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Automatic mesh generation

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AUTOMATIC MESH GENERATION

by

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Title of Thesis - "Automatic Mesh Generation"

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ABSTRACT

The objective of this thesis project is a study of Pre-Processors and development of an Automatic Mesh Generator for Finite Element Analysis. The Mesh Generator developed in this thesis project can create triangular finite elements from the geometric database of Macintosh Applications. The user is required to give the density parameter to the program for mesh generation. The research is limited to Mesh Generators of planar surfaces. Delauny Triangulation method maximizes the minimum angles of a triangle. Watson's Delauny Triangulation method can mesh only the 'convex hull' of a set of nodes. This algorithm has been modified to create triangular elements in convex and non-convex surfaces. The surfaces can have holes also. A node generation algorithm to place nodes on and inside a geometry has been developed in this thesis project. The mesh generation is very efficient and flexible.

Geometric modeling methods have been studied to understand and integrate the Geometric Modeler with the Finite Element Mesh Generator.

Expert Systems can be integrated with Finite Element Analysis. This will make Finite Element Method fully automatic. In this thesis project, Expert Systems in Finite Element Analysis are reviewed. Proposals are made for future approach for the integration of the two fields.

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TABLE OF CONTENTS

LIST OF FIGURES

Flowchart

LIST OF TABLES

LIST OF SYMBOLS

[ ]	Matrix
$[\ ]^T$	Tranpose of a Matrix
Bold letters in equations	Vector
Bar over letters	Vector
FEM	Finite Element Method
FEA	Finite Element Analysis
PDA	
pc	Parametric Cubic
CAD	Computer Aided Design
SDRC	Structural Dynamics Research Corporation
DT	Delauny Triangulation
DIT	Dirichlet Tessellation

CHAPTER 1

INTRODUCTION

1.1 FINITE ELEMENT MODELING IN MECHANICAL DESIGN

Modern engineers face monumental tasks when developing designs required to survive a broad range of extreme conditions and consumer demands. Added to this task are concerns for product safety and competition in commercial markets.

A mechanical design involves the engineer's ingenuity and analysis expertise. Typically, the design of a product from concept to finalization goes through the steps shown in figure (1.1) [1].

These steps are as follows :

1. Concept of Candidate Design.
2. Development of Mathematical Models.
   a. Discretization. Approximation of Mathematical Models.
   b. Numerical Solution of Discrete Models. Finite Element Analysis.
   c. Presentation and Interpretation of Results. Error Evaluation.
   d. Repetition . Steps (a) to (c) obtaining acceptable Analytical Results.
3. Assessment of Candidate Design. Theoretical or Experimental Conformity.
4. Repetition of Steps (1) to (3) . Design Acceptance.

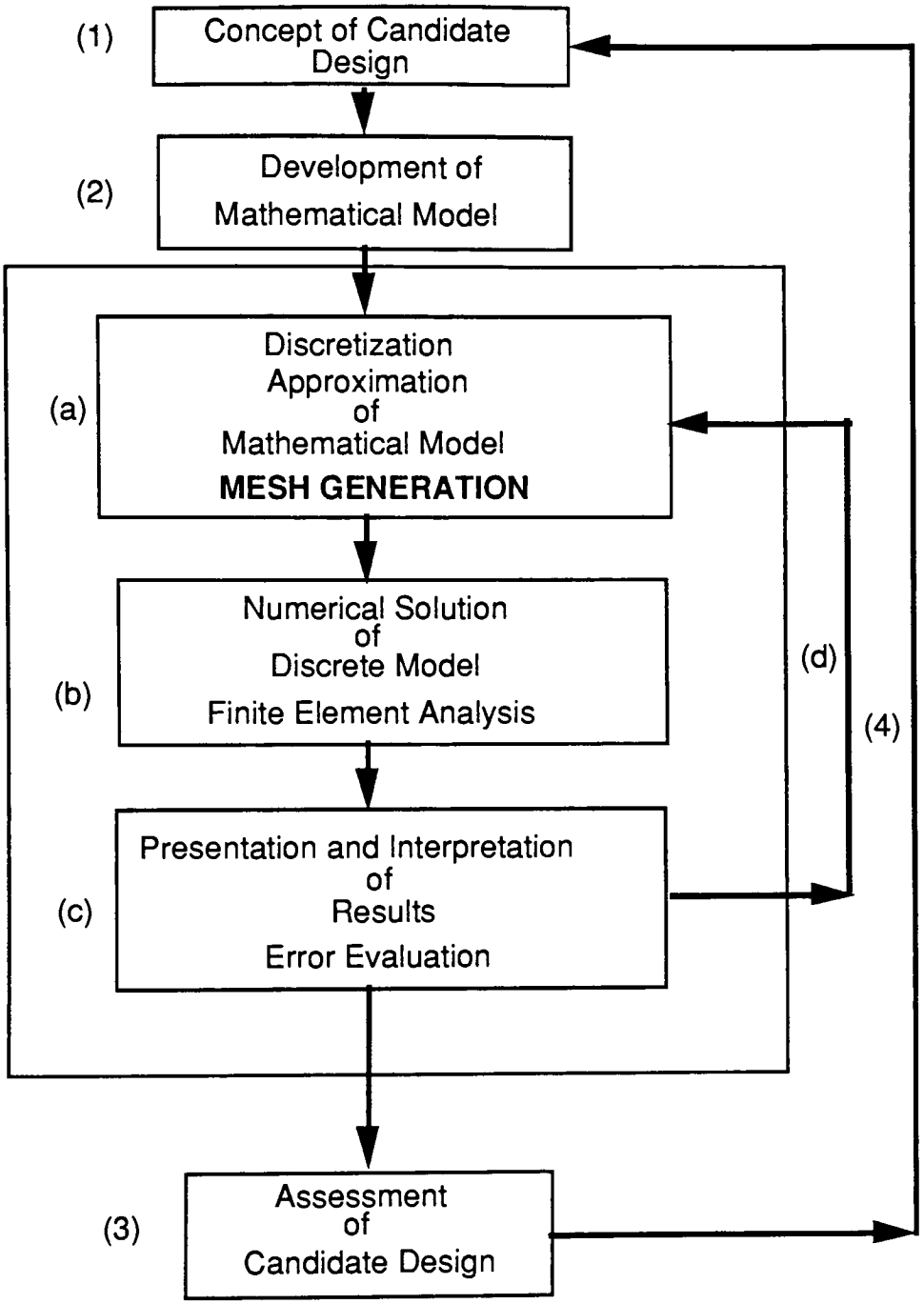


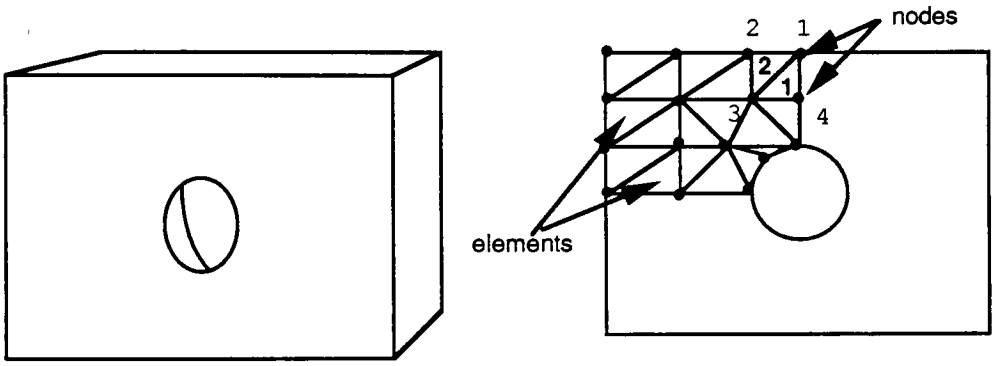
Figure1.1 Flow Chart Engineering System

Since mathematical models of field problems can become very complex, few can be analyzed through closed form analytical techniques. Therefore, step (2) in the design cycle, illustrated in flow chart of Figure (1.1) requires the iterative use of a numerical technique. The Finite Difference Method, the Finite Element Method (FEM), or the Boundary Value Method are used.

For design and analysis purposes, the finite element method is the most developed and versatile tool available today. It is a numerical procedure based on geometric discretization of mathematical models. It is the subdivision of the geometric domain of a problem into simple regions called finite elements. In other words, it can be said that the " FEM is a piecewise approximation, in the sense that the distribution of a field variable over a complicated geometry is approximated in terms of a series of relatively simple functions well defined within each finite element " [2]. Certain restrictions and conditions, boundary values, are imposed on these domains to provide useful results. Finite element analysis (FEA) is being applied successfully in many areas such as structural mechanics, fluid mechanics, bio-mechanics and thermodynamics.

A plate with a hole is shown as an object in Figure (1.2a), and a finite element mesh of a quarter of the plate is shown in Figure (1.2b). A finite element is constructed by a finite group of points known as nodes. In Figure (1.2b), nodes

1, 2, 3 form a two-dimensional triangular finite element. To produce a continuous body, nodes must be shared by adjacent elements in a particular order. For a topologically consistent model, nodes on common edges between two elements are shared by the elements. In Figure (1.2b) elements 1 and 2 share nodes 1 and 3. The collection of finite elements composing the model is the finite element mesh.



(a) Plate with a hole (b) Finite element mesh of quarter plate

Figure 1.2 Two-Dimensional Finite Element Model

1.2 THESIS OBJECTIVE - AUTOMATIC MESH GENERATION

The objective of this study is to develop an automatic mesh generation algorithm which can discretize planar surfaces in two dimensions by triangular finite elements. Watson's Delauny triangulation algorithm can mesh the 'convex hull' of a geometry. This algorithm is modified in this thesis project and

can mesh irregular shaped surfaces with or without holes. A node generation algorithm is also developed in this thesis project. The user defines only the mesh density parameter. The program then generates a valid finite element mesh consisting of triangular elements.

The nodes are placed such that they can be joined to give triangular elements with good shapes. The user gives commands to the program to generate boundary nodes, interior nodes and finally the mesh . One can edit the nodes or elements at any stage of the program. The program is kept modular so that changes can be made to enhance the efficiency and automation whenever a modification to any module is available. During this development, alterations were made on modules without disturbing the complete structure of the program. The program can be extended to mesh free-form surfaces.

To achieve the objectives of this study it was necessary to understand existing mesh generation and geometric modeling methods. Objectives for this research will become evident in chapters where these topics are discussed in detail. A chronological methodology of this study is given below :

Research efforts in the area of state of art in automatic mesh generation techniques.

Research in the area of geometric modeling methods.

Identification of current (thesis related) needs with respect to geometric modeling and selection of a geometric modeler.

Modification of the modeler for finite element mesh generation applications.

Algorithm for node generation on and in surfaces in two-dimensions.

Algorithm for automatic meshing of planar surfaces in two dimensions.

Integration of the mesh generation method with the geometric modeler using a common data base.

Discussion of expert systems and their application in finite element analysis.

Proposal made for future integration of mesh generation methods with expert systems.

To develop an efficient mesh generator it is imperative to have a good understanding of programming languages and data structures. To achieve this, work was done in collaboration with a graduate student from the Computer Science department at Rochester Institute of Technology. His contributions were as follows :

Implementation of the geometric modeler on Macintosh workstations.

Selection of programming language - Object Pascal.

Coding of the proposed algorithms.

1.3 PRE-PROCESSORS

Pre-processing comprises all the activities associated with the preparing, generating, checking and altering of data before the main structural finite element analysis is performed. Step 2 (a) of the flow chart in Figure (1.1) represents the Pre-processing stage. The functions of a Pre-processor are outlined in the flow chart of Figure (1.3).

General functions of a Pre-processor are as follows :

Generating Nodal Point Coordinates Nodes are generated on the boundary and interior of the geometry. Each node is unique by its location coordinates and number. Nodes can be generated manually by selecting their locations on the geometry one at a time, or interactively, by selecting the geometry and placing a desired number of nodes on it. Nodes can also be generated automatically if the elements are generated first. Section 3.3.1 explains this method of mesh generation.

Generating Element Connectivity This involves connecting the nodes to form finite elements. Finite elements can have different shapes. For example in two dimensions, triangles and quadrilaterals.

Generating Boundary Conditions This defines constraints on the model, essentially to simulate working conditions and to prevent rigid body motion. Constraints are placed on nodes, and can constrain up to three orthogonal translations and three orthogonal rotations in the working coordinate system.

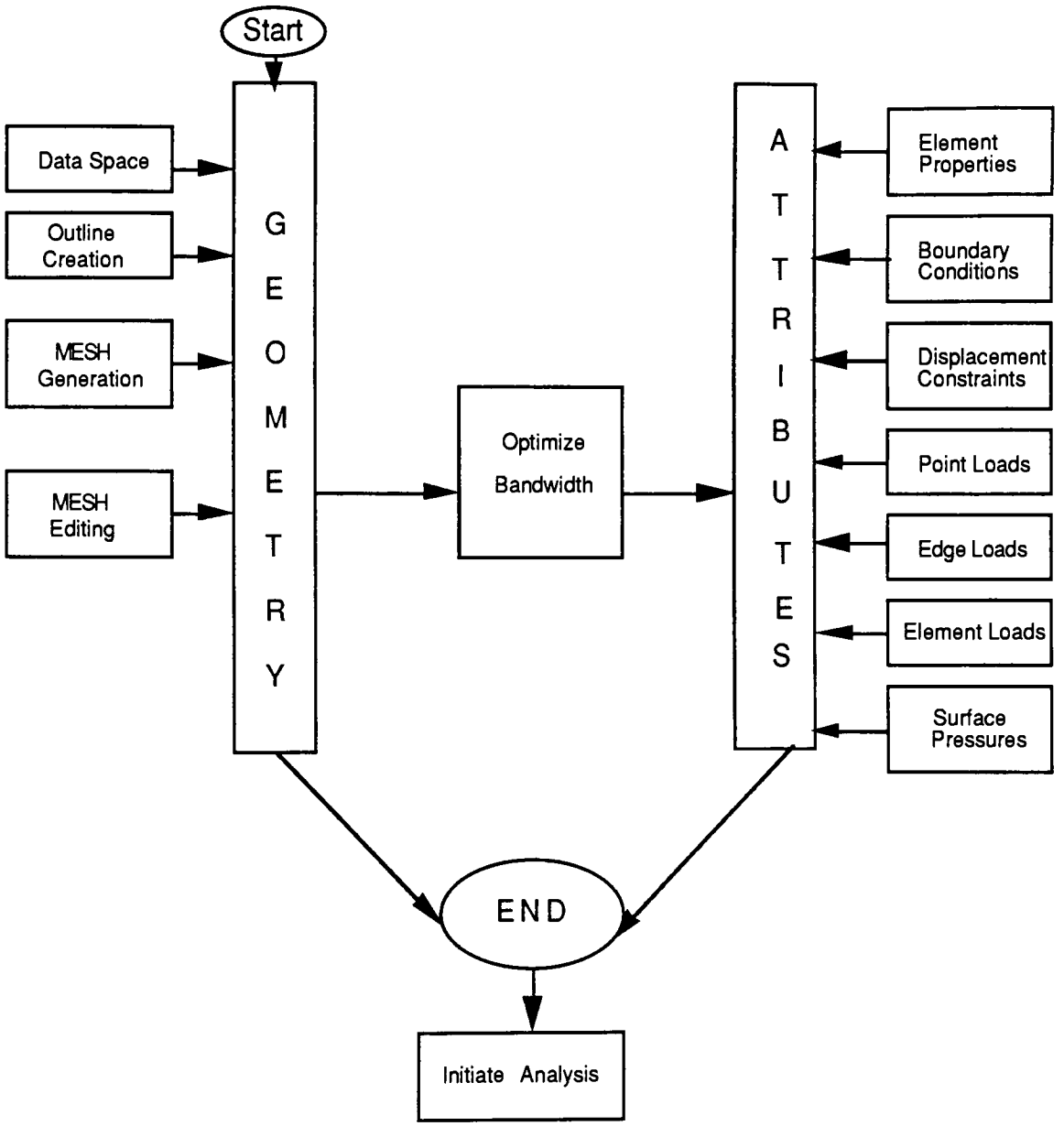


Figure 1.3 Pre-processor Flow chart

Generating Nodal, Element, and Distributed Loads Loads are modeled to simulate conditions of force and pressure. They can be distributed uniformly or vary over an edge or over a surface.

Generating Geometric and Elastic Data Each element in the finite element model must refer to an element property set. This set describes the geometric and material properties of the elements. The set can be defined before or after the mesh generation. Any number of different element types can be present in a finite element model and any number of property sets can be defined. The geometric properties can consist of cross-sectional areas, moments of inertia, shear factors, spring constants and thicknesses. Material description parameters include Young's modulli, Poisson's ratio, shear modulus, mass density, reference temperature and other parameters.

Checking Data for Syntax and Reasonableness Before an analysis is carried out, the element data supplied by the analyst is checked. This data matches the element specifications internal to analysis software .

Displaying and Plotting Data This is model verification. It allows the analyst to see the generated model through computer graphics. At this stage the flexibility of editing element properties and connectivities are provided. Remeshing can also be done at this time.

Renumbering Nodal Points to reduce Bandwidth In most programs a bandwidth or wavefront optimizer is necessary to reduce equation solution time.

The renumbering of elements or nodes is determined by the method of solution of the finite element analysis program. Renumbering assists in the reduction of storage space and solution time.

Preparing Connectivity Data for Coupling of Substructures The finite element mesh and all non-graphic parameters pertaining to analysis input data are associative. If a node is deleted, all the elements referenced by that node are deleted. When the nodes in a mesh are merged, the connectivity of all the affected elements is updated to reflect the retained node numbering. Element properties and material descriptions are attached to elements. Loads and constraints are attached to nodes. When nodes are merged, loads and constraint sets are referenced to retained nodes.

Transforming Input Data into a Format suitable to Analysis Programs When created and verified, the finite element model is prepared for analysis. A translator creates a text file for use by a finite element analysis program. The translator can be either customized or generalized for use with various programs.

1.4 DENSITY AND SHAPE OF FINITE ELEMENTS

General functions of a Pre-processor were presented in the previous section. It was assumed that the analyst creating a finite element mesh is aware of the importance of the 'shape' and 'density' of the elements of the mesh. The

definition of these two terms and their importance in terms of finite element analysis is presented below :

Shape - For a two-dimensional finite element, the ratio of the longest side to the shortest side, and the ratio of the largest angle to the smallest angle, determines the goodness of the shape of the element. These definitions are known as 'aspect ratio ' and ' skewness' of the element respectively. " It can be said that in finite element analysis the solution accuracy decreases with increase of element aspect ratio " [3]. "If the elements are too large, or have bad aspect ratios, or if the mesh as a whole does not obey the combinatorial sharing rules of FEM decompositions, inaccurate or inconsistent results will accrue because the mathematical conditions underlying finite element methods will have been violated " [4]. An ideal aspect ratio is 1.

For finite element analysis, element sides and interior angles of equal magnitude are most desirable. A square quadrilateral element and an equilateral triangular element have the best finite element shapes. Two sets of finite element mesh of a plate with a hole are shown in Figures (1.4a) and (1.4b). The plate is meshed with quadirlateral, four-sided finite elements. Elements with good shapes are shown in Figure (1.4a) and elements with poor shapes are shown in Figure (1.4b).

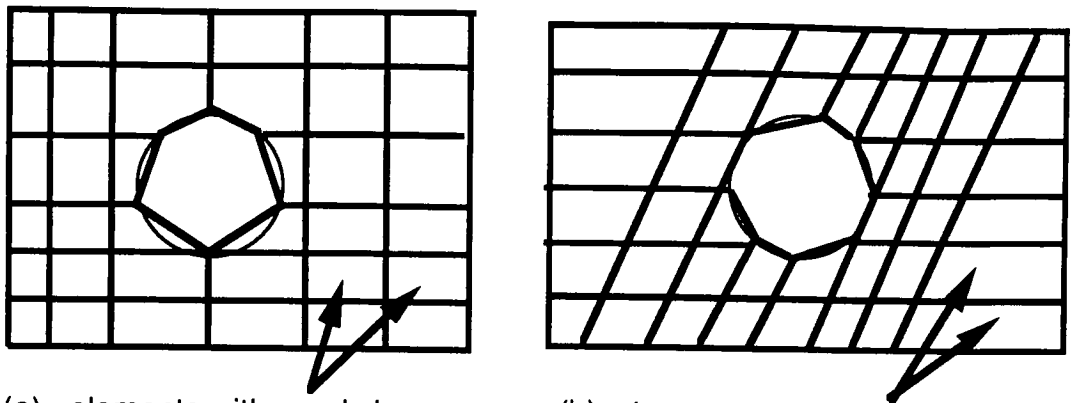


Figure 1.4 Quadrilateral Finite Elements

Density - The total number of elements in a finite element model is the density of the elements. Figure 1.5 (a) shows a coarse finite element mesh of a plate with a hole. Figure 1.5 (b) shows a fine mesh of the same object. If the element shapes are good, then the accuracy of the finite element analysis increases with the increase of element density. An increase of elements will also increase the computational cost of the analysis.

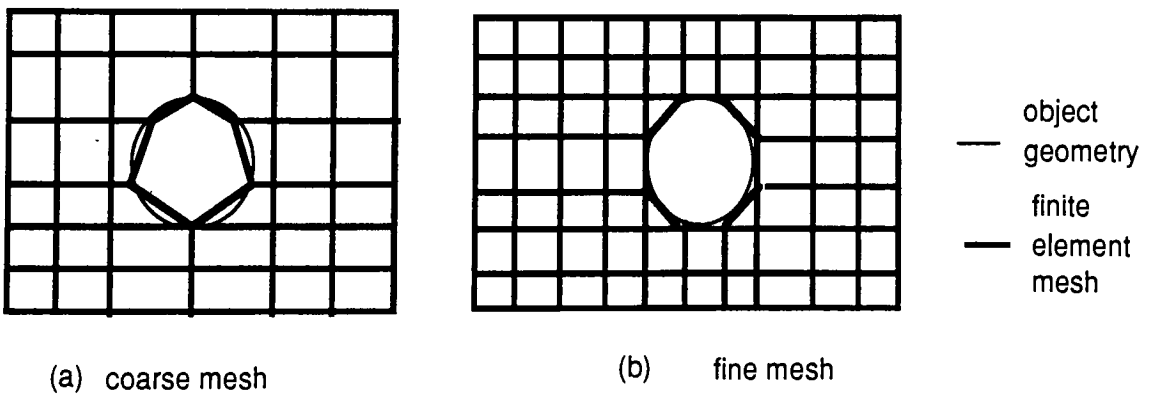


Figure 1.5 Finite Element Mesh Densities

1.5 EXPERIMENT ON SHAPES AND DENSITIES OF FINITE ELEMENTS

A thin rectangular plate in bending with moments distributed along the edges, shown in Figure (1.6), was modeled using the following finite element codes :

1. MSC/NASTRAN
2. Algor Supersap
3. Intergraph RandMicas

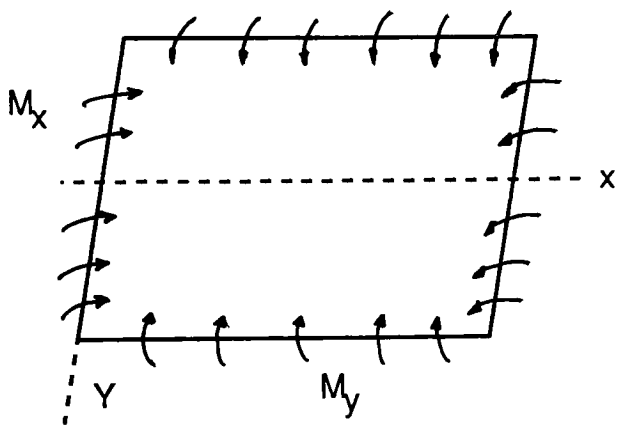


Figure 1.6 A Rectangular Plate with End Moments

The purpose of running the finite element analysis using the above mentioned codes was to see the variation of results when shapes and densities of the elements are varied. The plate was meshed by four-sided quadilateral elements in all cases. Table 1.1 shows the comparison of the bending moment

M_x in lbs/in., when the aspect ratio of the elements are varied. Table 1.2 shows the comparison of the bending moment M_x in lbs/in., when the element densities are varied.

Aspect ratio		Classical	Nastran	error %	RandMicas	error %	Supersap	error%
1	**1.43**	92.88	93.66	**1.16**	101.39	**9.16**	92.8	—
2	**2.8**	"	92.99	**1.19**	100.72	**8.44**	91.17	**1.18**
3	**16**	"	94.39	**1.62**	11.67	**87.43**	15.63	**83.17**

Table 1.1 Varying Aspect Ratio on Finite Element Codes

Density		Classical	Nastran	error %	RandMicas	error %	Supersap	error %
1	**77**	92.88	93.96	**1.16**	101.39	**9.16**	92.8	_
2	**100**	"	93.90	**1**	100.6	**8.31**	92.6	**-.3**
3	**140**	"	93.86	**1**	100.00	**7.65**	92.16	**-.7**

Table 1.2 Varying Element Density on Finite Element Codes

It can be concluded from Tables 1.1 that without experience with a finite element code, it is difficult to predict the accuracy of the results in terms of aspect ratios. Each of the three software packages have different levels of accuracy for the same aspect ratio. These differences can be attributed to the mathematical formulation of the plate finite element of these programs. The mathematical formulation of the finite elements is unrelated to the mesh generation process, therefore it will not be discussed here. At the same time it is difficult to determine the upper limit of the aspect ratio and the skewness factor for which the finite element results are within reasonable error. The solution to this dilemma in the finite element method is to design Pre-processors which give elements good shapes. This implies that aspect ratios are as close to unity as possible, and that they maximize the minimun angle. If the elements have good shapes then the increase of density will reduce the discretization error in the finite element method.

Table 1.2 shows that beyond a certain density the gain in accuracy is negligible. An increase in the number of elements increases the analysis cost. The solution to this problem is to start the analysis with a moderate density of elements. With the increase of density of every analysis, the analyst looks for convergence of results. This approach makes the computation cost effective.

"Minimizing the number of elements and reducing their element distortion in the model, while still accurately modeling the design is the most important objective of a finite element mesh generator" [5]. This thesis project was originally planned as research into the state of art in automatic mesh generation in two dimensions, and to develop a mesh generation program which works through geometric database. As the thesis project progressed and the results of the experiments were deduced, it was evident that automatic mesh generation was not a sufficient criteria for cost effectiveness. Added to this criteria is the demand for well shaped elements. The effort now is two fold, to design an automatic mesh generator from a geometric database, discussed later, and to create elements with good shapes.

1.6 HISTORY OF PRE-PROCESSORS

In the early days of finite element analysis, the analyst worked from hand produced drawings, and was all together responsible for the mesh and element integrity. This was a tedious and time-consuming job. Moreover, the finite element analysis is a batch process and no feedback was available to indicate error during model construction. It was only after the analysis was run and the results became suspect that the analyst would go back to check the validity of the model. It was therefore natural to attempt to improve the application of the finite element method by :

(a) Providing a graphics interface during mesh generation

(b) Automating the mesh generation process

Providing the combination of (a) and (b) saved time and cost, reducing human effort and chance of error. In the early 1970's Pre-processors with a graphic interface emerged. In the late seventies a dramatic change took place in the Pre-processor software with the introduction of low cost machines with high resolution graphic display which provided a means to view the model being created. Therefore, this feature overcame the problems connected with checking the validity of the mesh. Some of the first geometric and finite element modelers were PDA Engineering's "PATRAN" and SDRC's "GEOMOD/SUPERTAB". The graphic display of the model helped the analyst to create, edit and accept/reject the model before expensive and time consuming analysis was carried out.

Graphic display problems are not the immediate concern of today's finite element analysts. The problem is the lack of knowledge of the finite element method, and the lack of experience with the innumerable finite element analysis codes in todays commercially available software. A solution to this problem is to eliminate the requirement of an expert for performing the finite element analysis. This may be achieved by :

1. Eliminating the user interface as much as possible, by letting the Pre-processor create a valid finite element mesh automatically on a user defined geoemetry. The information required for creating the finite element mesh should be minimum, and easily obtainable by a new user.
2. Developing knowledge based expert systems to take the place of an experienced and knowledgable expert.

Essentially, both suggestions given above are the same. The difference is that while 1, is a program with no decision making abilities, the other , that is 2, is an expert system which mimicks an expert, and capable of making desicions. This thesis project aims at achieving the goal mentioned in 1, keeping in mind the necessity of good element shapes and flexible density of elements. Chapter 3 discusses in detail the automatic mesh generation methods. Chapter 4 discusses and presents the results of the automatic mesh generation program developed in this study. Chapter 5 presents the expert systems and their applications in finite element analysis.

1.7 AUTOMATIC PRE-PROCESSORS

All of the Pre-processors today are semi-automatic. During mesh generation the user interacts frequently with the program. As discussed above, the need is to advance mesh generation concepts towards user-transparent finite

element analysis systems. This will improve the robustness of the entire finite element analysis, so it can be used reliably by designers who are not finite element experts. The only way to do this is to automate the FEM process. This means that finite element software accepts a geometric description of the problem with analysis attributes tied to it as input and produces results to a pre-specified level of accuracy. "One factor contributing to this interest is the availability of advanced geometric modeling systems which have greatly increased the efficiency of the design process, thus making finite element mesh generation portion of the analysis process an even more obvious bottleneck" [6]. The desirable features of an automatic Pre-processor are as follows :

Precise Modeling of Boundaries No error beyond the discretization error inherent to the chosen finite element model. Boundary nodes precisely on the boundary of the structure. No limitation to the forms of geometry that can be modeled.

Good Information between the Mesh Interior and the Boundary The curvatures and the node spacing on the boundaries of the region be well represented in the interior of the mesh. This allows the analyst to control the shape of the elements in the interior of the region in a predictable fashion. It also permits the analyst to refine the spacing of the mesh, where accurate discretization is required. Unnecessary refinement of the mesh leads to wasted computations.

Minimal Input Effort Reduction of analyst time and the effort required to set up a finite element model. Minimizing chances of human error in the analysis.

Broad Range of Applicability Minimizing user learning time, program development time and program size. Desirablility of small sets of mesh generation techniques applicable to a broad range of structural topologies, replacing large sets of special purpose mesh generators.

General Topology The method of meshing unrestrictive to the topology of the mesh within the region.

Automatic Topology Generation Mesh generation creating element connectivity without user intervention. Reduction of user input sometimes clashing with features of general topology.

Favorable Element Shapes Elements produced by automatic mesh generations with good shapes.

Optimal Numbering Patterns Numbering of nodes and the elements within the structure arranged so that favorable conditions are obtained for solving equations. Favorable conditions depend on the method of solution used in the analysis program. Minimum bandwidth or wavefront are common desirable features.

Computational Efficiency Mesh generations making efficient use of the computer resources, minimizing expenses and providing acceptable results with minimum interaction.

1.8 THESIS ORGANIZATION

The following chapters have been organized as outlined below :

Chapter 2 Study of the geometric modeling methods for generating curves and surfaces in two dimensions and in three dimensions. Mathematics behind the development of curves and surfaces given for clear understanding of capabilities and limitations of various geometric modeling methods.

Chapter 3 Study of the state of art in mesh generation methods in finite element analysis. A brief discussion of the merits and drawbacks of each method.

Chapter 4 Discussion of the node generation algorithm. Watson's Delauny Triangulation Method, and modifications of Watson's Delauny Triangulation Method presented by this study. Development and presentation of results of the automatic mesh generator.

Chapter 5 Artificial intelligence and expert systems. Expert systems in finite element analysis and suggestions for integrating expert systems with finite element methods.

CHAPTER 2

GEOMETRIC MODELING

2.1 INTRODUCTION

Geometric modeling is the technique used to describe the shape of an object or to simulate a dynamic process. It provides a mathematical description of the object or the process. The model is created because it is a convenient and an economical substitute for the real object or process. It is often easier and practical to analyze a model than to test or measure or experiment with the real object Thus, geometric modeling is finding wide applications and acceptenace in engineering and scientific applications.

If a geometric model is 'good', it will respond to simulations as a real object. 'Good' is a relative term here and application dependent. For example, in some applications, the geometric model of a physical object may require the complete description of surface properties, texture, color, or it may include only information on elastic properties of the object's material. These essential details and properties in a model are determined from the operations the application is intended to perform. If the model can provide these details it is considered 'good'. The following sections discuss various types of geometric modelers and methods of generating curves and surfaces in three dimensions.

When finite element analysts were looking for ways to provide better representation of the finite element model, geometric modelers came as a boon. In the first chapter one of the requirements of automatic mesh generation is to create a mesh from a geometric database. It means that the Pre-processor takes the information stored in the database of the geometric modeler and generates a finite element mesh from it. This integration of the geometric modeler and the finite element modeler relieves the analyst of the painful and time-consuming process of creating a mesh of the object manually or interactively. Barring few exceptions, where the geometric model and the finite element model are not represented by same boundaries, this is the most efficient procedure of generating a finite element mesh.

In the present study Macintosh applications database is integrated with a modified Watson's Delauny Triangulation code for automatic mesh generation. The user creates the geometry of the object on which the FEA is desired. Prior to meshing the user is required to give the mesh generation program the density of the elements. The program then generates a mesh automatically from the information stored in the geometry database.

To create a finite element mesh from a geometric database it is imperative to understand the methods of geometric modeling, and the proposed algorithms

for meshing these geometric domains as discussed in chapter 3. The geometric modeler with which the finite element mesh generator is integrated is important. This is because the computational effort required for needed geometric operations, and the difficulty of those operators are a function of both the geometric modeler and the finite element mesh generator. In general, some geometric modelers will not currently support the geometric operations needed by some meshing algorithms [6]. In such a case it would become necessary to develop a new algorithm for required geometric operations and incorporate it in the geometric modeler. Macintosh applications database supports all the operations required for the finite element mesh generator developed in this thesis and is discussed in chapter 4.

