dot{V} - 1.44 V \dot{V} = 0 \quad (\text{differentiating})$$

$$V = 1.389 \dot{V} + .8456, \quad \sqrt{1/1.389} = .8485$$

$$\therefore V = A e^{.8485t} + B e^{-.8485t}$$

$$\dot{V} = .8485 (A e^{.8485t} - B e^{-.8485t})$$

$$\ddot{V} = .72 (A e^{.8485t} + B e^{-.8485t})$$

$$\text{but } V_0 = .5001 = A + B + .8456$$

$$\dot{V}_0 = .1878 = .8485 (A - B)$$

$$\therefore A + B = -.3455$$

$$\underline{A - B = .2213}$$

$$2A = -.1242, \quad A = -.06208$$

$$B = -.3455 - A = -.2834$$

$$\text{so, } V = -.06208 e^{.8485t} - .2834 e^{-.8485t} + .8456$$

$$\dot{V} = -.05267 e^{.8485t} + .2405 e^{-.8485t}$$

$$\ddot{V} = -.0447 e^{.8485t} - .204 e^{-.8485t}$$

$$\text{and since } \ddot{V} + 1.2 \dot{V} + .36V = .4R,$$

$$R = 2.5 \ddot{V} + 3 \dot{V} + .9V$$

$$R = -.3256 e^{.8485t} - .043 e^{-.8485t} + .761$$

$$\dot{V}(T_F) = -.05267 e^{.8485 T_F} + .2405 e^{-.8485 T_F} = 0$$

$$e^{1.697 T_F} = \frac{.2405}{.05267} = 4.566$$

$$T_F = .8949$$

$$\therefore \text{total transient time is } 2.8 + .895 = 3.695$$

However, for true time optimal switching,

$$T_F = \frac{1}{.04} \ln \left(\frac{15.5 \dot{V} + 9(V+1)}{14.5 \dot{V} + 9(V+1)} \right), \text{ from Appendix 4.}$$

$$V_F = -1 + \dot{V}(-25 e^{-.62t} + 25 e^{-.58t}) + (V+1)(-14.5 e^{-.62t} + 15.5 e^{-.58t})$$

$$V_F = .58 \text{ at } t = 3.12 \text{ by trial and error.}$$

$$T_F = .26 \text{ by trial and error.}$$

$$\therefore \text{total optimal transient time is } 3.12 + .26 = 3.38$$

Thus the controlled response is 9% greater than optimal. These results are plotted in Figure 34.