2.2 GEOMETRIC MODELERS

The three basic types of geometric modelers are wireframe, surface and solid modelers. They are defined below.

Wireframe Modelers In the geometric modeler the object is represented by its edges. No information is present regarding the surfaces or the space occupied by the object. The object is defined as if a wire is placed on every edge, shown in Figure (2.1a). The resulting representation is a wireframe. The model does not completely represent an object with non-planar surfaces. Even with planar

surfaces there is no information regarding inside or outside of surfaces. This lack of information can lead to ambiguous objects as shown in Figure (2.1b).

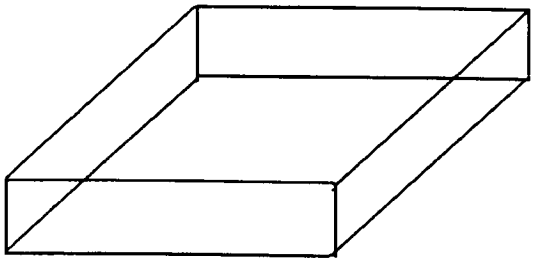
Surface Modelers In this geometric modeler the object is represented by surfaces. The modeler does not store any information on the inside or outside of the object. Therefore, it is not possible to compute volume or mass properties of the object. Surfaces modelers can represent curved surfaces and the objects are unambiguous as shown in Figure (2.1c).

Solid Modelers In this geometric modeler the object can be represented in various ways and it is defined unambiguously. It is possible to compute volume, mass, moments of inertia and other object-related physical properties. There is information regarding the inside and outside of the object, shown in Figure (2.1d). The object representation is complete. Most solid modelers store the geometry data so that it can be classified in one of three categories :

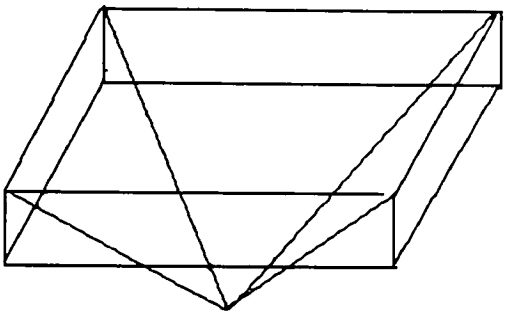
a) Constructive Solid Geometry (CSG), shown in Figure (2.2a)

b) Boundary Representation (bRep), shown in Figure (2.2b)

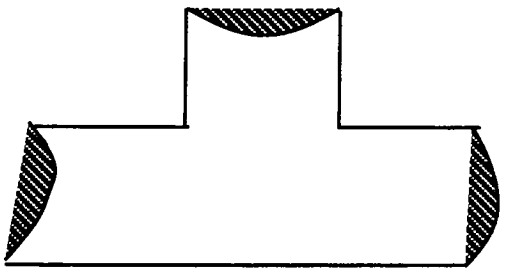
c) Cell decomposition method .



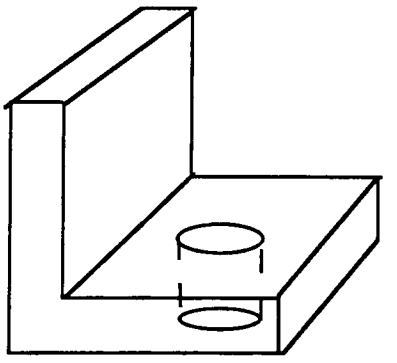
(a) Unambiguous wireframe model



(b) Ambiguous wireframe model

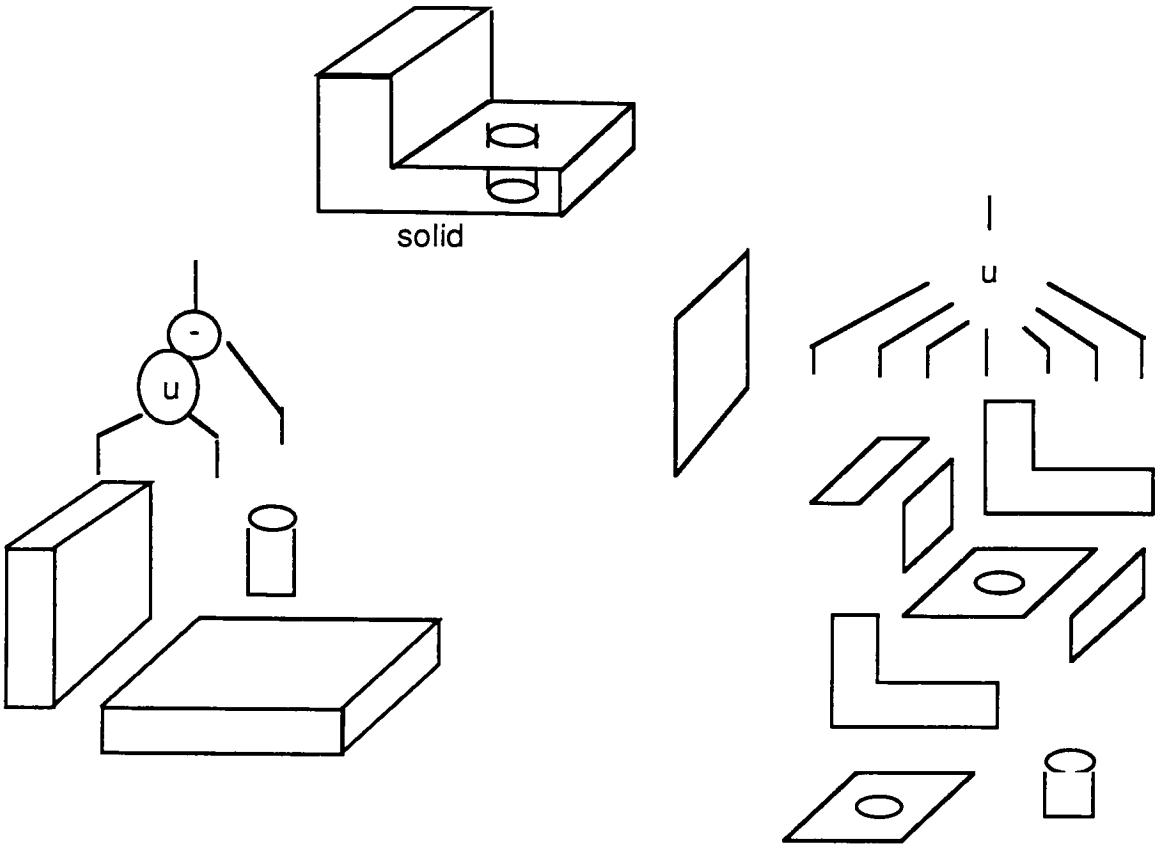


(c) Surface model



(d) Solid model

Figure 2.1 Geometric Modeling Methods



(a) CSG representation (b) Boundary representation

Figure 2.2 Solid Modeling Methods

2.3 GEOMETRIC MODELING

Geometric modelers build the geometric model from curves and surfaces. Curves and surfaces act as building blocks for object representation. The present interest is in geometric modeling and mesh generation of planar surfaces. Therefore, the discussion here will be limited to mathematical formulation of planar surfaces in space.

Generation of surfaces can be explained in two steps.

The first step is to represent a three dimensional curve.

The second step is the mathematical development of the three dimensional curves into three dimensional surfaces.

The following sections present the mathematics associated with curves and surfaces.

2.3.1 CURVE GENERATION

There are two basic methods of representing curves in space

Method one, as functions of orthogonal coordinates x, y and z.

Method two, as functions of a parameter.

In method one, functions are of the following form :

$$x = x, \qquad y = f(x), \qquad z = g(x)$$

This form defines a point on a curve in terms of its location in space, for example a point on a two-dimensional curve can be represented in the following way :

$$x = x, \qquad y = ax^3 + bx^2 + cx + d, \qquad z = 0$$

If the curve is parallel to one of the coordinate axis, for example the y axis, the slope of the curve is infinite. Mathematically, $dy/dx = \infty$. This means

$$dy/dx = 3ax^2 + 2bx + c = \infty$$

It is mathematically impossible to define ∞ by a non parametric equation such

as $3ax^2 + 2bx + c$. This, leads to two problems. Firstly, the parallel curve will be approximated by a non-parallel curve. Secondly, the division by a large number may lead to computational errors or even failures.

Parametric representation of the curve overcomes the above mentioned problem. A two-dimensional parametric cubic curve can have the following form :

$$x = au^3 + bu^2 + cu + d$$

$$y = eu^3 + fu^2 + gu + h$$

Tangent vectors at a point are defined as dy/du and dx/du and can be used to give the slope dy/dx = (dy/du)/(dx/du). An infinite slope is readily defined by having dx/du = 0. Representation of infinite slopes is one of the many benefits of geometric modeling in parametric co-ordinates. All present day geometric modelers take advantage of parametric representation of curves, surfaces and solids. [7]

In a parametric cubic curve the orthogonal coordinates x, y, and z are represented as a third order polynomial of a parameter u. When one deals with finite segments of a curve, it is convenient to normalize the parametric variable, limiting the parametric value to the closed interval, $0 \leq u \leq 1$. Cubic functions

are chosen because a lower order representation of curve segments cannot provide continuity of position and slope at the point where curve segments meet and at the same time ensure that the ends of the curve segments pass through given points [8]. Since the aim is to represent a curve by a series of curve segments, the continuity of slopes at common points is necessary. In Figure (2.3) curve segments 1 and 2 are joined together at point j. For segment 1, u_i = 0 at point i and u_j = 1 at j. For segment 2, u_j = 0 at j and u_k = 1 at k.

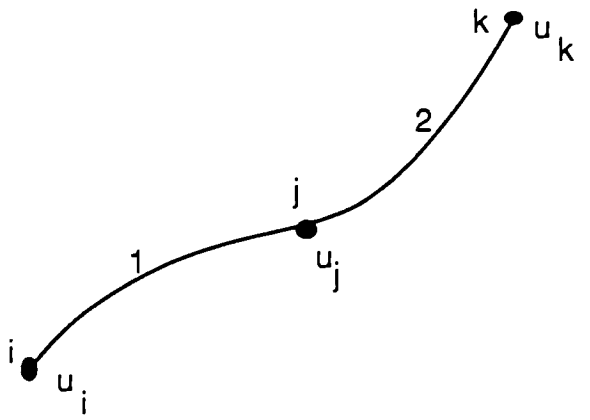


Figure 2.3 Two Parametric Curve Segments

Parametric cubic (pc) splines are the most popular curve generators. A point on a pc spline can be defined as :

$$\mathbf{p}(u) = \mathbf{a}_3u^3 + \mathbf{a}_2u^2 + \mathbf{a}_1u + \mathbf{a}_0 \qquad (1)$$

This is the algebraic form of a cubic spline. Here the limits of the parameter u are between 0 and 1, both values inclusive. In equation (1) there are four

unknowns. They are $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$. Therefore, if 4 geometric or boundary conditions are known then all of the unknowns can be found. Substitution of the end points and their slopes in parametric space gives the following identities :

$$\begin{aligned} \mathbf{p}_0 &= \mathbf{a}_0 \\ \mathbf{p}_1 &= \mathbf{a}_0 + \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \\ \mathbf{p}_0{}^u &= \mathbf{a}_1 \\ \mathbf{p}_1{}^u &= \mathbf{a}_1 + 2\mathbf{a}_2 + 3\mathbf{a}_3 \end{aligned} \qquad (2)$$

The vectors $\mathbf{p}_0$, $\mathbf{p}_1$ are the end points and $\mathbf{p}_0{}^u$, $\mathbf{p}_1{}^u$ the derivatives or the tangent vectors at these points. In matrix form these vectors are stored as

$$\mathbf{B} = [\,\mathbf{p}_0 \;\; \mathbf{p}_1 \;\; \mathbf{p}_0{}^u \;\; \mathbf{p}_1{}^u\,]^T \qquad (3)$$

Substituting the values of the constants $\mathbf{a}_0$, $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ from equation (2) in equation (1) and rearranging the terms gives an equation for a curve in space in terms of the end points, slopes at the end points and the parameter u.

$$\mathbf{p}(u) = (2u^3 - 3u^2 + 1)\mathbf{p}(0) + (-2u^3 + 3u^2)\mathbf{p}(1) + (u^3 - 2u^2 + u)\mathbf{p}^u(0) + (u^3 - u^2)\mathbf{p}^u(1) \qquad (4)$$

or, $$\mathbf{p}(u) = F_1(u)\mathbf{p}_0 + F_2(u)\mathbf{p}_1 + F_3(u)\mathbf{p}_0{}^u + F_4(u)\mathbf{p}_1{}^u \qquad (5)$$

note : alphabets in bold represent vectors

In equation (5) let $\mathbf{F} = [F_1 \ F_2 \ F_3 \ F_4]$ and $\mathbf{B} = [\mathbf{p}_0 \ \mathbf{p}_1 \ \mathbf{p}_0{}^u \ \mathbf{p}_1{}^u]^T$ where,

$$\bar{F} = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{6}$$

Therefore,

$$\mathbf{p}(u) = [F_1 \ F_2 \ F_3 \ F_4] [\mathbf{p}_0 \ \mathbf{p}_1 \ \mathbf{p}_0{}^u \ \mathbf{p}_1{}^u]^T \tag{7}$$

or,

$$\mathbf{p}(u) = \mathbf{FB} \tag{8}$$

where, F_1, F_2, F_3, F_4, are called the shape functions. Equation (5) is the geometric form of a parametric cubic curve in space, and $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_0{}^u$, $\mathbf{p}_1{}^u$ are called **geometric coefficients**. Figure (2.4) represents the basic elements of the vector geometric expression of a pc curve in space.

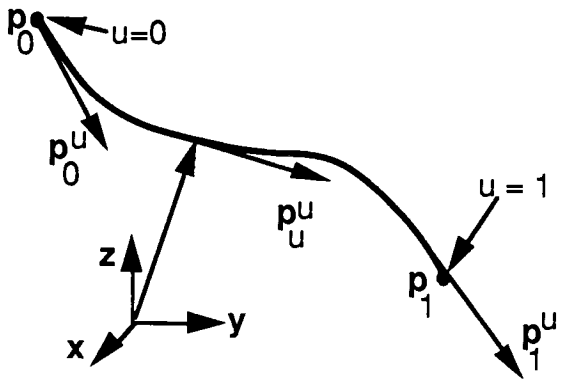


Figure 2.4 Elements of a Space Curve

Equation (1) can also be represented in a matrix form as :

$$\mathbf{p}(u) = [\, u^3 \;\; u^2 \;\; u \;\; 1 \,][\, \mathbf{a}_3 \;\; \mathbf{a}_2 \;\; \mathbf{a}_1 \;\; \mathbf{a}_0 \,]^T \qquad (9)$$

Let $\mathbf{U} = [u^3 \;\; u^2 \;\; u \;\; 1]$ and $\mathbf{A} = [a_3 \;\; \mathbf{a}_2 \;\; \mathbf{a}_1 \;\; \mathbf{a}_0]$. Equation (9) is called the algebraic representation of the pc space curve and can be rewritten as :

$$\mathbf{p}(u) = \mathbf{UA} \qquad (10)$$

'A' is the matrix of algebraic coefficients and **'B'** is the matrix of geometric coefficients or boundary conditions. Each can be readily converted into the other through a universal tranformation matrix,' **M** ' [8] .

$\mathbf{A = MB}$ and conversely $\mathbf{B = M^{-1}A}$

The universal transformation matrix **M** is :

$$\bar{M} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

It is often convenient and intuitive to work with the geometric form of pc- curves. The assembled matrix is of the following form

$$\bar{p}(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_0^u \\ p_1^u \end{bmatrix} \qquad (11)$$

or,

$$\mathbf{p} = \mathbf{UMB} \qquad 0 \le u \le 1 \qquad (12)$$

Cubic curves of this form, defined by the coordinates and tangent vectors (slopes) at their end points, are known as Hermite curves. They are different from Bezier and B-spline curves, first discussed in section 3.2.3. The matrices **U**, **F** and **M** are identical for all the pc curve generation methods. Only the

algebraic matrix **A** and the geometric matrix **B** vary between the different methods. Therefore, other curves will be discussed in terms of their **A** and **B** matrices.

The shape of Hermite curves is controlled by the magnitude of the tangent vector $\mathbf{p}^u$. If **'t'** is a unit tangent vector = $\mathbf{p}^u / \mathbf{p}^u$, then $\mathbf{p}^u = K\,\mathbf{t}$, where K repesents the magnitude of $\mathbf{p}^u$. Larger the K, the stronger the curve is 'pulled' in the direction of the vector before it begins to move towards the opposite endpoint. Another variation of the shape of this curve is achieved by constant tangent length but with changing tangent direction. Thus, the tangent vectors give shape and direction to curves passing through given points.

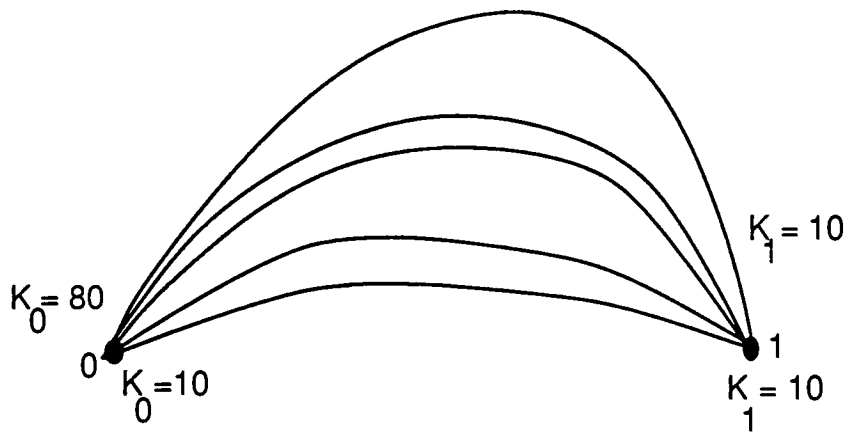


Figure 2.5 Effect of Tangent Vector Magnitude on Curve Shape

2.3.2 BEZIER FORM OF PC CURVE [9]

The curve generation method developed by Bezier differs from that of Hermite. The difference is in the definition of the tangent vector at the end points. The general form of Bezier curves is based on the principle that any point on a curve segment must be given by a parametric function of the following form :

$$\mathbf{p}(u) = \sum_{i=0}^{n} \mathbf{p}_i f_i(u) \qquad 0 \le u \le 1 \qquad (13)$$

where, the vectors $\mathbf{p}_i$ represent the n+1 vertices or control points of the characterstic polygon. As shown in figure (2.6), the characteristic or control polygon is the boundary formed by joining the points used to define the curve.

There are certain properties that the blending functions $f_i(u)$ should possess [9] . Bezier used a family of functions called the Bernstein polynomials which have desired properties and approximate the curve for given points. The shape of these functions depend on the number of vertices used to specify a particular curve [10]. A pc form of Bezier curves is obtained by four points in space. The shape of these curves is varied by keeping the end points fixed and moving the intermediate points, shown in Figure (2.6).

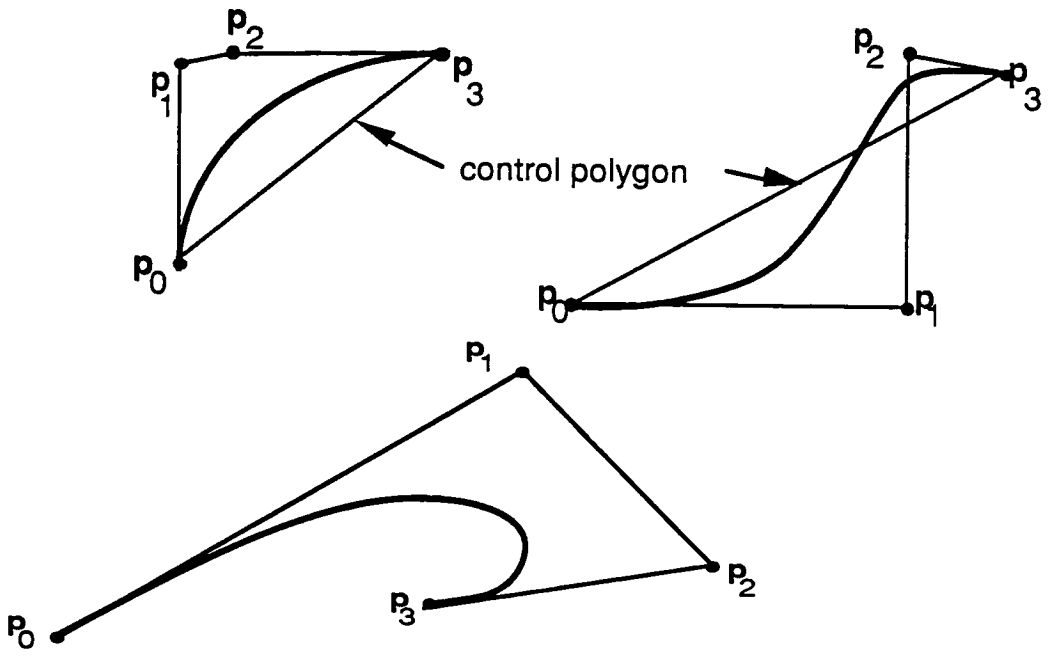


Figure 2.6. Variation of Bezier Curves by Intermediate Points

Equation 13 can be written as :

$$\mathbf{p}(u) = \sum_{i=0}^{n} \mathbf{p}_i B_{i,n}(u) ; \qquad 0 \le u \le 1 \tag{14}$$

where,

$$B_{i,n}(u) = C(n,i)u^i(1-u)^{n-i} \tag{15}$$

with, C(n,i) being the binomial coefficient given by :

$$C(n,i) = \frac{n!}{i!\,(n-1)!} \tag{16}$$

For the pc form of Bezier curves n = 4 and the polynomial form produced from equations 14, 15 and 16 is

$$\mathbf{p}(u) = [\,(1 - 3u + 3u^2 - u^3)\;(3u - 6u^2 + 3u^3)\;(3u^2 - 3u^3)\;u^3\,][\mathbf{p}_0 \;\; \mathbf{p}_1 \;\; \mathbf{p}_2 \;\; \mathbf{p}_3]^T$$

Rewriting this in matrix form

$$\bar{p}(u) = [\,u^3 \; u^2 \; u \; 1\,]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{p}_0 \\ \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \end{bmatrix} \qquad (18)$$

The two interior points $\mathbf{p}_1$ and $\mathbf{p}_2$ contribute to the required tangent vectors in the following way:

$$\mathbf{p}^u(0) = K(\mathbf{p}_1 - \mathbf{p}_0) \qquad (19)$$

$$\mathbf{p}^u(1) = K(\mathbf{p}_3 - \mathbf{p}_2) \qquad (20)$$

Where K is an arbitrary scale factor introduced to control the scale of the polygon. Figur (2.7) gives the parametric cubic equivalent of a cubic Bezier curve in space.

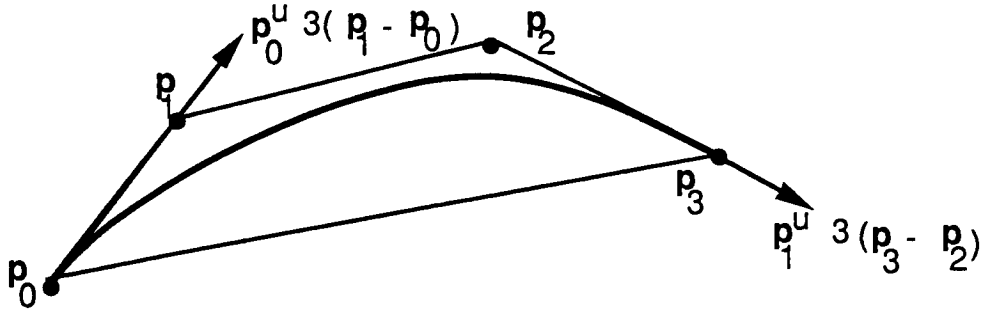


Figure2.7 PC equivalent of a Cubic Bezier Curve

The advantage of Bezier curves is that a higher order continuity between segments of compound curves can be achieved [11].

2.3.3 B-SPLINE CURVES

This curve fitting technique provides for local control of the shape of curves. Unlike the Bezier and Hermite methods where a small change in the position of the points of the characterstic polygon is propagated globally, the B-spline curve avoids this problem by using a special set of blending functions that have only local influence and depends only on a few neighboring control points. B-spline curves are similar to Bezier curves in that a set of blending functions combines the effects of n+1 control points, $\mathbf{p}_i$ given by

$$\mathbf{p}(u) = \sum_{i=0}^{n} \mathbf{p}_i N_{i,k}(u) \qquad (21)$$

The difference between Bezier and B-spline curves lies in the way the blending functions $N_{i,k}(u)$ are formulated. For Bezier curves, the control points determine the degree of the polynomials, while for B-splines curves the blending functions are controlled by a parameter k which are usually independent of the number of control points. The polynomial degree is instead controlled by the 'knot' values t_i described later. The B-spline blending functions are defined recursively by the following expressions. [8]

$$N_{i,1}(u) = 1 \qquad \text{if } t_i \le u < t_{i+1} \tag{22}$$

$$= 0 \qquad \text{otherwise}$$

$$N_{i,k}(u) = \frac{(u - t_i)N_{i,k-1}(u)}{t_{i+K-1} - t_i} + \frac{(t_{i+k} - u)N_{i+1,k-1}(u)}{t_{i+k} - t_{i+1}} \tag{23}$$

Where, k controls the degree (k - 1) of the resulting polynomial in u and also controls the continuity of the curve. The t_i are called knot values. They relate the parametric variable u to the $\mathbf{p}_i$ control points. For an open curve, the t_i are :

$$t_i = 0 \qquad \text{if } i < k$$

$$t_i = i - k + 1 \qquad \text{if } k \le i \le n \tag{24}$$

$$t_i = n - k + 2 \qquad \text{if } i > k$$

with $0 \le i \le n + k$

The range of the parametric variable u is

$$0 \leq u \leq (n - k + 2) \tag{25}$$

The parametric cubic analogous form of cubic B-splines (k = 4) is : [8]

$$\bar{p}_i(u) = \frac{1}{6}[\, u^3 \;\; u^2 \;\; u \;\; 1\,] \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{p}_{i-1} \\ \bar{p}_i \\ \bar{p}_{i+1} \\ \bar{p}_{i+2} \end{bmatrix} \quad \text{for } i \subset [1:n-2] \tag{26}$$

This cubic representation does not pass through any control points, but is continuous and also has continuity of tangent vector and of curvature. Hermite and Bezier forms of pc curces have first derivative continuity at the endpoints and pass through the endpoints. The B-Spline form is smoother than the other forms.

2.4 SURFACE GENERATION [8]

A surface is a two-dimensional region embedded in three-dimentional space. The simplest mathematical element of a surface is a **patch**. It is a curve bounded by a collection of points whose coordinates are given by continuous, two parameter, single valued mathematical functions of the form :

$$x = x(u,w) \quad y = (u,w) \quad z = (u,w) \tag{27}$$

The parametric variables are constrained in the intervals $0 \leq u,w \leq 1$.

Fixing the value of one of the parametric variables results in a curve on the patch in terms of the other variable, which remains free. By continuing this process first for one variable then the other for any number of arbitrary values in the interval [0,1], a parametric net of two one-parameter families of curves on the patch are formed. Only one curve of each family passes through a given point **p**(u,w). Associated with every patch is a set of boundary conditions, shown in Figure (2.8). The simplest of these are the four corner points and four curves defining the edges. Other boundary conditions are the tangent and twist vectors at the corner points.

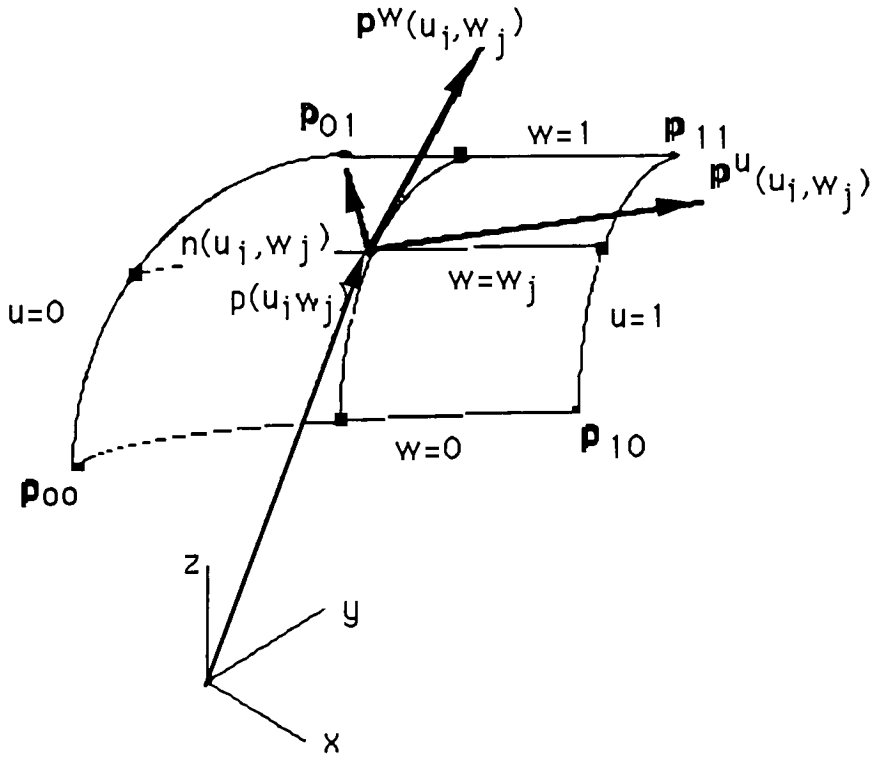


Figure 2.8 Parameters of a Bi-Cubic Patch

The algebraic form of a bi-cubic patch is given by

$$\bar{p}(u,w) = \sum_{i=0}^{3}\sum_{j=0}^{3} \bar{a}_{ij} u^i w^j \qquad (28)$$

where, $0 \le u,w \le 1$ and i, j denote the degree of the polynomial chosen to represent the bi-cubic patch.

Expanding and rearranging equation (28) gives :

$$\mathbf{p}(u,w) = \mathbf{a}_{33}u^3w^3 + \mathbf{a}_{32}u^3w^2 + \mathbf{a}_{31}u^3w + \mathbf{a}_{30}u^3 + \mathbf{a}_{23}u^2w^3 + \mathbf{a}_{22}u^2w^2 + \mathbf{a}_{21}u^2w + \mathbf{a}_{20}u^2 + \mathbf{a}_{13}uw^3 + \mathbf{a}_{12}uw^2 + \mathbf{a}_{11}uw + \mathbf{a}_{10}u + \mathbf{a}_{03}w^3 + \mathbf{a}_{02}w^2 + \mathbf{a}_{01}w + \mathbf{a}_{00}. \qquad (29)$$

This 16-term polynomial in u and w defines the set of all points lying on the surface. It is the algebraic form of the bi-cubic patch. Since each of the vector coefficients $\mathbf{a}_{ij}$ has three independent components, there are a total of 48 algebraic coefficients or 48 degrees of freedom.

In matrix form the algebraic equation (29) can be written as

$$\mathbf{p} = \mathbf{UAW}^T \qquad (30)$$

where,

$\mathbf{U} = [\, u^3 \;\; u^2 \;\; u \;\; 1 \,]$ and $\mathbf{W} = [\, w^3 \;\; w^2 \;\; w \;\; 1 \,]$ and matrix **A** is

$$\bar{A} = \begin{bmatrix} \bar{a}_{33} & \bar{a}_{32} & \bar{a}_{31} & \bar{a}_{30} \\ \bar{a}_{23} & \bar{a}_{22} & \bar{a}_{21} & \bar{a}_{20} \\ \bar{a}_{13} & \bar{a}_{12} & \bar{a}_{11} & \bar{a}_{10} \\ \bar{a}_{03} & \bar{a}_{02} & \bar{a}_{01} & \bar{a}_{00} \end{bmatrix} \qquad (31)$$

The **A** matrix is a 4x4x3 array, implying that each element of the matrix has three compoments, one in each of the x, y and z coordinate direction. Figure (2.9) gives the nomenclature of a bi-cubic surface

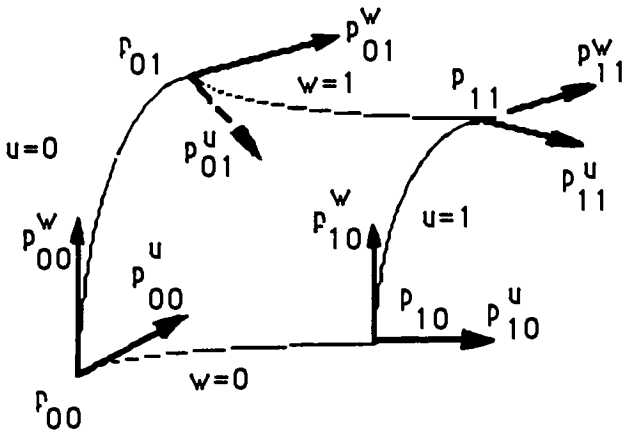


Figure 2.9 Nomenclature of a Bi-Cubic Surface

As it is for curves, algebraic coefficients are not the most convenient way of defining and controlling the shape of a patch, and they do not contribute to the understanding of surface behavior. Therefore, geometric forms are selected for

surface modeling. Of the 16 vectors required to define a bicubic patch, 12 of them are $\mathbf{p}_{00}$, $\mathbf{p}_{10}$, $\mathbf{p}_{01}$, $\mathbf{p}_{11}$, $\mathbf{p}_{00}^{u}$, $\mathbf{p}_{00}^{w}$, $\mathbf{p}_{10}^{u}$, $\mathbf{p}_{10}^{w}$, $\mathbf{p}_{01}^{u}$, $\mathbf{p}_{01}^{w}$, $\mathbf{p}_{11}^{u}$, $\mathbf{p}_{11}^{w}$ – the four corner points and the eight tangent vectors. The four additional vectors are provided by the twist vectors, one at each corner point- $\mathbf{p}_{00}^{uw}$, $\mathbf{p}_{10}^{uw}$, $\mathbf{p}_{01}^{uw}$, $\mathbf{p}_{11}^{uw}$. The geometric matrix **B** for the surface patch will take the form :

$$\overline{B} = \begin{bmatrix} \bar{p}_{00} & \bar{p}_{01} & \bar{p}_{00}^{w} & \bar{p}_{01}^{w} \\ \bar{p}_{10} & \bar{p}_{11} & \bar{p}_{10}^{w} & \bar{p}_{11}^{w} \\ \bar{p}_{00}^{u} & \bar{p}_{01}^{u} & \bar{p}_{00}^{uw} & \bar{p}_{01}^{uw} \\ \bar{p}_{10}^{u} & \bar{p}_{11}^{u} & \bar{p}_{10}^{uw} & \bar{p}_{11}^{uw} \end{bmatrix} \qquad (32)$$

As for the curve segment, the matrix representation of a surface patch in parametric space is :

$$\mathbf{p}(u,w) = \mathbf{F}(u)\mathbf{B}\mathbf{F}(w)^{T} \qquad (33)$$

Eliminating the functional notation $\mathbf{F}(u)$ and $\mathbf{F}(w)$ by $\mathbf{F}(u) = \mathbf{UM}$ and $\mathbf{F}(w)^{T} = \mathbf{M}^{T}\mathbf{W}^{T}$, as it was done with the curves, equation (33) transforms to:

$$\mathbf{p}(u,w) = \mathbf{UMBM}^{T}\mathbf{W}^{T} \qquad (34)$$

This form of surface patch is also known as a Hermite patch. Hermite patches

differ from the Bezier and B-spline patch generation schemes essentially in the manner in which the tangent and the twist vectors are calculated. These differences are analogous to the curve generation techniques.

Surfaces in three-dimensions are of two types :

1. Quadric Surfaces
2. Free-Form Surfaces

2.4.1 QUADRIC SURFACES

The general algebraic equation for a quadric surface is of the following form :

$$ax^2+ by^2+ cz^2 + 2hxy+ 2gzx + 2fyz + 2ux + 2vy + 2wz + d = 0 \qquad (35)$$

Spheres, Cylinders, Cones, Ellipsoids, Paraboloids, Hyperboloids are forms of quadric surfaces. All of them can be represented by some variation of the above equation.

$$x^2 + y^2 + z^2 - a^2 = 0 \qquad \text{sphere} \qquad (36)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = k \qquad \text{ellipsoid} \qquad (37)$$

Some Quadric surfaces can be further classified as developable surfaces. Developable surfaces are the ones which can be unrolled onto a plane without distortion and their topology stays the same, for example cylindrical, and conical surfaces. In other words these curved surfaces can be transformed into planar surfaces. These developable surfaces are of interest as major engineering parts are formed by these surfaces, such as in sheet metal operations.

In the present context of mesh generation on planar surfaces, only developable surfaces will be discussed. The condition that a surface is developable is that its Gaussian curvature should be zero [11]. mathematically :

$$(\mathbf{p}^{uu}.\mathbf{n})(\mathbf{p}^{ww}.\mathbf{n}) - (\mathbf{p}^{uw}.\mathbf{n})^2 = 0 \qquad (38)$$

There are various methods of generating a planar or a developable surface. Revolution of a curve, extrusion of a curve, [11] locus of a straight line as it moves along a curve [8] are some of the efficient and popular methods. In all the methods the geometry matrix **B** of equation (32) of the surface patch has the twist vectors $\mathbf{p}_{00}^{uw}$, $\mathbf{p}_{01}^{uw}$, $\mathbf{p}_{10}^{uw}$, and $\mathbf{p}_{11}^{uw}$, as zero.

Free form surfaces do not possess any standard mathematical representation. They are constructed by joining two or more patches together with continuity at

common boundaries. Transfinite form of surfaces, discussed in section 3.1.2 is one of the many free-form surface representation schemes. Methods for producing continuous composite surfaces have been suggested by Hermite, Bezier, Coons, and many more mathematicians. Referances in the end cite some of the papers. [9] [10] [18] [19] Since this thesis does not involve free-form surfaces, their development will not be discussed.

CHAPTER 3

AUTOMATIC MESH GENERATION

REVIEW OF RELATED LITERATURE

Today, there is an abundance of publications on mesh generation methods. [6] As computing capabilities of modern computers have increased, so have efforts of researchers to make full use of these capabilities. In matters of efficient mesh generation techniques, academicians have been able to propose useful methods.

The objective of this thesis project is to study the state of art in mesh generation methods in two-dimensions, and develop or modifiy an algorithm to automatically mesh a geometry with triangular finite elements.

First, the chronological development of meshing techniques is discussed. Then, some of the popular and new methods are reviewed along with their capabilities and shortcomings. Finally, the technique for Delauny Triangulation is presented. Watson's method for Delauny Triangulation is limited to triangulation of the convex hull of a set of given points or nodes. An algorithm is added to this method, to triangulate convex, as well as concave shapes. Watson's method and the modifications are presented in chapter 4.

The selection of this meshing method was complex. Other schemes were selected and discarded because of the use of unreasonable computer memory space and inability to support extension into three-dimension. Following extensive research, Watson's method was selected and modified.

3.1 MESH GENERATION METHODS

The development of meshing techniques can be chronologically classified into three categories :

Manual

Interactive

Automatic

Of the three above mentioned approaches, the manual methods take large data, and the mesh construction consumes about 85% of the total computational cost of a typical analysis. [12] Manual methods have become completely outdated. This method will not be discussed here.

3.2 INTERACTIVE SCHEMES

Interactively controlled meshing has been incorporated in most commercially available CAD systems. In the late 1960's, methods were suggested for automatically determining the coordinates of interior nodes via interpolation

schemes. In the second half of the 1970's computer graphics was used successfully to enable real time interactive finite element mesh generation. [13] [14] [15] In these schemes the analyst defined the object geometry via the boundary definition of the domain. In most of the mesh generators the nodes on each curve of the domain were user supplied. The mesh was then generated in the interior of the domain based upon analyst supplied information of the geometry and topology. The analyst controlled the process at the following stages :

Subdivision of the object into simple mapable regions.

Node placement on the boundary of, and inside the region to get the desired mesh generation.

Ensure field variable continuity across the regions by proper placement of nodes on the common edges between regions.

Selection of the appropriate mapping function for each region.

Some of the popular interactive schemes are now listed and discussed.

1. Laplacian Method
2. Isoparametric Method
3. Transfinite Mapping Method
4. Discrete form of Transfinite Mappings

3.2.1 LAPLACIAN METHOD

If the locations of all the boundary nodes of a geometry are known, the interior nodes can be generated automatically by placing each interior node at the average of its neighbour's positions. The position vector, $\mathbf{p}_i$, of each interior node ' i ' satisfies the following equation :

$$P_i = \frac{1}{2N_i} \sum_{n=1}^{N_i} (P_{nj} + P_{ni}) \qquad (1)$$

Where, N_i is the number of elements connected to node ' i '. P_{nj} and P_{ni} are the position vectors of the adjacent nodes in the neighboring element n, as shown in the Figure (3.1).

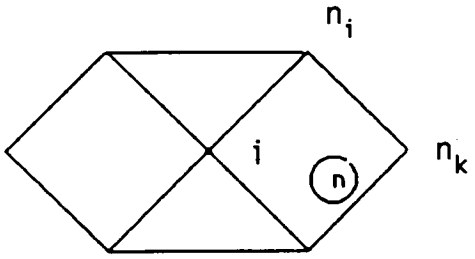


Figure 3.1 Point location by Laplacian method

The name of this method is derived from the fact that equation (1) can be interpreted as the Laplacian finite difference operator for the unknowns $\mathbf{p}_i$.

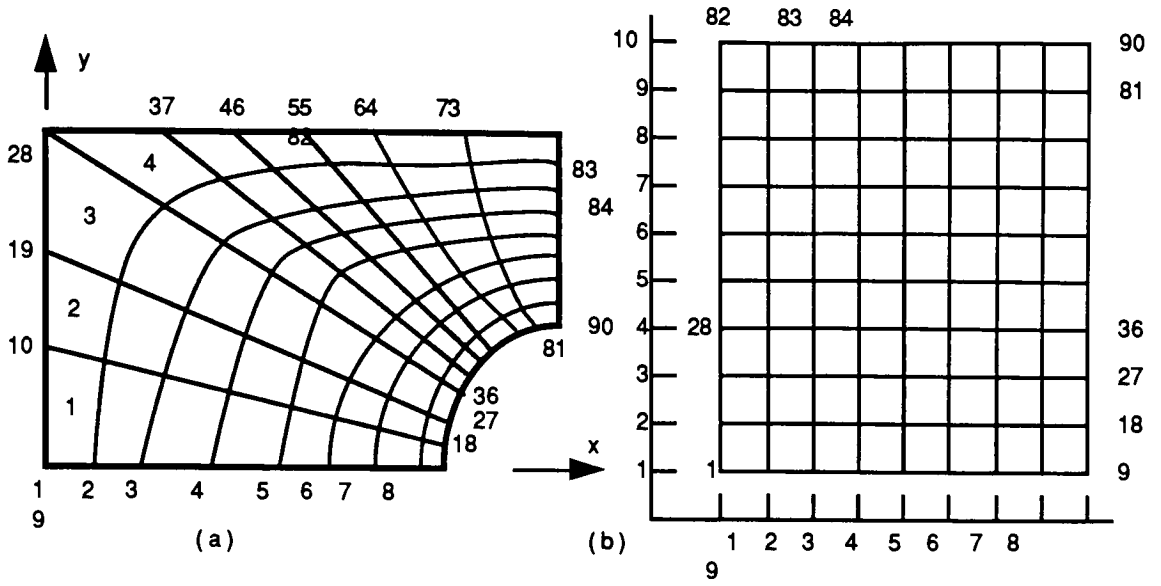


Figure 3.2 (a) Laplacian Mesh (b) Rectangular mesh

In the calculations from equation (1), interior nodes in Figure (3.2a) are identified by their location in the recangular mesh of Figure (3.2b).

A Laplacian grid has its drawbacks. Since the equation (1) represents a set of non-linear simultaneous equations for all the unknowns $\mathbf{p}_i$, the solution is best achieved via an iterative technique such as Gauss-Siedel or the Jacobi method. This represents two problems. The first problem of a relatively large computation for mesh generation. The second problem where elements may become undesirably distorted, interior nodes may pile up along curved edges, sometimes to the extent that they fall outside the edge. [2] The second problem is alleviated by a generalization of equation (1). In this method the interior

points are located by :

$$P_i = \frac{1}{N_i(2-w)} \sum_{n=1}^{N_i} (P_{nj} + P_{ni} - wP_{nk}); \qquad 0 \leq w \leq 1 \qquad (2)$$

Where N_i is the number of elements that share node i, subscripts nj and ni refer to adjacent nodes and nk to diagonally opposite node in element n. 'W' is an arbitrary factor between 0 and 1. Different values of 'w' produce a family of schemes called Laplacian-isoparametric schemes. When 'w' is set to zero, equation (1) is recovered, when 'w' is set to unity, the pure isoparametric Laplacian scheme is produced, these schemes should not to be confused with isoparametric schemes which are discussed later. This method requires higher Gauss-Siedel iterations for convergence with higher values of 'w'. For good shapes 'w' is typically 0.85. There is a trade off between mesh quality and computational efficiency. Denayer [16] has suggested methods for overcoming some of these difficulties.

Since interior nodes are placed at the average positions of the neighboring nodes, the Laplacian methods have been used in conjunction with other meshing methods for smoothing steps. This helps to improve the uniformity of the final mesh.

3.2.2 ISOPARAMETRIC MAPPING METHOD

The isoparametric mapping method was described by Zeinkiewicz and Phillips.[17] In this method, polynomial interpolation functions or shape functions are choosen to provide a unique mapping between curvilinear coordinates and cartesian coordinates

$$P = \sum_{i=1}^{n} N_i P_i (u,v) \qquad (3)$$

This represents mapping between a simple polygon, generally a unit square or triangle and the actual region. The resulting region boundaries are modeled by simple Lagrange polynomials. Considering the particular case of a parabolic quadilateral shown in Figure (3.3), where the x, y, (and z) coordinates of the eight edge nodes are known, one can write

$$x = \sum_{i=1}^{8} N_i X_i$$

$$y = \sum_{i=1}^{8} N_i Y_i \qquad (4)$$

$$z = \sum_{i=1}^{8} N_i Z_i$$

N_i are the shape functions associated with each node defined in terms of the curvilinear coordinates u and v, which have values ranging from 1 to -1 on opposite sides. Typical shape functions for an element are

$$N_1 = -\frac{1}{4}(1-u)(1-v)(u+v+1)$$

$$N_2 = \frac{1}{2}(1-u)(1-v^2)$$

If the coordinates of the nodal points are known then the cartesian coordinates of any specified point p_i can be found by equation (4).

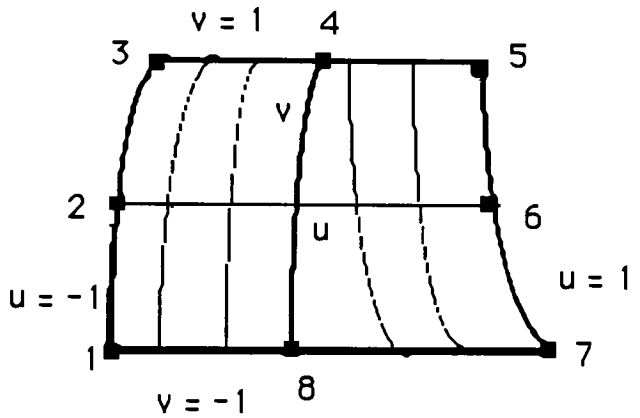


Figure 3.3 Curvilinear coordinates for Quadrilateral Isoparametric Mapping

The curvilinear coordinate system produced by this method provides a natural method of producing element topology automatically. Node points may be located at intersections of constant coordinate curves in u and v directions. This mapping produces quadilatral elements. Each quadrilateral may be diagonalized to produce triangular elements.

This method imposes restrictions on the topology of the mesh. There must be an equal number of elements along opposite sides of the region. Creation of mesh connectivity does not require any effort on the part of the analyst. Points of inflection and slope discontinuties cannot be incorporated in the boundary of a single region. Multiple regions are required to model complex geometries. Higher order shape functions can also be used to produce more complex boundaries at the expense of increased computations. Slope discontinuities cannot exist within a single region. An additional problem with isoparametric mappings is that curve fitting errors are introduced in the description of structures whose boundaries cannot be exactly described by polynomials of the same order as those appearing in shape functions.

3.2.3 TRANSFINITE MAPPING METHOD

Transfinite mapping techniques are a class of methods for establishing curvilinear coordinate systems in arbitrary domains. The method was developed at General Motor research labratory by Gordon, Hall and Associates in the 1970's. [18] It can approximate complex surfaces and volumes. Only surfaces will be discussed here. The general transfinite method describes an approximate surface which will match the desired or true surface at a non-denumerable or infinite number of points. It is this property that gives rise to the term 'transfinite mapping'.

This method of mapping contrasts with isoparametric mappings described in the previous section. The isoparametric mappings match the true surface only at points used for interpolation. In the case of planar surfaces, the transfinite mapping can be made to model all region boundaries exactly and no geometric errors are introduced by the mapping.

To understand transfinite mappings it is important to understand the concept of the projector ***P***. A projector is any linear operator which approximates a true surface, subject to certain interpolation constraints. There is a wide variety of projectors. We will look at a simple projector, the linear lofting projector. As shown in Figure (3.4) this lofting projector ***P*** performs a linear interpolation between two boundary curves Ω_1 (u), and Ω_2 (u).

$$\boldsymbol{P}\,[F] = P(u,v) = (1-v)\Omega_1(u) + v\Omega_2(u); \qquad 0\le u\le 1,\ 0\le v\le 1. \tag{6}$$

Where, u is a normalized parametric coordinate along Ω_1 and Ω_2, and v is a normalized coordinate which has a value of zero on Ω_1 and unity on Ω_2. This in effect means that the projector ***P*** maps the surface truly in the coordinate u direction, and approximates it linearly in v direction.

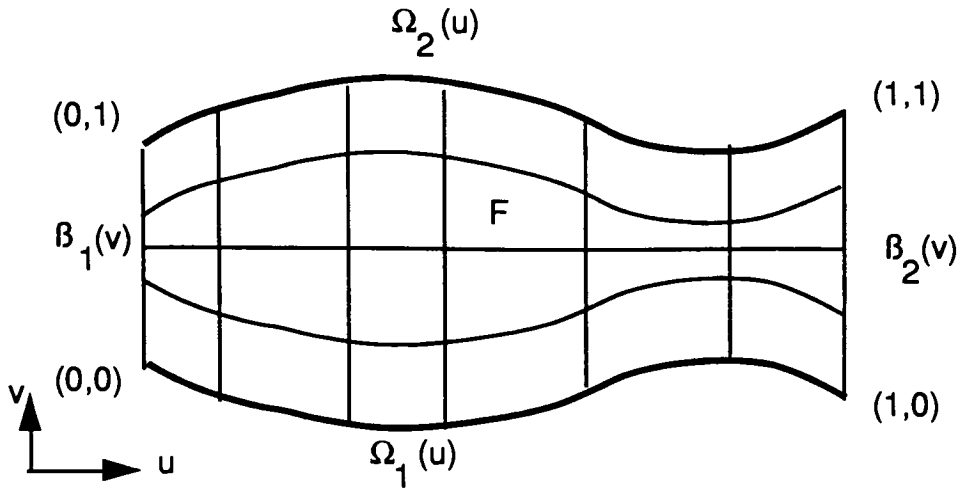


Figure 3.4 Linear Lofting Projector *P*

Sets of these simple linear projectors can be 'blended' to form more complex projectors which will match the boundary of a region F at all points. A bilinear projector which can represent region F bounded by four curves $\Omega_1(u)$, $\Omega_2(u)$, $\beta_1(v)$, $\beta_2(v)$ shown in the Figure (3. 5), will be developed here from two linear projectors.

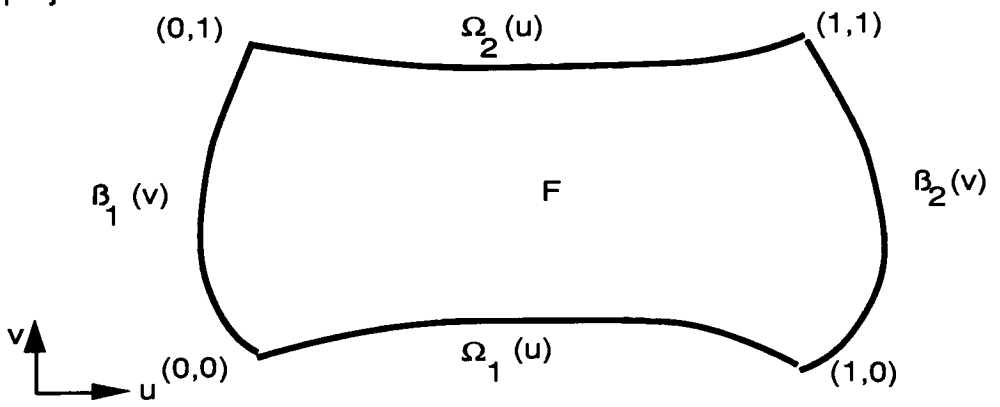


Figure 3.5 Region F to be Mapped by Bilinear Projector

Two linear projectors, one interpolating in the u direction and one in the v direction are formed :

$$\boldsymbol{P}_1[F] = P(u,v) = (1-v)\Omega_1(u) + v\Omega_2(u); \qquad 0 \le u \le 1$$

$$\boldsymbol{P}_2[F] = P(u,v) = (1-u)\beta_1(v) + u\beta_2(v); \qquad 0 \le v \le 1 \qquad (7)$$

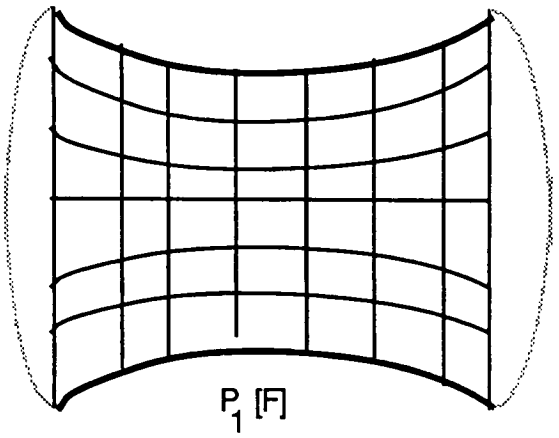
Figure (3.6a) and Figure (3.6b) show the linear projectors $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ respectively. Figure (3.6c) shows the product $(\boldsymbol{P}_1 . \boldsymbol{P}_2)[F] = (\boldsymbol{P}_2 . \boldsymbol{P}_1)[F]$, of the two operators. The Boolean sum of the two projectors is defined as :

$$(\boldsymbol{P}_1 + \boldsymbol{P}_2)[F] = \boldsymbol{P}_1[F] + \boldsymbol{P}_2[F] - (\boldsymbol{P}_1 . \boldsymbol{P}_2)[F] = (1-v)\Omega_1(u) + v\Omega_2(u) +$$

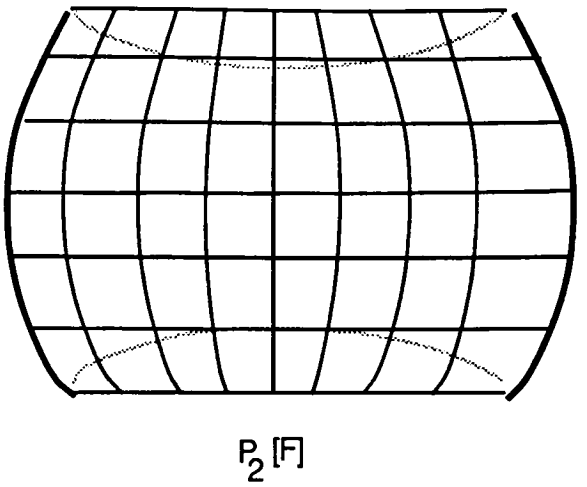
$$(1-u)\beta_1(v) + u\beta_2(v) - (1-u)(1-v)F(0,0) - (1-u)vF(0,1) - uvF(1,1) - u(1-v)F(1,0).$$

$$0 \le u \le 1, \quad 0 \le v \le 1 \qquad (8)$$

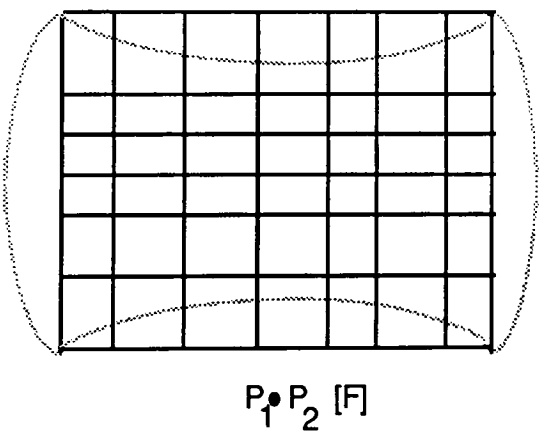
The Boolean sum of the projectors represents the region F of the Figure (3.5). Figure (3.6d) shows the representation of the region F by bilinear projectors. This Boolean projector represents a curvilinear coordinate system created by a mapping of the unit square on F. It may be called the transfinite bilinear Lagrange interpolant of F, and is identical to simplest form of Coon's patch. [19] If the boundary curves are defined by Lagrange polynomials, the isoparametric mapping is obtained as a special case of transfinite mappings. Higher order interpolants may be used to force the coordinate curves to pass through specified curves in the interior of F.



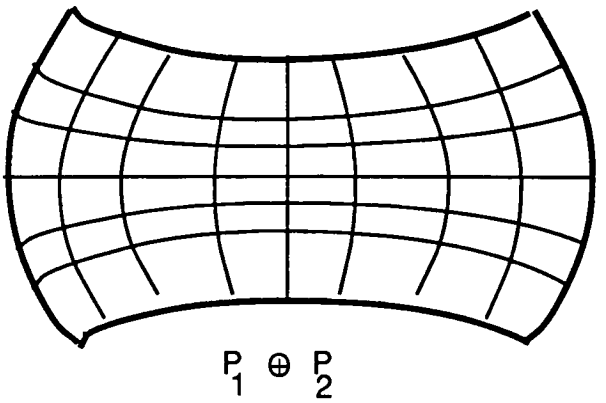
(a) Bilinear Projector P_1



(b) Bilinear Projector P_2



(c) Bilinear Projector $P_1 \bullet P_2$



(d) Bilinear Projector $P_1 \oplus P_2$

Figure 3.6 Sum and Product of a Bilinear Projector

Another useful projector is based on a mapping of a unit triangle to a region defined by three boundary curves. For further reading Barnhill [20] is suggested. Transfinite mappings can be used to map any mesh in an appropriate primitive polygon to the actual region. Any mesh topology can be used. It is generally convenient to sacrifice topological generality by choosing intersections of the constant coordinate curves for use as node points in order to minimize user input. The major drawback with this method is that it requires large computer memory space and time.

3.2.4 DISCRETE FORM OF TRANSFINITE MAPPINGS [21]

Transfinite or continuous representations allow a curve or surface to be evaluated at all points on the geometry. Discrete representations consist of a finite list of points located on the geometry with a unique coordinate associated with each point in the list. The curve/surface position can be evaluated at points contained in the finite list and is undefined elsewhere. This method of mesh generation corresponds to the geometric modeling of a composite surface by a rectangular network of bicubic patches [18] .

Two families of intersecting curves $\mathbf{q}_i$ and $\mathbf{r}_j$ with $1 \leq i \leq m$ and $1 \leq j \leq n$ combine to form a wireframe surface, shown in Figure (3.7). There are m x n intersection points.

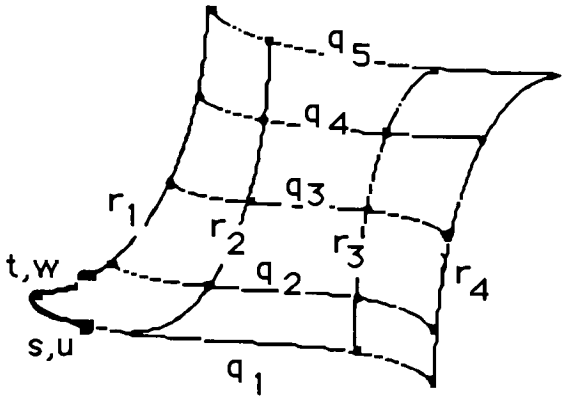


Figure 3.7 Rectangular Network of Intersecting Curves

The $\mathbf{q}_i$ family of m curves is expressed in terms of the parameter s, and the $\mathbf{r}_j$ family of m curves in terms of the parameter t . Then for a typical curve,

$$\mathbf{p}(s) = \mathbf{q}_i(s) \qquad 0 \le s \le S_i \tag{9}$$

$$\mathbf{p}(t) = \mathbf{r}_j(t) \qquad 0 \le t \le T_j \tag{10}$$

Here, $\mathbf{p}$ is the position vector to a point on the indicated curve. For the curves $\mathbf{q}_i(s)$ the range of the parameters S_i is not identical . This is also true for the set $\mathbf{r}_j$, where the range of the parameters T_j is not identical. Therefore, S_i and T_j are normalized and defined in terms of a second set of parameters u and w respectively.

$$u_j = \frac{1}{m} \sum_{i=1}^{m} \frac{s_{ij}}{s_{in}} \qquad (11)$$

$$w_i = \frac{1}{n} \sum_{j=1}^{n} \frac{t_{ij}}{t_{mj}} \qquad (12)$$

The double subscript on s and t denotes their value at the intersection node indicated. The parameters u and w map the entire surface into a unit square in u, w, space. This procedure has the effect of defining n functions $t = T_j(w)$, with $1 \le j \le n$ and m functions $s = S_i(u)$, with $1 \le i \le m$. Therefore equations (9) and (10) can be written as :

$$\mathbf{p}(u) = \mathbf{q}_i [S_i(u)] \qquad 1 \le i \le m, \quad 0 \le u \le 1 \qquad (13)$$

$$\mathbf{p}(w) = \mathbf{r}_j [T_j(w)] \qquad 1 \le j \le n, \quad 0 \le w \le 1 \qquad (14)$$

The interpolation of the family of curves $\mathbf{q}_i$ and $\mathbf{r}_j$ is carried out separately. Consider the m curves $\mathbf{q}_i(u)$. Each of these curves corresponds to a distinct value of the parameter w, let $F_i(w)$ be the blending or the interpolating functions, where

$$F_i(w_j) = \partial_{ij} \qquad (15)$$

where, $\partial_{ij} = 0$ if $i \neq j$

$\partial_{ij} = 1$ if $i = j$

Therefore,

$$\mathbf{p}(u,w) = \sum_{i=1}^{n} F_i(w)\mathbf{q}_i(u) \qquad (16)$$

Any function F that satisfies equation (15) can be used in equation (16)

Since the interpolation curves are assumed to be defined in discrete form, it is possible to write highly efficient and completely general subroutines to perform transfinite mappings. An apparent drawback of the discrete form of curve representation is the extensive data required to describe an interpolation curve in discrete form.

3.3 AUTOMATIC MESH GENERATION

The definition of an automatic mesh generation is subjective. It is time as well as human interpretation dependent. Many software developers call the interactive methods automatic. Compared to manual methods of early seventies these programs can be classified as semi-automatic. Given the computer capabilities of today, a fully automatic meshing procedure should discretize the object geometry into a finite element mesh without the analyst's interaction. The input from the analyst should only consist of the following definitions :

a) Object geometry.

b) Mesh density.

c) Attributes of the domain.

In this thesis project a mesh generation program has been developed which works with inputs (a) and (b) from an analyst. The program does not have the capability of storing attributes. This part of the Pre-processor development is relatively simple, once the mesh is generated. Some of the existing automatic programs are discussed first. One can classify the existing automations into three categories :

1. Topology based algoritms.
2. Hierarchical algorithms.
3. Triangulation algorithms.

3.3.1 TOPOLOGY BASED ALGORITHMS

These algorithms extract one element at a time from the object boundary representation by operating on the topological description. [22] [23] [24] The algorithms extract individual elements one at a time, and update the boundary representation of the resultant object. The process terminates when the whole geometry has been exhausted.

Element removal meshing procedures employ a specific set of element removal operators that are capable of removing a single element from the object. They operate by first examining the topological features of the object, and testing a specified set of geometric measures to see if any of the element removal

operators can be applied. There is a hierarchy in which the various operators are applied. [22] [24]

Such a meshing procedure can be most directly integrated with boundary based modeling systems that maintain the object geometry by boundary representation, discussed in chapter 2. In this method the element removal procedure can operate by invoking a set of operators that already exist for the geometric modeler, explained later. If the meshing procedure is to be integrated with geometric modelers that do not support a boundary representation and the required geometric operators, it will be neccessary to provide capabilities for element extraction. This would require major algorithmic and software development. Currently, efforts are concentrated in developing a two-dimensional prototype which extracts triangular elements to triangulate the object geometry.

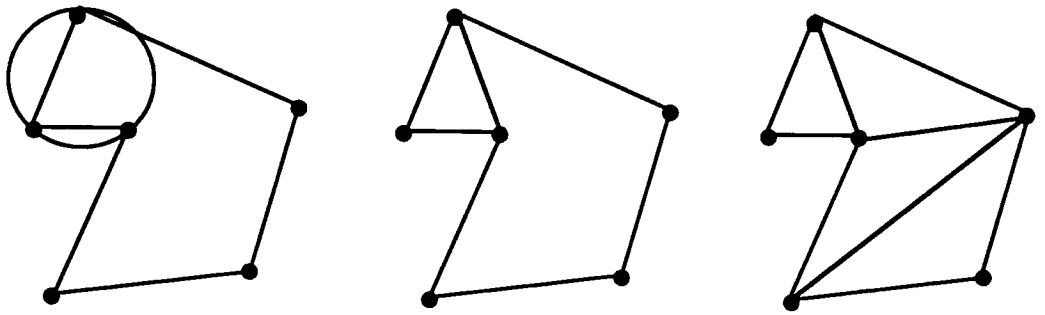
An important factor in the operation of this algorithm is the shape control parameter used for controlling the element shape. For triangles the ratio of lengths of the diameters of the circumscribed circle to that of the inscribed circle is used. The ratio is 2 for an equilateral triangle and 2.414 for a right angle isosceles triangle. For higher ratios the aspect ratio increases. The ratio is denoted as RBYR with a limiting value, as an input parameter, RBYRCT.

The basic algorithm for meshing a simply connected two dimensional region works by applying three operators for element removal until the region is exhausted. The first operator is a two dimensional version of the vertex removal operator of the Woo and Thomasama algorithm. [24] It is applied to one vertex at a time in a polygon. The vertex on which the operator is applied is called the 'before ' vertex, and the vertices with which it chooses to form a valid triangle are called ' after ' vertices. The following conditions have to be met before a triangle is extracted.

1. The triangle formed by joining the before and after vertices has an RBYR ratio less than or equal to the control parameter RBYRCT.
2. None of the vertices in the polygon other than the 3 vertices involved in the triangle being considered, lie inside the circle passing through the vertices, shown in Figure (3.8a).
3. None of the edges of the polygon intersects the triangle created by joining the before and after vertices. This is automatically satisfied if the above two conditions are satisfied.

Vertex Removal is applied to the vertex and a triangle is formed joining the three vertices, as shown in Figure (3.8b). Two edges are removed from the object representation and one new edge joining the two previously unconnected vertices is introduced in the object representation. A region can

be meshed with only the Vertex Removal operator under some favourable conditions, Figure (3.8c).

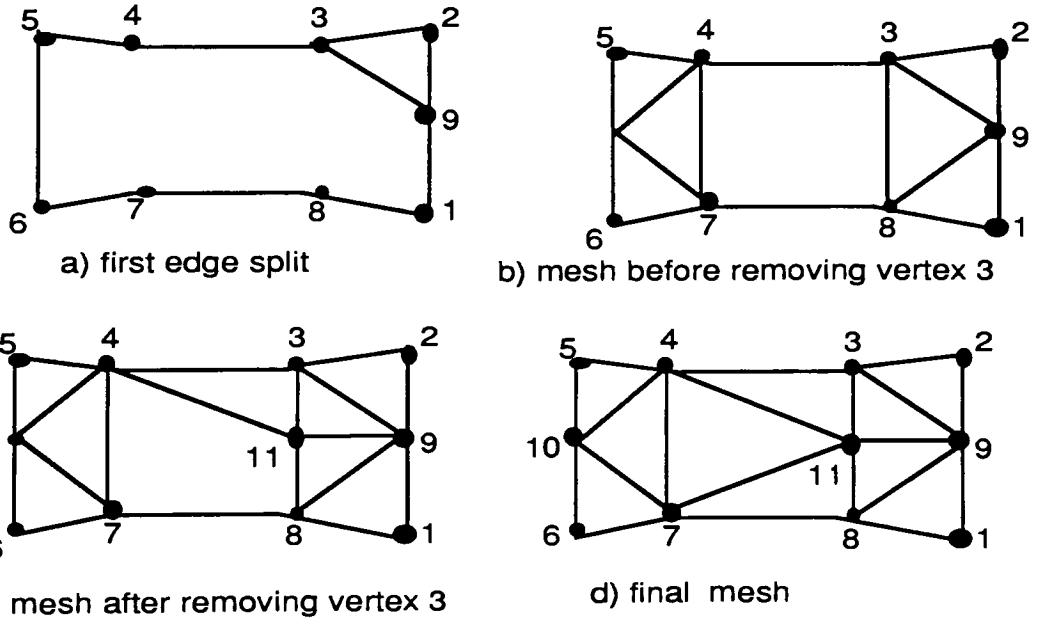


(a) Geometry (b) First Element (c) Final Mesh

Figure 3.8 Vertex Removal Operator

If for a given vertex all the RBYR ratios exceed the limit RBYRCT, a triangle cannot be extracted by the application of the Vertex removal operator. In such situations a second operator, 'Vertex Removal with Edge Split' is attempted. Tests made for this operator are similar to those of the Vertex removal operator. The difference is that this operator examines the edges on both sides of the vertex being inspected, and attempts to introduce a node at an optimal location along the longer edge. The location for node insertion is determined such that an acceptable RBYR is produced. If the tests are passed, the element is defined with the shorter edge being removed from the region, the longer edge is

redefined as being connected to the new vertex and the uncommon vertex. The triangle is formed by joining the vertex of the shorter edge and the new vertex on the larger edge, shown in Figure (3.9a). The process of meshing a geometry by the described operators is shown in Figure (3.9). Since the minimum RBYR for all vertices is higher than the given RBYRCT, the vertex 2, required the splitting of edge 1-2 into edges 1-9 and 9-2 shown in Figure (3.9a). Both the operators continued to be applied to produce a mesh shown in Figure (3.9b). At this point the removal of vertex 3 requires the splitting of edge 3-8 which is the side of existing element 3-8-9. This edge can be split if element 3-8-9 can be split into two elements as the edge is split. This operation is allowed as long as the resulting elements have RBYR values less than RBYRCT , shown in Figure (3.9c). Figure (3.9d) shows the final mesh.



a) first edge split

b) mesh before removing vertex 3

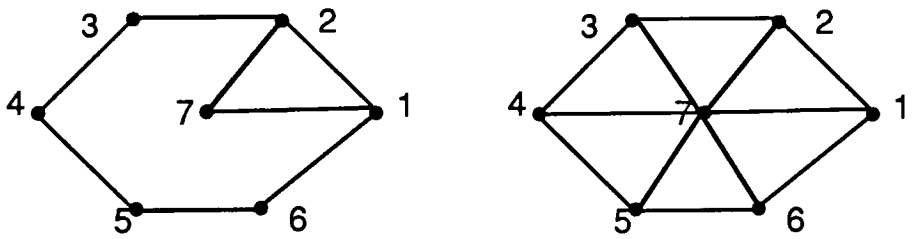
c) mesh after removing vertex 3

d) final mesh

Figure 3.9 Vertex Removal with EdgeSplit

Situations arise when neither of the vertex removal operators can be applied. In these cases, a third operator, 'Edge Removal' is applied. This operator is applied to an edge, when two angles subtended by other edges at the current edge, are higher than the upper angle limit decided by the RBYR ratio. The application of the Edge Removal operator introduces a new node inside the domain being meshed. If all the above conditions are satisfied, Edge Removal is applied by forming a triangle with the inserted vertex and the edge. Two new edges and one new vertex is introduced in the region definition.

Consider the meshing of a hexagon with an RBYRCT ratio of 2.1. Because of the low value of RBYRCT ratio, neither vertex removal operator can be applied. Edge Removal is applied at the edge connecting nodes 1 and 2 and a new node 7 is generated, see Figure (3.10a). Once node 7 is generated, Vertex Removal can be applied, producing the mesh shown in Figure (3.10b).



a) removal of edge 1-2

b) final mesh

Figure 3.10 Edge Removal Operator

3.3.2 HIERACHICAL DECOMPOSITION ALGORITHMS [25] [26] [27]

These algorithms are based on quadtree/octree decomposition of an object . The idea underlying this recursive spatial subdivision scheme is to approximate the object to be meshed with a union of disjoint, variably sized rectangles in two dimensions or blocks in three dimensions. [1] These rectangles are generated by subdividing recursively a spatial region enclosing the object, rather than the object itself. Figure (3.11) provides a two dimensional example. The object, a bracket with a hole, 'is boxed', to establish a convenient minimal spatial region. The box is then decomposed into quadrants. When a quadrant can be classified as wholly inside or outside the object, the subdivision ceases; when the quadrant cannot be classified, it is subdivided into quadrants and this process continues until some minimal resolution level is reached.

Approximations produced in this manner can be represented by logical trees whose nodes have four sons, shown in Figure (3.12). Hence the name quadtree. The tree structures in Figure (3.11) result from the subdivision rule used to produce the decomposition. The tree structure can be thought of as organizing or cataloging structure for data describing particular regions of space, Figure (3.13a). Pertinent information that might be stored in such a data record for automatic mesh generation are shown in Figure (3.13b). The record includes classification of the spatial region represented by the node as inside,

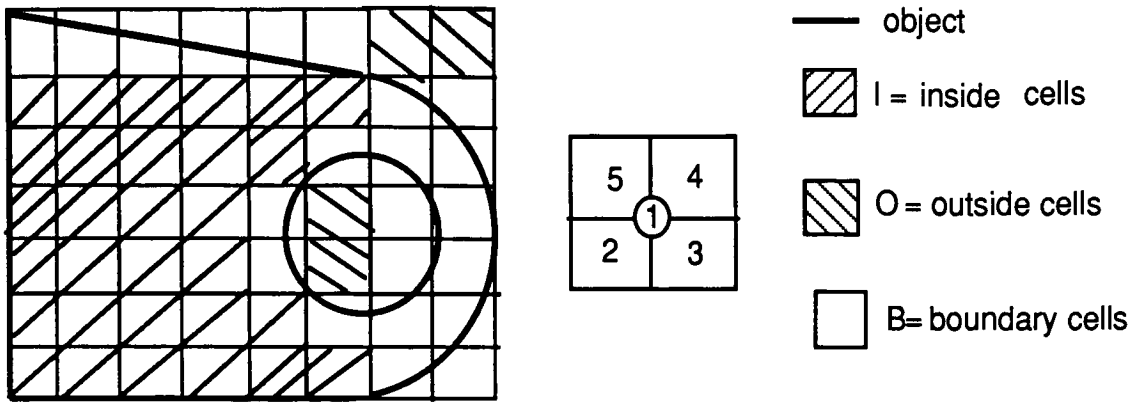


Figure 3.11 Hierarchical Approximation of an Object

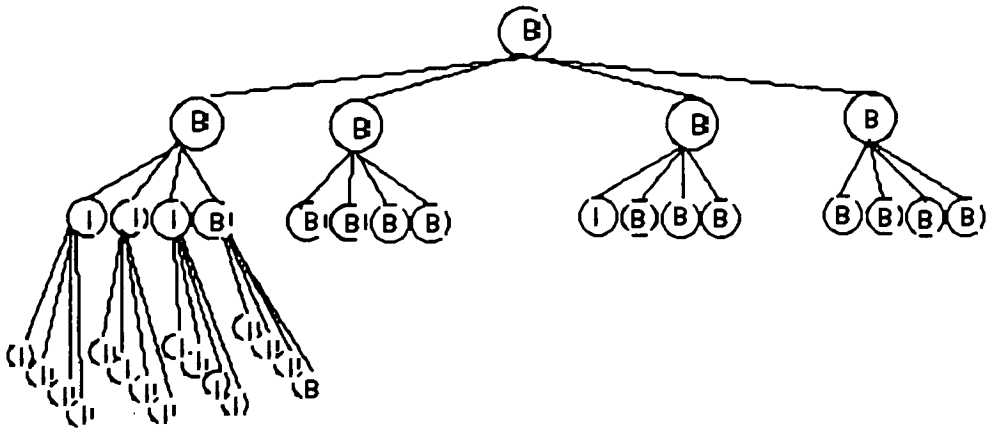


figure 3.12 Tree Structure of Quadtree Approximation

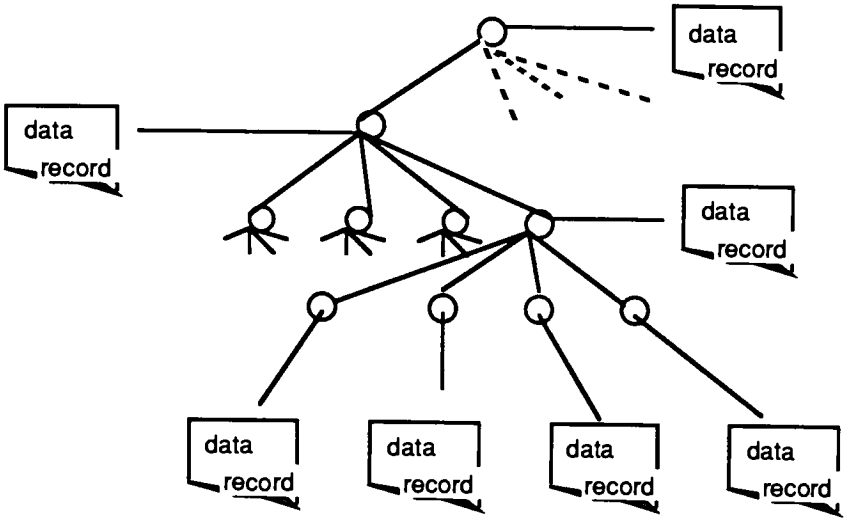


Figure 3.13a Tree Structure of Data Records

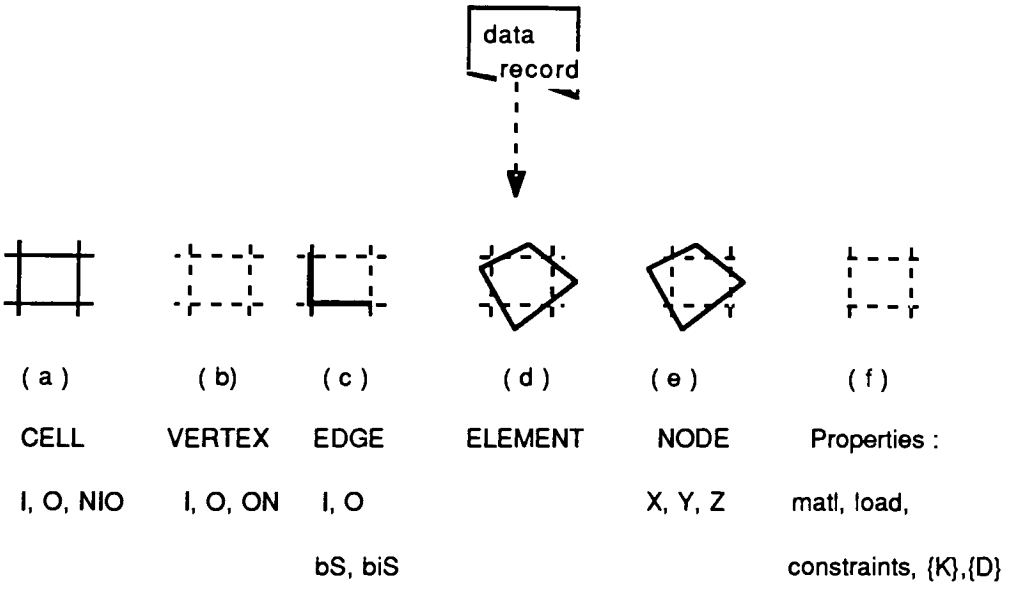


Figure 3.13b Hierarchical Structure for a Finite Element Model

outside, or on the boundary, shape functions for the finite element associated with the region, and properties associated with the finite element, such as stifness matrix, boundary conditions and such others.

At the lowest level of the tree are the smallest spatial regions and the simplest finite elements. As one ascends the tree the regions become larger, encompassing multiples of four elemental regions, and the finite elements become superelements with associated "assembled" stiffness matrices, and collected constraints. Such an organization is ideally suited for mesh refinement by subdivision. 'Inside' cells of the spatial decomposition can be converted easily into finite elements or substructures, but 'Boundary' cells require further processing to produce valid elements that closely approximate the object's boundary.

The interior mesh is built with regular cells. This saves substantial computational costs when deriving the stiffness matrix for the interior elements and substructures, shown in Figure (3.11). The major disadvantage of this method is the difficulty of identifying the boundary cells.

3.3.3 TRIANGULATION [28] [29]

Except for the heirarchical mesh generation technique, all the existing techniques produce meshes composed of triangles and tetrahedra in two dimensions and three dimensions respectively. For the topologicaly based algorithms, the rules for element extraction are much more direct for simplex topologies, triangles, tetrahedrons than for non-simplex ones, such as quadrilateral or hexahedron. Spatial decomposition schemes as applied by Yerry, [25] [26] reduce all elements to triangles or tetrahedra to accomodate quaternal or octal cells. Yerry avoids ill-formed elements on the boundary, via a smoothing operation that works best for simplex topologies.

For finite element analysis in two dimensions it is generally recognized that the triangle can best adapt the boundaries of a domain. Very often only triangular elements are able to fill domains with irregular boundaries and openings. Triangular elements also allow a progressive change of element size without serious distortions. Since most of the triangular elements situated in the interior part of the domain have straight edges, explicit expressions of the elementary stiffness matrices can be obtained without numerical integration. This gives high computational efficiency. A comparision can be found between an isoparametric eight node element and a linear strain six node element. [30]

Given a set of vertices (nodes) there exist many valid triangulations. Different approaches have been used for finding a satisfactory triangulation. The criterion for judging triangulation algorithms are :

1 Speed. The algorithm should be quick, but more importantly, the time taken should increase as slowly as possible as the number of points (n) increases.

2 Quality of triangulation. The shape of the triangles should as close to equilateral as possible.

3 Generality. The algorithm should be capable of dealing efficiently with complex geometry, including re-entrant boundaries and multiply connected regions.

4 Extensibility. The algorithm should be capable of extension into higher dimensions without violating any of the above criterion.

Over the years several techniques have been developed for triangulation of planar surfaces. Some of them will be discussed, and finally the Delauny Triangulation method, which is used in this thesis project will be discussed in detail.

3.3.3.1 LEWIS AND ROBINSON METHOD [31]

This algorithm operates by applying a 'problem reduction' technique to the triangulation problem. The technique consists of dividing the original data

space into disjointed segments, and then solving the problem for each of these smaller segments. The technique is applied recursively on each segment and its sub-segments until each data space is sufficiently small for the application of a very simple algorithm. The triangulation of region R of Figure (3.14) can therefore be achieved by:

a) Splitting R into two sub-regions, R_1 and R_2, by creating a new boundary across the region.

b) Solving the triangulation problems for R_1 and R_2 separately.

Usually, when a region has to be divided there are numerous possibilities. The best of them is selected by the measure of a product called ' Π '. The factors of Π are signed meaning positive or negative distances of the boundary points from the proposed split line. Π is formed by calculating two partial products, one for each half of the split line as shown in Figure (3.15). Here two partial products are $\Pi_1 = d_1 d_2$ and $\Pi_2 = S_1 S_2$, therefore, $\Pi = \Pi_1 \Pi_2$. If , in forming the partial products, one of the elements is opposite in sign to that of its predecessor, then the split under consideration is rejected. All possible splits of the region are not considered, since a boundary of n points would require $O(n^2)$ evaluations of product Π. Examinations of splits is stopped when several valid results have been computed. The splits between ' opposite ' boundary points

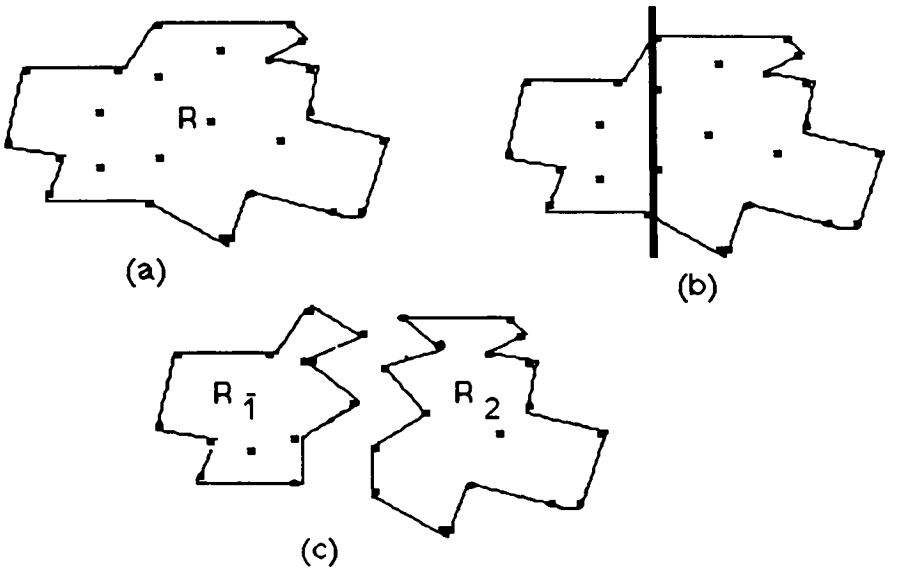


Figure 3.14 Subdivision of a Domain

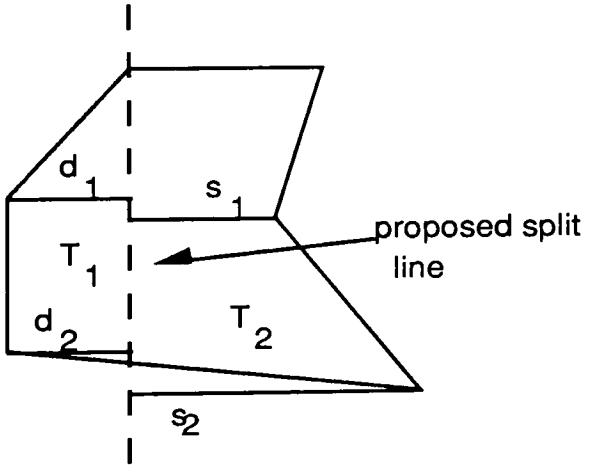


Figure 3.15 Information used for calculating Π

are considered first as these are more likely to give high values of Π. From the valid splits a choice is made for the split to be used. The selection is made for the split which maximizes the expression :

$$\Pi \cdot E^b$$

where b is the number of boundary points and E is the minimum of

a) Half of the average distance between the boundary points, an

b) The distance from the split line of the nearest interior points contained within any rectangle having the split line as the side.

The use of the above mentioned method tends to force the split to lie perpendicular to, and cut in half, the line joining the two boundary points that lie furthest apart. This eliminates the formation of elements with large aspect ratios.

When a particular spilt of a region has been chosen any interior point which lies close to the proposed split line is included as part of the new boundaries. This gives the zigzag effect, shown in figure (3.14c). The inclusion of points is to avoid the formation triangles with bad shapes. This is a very powerful and efficient algorithm which can automatically triangulate multiply connected regions, but its use is limited to two dimensions and axisymmetric shapes [31].

3.3.3.2 TRIQUAMESH ALGORITHM [32]

'TRIQUAMESH' is a technique for generating finite elements on arbitrarily shaped curves, areas and volumes. It is currently used with SDRC's 'I-DEAS' geometric modeling program and 'Supertab' finite element pre/post processing program.

This technique requires :

1. Boundary file of the geometry.
2. Mesh density values at various locations in the geometry.

The algorithm creates nodes on the boundaries based on element densities. This results in a polygon, each vertex of which is a node. This polygon is spilt into two by generating nodes along the "best splitting line" (discussed later). This results in two sub-polygons. The process is continued until all polygons are reduced to trivially simple polygons, which are triangles. These are the finite elements. The nodes are then repositioned to give a smoother mesh.

A surface with a hole can also be reduced to a polygon by introducing cuts. Figure (3.16a) shows a surface with two holes reduced to a polygon by introducing two cuts. The surface is then transformed to a plane and nodes are generated on the boundaries according to user defined densities. The nodes are joined to form a polygon. The polygon is split into two parts in such a way

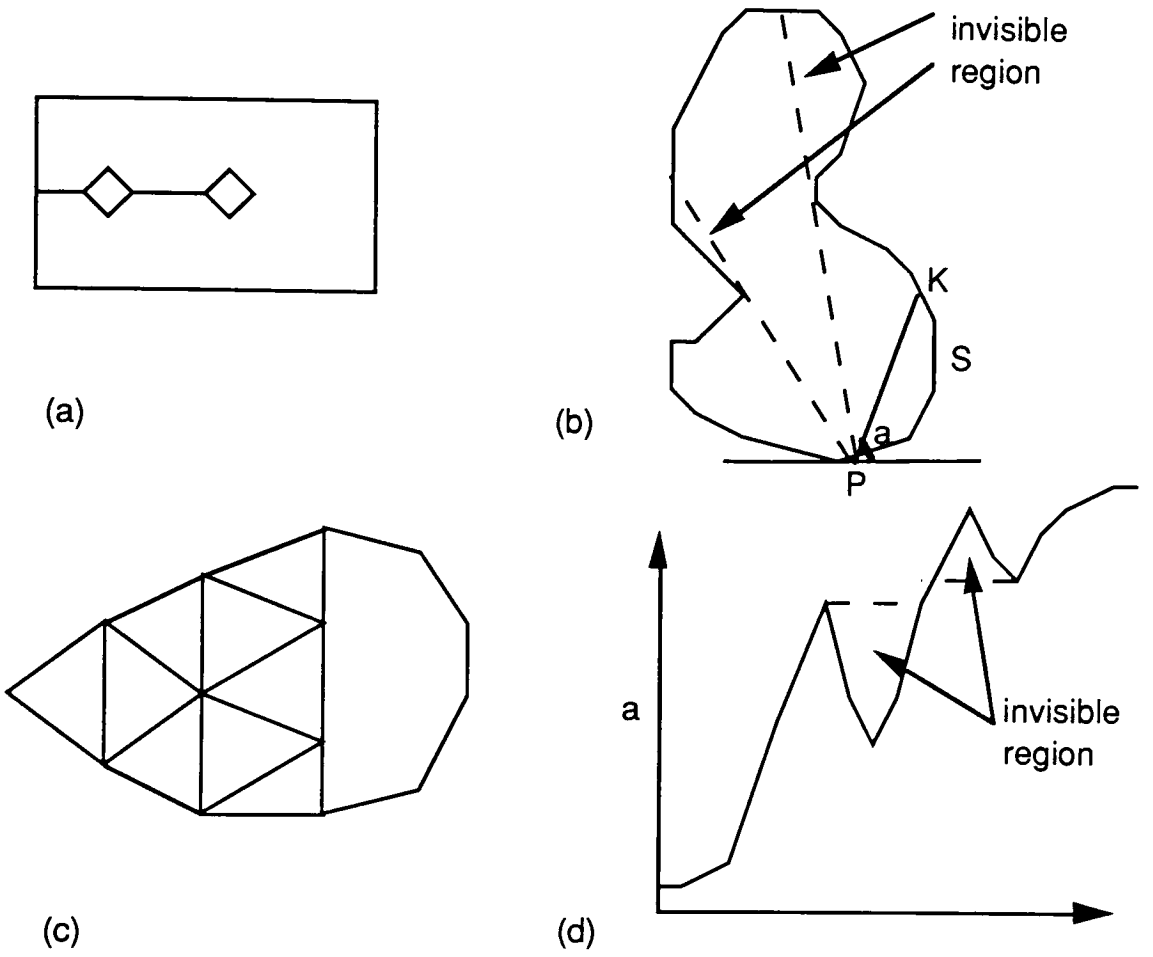


Figure 3.16 Triangulation by Triquamesh Technique

that the triangular elements formed have good triangular shapes.

The rules for spliting the polygon are as follows :

1. Split line, or the line dividing the polygon, should first be created through concave vertices in the polygon. The other vertices should be considered only if there are no concavities.
2. The split line should make angles as close as possible, to angles of 60 and 120 degrees with the boundary.
3. The split line should have the smallest possible length.
4. When creating nodes on the split line, based on the element density in this region, the line should have minimum number of nodes to satisfy the density criteria.

For every division there is more than one split line. Weighting or priority factors for each of the rules 1 to 4 are determined. They are used to select the optimum spilt line from among all the available lines. If the polygon has concavity, then only some of the split lines are valid. This is because some of them may not be wholly contained in the polygon.

The valid split lines are determined by creating a plot of visible points from a given starting vertex P on the polygon, shown in Figure (3.16b). The angle 'a' from P to any location K on the polygon is plotted against 's', the length of the

polygon from P to K. If the polygon is convex then the plot keeps rising. In case of concavities, the plot dips at such places, and the invisible areas (any vertex in these areas cannot form a split line), in the polygon are determined shown in Figure (3.16d). The process of splitting continues till one of the following conditions occur :

1. The polygon has only 3 vertices.
2. A polygon with a sharp angle is encountered. An angle of less than 80 degrees is a sharp angle. In this case, creating a split line through this vertex will at best yield an element with a 40 degree angle, which is considered poor. This possibility is ruled out. Then the split line has to pass through two neighboring vertices, one on each side. The split line is created without further search. This generates an element and at the same time reduces the number of vertices in the polygon by 1, shown in Figure (3.16c)

3.The remaining polygon is split using the method described earlier.

This technique is computationaly inefficient in three dimensions.

3.3.3.3 LO'S METHOD [33]

This mesh generation scheme can generate a triangular mesh within any simply or multiply connected planar domain. It can also mesh regions with

holes. The mesh generation time is independent of the topology and geometry of the given domain, but depends on the number of elements to be generated and the number of nodes present.

The boundary of the domain is represented by a disjoint union of simple closed loops of straight line segments. For simply connected regions there is only one loop, whereas for multi-connected regions there may be as many internal loops as the number of openings inside the domain.

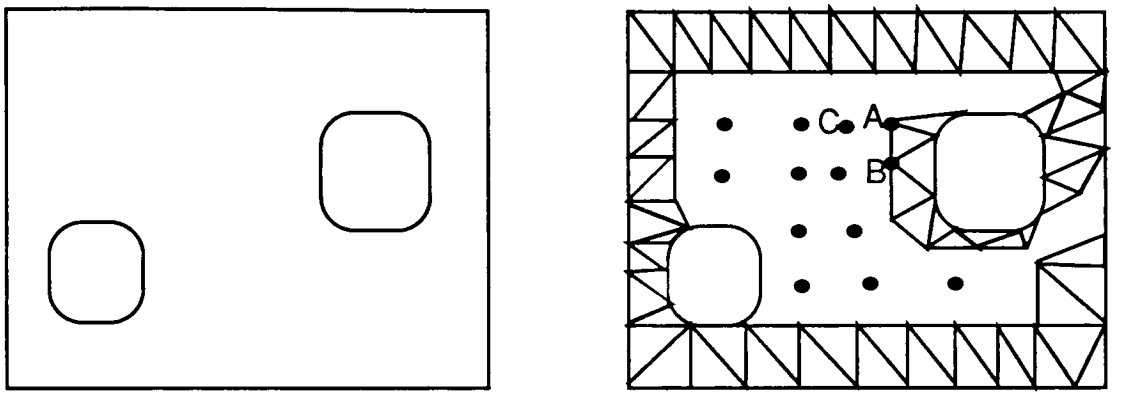
The node numbering on the exterior boundary is generated in counterclockwise order while the node numbering on the interior boundary or holes is generated in clockwise order. More than one connected regions can be meshed. It is not necessary to follow any sequence of boundaries while generating nodes on them. This flexibility allows one to generate a triangular element mesh from one region to another without concern for the common boundaries between them.

The algorithm first generates additional interior nodes according to the average nodal spacing of all boundary segments which make up the region to be triangulated. The program then connects all the nodes to form triangular elements with good aspect ratios. The triangulation scheme is discussed as follows :

The triangulation is designed to produce elements as near to equilateral triangles as the set of nodes permits. The complete set of boundary nodes is called generation front Π, and the set of all interior nodes is defined as Ω. When a scheme of counterclockwise order or numbering is assigned to the nodes on the exterior boundary and clockwise order assigned to the nodes on the interior boundaries, then the domain to be triangulated has an interior area always situated to the left of all connecting straight lines. This process has to take place between two consecutive boundary nodes in belonging to Π, as shown in Figure (3.17). In the begining of the triangulation the generation front Π is equal to the collection of the domain boundaries. While the given domain boundary always remains the same, the generation front changes throughout the process of triangulation, and has to be updated whenever a new triangle is formed.

The triangulation starts by selecting the last segment AB C Π, shown in Figure (3.17). The goal is to determine a node C c Π u Ω such that it lies to the left of directed line segment AB and ABC is in some sense optimal. Every time a node from either Π or Ω is used for triangulation the respective set is updated. Both the sets Π and Ω change continuously during the triangulation, and both are reduced to zero at the end, indicating the termination of the process.

A check to see if triangle ABC overlaps any previously generated triangles or any fixed boundary segments is unnecessary. This advantage not only gains great deal of computer time but also produces elements with good aspect ratios. This algorithm cannot be extended to mesh solids in three-dimensions.



Boundary to be triangulated Intermediate stage of triangulation

Figure 3.17 Triangulation by Lo's Method

3.3.3.4 SADEK'S ALGORITHM [34]

Given a domain of Figure (3.18a), with a number of nodes around its boundary the most well conditioned elements or triangles which can be formed at each corner of the domain are determined, shown in Figure (3.18b) . A corner is a boundary point at which the boundary angle Ø, is not equal to 180^0, shown in Figure (3.19a) Having computed this step, the elements formed are considered cut out from the original area and the same concept is applied to the remaining.

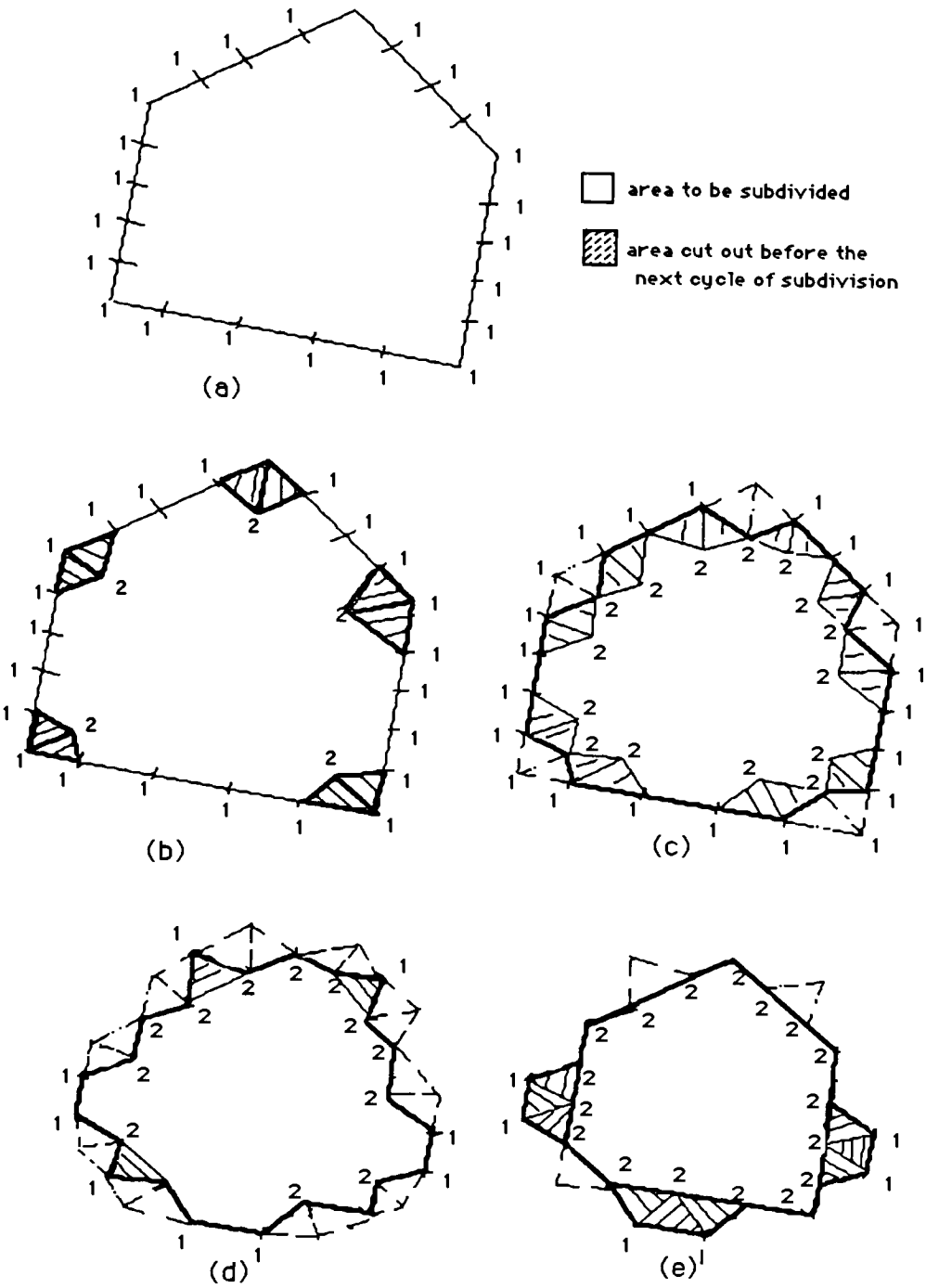


Figure 3.18 Stages of Element Generation in Sadek's Method

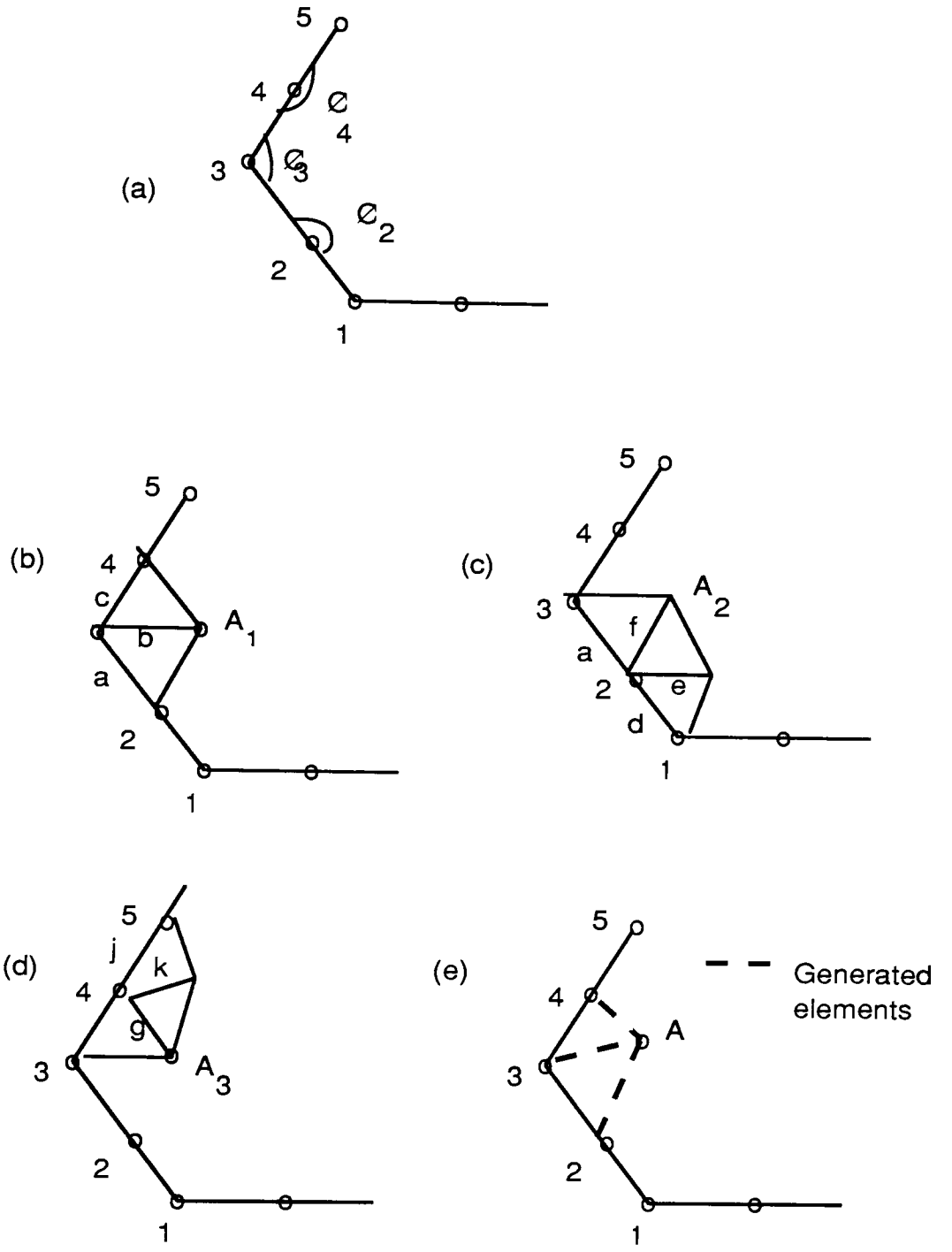


Figure 3.19 Generation of a Node by Sadek's Method

The nodes around the original boundary of the domain are of level unity. Subdivision at a corner node of level unity generates nodes of level two. Figure (3.18b) to Figure (3.18e) show some steps of subdivision, and the levels of the different nodes of the domain.

The subdivision of a domain is performed in succesive stages. In each stage a continuous boundary layer is cut out. Elements cut out to form level two belong to first boundary layer. The first stage is complete when all nodes bounding the area to be subdivided reach level two, shown in Figure (3.18e). In case of a domain with a number of inside holes, the nodes around the boundary of the domain and those around the boundaries of the holes are all considered of level unity. Subdivision is performed once by cutting a boundary layer around the outer boundary of the domain and once by cutting a boundary layer around the boundary of each hole.

The node generation procedure can be understood by considering the geometry of Figure (3.19a). The number of triangular elements that can be generated at a node ' i ' with an internal boundary angle $\emptyset_i$ is taken as nearest integer of $\emptyset_i/(\pi/3)$. The parameter $\emptyset_i/(\pi/3)$ is used to ensure that a suitable number of triangles are generated and each of them is as equilateral as

possible. Assume that the angle $\emptyset_3$ at node 3 is such that two triangular elements can be generated. Well conditioned triangles will be produced if the ratio a/b is equal to b/c. In other words length b is $(ac)^{0.5}$. Therefore, length b taken along the angular bisector of $\emptyset_3$ is the position A_1, of new node, shown in Figure (3.19b). This is the position of the newly generated node if the situation at node 3 is only taken into consideration. But the position of the new node will influence the shape of the elements formed at adjacent nodes, namely nodes 2 and 4. The procedure is repeated to find out the number of triangles and the new node position. The angle Ø2 at node 2 is 180°, thus three triangles can be formed at equal divisions of 60°, shown in Figure (3.19c). The condition that makes the three triangles well conditioned is

$$d/e = f/g = g/a$$

or

$$e = (d^2a)^{1.5} \text{ and } g = (da^2)^{1.5}$$

The length g determines another position A_2. This is the position of the new node if the situation only at node 2 is taken into consideration. The situation at node 4 indicates a third position A_3 (see figure 3.19d), with lengths

$$h = (j^2 k)^{1.5} \text{ and } l = (jk^2)^{1.5}$$

The best position is determined from A_1, A_2 and A_3 by finding the centroid of the triangle formed by them.

This algorithm gives elements with good shapes, but has no extension in three-dimensions.

3.3.3.5 BYKAT'S METHOD [35]

The algorithm is a recursive shape controlling the triangulation method. The method produces a node numbering which implies reduced bandwidth in the matrix assembly of the solution equations. This approach saves a substantial amount of time usually spent on renumbering nodes. The suggested data structure allows simple implementation of a nested dissection algorithm for an irregular simply- connected domain. The algorithm can be applied to polygonal regions of any shape provided they are convex. If the regions are non-convex, they are subdivided into convex shapes. This algorithm does not support extension in three-dimensions.

3.4 DELAUNY TRIANGULATION

In this thesis, Delauny triangulation (DT) has been chosen as the method for automatic meshing. Delauny triangulation can mesh surfaces in two-dimensions and can be extended to three dimensions [36]. To understand the

method of construction of DT it is necessary to understand the definitions of the terms ' tessellations ' and ' convex hull '. They are explained as follows :

Tessellations of a Plane

Subdivision of a plane into smaller regions is known as Tessellation of the region. The region to be divided, and the subdivisions, can be of any polygonal shape. The bold line of Figure (3.20) is the region to be subdivided and the light lines are various Tessellations. Dirichlet has done useful work in Tessellations [36]. In the rest of the text, Dirichlet Tessellations will be referred to as DIT. Delauny worked further on DIT to show that a non -triangular Tessellation of a convex hull can be triangulated. The DT can only triangulate a convex region.

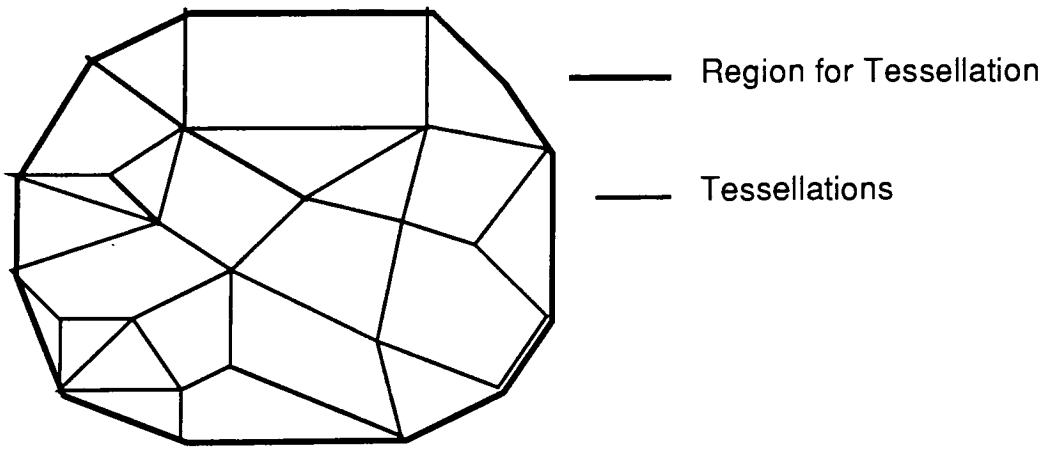


Figure 3.20 Tessellations of a Plane

Delauny triangulation is the dual of the Dirichlet Tessellation construct. Given the data for either the triangulation or tesselation, the unknown can be computed. The DIT construct has been known to mathematicians for 130 years and has been implemented in geography, geology, crystallography and other fields. In the early 1980's Cavendish, David Field, Frey [29] and group, started the development of a pre-processor by Delauny triangulation, from a CAD database. DT is ideally suited to finite element applications because a mesh can be created or modified dynamically, one node at a time. Each node operation produces only a local change in the mesh. DT can generate triangular elements that are ' best ' shaped for the given set of nodes.

Convex Hull [37]

Let S be a set of points and E^d be a 'd' dimensional space. The 'convex hull' of a set of points S in E^d is the boundary of the smallest convex domain in E^d containing S. In two dimensional space E^2 (d=2), a polygon is defined by a finite set of segments such that every segment extreme is shared by exactly two edges and no subset of edges has the same property, shown in Figure (3.21). The segments are the edges and their extremes are the vertices of the polygon. The number of vertices and edges are identical. A 'n-vertex' polygon is called an 'n-gon'.

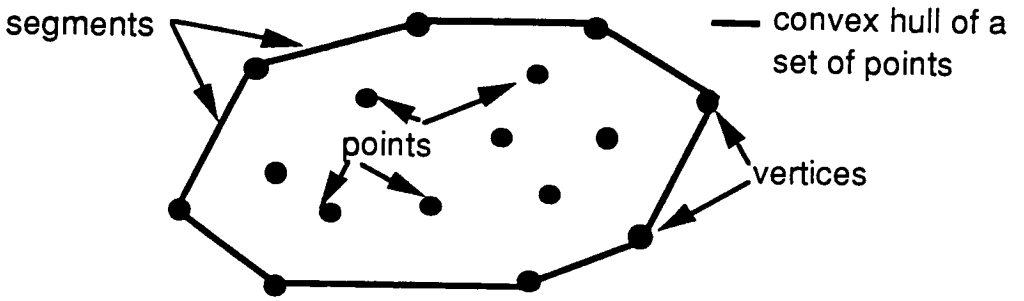


Figure 3.21 Convex Hull of a Set of Points

3.4.1 NEAREST NEIGHBOR SEARCH BY LOCI PROXIMITY [37]

Given a set S of N points in a plane. Let the universal set be U, then the set $M = U \cap S$. For any two points $\mathbf{p}_i$ and $\mathbf{p}_j$ of the set S, the set of points in the set M, closer to $\mathbf{p}_i$ than to $\mathbf{p}_j$ is the half plane including $\mathbf{p}_i$. This half plane is defined by the perpendicular bisector of the vector $\mathbf{p}_i\,\mathbf{p}_j$, shown in Figure (3.22a). Let us denote the half plane between the points $\mathbf{p}_i$ and $\mathbf{p}_j$ by $H(\mathbf{p}_i,\mathbf{p}_j)$. A point $\mathbf{p}_i$ is the nearest neighbor of a subset of set M if no other point $\mathbf{p}_k$ belonging to the set S lies in the half plane $H(\mathbf{p}_i,\mathbf{p}_j)$. The locus of points closer to $\mathbf{p}_i$ than any other point of the set S, which we denote by V(i), is the intersection of N -1 half planes, and is a convex polygonal region having no more than N -1 sides, shown in Figure (3.22a). Mathematically, the locus is defined as :

$$V(i) = \prod_{i \neq j} H(p_i, p_j). \qquad (17)$$

The half -planes partition the plane into convex polygons known as Dirichlet, Thiessen, or Voronoi polygons [38] , shown in Figure (3.22b). From now on, the given points in the set S, will be called nodes. Dirichelet tessellations assign to any point in the plane a subdivision containing a node which is its nearest neighbor. The point of intersection of the half-planes is called the vertex of DIT, shown in Figure (3.22b).

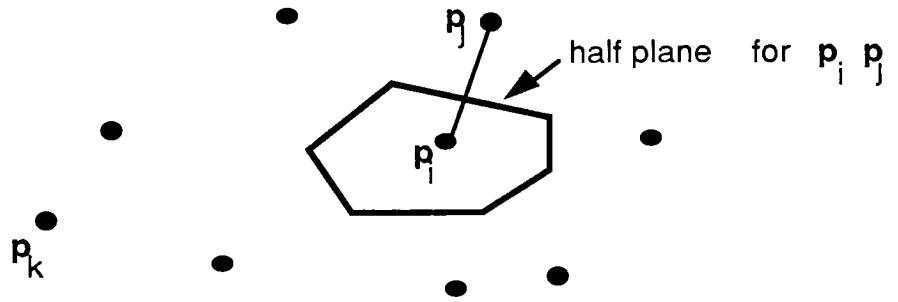


Figure3.22a Half Planes and Locus of Intersection of Half Planes

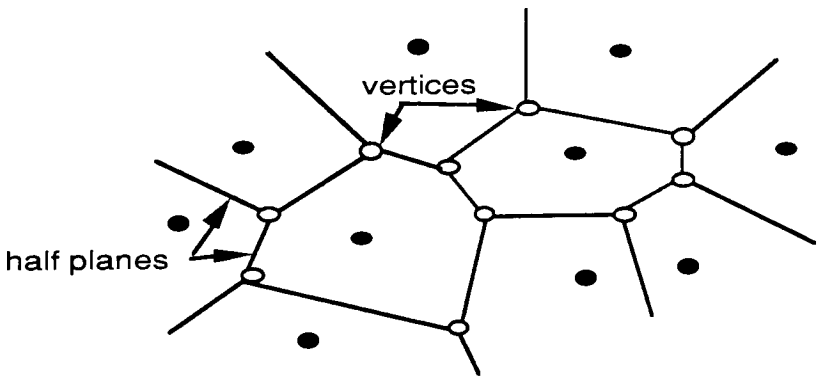


Figure 3.22b Vertices and Half Planes of Dirichlet Tessellations

The light lines in Figure (3.23) outline the half-planes induced by a set of 16 nodes. The intersections of these half-planes form the Dirichlet tessellation of the entire plane. If all the node pairs which have common half planes, are joined by edges, the result is the **Delauny triangulation of the convex hull of the nodes**, shown as dark lines in Figure (3.23). Node pairs joined by lines in the triangulation are contiguous. The dual existence of the two geometric constructs is obvious from the Figure (3.23). There is correspondence between the points (vertices, nodes) of one, and the subdivisions (tessellation, triangles) of the other. Given a set of nodes, and either tessellation, it is straight forward to compute the other. To date, most computation methods have focussed on the Dirichlet tessellation , however, as Watson [39] , has emphasized, the DT is easier to compute in two dimensions, and in three dimensions the method has an analogical extension.

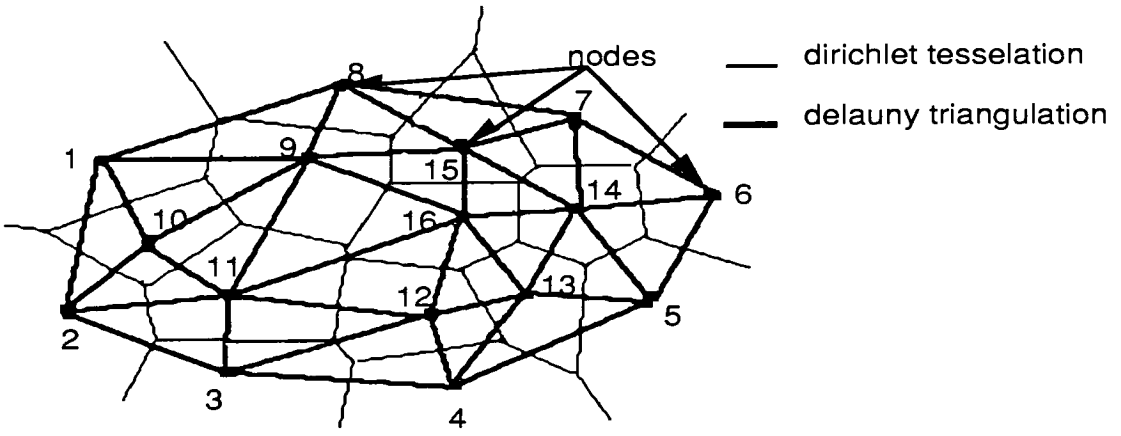


Figure 3.23 Delauny Triangulation with Dirichlet Tesselation

3.4.2 PROPERTIES OF DELAUNY TRIANGULATION

The properties of Delauny triangulation which make it a efficient algorithm are presented below :

3.4.2.1 OPTIMAL EQUIANGULARITY PROPERTY

A set S, of nodes may be triangulated in different ways. Figure (3.24) contrasts DT with another possible triangulation of the same set of nodes. In finite element applications it is desirable to have a triangulation which maximizes the equiangularity of the triangles where all the angles are as close to being equal as possible. Equiangularity of the triangles leads to finite elements with an aspect ratio close to unity. It has been shown that of all the possible triangulations of a given set of nodes, the Delauny triangulation forms triangles which are as equiangular as can be achieved [40]. This is called the optimal equiangularity property.

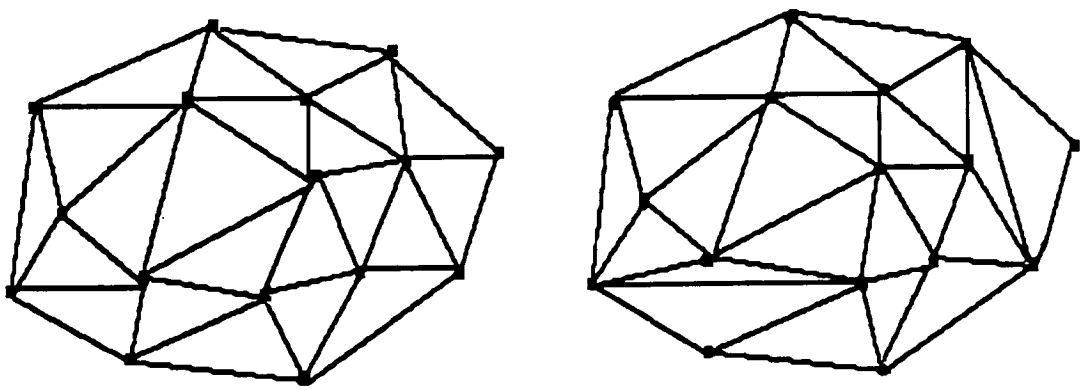


Figure 3.24 Delauny Triangulation and General Triangulation

3.4.2.2 EMPTY CIRCUMCIRCLE PROPERTY

Let a vertex of the Diriichlet tessellation be v. Assuming that no four nodes of the original set of nodes S, lie on a circle, it can be proved that :

For every vertex v of the Dirichlet tessellation of the set S, the circle C(v) contains no other nodes of the set shown in Figure (3.25), or in other words every vertex of the Dirichlet tessellation is the common intersection of exactly three half planes [37].

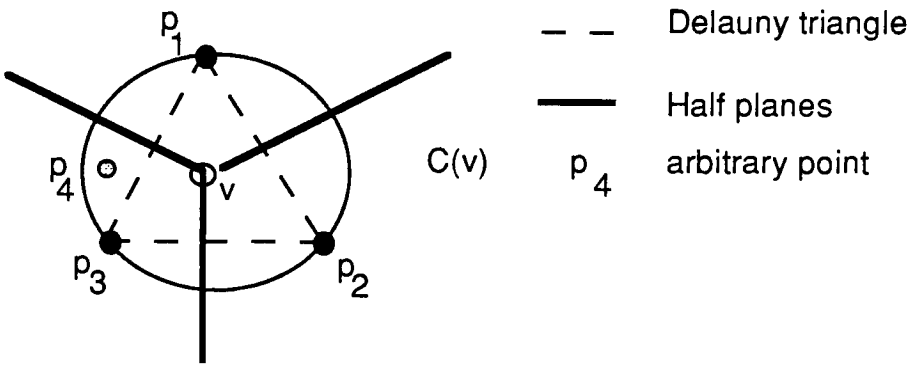


Figure 3.25 Empty Circumcircle of a Delauny Triangle

The vertex v is associated with a given Delauny triangle, namely, the triangle constructed from the nodes whose half-planes come together at that vertex. The three half planes of the Dirichlet tessellation which intersect at the vertex v are the perpendicular bisectors of the sides of the Delauny triangle. Therefore, the vertex is at the circumcenter of the DT, $\mathbf{p}_1\,\mathbf{p}_2\,\mathbf{p}_3$. The vertex is also equidistant from all three nodes of the triangle. Furthermore, since $\mathbf{p}_1$, $\mathbf{p}_2$, and

$\mathbf{p}_3$, are the three nodes of the set S, if the circumcircle C(v) contains some other node, say $\mathbf{p}_4$, then $\mathbf{p}_4$ is included in the half plane H($\mathbf{p}_3$, $\mathbf{p}_1$) . By definition of Dirichlet tessellation this is a contradiction. Therefore, in Delauny triangulation there is no node in the open circumdisk, the interior of the circumcircle of any triangle. This is a general property of a DT, and provides the basis of a convenient and efficient method of construction. The empty circumdisk also has an analogue in three dimensions and has been described by Watson. [41] Figure (3.26) shows the circumcircles associated with a set S. It was assumed that only three nodes of the set S can lie on a circumcircle. The case where more than three nodes lie on the circumcircle is discussed in the following section.

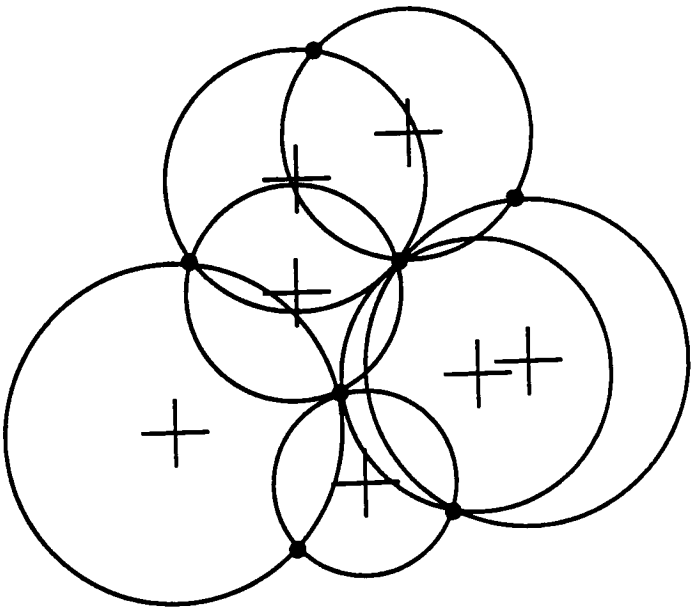


Figure 3.26 Empty Circumdisks of a Set of Nodes

3.4.3 DEGENRACIES OF REGULAR DELAUNY TRIANGULATIONS

When four or more nodes lie on the same circle, degenracy occurs, and the empty circumdisk property leads to no unique DT triangulation.

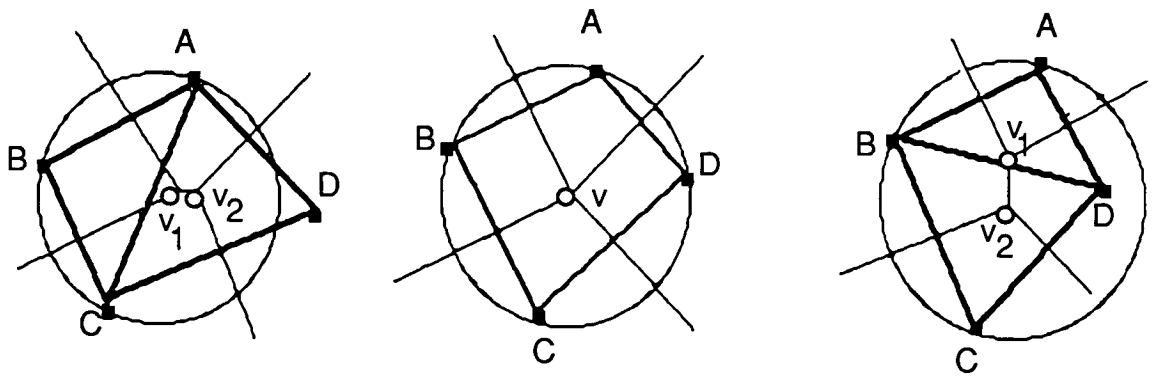


Figure 3.27 Degeneracy of Delauny Triangulation

The degenrate situation can be understood by studying Figure (3.27) above. The left side of the figure shows a normal Delauny triangulation of nodes ABCD (dark lines). The corresponding DIT are shown by light lines. The circumcircle associated with the nodes ABC is also shown. The node D lies outside the circumcircle. The DIT has two distinct vertices, each of which is the intersection of the three half planes. The right hand side figure is a DIT with two distinct vertices, but their contiguities are different from the case on the left. Consequently the Delauny triangulation is different. In both cases the DT is unique. In the quadilateral ABCD the placement of the fourth node relative to the circumcircle determines which of the two possible diagonal connections can

be made to form the DT. In the center figure, the fourth node lies precisely on the circle. Here the two vertices coalace into a single degenerate vertex which is the intersection of the boundary segments and the corners of the four half planes. The Dirichlet Tesselation is still well defined but the DT is not unique as the triangulation can be done by joining any of the two diagonals. The choice is made by determining which diagonal generates 'good' set of triangles.

Degenracies can occur in a large set of nodes whenever a localized cluster of nodes lie on a common circle. In such cases DT is done by breaking down the existing polygon.

In general, the number of boundary segments of the DIT which intersect at a given vertex is equal to the number of nodes which lie on the circle. The degenerate Delauny tesselation is the polygon formed by connecting the nodes in order around the circle.

3.4.4 DELAUNY TRIANGULATION ALGORITHMS

Classical algorithms for constructing Delauny triangles or, equivalently, its dual graph DIT can be classified into :

Incremental Algorithms [41] [42] [43] These algorithms construct the tessellation by starting from any interior node or from the boundary of the domain, and

stepwise add the remaining nodes of the given data set to the subdivision.

Divide and Conquer Algorithms [31] [44] These algorithms recursively split the set of the data points into equally sized subsets until elementary subsets are obtained, and then merge the resulting tessellations piecewise.

These algorithms can be further classified into one-step or two-step methods depending on whether they produce the final optimal triangulation in single step, or they build an arbitrary triangulation first, and then modify it by an iterative application of an optimization criterion.

Another approach to the construction of DT consists of a two step process. First, Dirichlet tessellation of the given data set is carried by employing either the divide and conquer or incremental approach. Secondly, the Delauny triangulation of the DIT is done. This scheme is space inefficient due to the temporary data structure for the DIT. This structure contains more information than needed to compute the triangulation.

The method selected in the present work is the dynamic incremental algorithm known as Watson's algorithm. This method can triangulate only the convex hull with no internal holes. In this thesis the algorithm is modified to work with non convex shapes with or without holes. Watson's algorithm and the modifications are discussed in chapter 4.

CHAPTER 4

AUTOMATIC MESH GENERATION BY MODIFIED WATSON'S METHOD

A mesh generator is a Pre-processor with limitations. The Pre-processor has all the functions required to prepare a finite element model for analysis as discussed in Chapter 1. A mesh generator can only discretize the geometric model into finite elements. It does not have the capabilities of modeling boundary conditions and physical properties of the finite element mesh. In Chapter 1 it was stated that the objective of this thesis is to develop an automatic mesh generator. To achieve this goal extensive research was conducted. The first three chapters reflect this research effort. To recapitulate the contents of the previous chapters a brief outline is given below.

Chapter 1 reviews the history and functions of Pre-processors.

Chapter 2 reviews geometric modeling and the geometric modeling methods.

Chapter 3 reviews the existing automatic mesh generation methods.

Watson's algorithm for Delauny triangulation can mesh the 'convex hull' of a given set of nodes. It cannot mesh non-convex surfaces or surfaces with holes. This algorithm has been modified to mesh convex and non-convex surfaces in two dimensions. The modified algorithm can also mesh surfaces with or without

holes. The mesh generator developed in this thesis can create triangular elements which have acceptable shapes as finite elements. Since the finite elements created by the algorithm in this thesis are triangular in shape, the terms 'triangle' and 'element' are synonymous.

To mesh a geometry, first the nodes have to be generated on and inside the geometry boundaries. These nodes are then combined to give elements. The possibility of generating good elements is dependent on the placement of nodes. A node generation algorithm has been written which can give nodes for potentialy well shaped elements. The modified Watson's algorithm then generates a mesh from the set of nodes. The density of the elements is controlled by a density parameter in the node generation algorithm. In this chapter first the node generation algorithm is presented, followed by Watson's Delauny triangulation algorithm and the modifications applied to it. Finally, some results and conclusions are presented.

4.1 NODE GENERATION

The following steps provide a discription of the pseudo code for the generation of boundary and interior nodes on and in a given domain, shown in Figure (4.1).

1. Sort out y_{min} and y_{max} of the domain.

2. Imaginary horizontal lines at a distance y_{max} - mD are drawn between y_{min}

and y_{max} across the domain. Here 'D' is the element 'spacing' or density parameter specified by the user. The parameter m is defined by

$$m = 0, n \text{ by } 1$$

where n is an integer defined by

$$n = (y_{max} - y_{min})/D$$

let

$$Y_L = y_{max} - mD$$

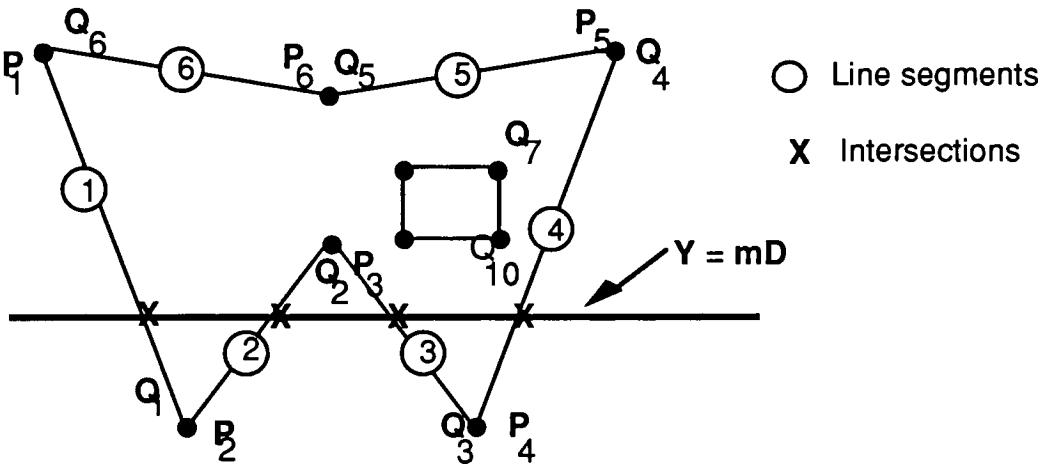


Figure 4.1 Node Generation

3. Determine the intersections of each horizontal line with the region. Let S $\{P_iQ_i , i = 1,N\}$ be the set of line segments representing the region boundaries. The intersection points between each horizontal line and the region are determined by considering the line segments of the set S one by

one. As an example consider line segment P_1Q_1 of Figure (4.1). Let $X = X(P_1)$, $Y_1 = Y(P_1)$, and $X_2 = X(Q_1)$, $Y_2 = Y(Q_1)$. There is intersection only if

(i) $(Y_1 - Y_L)(Y_2 - Y_L) < 0$

or,

(ii) $(Y_1 - Y_L)(Y_2 - Y_L) = 0$ and $(Y_L > Y_1$ or $Y_L > Y_2)$

4. Each horizontal line must cut the domain in an even number of points, and the intersection points are arranged in ascending magnitude of X.
5. Assume there are 2n cuts between a particular horizontal line and the segment set S. The cuts are considered two by two, begining with the first and second cuts. No nodes are placed at the intersections, interior nodes are generated between each pair of cuts by placing them a distance D from each other.
6. After the first two cuts, the process is continued with the third and the fourth cut, till 2n-1 th and 2nth cuts. Steps 1-6 are repeated for all horizontol lines.
7. Boundary nodes are generated by placing them at and between vertices of each line segment of the set S. The spacing for boundary nodes can be user defined or taken as D by default.

To avoid triangular elements with small interior angles, a check is made

while generating boundary nodes to see if an interior node is less than :

$$(X - P_i)^2 + (Y - Q_i)^2 < C^2$$

where (X,Y) is the node to be generated, (P_i , Q_i) is an interior node, and C is a constant dependent on the average element size D of the interior region. In this algorithm C = 0.7 D. If there is an interior node closer than the above criteria, it is deleted in preferance to the boundary node. This insures that no interior node is too close to the boundary to give an element small interior angles.

4.2 WATSON'S DELAUNY TRIANGULATION ALGORITHM

First, Watson's algorithm for creating Delauny triangles is presented. Then the advantages and disadvantages of this algorithm are discussed followed by the modifications to overcome the disadvantages. For a given set of nodes the following steps are carried out :

Step1 Create a triangle which encloses all the given nodes, see Figure (4.2). This triangle is called the 'enclosing triangle'. In this thesis the enclosing triangle is created by bounding the set of nodes by a rectangular box. The vertices of the box are at a relatively large distance from the extreme nodes. The mid point of the top side is joined to the ends of the bottom side to form the enclosing triangle.

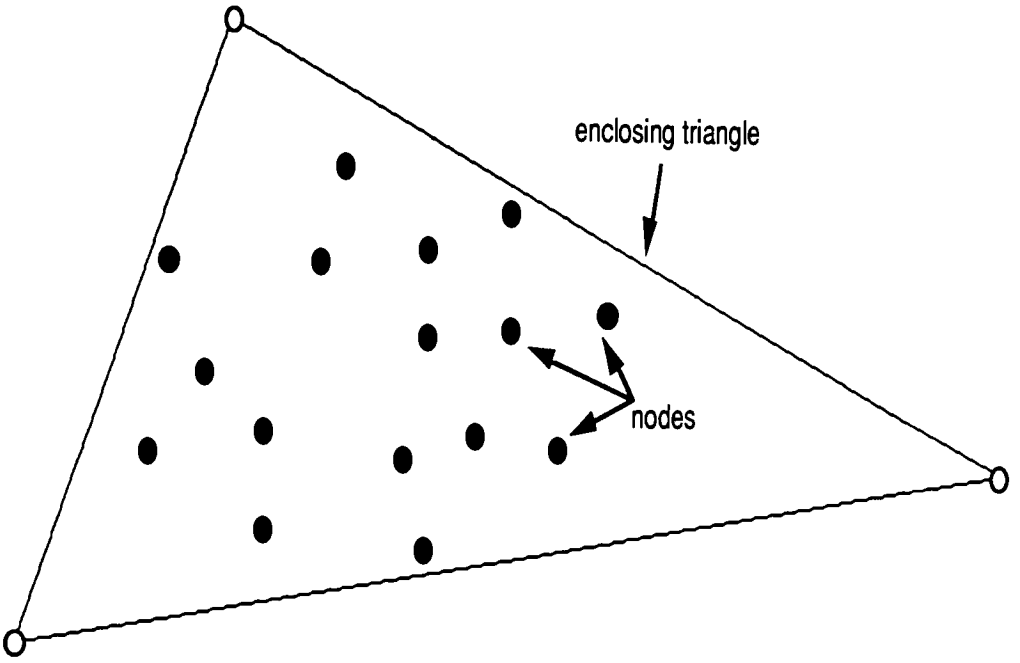


Figure 4.2 The Enclosing Triangle of Nodes

Step2 Construct the Delauny triangulation of the vertices of the enclosing triangle and one of the boundary nodes, shown in Figure (4.3). Maintain a master list of the triangles and their circumdisks (circumdisks are explained in section 3.4.2.2). The circumcenter and circumradii of each circumdisk is calculated and stored with the master list.

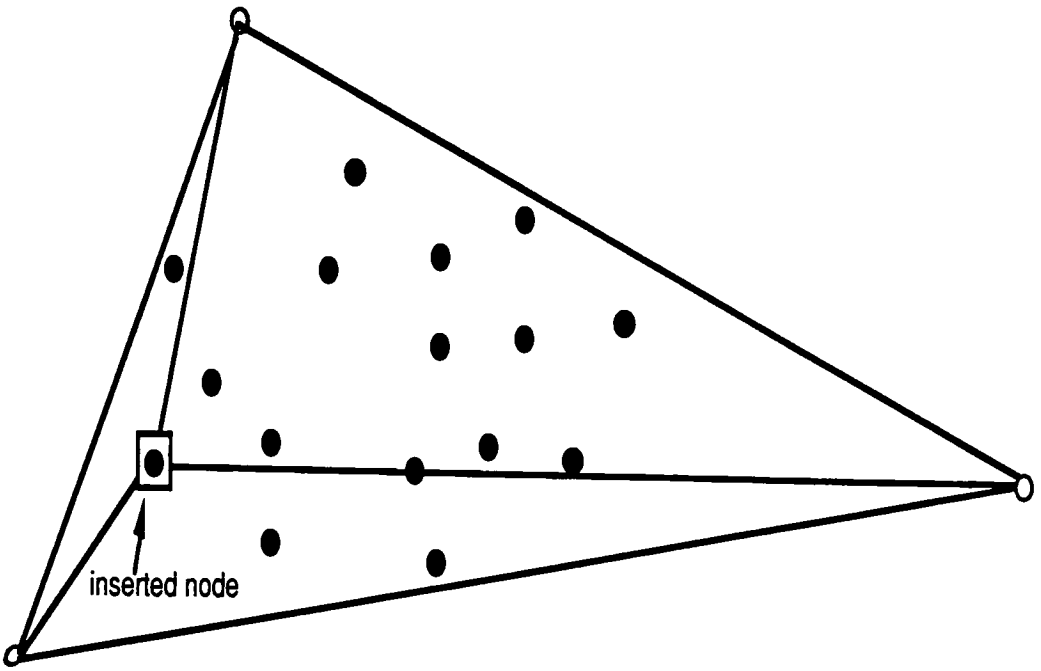


Figure 4.3 Delauny Triangulation of a Node

Step3 Insert the rest of the boundary nodes into the triangulation one at a time as follows:

a) Determine which existing circumdisks contain the given node. This is done by comparing the circumradius of each circumcircle with the distance between the node to be inserted and the circumcenter of that circumcircle, as shown in Figure 4.4.

b) Create a list of triangles associated with the circumcircles determined in step 3a. The union of these triangles is called the 'insertion polygon', see Figure (4.5).

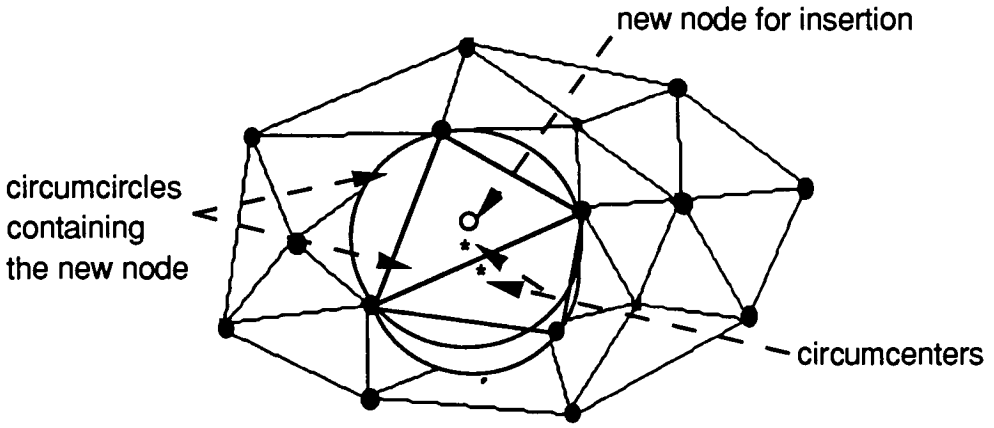


Figure 4.4 A Node for Insertion and Empty Circumcircles

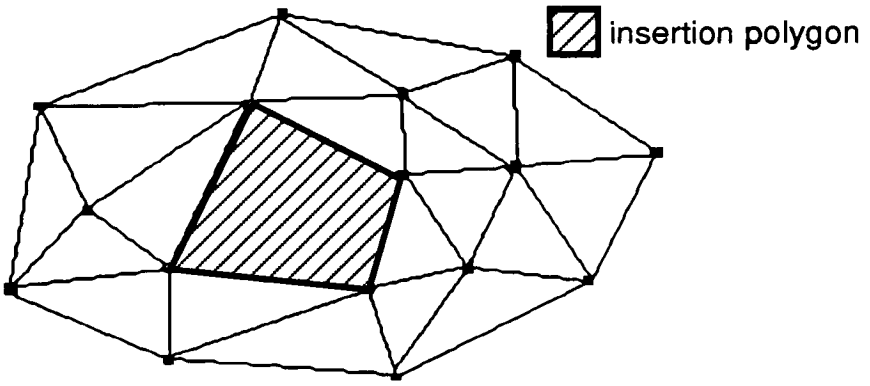


Figure 4.5 The Insertion Polygon

c) Create a list of boundary edges of the insertion polygon.

d) Create new triangles filling the insertion polygon by connecting the new node to the boundary vertices of the insertion polygon.

e) Delete from the master list the triangles and the circumdisks determined in step 3a and step 3b, see figure 4.6.

f) Add to the master list all triangles and their associated circumdisks created in step 3d.

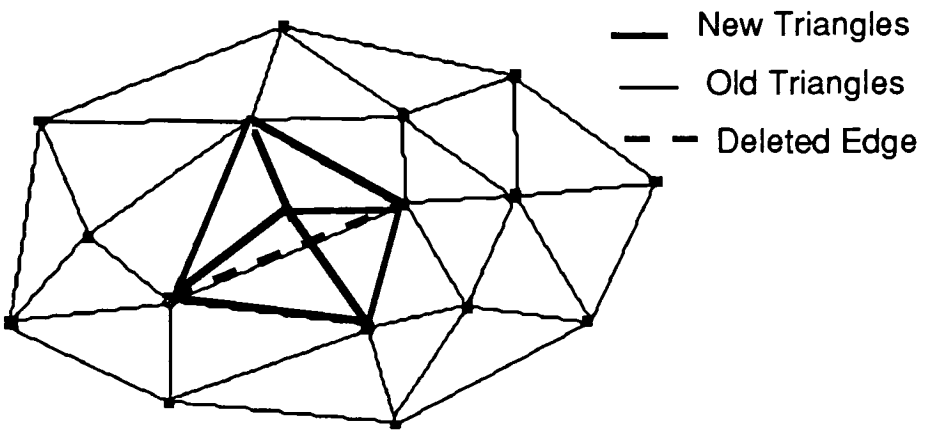


Figure 4.6 New Triangulation

Step 4 Repeat step 3 with the remaining boundary nodes followed by the interior nodes.

Step5 Remove from the master list all triangles which share a vertex with the 'enclosing triangle'. The union of the remaining triangles in the master list is the Delauny triangulation of the convex hull of the nodes.

Watson's algorithm cannot triangulate a non-convex surface and a surface with holes, shown in Figure (4.7). The order of compute time for a two dimensional mesh is 0(nlogn), where n is the number of nodes.The algorithm triangulates the convex hull of the nodes. As seen in Figure (4.8), this algorithm cannot identify which of the Delauny triangles are not part of the geometry. In Figure (4.8) triangles 1and 2 are inside the hole and triangle 3 is outside the

geometry. Therefore, all three are 'external triangles'. Triangles lying entirely outside the geometry or inside the holes are external triangles. A modification is done on this algorithm in order to identify external triangles to and discard them. The modified algorithm can triangulate geometries with irregular shapes and with holes.

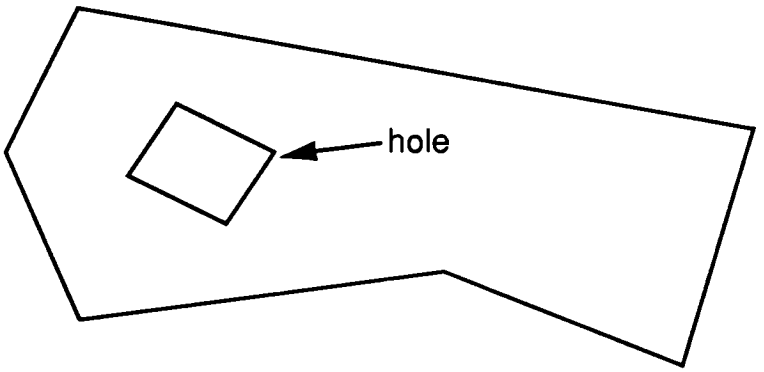


Figure 4.7 A Non-Convex Geometry with Hole

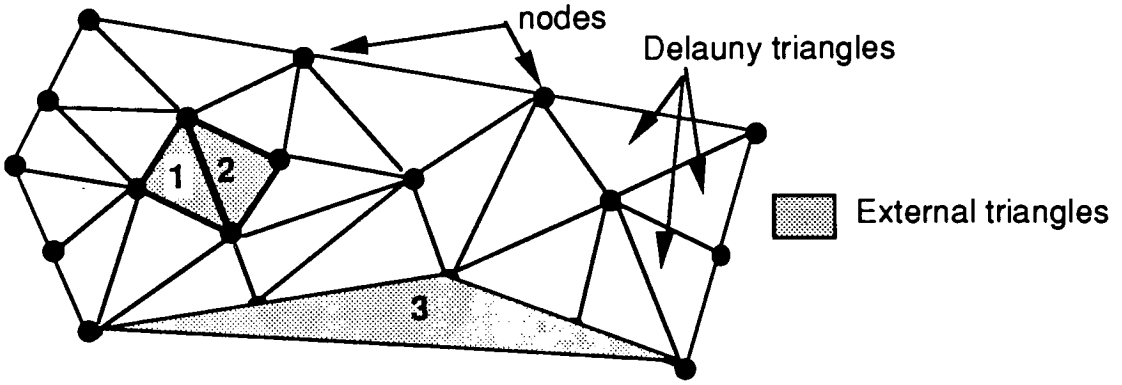
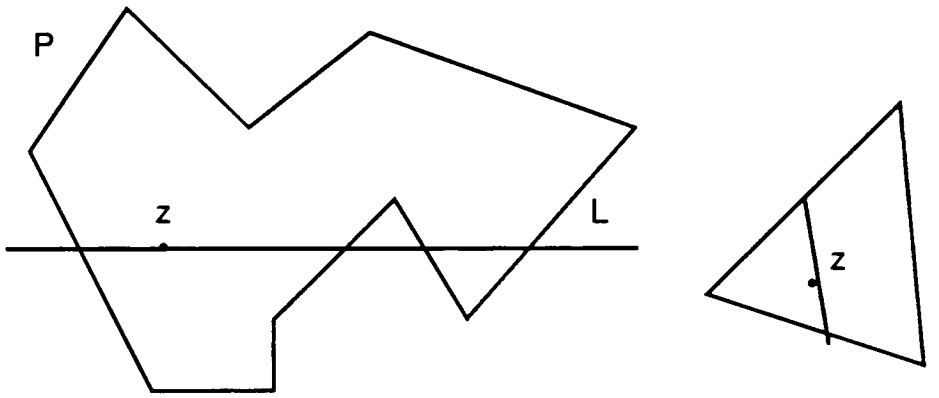


Figure 4.8 Delauny Triangulation by Watson's method

4.3 MODIFICATIONS OF WATSON'S ALGORITHM

Delauny triangulation of the geometry by Watson's method is carried out first. After the convex hull of the set of nodes is meshed or triangulated, 'external triangles' are identified and removed by application of a simple procedure.

Consider a polygon P made up of line segments, shown in Figure 4.9. Whether a point 'z' is external or internal to the polygon P can be cheked as follows:



(a) Location of a point in a polygon (b) Placement of point z in a triangle

Figure 4.9 Identifying External Triangles

Let a horizontol line L pass through the point z. If L does not intersect P, then z is external. Assume that L intersects P and consider that L does not intersect any vertex of P. Let M be the number of intersections of L with the segments of P to the left (or right) of z. Since P is a closed or bounded polygon, the left

extremity of L lies in the exterior of P. Consider moving right on L from $-\infty$, towards z . At the left most crossing with the boundary of P, one moves into the interior. At the next crossing one is outside. Therefore, z is internal only if M is odd and vice-versa. The pseudo code of the algorithm is as follows :

```
begin  M : = 0 ;
        for i : = 1 until  N do if edge (i) is not horizontal then
                    if ( lower extreme of edge (i) intersects L to the left of z )
                       then  M : =  M + 1;
       if ( M is odd ) then z is internal else z is external
end.
```

In the modified algorithm of this thesis the polygons are the external and internal boundaries of the geometry. Triangular elements associated with the external and internal boundaries are saved in two seperate lists. The z point of each triangular element is located at the mid-point of the line joining the mid-points of any two sides, as shown in Figure (4.9b). Elements associated with the external boundary are checked to determine if they lie outside the geometry, and discarded. Elements associated with the internal boundaries are checked to determine if they lie inside the geometry, and discarded.

4.4 RESULTS

In the automatic mesh generator developed in this thesis the node density and consequently the element density are defined by the parameter 'D' explained in section 4.1 of this chapter. This parameter 'D' is called 'spacing parameter' in the algorithm. The spacing parameter has a range between 0 and 100. A default value of 35 is given to 'D' in the node generation algorithm. The user has the option of changing the default value while generating boundary nodes and interior nodes. Some results of the modified Watson's Delauny triangulation method are presented now.

A square with nodes at the corners is shown in Figure (4.10a). This is a case where four nodes lie on a circumcircle. The geometry is meshed with elements 1 and 2 by connecting nodes 2 and 4. Nodes 1 and 3 can also be connected to form a valid mesh, shown in Figure (4.10c). There is no unique Delauny triangulation, as was discussed in chapter 3. The fourth node on the circumcircle is known as the degenerate node. The modified algorithm in this thesis is biased to consider the degenerate node as 'inside'. The fourth node is inside the circumcircle as shown in Figure (4.10b). The mesh is generated by connecting nodes 2 and 4. The fourth node is 'outside' the circumcircle in Figure (4.10c). Therefore, the mesh is formed by connecting nodes 1 and 3.

Delauny triangulation of a 'convex hull' of a set of nodes is shown in Figure (4.11). The circumcircles associated with the Delauny triangles of Figure (4.11a) are shown in Figure (4.11b). Each circumcircle is 'empty'. The 'cross hair' show the circumcenters of these circumcircles.

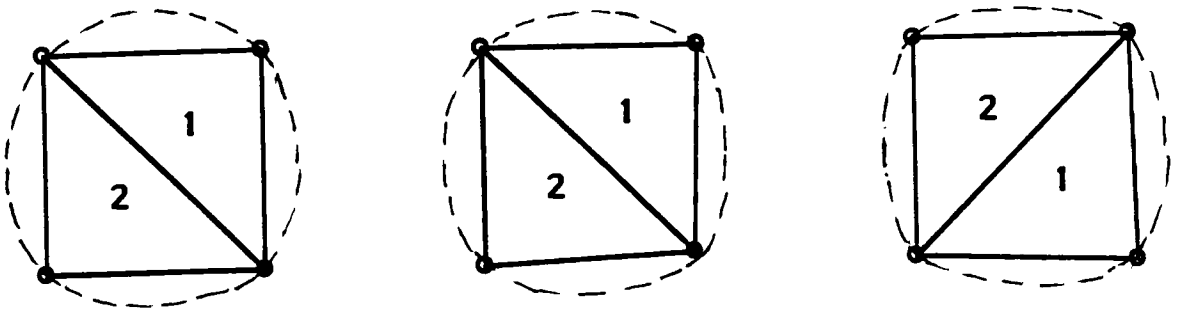


Figure 4.10 Delauny Triangulation of a Square

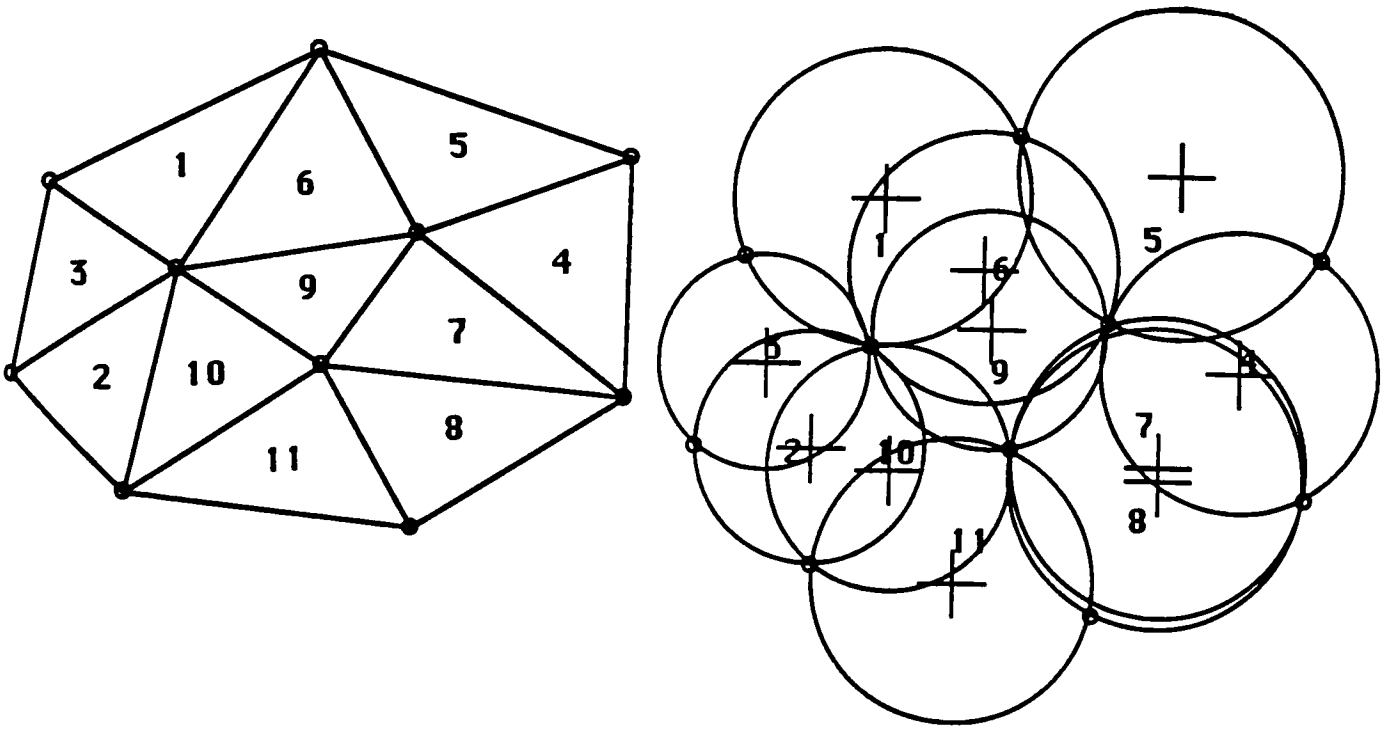
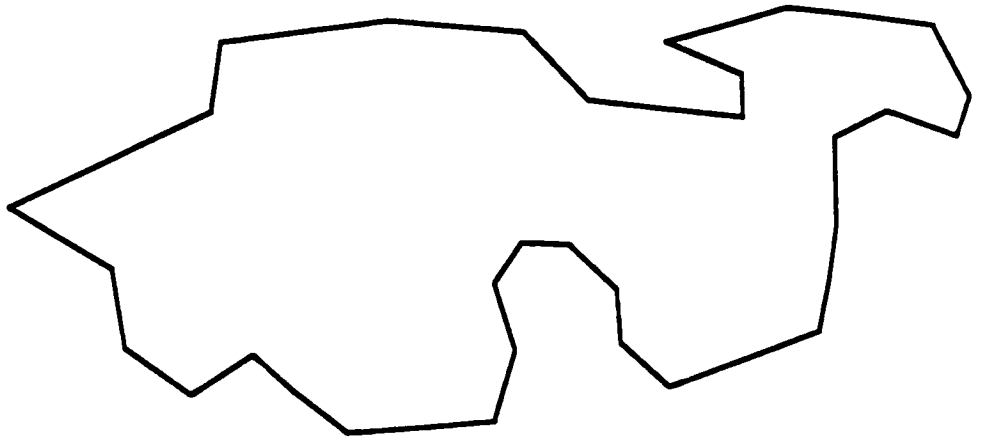


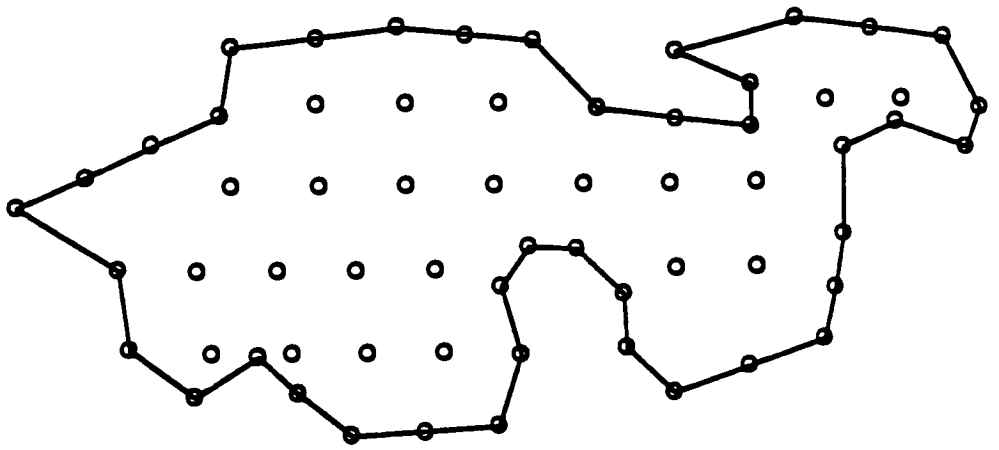
Figure 4.11 Triangulation of a 'Convex Hull'

A geometry with concavities is shown in Figure (4.12a). Boundary nodes and interior nodes are generated with the default spacing parameter 'D', shown in Figure (4.12b). The mesh of the set of nodes is generated by 'create mesh' command in the program. The corresponding mesh is shown in Figure (4.13a). Undesired nodes can be deleted before the mesh generation is carried out. The algorithm also has the flexibility of deleting interior nodes after the mesh generation. The high-lighted node of Figure (4.13a) is deleted after the mesh is generated. Re-meshing is done in the local region of the deleted node, shown in Figure (4.13b). A node can be added in the interior of the geometry after the mesh generation. The high lighted node of Figure (4.13c) is added to the set of nodes of Figure (4.13b). For deleting or adding a node after mesh generation the re-meshing algorithm finds the 'insertion polygon' without a node or with the new node as the case may be and creates a new set of triangles. The insertion polygon is explained in section 4.2 of this chapter. This feature of deleting and adding nodes after mesh generation provides the user the control over the final set of finite elements. The algorithm is time and cost efficient. It re-meshes a small set of local elements without disturbing the entire mesh, which can have a large set of elements.

A high density of a set of nodes and their triangulation are shown in Figure (4.14a) and Figure (4.14b) respectively.

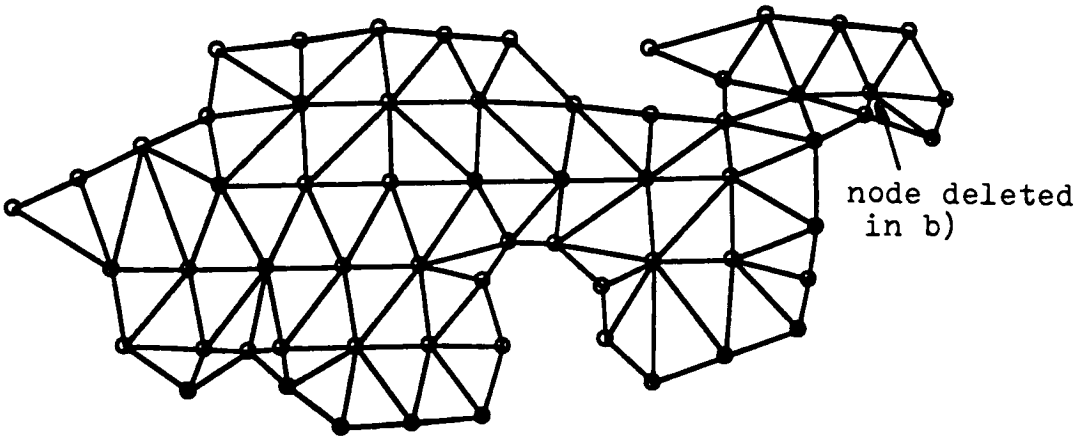


(a)

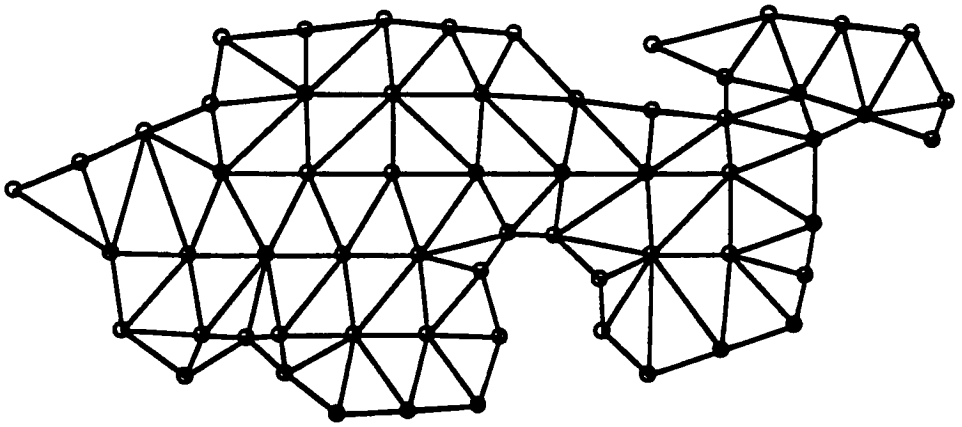


(b)

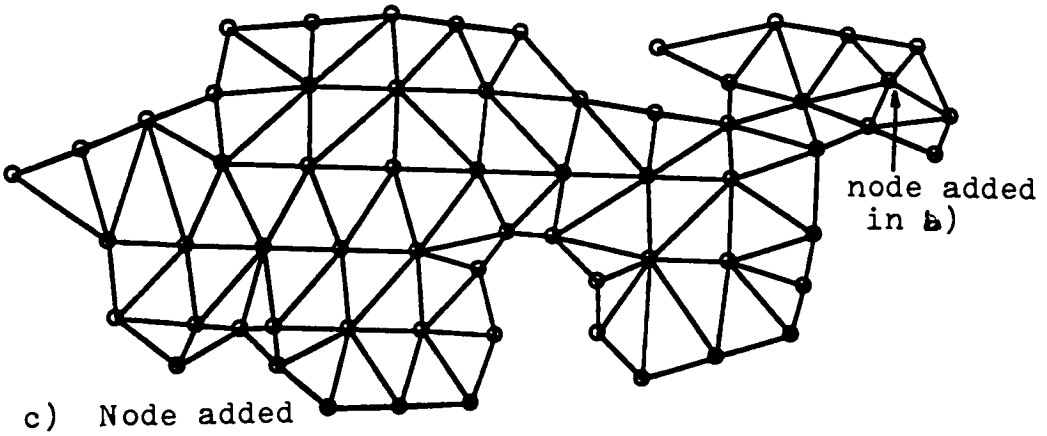
Figure 4.12 Node Generation on an Irregular Boundary



a) Automatic Mesh

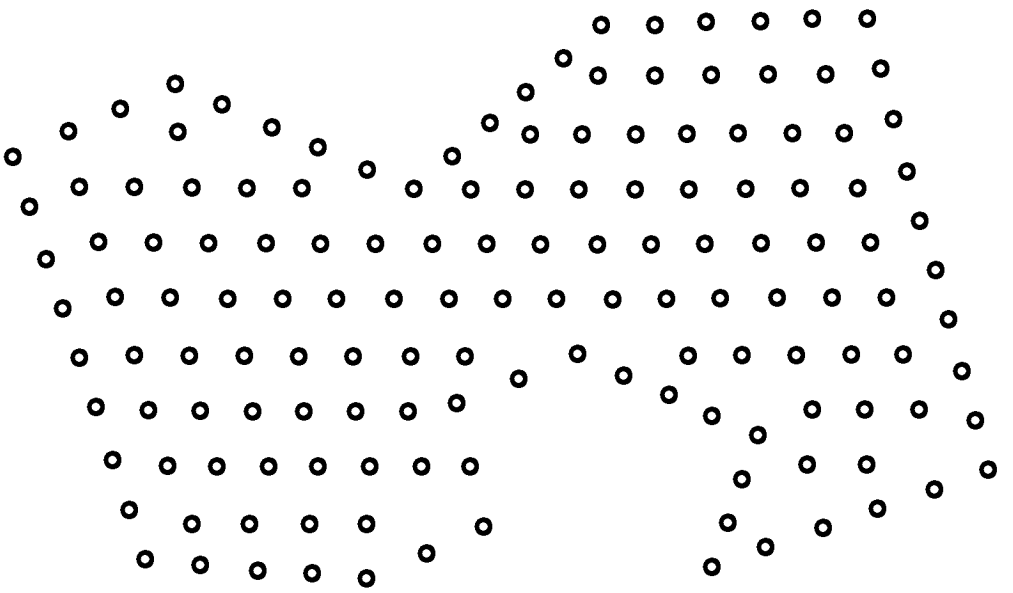


b) Node Deleted

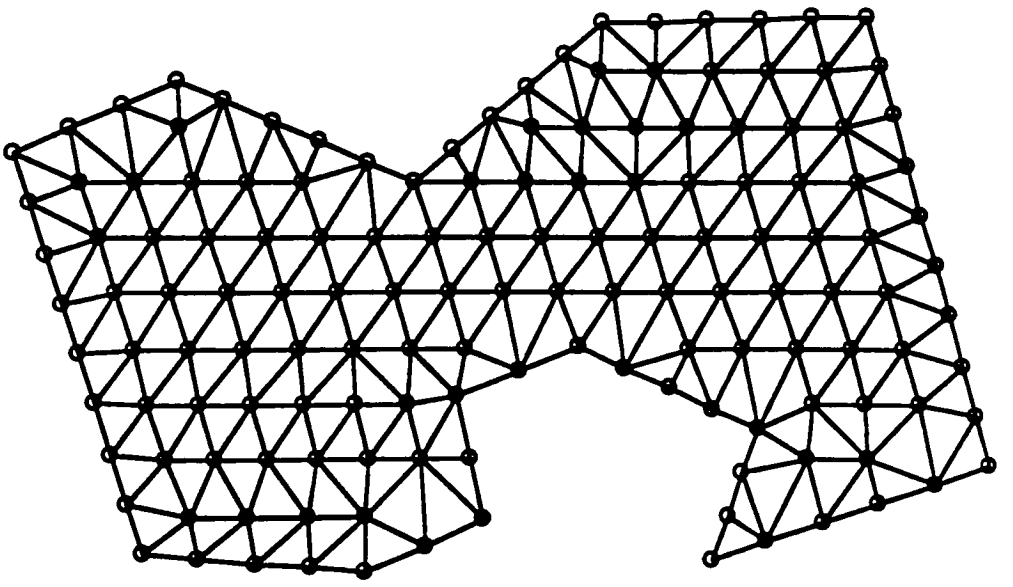


c) Node added

Figure 4.13 Automatic Meshed and Remeshed Elements



a) Generated nodes



b) Generated mesh

Figure 4.14 High Node and Mesh Densities

The modified algorithm can mesh surfaces with holes. In this case the convex hull of the geometry is meshed first, then the external triangles are identified correctly by the 'delete external triangle' algorithm discussed in section 4.3 of this chapter. The external triangles are discarded from the master list of the elements. An irregular geometry with a hole is meshed, shown in Figure (4.15). This geometry was discussed in section 4.2 of this chapter. It was shown that Watson's Delauny triangulation algorithm cannot identify the 'external triangles'. The modified algorithm can mesh any irregular shape with or without holes. An irregular geometry with four holes of different shapes is shown in Figure (4.16a). The final mesh of the geometry is shown in Figure (4.16b).

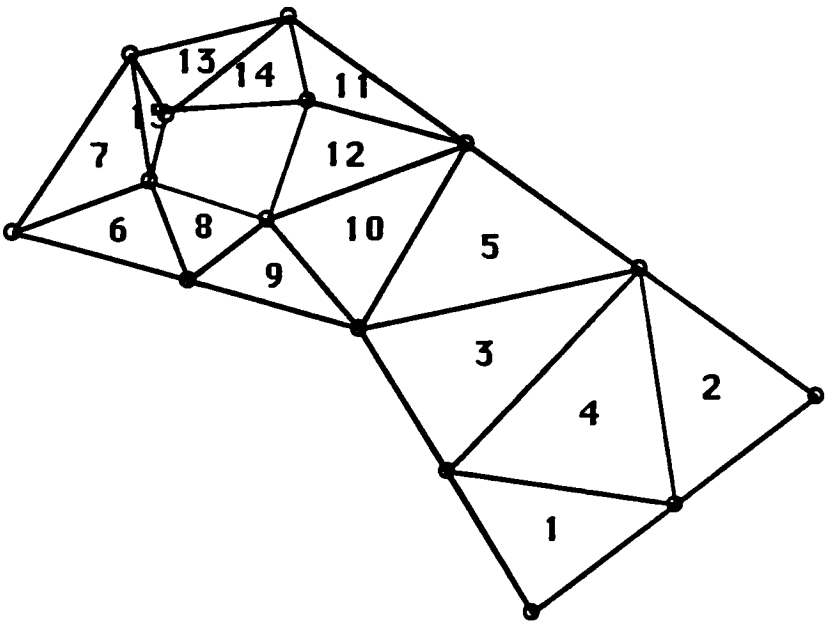


Figure 4.15 Mesh of a Geometry with a Hole

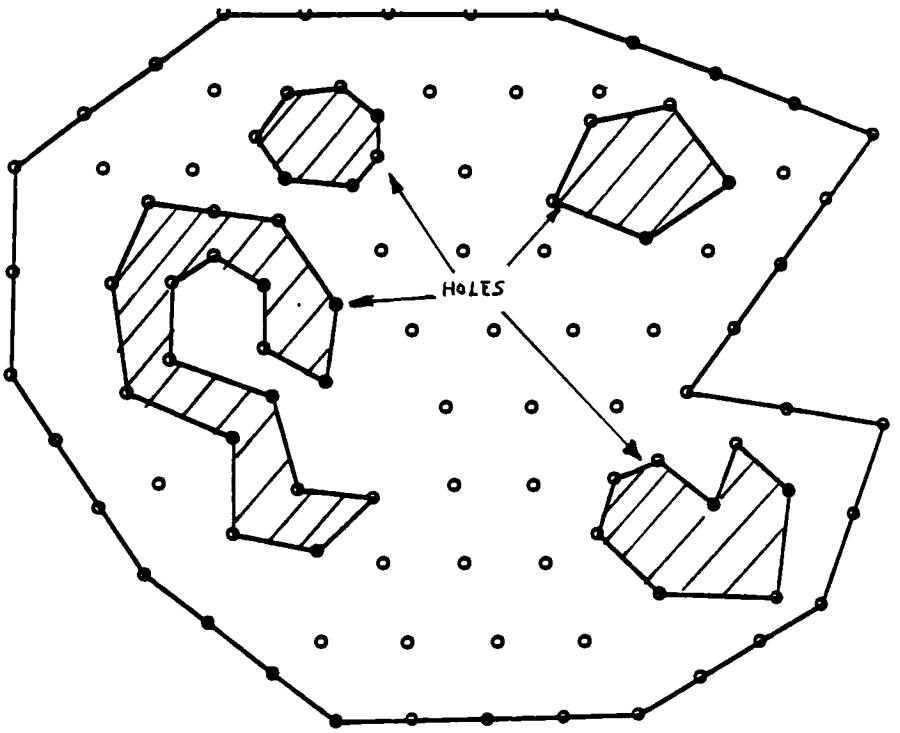


Figure 4.16 An Irregular Geometry with Holes

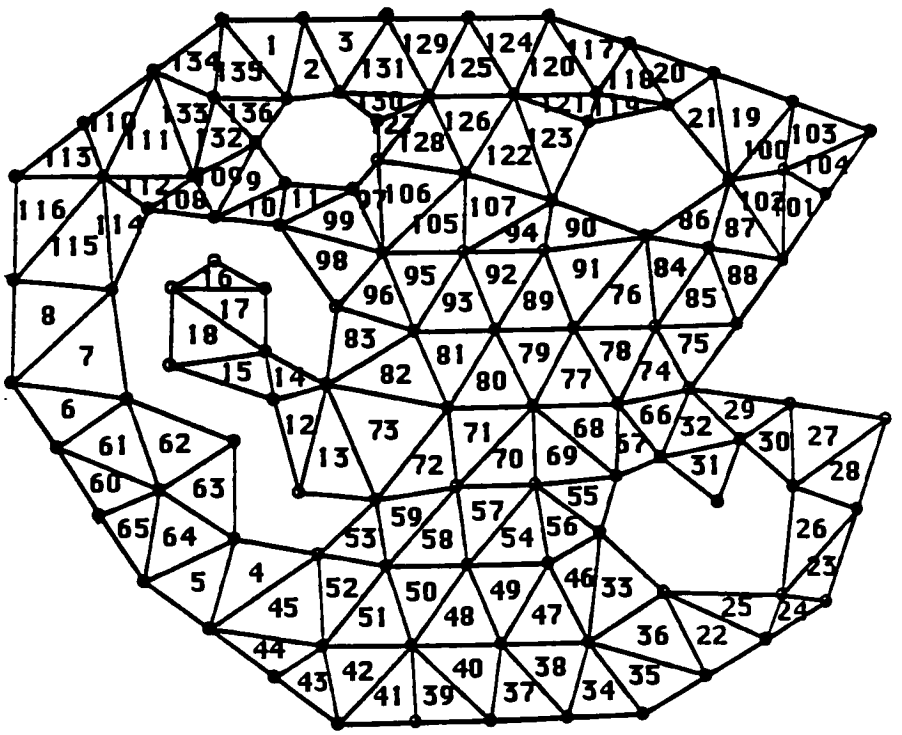


Figure 4.17 Mesh of an Irregular Geometry with Holes

4.5 DISCUSSIONS

If during the triangulation a node lies precisely on the bounding polygon boundary, a degenerate triangle is formed. All three nodes of this triangle will be collinear. The collinearity can be detected during the calculation of circumradii of the triangles. The circumradius of a degenerate triangle is zero. Any triangle with collinear nodes is excluded from the master list of triangles.

A degeneracy can exist because of the limited arithmetic precision of the computer. Due to round off errors, ambiguity can occur whenever a new node is close enough to a circumcircle to be indistinguishable from one that lies exactly on the circle. This can lead to situations which influence the assessment of whether the node is inside, on, or outside a circumcircle. Inconsistent decisions with respect to the triangles that are candidates for the insertion polygon can result which will yield an insertion polygon that is not connected. This will ultimately leave gaps or overlapping triangles. A practical way to circumvent the degeneracy problem is to bias the in/out decision by a amount slightly larger than the maximum possible truncation error. In the developed algorithm a parameter 'disk spacing' controls this error. For the tested geometries the truncation error has not been encounterd.

4.6 CONCLUSIONS

This algorithm is a step towards creating a true automatic Pre-processor. Some of the desired capabilities of an automatic pre-processor, that is, minimum user interaction, mesh generation from the geomtric database, elements with good aspect ratios, are all addressed in this program. Other desirable features, as discussed in chapter 1, can be added in the future. The modular structure of the algorithm provides access to programming additions and modifications. The node generation algorithm can also be enhanced to add flexibility to the program. For example, instead of generating nodes at a uniform spacing along a segment, the algorithm can place nodes at a changing ratio of distances between consecutive nodes. In this manner, density of elements within a region can be varied. This is a desirable feature and provides for good approximation in the region of interest, avoiding extra elements elsewhere. The algorithm can also be modified to check the curvatures of curve segments. The programme can then be modified to add extra nodes on curves.

Some popular and efficient algorithms for triangulating non-convex planar surfaces were presented in chapter 3. These algorithms split a non-convex surface with or without holes into a number of convex regions. Each algorithm has a different criteria for determining the split regions. For an irregular geometry with holes the spliting operation has to be performed more than once.

These operations require extra computer time for generating good meshes. In the modified Watson's algorithm presented in this thesis, the 'delete external triangle' algorithm can be made more efficient. At the time of meshing the convex hull of the nodes, triangles with one or more external boundary segments as their sides can be saved in an 'external' list. Similarly, triangles associated with the internal segments can be saved in an 'internal' list. Triangles in the 'external' and 'internal' lists should be checked by the 'delete external triangle' algorithm. This new scheme of determining valid triangles will save computer search time and enhance the algorithm of this thesis.

Delauny triangulation method is an efficient method for mesh generation in two dimensions as well as three dimensions. A recent paper [45] has presented a mesh generator which creates tetrahedrons in three dimensions through Delauny triangulation method.

Expert systems were suggested in chapter one. Adding 'intelligence' to Pre-processors will be beneficial to finite element analysts. Research in this area indicates that various academic institutes as well as industrial research centers are working towards adding intelligence to FEA.. Contemporary expert systems in FEA are discussed in the following chapter. Proposals for integrating Expert Systems with a Pre-processor are also presented.

CHAPTER 5

EXPERT SYSTEMS IN FINITE ELEMENT ANALYSIS

An analyst's interaction is essential even for automatic finite element pre-processors. The automatic mesh generator developed in this thesis project requires the user to decide on the density of elements. Curved areas and sharp corners need more elements around them than straight edges. High density of elements in the whole region can be expensive. One has to use knowledge and experience to decide upon an optimum density.

This is one of the many occasions where the analyst's knowledge and experience in finite element analysis is required to make a decision in preparation of a finite element model. Expert Systems can relieve the analyst of this burden and make finite element modeling less costly and less time consuming. Expert Systems are part of the field of 'Artificial Intelligence'. In this chapter Artificial Intelligence, Expert Systems, and their integration with finite element analysis is discussed.

5.1 ARTIFICIAL INTELLIGENCE

Artificial intelligence (AI) is the study of methods by which computers are made to perform intelligently, in an effort to mimic the human brain. The history of

Artificial Intelligence is about thirty years old. During these years significant advances have been made in research laboratories and academic institutions, in order to imitate the functions of the human mind. In the late 1970's the computational power of computers advanced by leaps and bounds, and commercial applications of Artificial Intelligence emerged. Modern engineers and computer scientists are actively involved in the development of various applications of Artificial Intelligence. The primary areas of research are :

1. Natural languages.
2. Vision systems.
3. Expert systems.

Expert Systems research can claim responsibility for the current heightened awareness of Artificial Intelligence. Expert Systems are one of the first Artificial Intelligence technologies to be used commercially and hold a tremendous promise for the future.

5.2 EXPERT SYSTEMS [46]

An Expert System contains knowledge about a particular field and assists human experts in providing specialized information to people who do not have expertise in given field. Human experts in any field are frequently in great demand and unsolved problems far exceed the number of human experts who can solve them. Expert Systems are a solution to this dilemma.

Although both Expert Systems and database programs feature the retrieval of stored information, the two types of programs differ greatly. Database programs contain knowledge of their particular domains and areas of expertise. This knowledge is declarative or factual and a database program cannot draw conclusions by the rationalization of the facts within its domain. In contrast, Expert Systems contain expertise consisting of both declarative and procedural knowledge which allows them to emulate the reasoning process of the human experts.

5.3 COMPONENTS OF AN EXPERT SYSTEM

Currently there are no standard Expert Systems. Because of a variety of techniques used to create these systems, they differ as widely as the programmers who develop them, and the innumerable problems they are designed to solve. The principal components of most Expert Systems are as shown in Figure (5.1). They are as follows :

1. Knowledge Base
2. Inference Engine
3. User Interface

5.3.1 KNOWLEDGE BASE

The component of the Expert System that contains the system's knowledge is called the knowledge base. A knowledge base contains both declarative knowledge, facts about objects, events, and situations, and procedural knowledge, consisting of information about courses of action. Depending on the two forms of knowledge chosen, knowledge can be separate or integrated. Although many knowledge representation schemes have been used in expert systems, the most prevalent form is the 'rule based production system'. [46]

In a rule based system, the procedural knowledge, in the form of heuristic "if-then" is completely integrated with declarative knowledge. Not all rules pertain to the system's domain. Some production rules called meta-rules pertain to other production rules or even to themselves. A meta-rule is a "rule about a rule". It helps in guiding the execution of an Expert System in determining conditions under which it is considered.

5.3.2 INFERENCE ENGINE

Access to a great deal of knowledge does not make an expert. One must know how and when to apply the appropriate knowledge. Similarly, having a knowledge base does not make an expert system intelligent. The system must have another component that directs the implementation of the knowledge.

That element of the system is known variously as the control structure, the rule interpreter, or the Inference Engine.

The Inference Engine is the key to the workings of the entire system. It makes high level decisions that were until now made by the expert. Ideally, it possesses all the knowledge held by its human counterpart and is able to make the same decisions, only faster. In reality it can possess only a small subset of the knowledge of the human expert and is much less flexible. It still can, perform many if not all, human tasks. When unable to reach a decision the Inference Engine transfers control to the user.

An advantage of having an Inference Engine is that it provides a framework on which a fully automated system can be built. It also creates a situation where all quantifiable knowledge can be quickly entered into the system, leaving only the undefinable intuition decisions for people. The Inference Engine processes information in the form of value parameters. Parameters can be numerical values, Booleans, strings, or taken from a list. They quantify what is known about a goal and what needs to be decided. The value of the parameters are determined through the value of other parameters and rules. Rules tell how to determine new parameter values from existing values. The rules have two parts, the predicate and the antecedant.

The predicate tells whether or not a rule is applicable in a given situation. The antecedant is the action part of the rule. It tells what new information can be deduced from the fact that a predicate was satisfied.

The Inference Engine uses two schemes for finding the sequence of deductions. They are called forward chaining and backward chaining. In forward chaining the Inference Engine scans the rule base for applicable rules, whose predicate is satisfied by known information. When such rules are found the new information contained in the rule antecedant is added to the pool of knowledge. It is hoped that by continually increasing known facts, the desired fact or goal will eventually be deduced.

Backward chaining works from facts that need inference and tries to find rules whose antecedant will determine this fact. As such rules are found, their predicates are checked for truth. If they are not satisfied, then the rules are backward chained for determination. Eventually, it is hoped that all the predicates of the rules needed to show result by a given scheme will be satisfied by the facts that are known initially.

Although each method operating on its own is merely a blind search for a valid chain of deductions, a combination of these methods gains advantages.

5.3.3 USER INTERFACE

The component of the expert system that communicates with the user is known as the User Interface. The communication performed is bi-directional. At the simplest level the user must be able to define the problem to the expert system, and the system must be able to respond with its recommendations. There are several other tasks performed by the User interface. The user may 'question' the system's decisions and ask for its 'reasoning'. The system may prompt the user for more information.

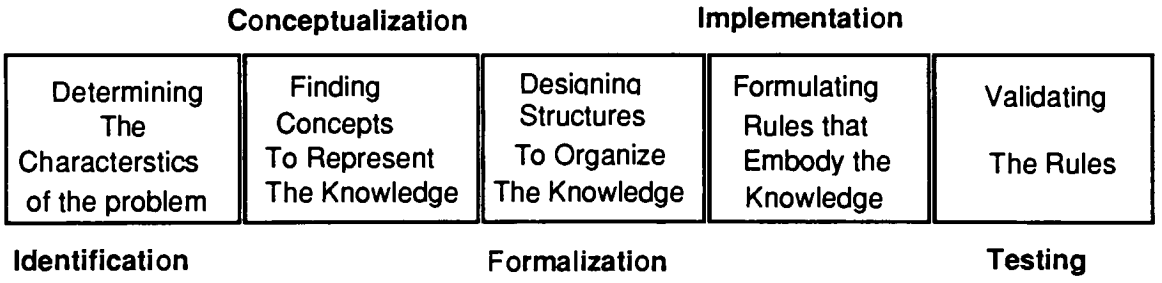
Although the designers of expert systems have generally a great deal of experience with computers, potential users of expert systems are frequently novices. Therefore, it is important to ensure that an expert system is especially easy to use. Most User Interfaces make use of techniques developed in the natural language discipline of Artificial Intelligence. Natural language techniques allow the user to communicate with the expert system in ordinary English and enable the computer to respond in the same language. This type of interface is sometimes called a natural language front end.

5.4 DEVELOPING AN EXPERT SYSTEM

There are two types of people involved in developing an expert system. Knowledge engineers and domain experts.

Knowledge engineers are Artificial Intelligence specialists. A computer scientist or a programmer who is skilled at developing expert systems can be a knowledge engineer. A domain expert is an individual who has significant expertise in the domain of the Expert System being developed. It is not critical that the domain expert understand Artificial Intelligence or Expert Systems. It is one of the functions of the knowledge engineer.

Typical development stages of an Expert System are as shown in Flow chart (5.1)



Flowchart 5.1 Development Stages of an Expert System

5.4.1 IDENTIFICATION

The domain expert describes several typical problem situations. The knowledge engineer attempts to extract fundamental concepts from the case presented in order to develop a more general idea of the purpose of the expert system.

Results are evaluated at each stage and the domain expert suggests corrections.

5.4.2 CONCEPTUALIZATION

Once the problem is formally identified it is analyzed further to ensure that its specific and general requirements are understood. The knowledge enginner frequently creates a diagram of the problem to depict graphically the relationships between the objects and the processes in the problem domain. As in the identification stage, this stage also evolves iteratively.

5.4.3 FORMALIZATION

During this stage the problem is connected to its proposed solution, that is an expert system. It analyzes the relationships depicted in the conceptualization stage. The knowledge engineer begins to select the development techniques that are appropriate to this particular system. The analyst becomes familiar with :

1. The various techniques of knowledge representation and heuristic search.
2. The Expert System " tools " that can expedite the development process.
3. Other Expert Systems that may solve similar problems .

If a rule based system is being developed, the knowledge engineer develops a

set of rules for review of domain experts. Rules are revised until there is full agreement. To facilitate the formalization stage, Artificial Intelligence researchers are looking for ways to achieve time-effectiveness in entering information into the knowledge base.

5.4.4 IMPLEMENTATION

During the implementation stage, a prototype of the expert system is developed to help the knowledge engineer determine if correct techniques were chosen. Once the prototype has been refined sufficiently to allow it to be executed, the system is ready to be tested thoroughly to ensure that it executes correctly.

5.4.5 TESTING

Testing is the last stage of the development process. Here the knowledge engineer revises the structure and implementation of the expert system until the system provides solutions as valid as those of a human expert.

5.5 EXPERT SYSTEMS IN FINITE ELEMENT ANALYSIS

Currently in finite element methods the finite element model is created on a computer aided design (CAD) system. Then the finite element analysis is carried out on the computer by a FEA code such as, Nastran, Ansys, etc. The finite element model is generated interactively. Because of the time required

for the generation of the analysis model, "the analysis is used primarily for verification rather than feedback, to facilitate iterative design improvement ". [47] "The reason for this bottleneck in the FEA process is the need for expertise in the finite element model generation stage of the analysis." [47] Answers from a FEA are only as good as the model itself, and a FEM model is as good as the engineer's understanding of the physics of the problem. [48] If a young engineer sets out to model a plate structure without understanding the underlying mechanics of how the structure may behave, he may make incorrect modeling decisions leading to an incorrect analysis and interpretation of the results.

A second major stumbling block in automating the FEA process is the host of commercially available packages to perform the analysis. Understanding the myriad of details necessary to effectively use the tools of a given software package is an art itself. According to Henry Fong [49] , "The name of the game is confusion. The user needs help from the code developers to decide the suitability of certain elements in each situation. It is confusing for him to look through all the users manuals and find that the same element type is called different names in each code". There are about 88 elements catalogued for plate bending alone.

Decisions concerning the idealization of an actual problem are most important in FEM. They refer to proper selection of elements to represent the physical state. The element type is characterized by geometric properties such as truss, beam, membrane plate; mathematical formulation, such as linear, quadratic, and others; and shapes such as triangular, quadrilateral. The selection of the correct elements for the problem at hand requires understanding of the physical problem and the characteristics of elements available within the program library. This includes the recognition of the failure modes expected under the prescribed loads, the presence of loads, geometry of the domain, and capabilities and limitations of each finite element. No general guidelines for choosing the right element can be obtained, since the type of element that yields good accuracy with low computing time is problem dependent. Whether the mathematical model of a problem is represented by a beam of one dimension, plate of two dimensions or a solid brick of three dimensions, is also analyst's decision.

Dimensionality of the problem also plays an important role in element selection. Current finite element pre-processors are designed for the efficient conversion of geometric entities to the finite element model when the dimensionality of the two are the same. They do not effectively handle the case where the dimensionality of the finite elements are lower than the geometric entities they

represent. As a result of this shortcoming, experts are required to model the structures interactively to be able to capture the significance of the structure with a limited number of discrete elements.

The discretization of the structural or mechanical system involves decisions on the number, shape, size and distribution of the finite elements in the problem domain. Proper specification of node locations and element shapes is essential in order to have the discretized domain approximate the actual domain as close as possible. Due consideration must also be given to an accurate representation of concentrated loads, distributed loads with and without discontinuities, and material and geometric discontinuities.

It is evident from the foregoing discussion that a considerable amount of experience and engineering judgement is needed in establishing an optimum finite element model. The Expert System approach provides a useful tool to bring together the accumulated knowledge of an experienced user of finite element methods and the computational efficiency of modern computers.

5.6 THE EXPERT SYSTEM APPROACH

As discussed in earlier sections, expert systems can serve to replace an expert in finite element modeling to overcome the difficulties faced by current users.

This is the most promising direction towards truly automating FEA. The key to developing a robust system are a suitable knowledge representation scheme and inference reasoning technique. The sources of expert knowledge are the domain expert and the experiences of the domain experts that are documented in available literature. Several researchers have developed proto-types of Expert Systems. The systems have been partially successful in automating finite element analysis. Some of the systems developed are :

FEMOD (Chen, Hajela) [50] This Expert System contains rules for the selection of elements, node placement, mesh generation, selection of subdivision lines, and mesh refinement. The structure of knowledge base, knowledge representation, and inference engine is used in this system.

ADEPT (Holt, Narayana) [51] This Expert System is based on a description of the object's geometry, anticipated loading conditions, boundary conditions, material characterstics, and available analysis system. The system can

1. Recommend the most appropriate analysis package
2. Recommend the element type best suited to the given constraints
3. Create a mesh of the object using the available elements
4. Prepare a suitable input file and runs the analysis

FACS (Gregory, Shephard) [47] An expert system based procedure for the creation of airframe finite element models from the geometric model available in a computer-aided design system. The knowledge driving the expert system is variable and changes as the system evolves. The system is compatible with commercially available geometric and analysis packages. It is modular and improvement of individual modules can be carried out.

PLASTHRAN (Cagan, Genberg) [48] An expert system and a teaching aid for the users of MSC/NASTRAN finite element code in modeling process with two dimensional elements. It allows engineers to obtain recommendations while modeling interactively. The knowledge base is expandable and allows incorporation of new knowledge.

AMEKS (Blacker et al) [52] The Expert System mimics an experienced analyst's approach to meshing 2-D geometries using transfinite mapping technique.

5.7 SUGESSTIONS FOR THE KNOWLEDGE BASE OF AN EXPERT FINITE ELEMENT MODELER

Finite element pre-processors and the Expert Systems have been reviewed extensively in this thesis. The Expert Systems can be integrated with automatic pre-processors to provide 'expert finite element systems'. Suggestions for the

information which should be available in the knowledge base of an expert system to generate a finite element mesh are presented

5.7.1 ELEMENT SELECTION

In the previous sections the importance of selecting the right element type was presented. It was shown that it is difficult for a new user to make all the decisions correctly. The knowledge base can have the information which can take the burden of element selection from the user. The following queries should be made by the Expert System for element selection :

a) Whether the type of problem is a mechanism or a true load carrying structure.
b) Whether the structure can be visualized as a truss or needs to be as a beam frame assembly.
c) Whether the structure can be construed to be in plane stress or plane strain, which in turn determines if it can be modeled as a membrane, plate, or shell elements.
d) Whether the failure mode of the problem is yielding or buckling.
e) Whether the problem pertains to static loading only or if it requires dynamic load analysis and frequency computations.
f) Whether the problem is isotropic or anisotropic.

g) Special considerations for problems that have specific deflection, or are subjected to shear loading, torsional loading, axial loading, or in-plane/out-of-plane loading.

5.7.2 NODE PLACEMENT

In chapters 1 and 4 it was shown that the placement of nodes is an important factor in finite element model creation and analysis. Even after the analysis is carried out, the nodes are valuable in the areas where the results are important for review. The nodes should be generated from the following information in the knowledge base of an Expert System :

a) The location of concentrated loads in the structure. For distributed loads, the position where the load magnitude changes appreciably.

b) The existence of material inhomogeneity in the analysis domain.

c) The presence of irregular domain boundaries.

d) The presence of a geometric and loading symmetry in the structure.

e) Select locations in the domain in the analysis domain where analysis results are required.

f) Change in cross-sectional properties.

g) Change of boundary conditions.

h) Existence of holes and discontinuities in the analysis domain.

5.7.3 MESH GENERATION

Finite elements should be generated after the choice of the element type and the node locations have been made. In this thesis project it has been shown that the shape of finite elements is important for obtaining good results. For the size and distribution of elements the Expert System must consider the following

a) The existence of complicated domain boundaries.

b) The type of failure mode appropriate to the structure.

c) The type of loading the structure is subjected to - concentrated or distributed.

d) The existence of holes and discontinuities in the analysis domain.

e) An analysis that is primarily focussed on displacement or if both displacement and stresses are required.

f) The selection of shape of elements - triangular, quadilateral.

5.7.4 MESH REFINEMENT

To reduce the computation cost, the initial finite element analysis should be carried out with a coarse finite element mesh. Subsequently, the mesh should be refined to get good results. The mesh refinement information in the knowledge base should take care of the following :

a) The mesh arrangement should minimize the element distortion.

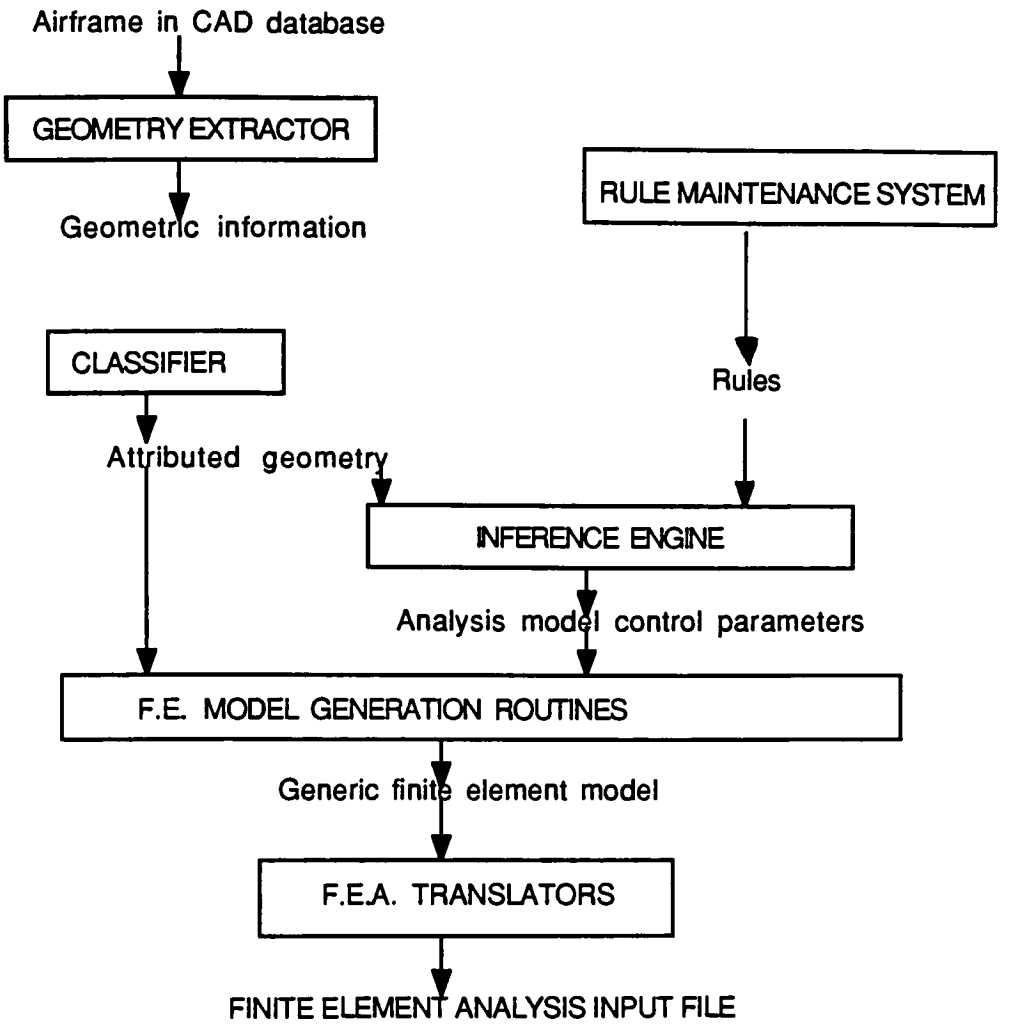
b) Whether the element aspect ratio is within a permissible range to achieve satisfactory results.

c) Whether renumbering of nodes would reduce the bandwidth.

d) Whether the analysis results display a smooth spatial variation over the domain.

5.8 OVERVIEW OF AN EXPERT SYSTEM

FACS [47] (Flexible Automation Conversion System) has been designed to automate the creation of airframe f.e. models by performing many tasks done by experts. The control sructure of the system is given in Flow Chart (5.2).



Flowchart 5.2 Data flow through FACS Finite Element Expert System

FACS works as follows :

For the geometry extractor the user defines the geometry one wants to create the analysis model for. Some initialization questions are also answered.

The geometry extractor validates the users input and locates and retrieves the desired geometry, converting it into a form that is understandable by the rest of the system. The "interna " geometry representation groups the geometry in sets delimited by individual aircraft components.

Each structural component is sent through the classifier which extracts useful information from the set of primitives in the geometric model. The information is extracted in the form of parameters that describe the geometry in terms that are meaningful to an airframe analyst.

For the next step, artificial intelligence is required, since this is the phase that normally requires human decision making. An expert system considers all the parameters that describe a component using a system of rules embodying airframe analysis knowledge. It chooses, from a set of possible modeling methods, a technique that an expert would have selected, given the same information. It also decides certain conversion parameters such as fineness of the finite element mesh.

A finite element model generation routine that carries out the chosen conversion method is invoked. It is supplied a template containing the conversion parameters and dictated by the inference engine along with the

geometric specification of the component. The finite element model generation routine produces generic finite elements.

Finally, a FEA translator is invoked to convert the generated finite elements into the form required by the target analysis package.

5.9 CONCLUSION

Expert systems as useful daily tools may not be as advanced as they should be, but the development is on the right course. Finite element analysis and its related fields of pre and post-processing offer fertile ground for research. The day is not far off when all the pre and post-processors will be marketed with some kind of 'expert' user interface. The demand exists today; all that is necessary is for the experts to unload their wisdom and the software developers to capture it !

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A P P E N D I X

```
PROCEDURE TPolygon.IPolygon;
{   Never Do this more than once for a polygon! !   }
Var i :INTEGER;
    pMCEntry:   MCEntryPtr;

BEGIN
    {Set the shape's color}
    IF gConfiguration.hasColorToolbox THEN
        {pick a random color from the Color menu}
        BEGIN
        pMCEntry := GetMCEntry(mColor,
                                ABS(Random MOD CountMItems(GetMHandle(mColor
                                                                    1);
        fColor := pMCEntry^.mctRGB2;
        END
    ELSE
        {set color to black}
        fColor := gRGBBlack;
    fModeled        := FALSE ;
    fPtsDefined     := FALSE ;
    fQDrawPoly      := NIL ;
    fClosedGeometry := TRUE;
{$D++}
    fPointList      := NewList ;
    fBoundaryNodes  := NewList ;
    fInteriorNodes  := NewList ;
    fBoundLists     := NewList ;
    FailNil(fPointList);
    FailNil(fBoundaryNodes);
    FailNil(fInteriorNodes);
    FailNil(fBoundLists);
        fPointList.SetEltType('TNode');
    fBoundaryNodes.SetEltType('TNode');
    fInteriorNodes.SetEltType('TNode');
    fBoundLists.SetEltType('TList');
    fMesh   := NIL;
    fShade  := cWhite;
    fColor  := gRGBWhite;
    fIsSelected  := FALSE;
    fWasSelected := FALSE;
END;



PROCEDURE TPolygon.AddVertex(pt : XYZExtPoint; itsID: INTEGER ) ;
    Var i : INTEGER;
        r : Rect;
        testNode : TNode;
BEGIN
{$D++}
    testNode := NIL;    { Initialize temporary TNode variable. }
    fModeled := FALSE;
    gNodeNumber := 0;
    { allocation and assignment of new point }
    New(testNode);
    FailNil(testNode);
    testNode.fPos := pt ;

    fPointList.insertLast(testNode) ;

    {   ^Modify this for use with moving a node.  To move a node,
    Delete an existing node, and insert the new one in its place. }

    testNode := NIL ;   { Remove any association }

    { Update the MinCoords and MaxCoords records }
```

```
    testNode := TNode(fPointList.First);
    fMaxCoords := testNode.fPos ;
    fMinCoords := fMaxCoords ;
    for i := 1 to fPointList.fSize Do
        MaxMinCoords(TNode(fPointList.At(i)),fMaxCoords,fMinCoords);

    { Define a Polygon and its Bounding Box with QuickDraw. }
    if (fQDrawPoly <> NIL) then
        KillPoly(fQDrawPoly);
    fQDrawPoly := OpenPoly;
    SELF.DrawOutline;
    ClosePoly;

    { call INHERITED initialization which updates fExtentRect. }
    IBox(fQDrawPoly^^.polyBBox, itsID)
END;    {TPolygon.AddVertex}

PROCEDURE TPolygon.EditBoundary;
    Var
        i, j, k, BLindex, start, finish, PLSize : INTEGER;
        done, NeedToCheckDistance : BOOLEAN;
        P1, P2, aNode, testNode: TNode;
        aLine : TLine;
        temp : XYZExtPoint;
        d, USpacing, slope, dummy: Extended;

    FUNCTION CheckDistance(existing: TNode): BOOLEAN;
    BEGIN
        if (gSpacing * 0.6 > NodeDistance(existing, aNode)) then
            CheckDistance := TRUE
        else
            CheckDistance := FALSE;
    END;

BEGIN
    fBoundaryNodes.DeleteAll;
    fBoundaryNodes.RemoveDeletions;
    j := 1;
    PLSize := fPointList.fSize + 1;
    for i := 2 to PLSize Do
    BEGIN
        P1 := TNode(fPointList.At(i - 1));
        if (i = PLSize) then
            P2 := TNode(fPointList.At( 1 ))
        else
            P2 := TNode(fPointList.At(i));
        New(aLine);
        FailNil(aLine);
        aLine.ILine(P1.fPos, P2.fPos);
        d := NodeDistance(P1, P2);
        k := Round(d / gSpacing);   { Find j = (# of nodes on a segment) +
        slope := aLine.fV.y / aLine.fV.x ;
        if ((slope < 1) AND (slope > -1)) then
            NeedToCheckDistance := TRUE
        else
            NeedToCheckDistance := FALSE;
        if (k > 1) then
        BEGIN
            USpacing := 1 / k ;     { Increment of U parameter is determine
            for j := 1 to k Do
            BEGIN
                aNode := NIL;
                New(aNode);
                FailNil(aNode);
                aNode.fPos := aLine.SolveDirect(Extended(j * USpacing), dum
                aNode.fAtt := NIL;
```

```
                if (NeedToCheckDistance OR (j = k)) then
                BEGIN
                    Repeat
                        testNode := TNode(fInteriorNodes.FirstThat(CheckDis
                        if (testNode <> NIL) then
                        BEGIN
                            {Remove the Interior Node which was too close.}
                            fInteriorNodes.Delete(testNode);
                            fInteriorNodes.RemoveDeletions;
                        END;
                    UNTIL (testNode = NIL);
                END;
                dot(aNode);
                fBoundaryNodes.InsertLast(aNode);
            END;
        END
        else
        BEGIN
            New(aNode);
            FailNil(aNode);
            aNode.fPos := P2.fPos;
            aNode.fAtt := NIL;
            dot(aNode);
            fBoundaryNodes.InsertLast(aNode);
        END;
        aLine.Free;
    END;
    SELF.Draw;
END;        { TPolygon.EditBoundary }

PROCEDURE TPolygon.Draw;
Var
    index, k: integer;
    aBoundaryList: TList;
BEGIN
    UpdateExtentRect;
    SELF.DrawOutline;
    IF gConfiguration.hasColorToolbox THEN
    BEGIN
        {Get the color of the menu item representing the shape's color}
        RGBForeColor(fColor);
        PaintPoly(fQDrawPoly);
        FillPoly(fQDrawPoly, gPat[fShade]);
        ForeColor(blackColor);
    END;
    if (fBoundaryNodes <> NIL) then
        fBoundaryNodes.Each(dot);
    if (fInteriorNodes <> NIL) then
        fInteriorNodes.Each(dot);
    if (fBoundLists <> NIL) then
        k := fBoundLists.fSize
    else
        k := 0;
    for index := 1 to k do
    BEGIN
        aBoundaryList := TList(fBoundLists.At(index));
        if (aBoundaryList <> NIL) then
            aBoundaryList.Each(XNode);
    END;
    if (fMesh <> NIL) then
        fMesh.Draw;
END;

PROCEDURE TPolygon.DrawOutLine;
VAR     aTNode : TNode;
        p: Point;
```

```
BEGIN
    { Draw lines from each node to the next. }
    if gShowPolygon then
    BEGIN
        if (fPointList.fSize <> 0) then
        BEGIN
            aTNode := TNode(fPointList.First);
            p := Real2Local(aTNode.fPos);
            MoveTo( p.h , p.v );
            fPointList.Each(Line2Node);
            LineTo( p.h , p.v );
        END;
    END;
END;

PROCEDURE TPolygon.UpdateExtentRect;
BEGIN
    fExtentRect.topLeft  := Real2Local(SELF.fMinCoords);
    fExtentRect.botRight := Real2Local(SELF.fMaxCoords);
END;

PROCEDURE TPolygon.Free;

BEGIN
    fPointList.Free;
    if (fMesh <> NIL) then
    BEGIN
        fMesh.fGeometry := NIL ;     { Remove reference to SELF from fMesh }
        fMesh.Free;
    END;
    INHERITED Free;
END;    {TPolygon.Free}

PROCEDURE TPolygon.InteriorNodes;
    Var d, Y, stepSize: Extended;
        i, q: Integer;
        N1, N2: TNode;
        dummy: TList;   { Only used to make this compile }

    FUNCTION X1Bigger(thisOne : TNode): BOOLEAN;
    BEGIN
        if (thisOne.fPos.y > N1.fPos.y) then
            X1Bigger := TRUE
        else
            X1Bigger := FALSE;
    END;
    FUNCTION X2Bigger(thisOne : TNode): BOOLEAN;
    BEGIN
        if (thisOne.fPos.y > N2.fPos.y) then
            X2Bigger := TRUE
        else
            X2Bigger := FALSE;
    END;

    {   ScanList takes aList of pairs of TNodes adjacent on a
        boundary and finds the intersection between the line segment
        connecting the pair of TNodes and the horizontal plane on which
        interior nodes are being generated.                              }

    PROCEDURE ScanList(aList: TList);
        Var i, j, k, testnum, ListSize: Integer;
            e: Extended;
            a, b, c, d: TNode;
            NodeList: TList;
            theLine: TLine;
            numberString: STR255;
```

```
        longNum: Longint;
        intersectNode1, intersectNode2: TNode;
        spot1, spot2: Point;
    Procedure removeOddIntersection(testNode: TNode);
        Var
            distance: Extended;
            PtListNode: TNode;
            index: INTEGER;
    BEGIN
        for index := fPointList.fSize downto 1 Do
        BEGIN
            ptListNode := TNode(fPointList.At(index));
            distance := NodeDistance(testNode, PtListNode);
            if (distance < gSpacing/1000) then
                NodeList.Delete(testNode);
        END;
    END;
BEGIN
    NodeList := NewList;
    New(theLine);
    FailNil(NodeList);
    FailNil(theLine);
    testnum := aList.fSize - 2;
    j := 1 ;
    i := 1 ;
    While i < testnum Do
    BEGIN
        a := TNode(aList.At(i));
        b := TNode(aList.At(i + 1));
        c := TNode(aList.At(i + 2));
        d := TNode(aList.At(i + 3));
        New(N1);
        New(N2);
        FailNil(N1);
        FailNil(N2);
        N1.fAtt := NIL;
        N2.fAtt := NIL;
        theLine.ILine(a.fPos,b.fPos);
        if NOT theLine.YZPlaneInt(Y, 0.0, N1.fPos, e) then
            WriteLn('TPolygon.InteriorNodes Broke');
        theLine.ILine(c.fPos,d.fPos);
        if NOT theLine.YZPlaneInt(Y, 0.0, N2.fPos, e) then
            WriteLn('TPolygon.InteriorNodes Broke');
        ListSize := NodeList.fSize;
        if ListSize = 0 then
            NodeList.InsertFirst(N1)
        else
            for k := 1 to ListSize Do
            BEGIN
                a := TNode(NodeList.At(k)) ;
                if X1Bigger(a) then
                BEGIN
                    NodeList.insertBefore(k,N1);
                    Leave;
                END
                else
                    if (k = ListSize) then { N1 hasn't been inserted }
                        NodeList.InsertLast(N1);
            END;    { for k to ListSize }

        {   Now Do the same for N2  }

        ListSize := NodeList.fSize;
        for k :=  NodeList.fSize Downto 1 Do
        BEGIN
            a := TNode(NodeList.At(k)) ;
```

```
            if X1Bigger(a) then
            BEGIN
                NodeList.insertBefore(k,N2);
                Leave;
            END
            else
                if (k = ListSize) then { N2 hasn't been inserted }
                    NodeList.InsertLast(N2);
        END;    { for loop }

        i := i + 4;
    END;
    testnum := NodeList.fSize DIV 2;
    if ((NodeList.fSize MOD 2) <> 0) then
    BEGIN
        for i := NodeList.fSize downto 1 Do
        BEGIN
            removeOddIntersection(TNode(NodeList.At(i)));
        END;
        NodeList.RemoveDeletions;
    END;
    SortNodesByX(NodeList);

    i := 1 ;
    While i <= testnum Do
    BEGIN
        LineNodeFill(TNode(NodeList.At(2*i - 1)),
                     TNode(NodeList.At(2*i)));
        longNum := 2*i - 1;
        NumToString(longNum, numberString);
        intersectNode1 := TNode(NodeList.At(2*i - 1));
        spot1 := Real2Local(intersectNode1.fPos);
        MoveTo(spot1.h, spot1.v);
        drawString(numberString);
        longNum := longNum + 1;
        NumToString(longNum, numberString);
        intersectNode2 := TNode(NodeList.At(2*i));
        spot2 := Real2Local(intersectNode2.fPos);
        MoveTo(spot2.h, spot2.v);
        drawString(numberString);
        i := i + 1;
    END;
    Y := Y - stepSize;
    NodeList.Each(freeNode);
    NodeList.DeleteAll;
    NodeList.Free;
END; {Procedure ScanList}

Function Node2Close2Boundary(aNodeList: TList): BOOLEAN;
    Var i, j, k , l: INTEGER;
        bNode, iNode: TNode;    {Boundary and Interior Nodes}
BEGIN
    Node2Close2Boundary := FALSE;
    l := fInteriorNodes.fSize;  {You may assume fInteriorNodes is <> NI
    for k := l downto 1 Do      { Do this for each interior Node }
    BEGIN
        iNode := TNode(fInteriorNodes.At(k));
        if (iNode <> NIL) then
        BEGIN
            if (aNodeList <> NIL) then
            BEGIN
                j := aNodeList.fSize;
                for i := 1 to j Do  { Do this for each boundary node. }
                BEGIN
                    bNode := TNode(aNodeList.At(i));
                    if (bNode <> NIL) then
```

```
                        BEGIN
                            if ((gSpacing * 0.6) > NodeDistance(bNode, iNod
                            BEGIN
                                fInteriorNodes.Delete(iNode);
                                fInteriorNodes.RemoveDeletions;
                                Node2Close2Boundary := TRUE;
                                Leave;
                            END;
                        END;
                    END;
                END;
            END;
        END;
    END;    { Node2Close2Boundary }

BEGIN   { Procedure InteriorNodes }
    d := fMaxCoords.y - fMinCoords.y;
    q := Round(d / gSpacing);
    if (q > 0) then
        stepSize := d / q ;
    SELF.SidesList;
    Y := fMaxCoords.y - stepSize;

    fSides.Each(ScanList);

    dummy := TList(fBoundLists.FirstThat(Node2Close2Boundary));
                                            { Delete Internal nodes
                                              which are too close to
                                              an internal boundary. }
    EraseRect(fExtentRect);
    Draw;
END;    { InteriorNodes }

PROCEDURE TPolygon.LineNodeFill(p1, p2: TNode);
    Var
        aLine: TLine;
        aNode: TNode;
        j, k: Integer;
        USpacing, d, dummy: Extended;
BEGIN
    New(aLine);
    FailNil(aLine);
    aLine.ILine(p1.fPos, p2.fPos);
    d := NodeDistance(p1, p2);
    k := Round(d / gSpacing);   { Find k = (# of nodes on a segment) + 1 }
    if (k > 1) then
    BEGIN
        USpacing := 1 / k ;     { Increment of U parameter is determined }
        k := k - 1 ;            { <- Stay away from the boundary }
        for j := 1 to k Do
        BEGIN
            aNode := NIL;
            New(aNode);
            FailNil(aNode);
            aNode.fPos := aLine.SolveDirect(Extended(j * USpacing), dummy);
            aNode.fAtt := NIL;
            dot(aNode);
            fInteriorNodes.InsertLast(aNode);
        END;
    END
    else
    BEGIN
        New(aNode);
        FailNil(aNode);
        aNode.fPos := p2.fPos;
        aNode.fAtt := NIL;
```

```
        dot(aNode);
        fBoundaryNodes.InsertLast(p2);
    END;
    aLine.Free;
END; { LineNodeFill }

PROCEDURE TPolygon.SidesList;
    Var i, j, ListSize, q, VertexCounter, ListCount, ListIndex: Integer;
        thisList: TList;
        p1, p2, testVertex: TNode;
        Y, d, stepSize: Extended;
        index, k: Integer;
        aBoundaryList, theNthBoundary: TList;
    FUNCTION IsVertex(PolygonNode: TNode): BOOLEAN;
    BEGIN
        if (PolygonNode.fPos.y = Y) then
            IsVertex := TRUE
        else
            IsVertex := FALSE;
    END;

    FUNCTION FindVertIntersect(theNodeList: TList;Var theIndex: integer): TI
    {       ** theIndex must be externally initialized !! }
        Var Count, i: INTEGER;
            vNode: TNode;
    BEGIN
        vNode := NIL;
        if (theNodeList <> NIL) then
        BEGIN
            Count := theNodeList.fSize;
            for i := theIndex to Count Do
            BEGIN
                vNode := TNode(theNodeList.At(i));
                if ((vNode <> NIL) AND (IsVertex(vNode))) then
                BEGIN
                    theIndex := i;
                    leave;
                END
                else
                    vNode := NIL;
            END;
        END;
        FindVertIntersect := vNode;
    END;


BEGIN
    fSides := NewList;                      { Initialize a new List object }
    FailNil(fSides);
    d := fMaxCoords.y - fMinCoords.y;       { d is the length of the range in Y
    q := Round(d / gSpacing);               { q is the number of horizontal
                                              y values on which nodes will be
                                              generated }

    if (q > 0) then
        stepSize := d / q ;                 { step size is the distance between
                                              the horizontal levels }
    Y := fMaxCoords.y - stepSize ;          { Y is the y coordinate of the firsi
                                              horizontal level. }


    for i := 1 to (q - 1) Do
    BEGIN
        thisList := NewList ;
        FailNil(thisList);
        fsides.insertLast(thisList);

        { Check to make sure that Y is not the y coordinate of some
```

```
  Vertex of the geometry.  If it is, add a dummy pair of intersecti
}
testVertex := NIL;
VertexCounter := 1;
testVertex := FindVertIntersect(fBoundaryNodes, VertexCounter);
While (testVertex <> NIL) Do
BEGIN
    thisList.insertLast(testVertex);
    thisList.insertLast(testVertex);
    thisList.insertLast(testVertex);
    thisList.insertLast(testVertex); {Enter two segment intersectio
    testVertex := FindVertIntersect(fBoundaryNodes, VertexCounter);
END;

if (fBoundLists <> NIL) then
    ListCount := fBoundLists.fSize
else
    ListCount := 0;
for ListIndex := 1 to ListCount Do
BEGIN
    theNthBoundary := TList(fBoundLists.At(ListIndex));
    testVertex := NIL;
    VertexCounter := 1;
    testVertex := FindVertIntersect(theNthBoundary, VertexCounter);
    While (testVertex <> NIL) Do
    BEGIN
        thisList.insertLast(testVertex);
        thisList.insertLast(testVertex);
        thisList.insertLast(testVertex);
        thisList.insertLast(testVertex); {Enter two segment interse
        testVertex := FindVertIntersect(theNthBoundary, VertexCount
    END;
END;    { testing the node against geometric vertices. }

ListSize := fPointList.fSize ;
for j := 1 to ListSize Do       { Check each adjacent pair of
                                  boundary vertices to see if
                                  they cross the plane y = Y. }
BEGIN
    p1 := TNode(fPointList.At(j)) ;
    if (j = ListSize) then
    BEGIN
        if NOT fClosedGeometry then
            leave;
        p2 := TNode(fPointList.First)
    END
    else
        p2 := TNode(fPointList.At(j + 1)) ;
    if (p1.fPos.y >= Y) then
    BEGIN
        if (p2.fPos.y < Y) then
        BEGIN
            thisList.insertLast(p1);
            thisList.insertLast(p2);
        END
    END
    else
    BEGIN
        if (p2.fPos.y > Y) then
        BEGIN
            thisList.insertLast(p1);
            thisList.insertLast(p2);
        END
    END;
END;
```

```
        { •••• DO EACH OF THE INTERNAL BOUNDARIES ••• }
        k := fBoundLists.fSize;
        for index := 1 to k do
        BEGIN
            aBoundaryList := TList(fBoundLists.At(index));
            if (aBoundaryList <> NIL) then
                ListSize := aBoundaryList.fSize
            else
                ListSize := 0 ;
            for j := 1 to ListSize Do       { Check each adjacent pair of
                                              boundary vertices to see if
                                              they cross the plane y = Y. }
            BEGIN
                p1 := TNode(aBoundaryList.At(j)) ;
                if (j = ListSize) then
                BEGIN
                    if (fBoundTypes[index] <> kClosedGeometry) then
                        leave;
                    p2 := TNode(aBoundaryList.First)
                END
                else
                    p2 := TNode(aBoundaryList.At(j + 1)) ;
                if (p1.fPos.y >= Y) then
                BEGIN
                    if (p2.fPos.y < Y) then
                    BEGIN
                        thisList.insertLast(p1);
                        thisList.insertLast(p2);
                    END
                END
                else
                BEGIN
                    if (p2.fPos.y > Y) then
                    BEGIN
                        thisList.insertLast(p1);
                        thisList.insertLast(p2);
                    END
                END;
            END;
        END;
        Y := Y - stepSize ; { Decrement Y and start over }
    END;
END; { Procedure TPolygon.SidesList }
```

```
PROCEDURE TMesh.IMesh(aGeometry : TPolygon);
Var
    A, B, C, D, Q1, Q2: XYZExtPoint;
    Okay: BOOLEAN;
BEGIN
    fElementList := NewList;
    ftempList := NewList;
    fTriangleList := NewList;
    fBoundPoly := NewList;
    fElementList.SetEltType('TTriangle');
    ftempList.SetEltType('TTriangle');
    fTriangleList.SetEltType('TTriangle');
    fBoundPoly.SetEltType('TNode');

    fGeometry := aGeometry;
    fGeometry.fMesh := SELF;
    A := fGeometry.fMaxCoords;
    C := fGeometry.fMinCoords;
                { Bounding Rectangle of the Entity: }
                {                                   }
                {      /|\  • A            • B      }
    B.x := C.x; {       |                           }
    B.y := A.y; {       |                           }
    B.z := A.z; {       | Q1                        }
                {       |                           }
    D.x := A.x; {       |                           }
    D.y := C.y; {       |                           }
    D.z := A.z; {       |   • D            • C      }
                {           ---------------->       }
                {                  Q2               }
    Q1.x := A.x - D.x ;
    Q1.y := A.y - D.y ;
    Q1.z := A.z - D.z ;

    Q2.x := C.x - D.x ;
    Q2.y := C.y - D.y ;
    Q2.z := C.z - D.z ;     { Bounding Rectangle Described }

    NewTriangle(fBoundingTriangle);

    New(fBoundingTriangle.V1);  {Allocate the V1 TNode}
    New(fBoundingTriangle.V2);  {Allocate the V2 TNode}
    New(fBoundingTriangle.V3);  {Allocate the V3 TNode}

    With ( fBoundingTriangle.V1.fPos ) Do
    BEGIN
        z := A.z;
        y := A.y + 15 * Q1.y;
        x := (A.x + B.x) / 2;
    END;

    With ( fBoundingTriangle.V2.fPos ) Do
    BEGIN
        z := C.z;
        y := C.y - 15 * Q1.y;
        x := C.x + 15 * Q2.x;
    END;

    With ( fBoundingTriangle.V3.fPos ) Do
    BEGIN
        z := D.z;
        y := D.y - 15 * Q1.y;
        x := D.x - 15 * Q2.x;
    END;
    Okay := Calculate(fBoundingTriangle);
    fElementList.InsertLast(fBoundingTriangle);
```

```
END;    { Initialize Mesh Object }

PROCEDURE TMesh.AddNode(Position: XYZExtPoint);
    VAR ListSize, j: Integer;
        theNode: TNode;
BEGIN
        New(fNewNode);
        fNewNode.fAtt := NIL;
        fNewNode.fPos := Position;
        ListSize := fElementList.fSize;
        for j := ListSize DownTo 1 Do
            DiskContainsNode(TTriangle(fElementList.At(j)));
        fElementList.RemoveDeletions;
        ConvertTriangleList;
        fElementList.RemoveDeletions;
END;

PROCEDURE TMesh.ConvertTriangleList;

{   This procedure takes the list of Triangles which have been
identified as having the new Node contained within thier CircumDisks.

From this list of Triangles, the Bounding Polygon is generated, and
fBoundPoly contains its vertices in the order of a walk around the
Bounding Polygon.  The input list of triangles is then deleted.

From this list of vertices, the appropriate triangles are added
to the fElementlist.

}

Var
    i, j : INTEGER;
    test : TTriangle;
    Yummy_Node: TNode;

    PROCEDURE DeleteCommonEdges(theTriangle: TTriangle);
    BEGIN
        FindDupEdges(test, theTriangle);
    END;    {Glue Routine to allow the use of TList.Each}

    FUNCTION HasEdge(aTriangle: TTriangle):BOOLEAN;
    BEGIN
        With aTriangle Do
        if (E12 OR E13 OR E23) then
            HasEdge := TRUE
        else
            HasEdge := FALSE;
    END;

BEGIN
    for i := fTriangleList.fSize DOWNTO 1 Do
    BEGIN
        test := TTriangle(fTriangleList.At(i));
        if (test <> NIL) then
        BEGIN
            fTriangleList.Delete(test);
            fTriangleList.Each(DeleteCommonEdges);
            ftempList.InsertLast(test);
        END
        else
            fTriangleList.RemoveDeletions;
    END;
    fTriangleList.RemoveDeletions;      { fTriangleList is EMPTY }

    for i := ftempList.fSize DOWNTO 1 Do
```

```
    BEGIN
        MakeNewTriangles(TTriangle(ftempList.At(i)));
    END;
    ftempList.DeleteAll;
    ftempList.RemoveDeletions;
END;

PROCEDURE TMesh.CreateMesh;
    Var i, j, ListSize,
        NumBoundNodes,
        NumInterNodes,
        index, boundaries: INTEGER;
        theNode: TNode;
        tossTriangle: TTriangle;
        aBoundaryList: TList;
        MeshString : Str255;
        aReply: SFReply;
        BoxSpot: Point;
        OkeyDokey: OSErr;
        DelayTime: Longint;
        r: Rect;

BEGIN
    r.Top := -500;
    r.Left := -500;
    r.Right := 400;
    r.Bottom := 500;
    fGeometry.fBoundaryNodes.RemoveDeletions;
    fGeometry.fInteriorNodes.RemoveDeletions;
    NumBoundNodes := fGeometry.fBoundaryNodes.fSize;
    NumInterNodes := fGeometry.fInteriorNodes.fSize ;
    EraseRect(fGeometry.fExtentRect);
    for i := 1 To NumBoundNodes Do
    BEGIN
        theNode := TNode(fGeometry.fBoundaryNodes.At(i));
        fNewNode := theNode;
        dot(fNewNode);
        ListSize := fElementList.fSize;
        for j := ListSize DownTo 1 Do
            DiskContainsNode(TTriangle(fElementList.At(j)));
        ConvertTriangleList;
    END;

    if (fGeometry.fBoundLists <> NIL) then
    BEGIN
        boundaries := fGeometry.fBoundLists.fSize;
        for index := 1 to boundaries Do
        BEGIN
            aBoundaryList := TList(fGeometry.fBoundLists.At(index));
            if (aBoundaryList = NIL) then leave;
            NumBoundNodes := aBoundaryList.fSize;
            for i := 1 To NumBoundNodes Do
            BEGIN
                theNode := TNode(aBoundaryList.At(i));
                fNewNode := theNode;
                dot(fNewNode);
                ListSize := fElementList.fSize;
                for j := ListSize DownTo 1 Do
                    DiskContainsNode(TTriangle(fElementList.At(j)));
                ConvertTriangleList;
            END;
        END;
    END;

    for i := 1 To NumInterNodes Do
    BEGIN
```

```
        theNode := TNode(fGeometry.fInteriorNodes.At(i));
        fNewNode := theNode;
        dot(fNewNode);
        ListSize := fElementList.fSize;
        for j := ListSize DownTo 1 Do
            DiskContainsNode(TTriangle(fElementList.At(j)));
        ConvertTriangleList;
    END;
    fElementList.RemoveDeletions;

    i := fElementList.fSize;
    for j := i DOWNTO 1 Do
    BEGIN
        if (ScrutinizeTriangle(TTriangle(fElementList.At(j)))) then
        BEGIN
            fElementList.Delete(TTriangle(fElementList.At(j)));
        END
        else
        BEGIN
            if (NOT inGeometry(TTriangle(fElementList.At(j)))) then
            BEGIN
                tossTriangle := TTriangle(fElementList.At(j));
                fElementList.Delete(tossTriangle);
            END;
        END;
    END;
    fGeometry.Draw;
    fElementList.RemoveDeletions;
END;

PROCEDURE TMesh.DiskContainsNode(theTriangle: TTriangle);
Var
    Rsquared : Extended ;
BEGIN
    With fNewNode.fPos Do
    BEGIN
        Rsquared := Sqr(x - theTriangle.fCircumCenter.x) ;
        Rsquared := Rsquared + Sqr(y - theTriangle.fCircumCenter.y) ;

        Rsquared := Rsquared + Sqr(z - theTriangle.fCircumCenter.z) ;
    END;
    Rsquared := theTriangle.fSquaredRadius - Rsquared;
    if (Rsquared >= 0.0 {gDisk ???}) then
    BEGIN   { the disk contains the node }
        fTriangleList.InsertLast(theTriangle);
        fElementList.Delete(theTriangle);
    END;
END;    {TMesh.DiskContainsNode}

PROCEDURE TMesh.Draw;
BEGIN
    PenMode(PatOr);
    fElementList.Each(DrawTriangle);
    if (gNumberElements <> 0) then
    BEGIN
        gNodeNumber := 1;
        fElementList.Each(NumberTriangle);
        gNodeNumber := 1;
    END;
END;

PROCEDURE TMesh.Free;
BEGIN
            if (fGeometry <> NIL) then
                if (fGeometry.fMesh <> NIL) then
                    fGeometry.fMesh := NIL;
```

```
                    fBoundingTriangle.Free;

                    fElementList.RemoveDeletions;

                    fElementList.Each(FreeTriangle);

                    fElementList.Free;      { List of Triangle Objects to keep


                    fTriangleList.Each(FreeTriangle);

                    fTriangleList.Free;     { List of Triangles to discard }

                    ftempList.RemoveDeletions;

                    ftempList.Each(FreeTriangle);

                    ftempList.Free;         { Another List of Triangles to
discard }

                    fBoundPoly.RemoveDeletions;

                    fBoundPoly.Free;        { List of Nodes }

                    INHERITED Free;
        END;

        FUNCTION TMesh.InGeometry(theTriangle: TTriangle): BOOLEAN;
            Var Q, R : XYZExtPoint;
                aSegment: TLine;
                aBoundary: TList;
                P1, P2, P3 : TNode;
                spots: array [1..15] of Point;
                i, pListSize, numIntersections,
                ListCount, ListIndex: INTEGER;
                t, u, v, param: Extended;
                errMsg: Str255;
        BEGIN
            EraseRect(fGeometry.fExtentRect);
            gDrawTriangleFunny := TRUE;
            drawTriangle(theTriangle);
            gDrawTriangleFunny := FALSE;
            numIntersections := 0;
            With theTriangle Do
            BEGIN
                Q.x := (V1.fPos.x + V2.fPos.x + V3.fPos.x)/ 3 ;
                Q.y := (V1.fPos.y + V2.fPos.y + V3.fPos.y)/ 3 ;
                Q.z := (V1.fPos.z + V2.fPos.z + V3.fPos.z)/ 3 ;
            END;
            New(aSegment);
            ListIndex := 0;
            ListCount := fGeometry.fBoundLists.fSize;
            for ListIndex := 0 to ListCount Do
            BEGIN
                if (ListIndex = 0) then
                    aBoundary := fGeometry.fPointList
                else
                    aBoundary := TList(fGeometry.fBoundLists.At(ListIndex));

                pListSize := aBoundary.fSize;
                for i := 1 to pListSize Do
                BEGIN
                    if (i = pListSize) then
                        P2 := TNode(aBoundary.First)
                    else
```

```
                    P2 := TNode(aBoundary.At(i + 1)) ;
                P1 := TNode(aBoundary.At(i)) ;
                t := Q.x;
                u := P1.fPos.x ;
                v := P2.fPos.x ;

                {   Q is the Midpoint of the Triangle.
                    t is the x coordinate of Q
                    u is the x coordinate of P1
                    v is the x coordinate of P2
                }
                if (u > v) then
                BEGIN
                    P3 := P1;       param := u;
                    P1 := P2;       u := v;
                    P2 := P3;       v := param;
                    P3 := NIL;
                END;            { Guarantee that P1.x <= P2.x and that u
<= v.  }
                                { Then if v <= t, we are sure that u <= t.
    }
                if (v > t) then
                BEGIN
                    aSegment.ILine(P1.fPos, P2.fPos);
                    if aSegment.YZPlaneInt(Q.y, Q.z, R, param) then
                    BEGIN
                        if ((param >= 0) AND (param <= 1)) then
                        BEGIN
                            {   There was an intersection with the
                                horizontal line through Q and the segment
                                between P1 and P2.
                            }
                            if (R.x > t) then   { R is Right of Q }
                            BEGIN
                                if ((P1.fPos.y = Q.y) AND (P2.fPos.y =
Q.y)) then
                                BEGIN
                                    errMsg := 'Collinear Line from Q:
InGeometry';
                                    LabelPoint(R, errMsg);
                                    SysBeep(1);
                                END
                                else
                                BEGIN
                                    numIntersections := numIntersections +
1;
                                    if numIntersections < 16 then
                                    BEGIN
                                        spots[numIntersections] :=
Real2Local(R);
                                    END;
                                END;
                            END
                        END;        { We got a parameter in [0.0, 1.0] }
                    END;
                END;                { Case of v > t }

            END;
        END;

        if  (numIntersections MOD 2 = 1) then { it is odd }
            InGeometry := TRUE
        else    { it is even }
        BEGIN
            P3 := NIL;
            New(P3);
```

```
            P3.fPos := Q;
            BigDot(P3);      { display the center of the triangle }
            P3.fPos := R;
            P3.Free;
            InGeometry := FALSE;
                             { display the points of intersection }
            for i := 1 to numIntersections Do
            BEGIN
                MoveTo(spots[i].h - 5, spots[i].v - 5);
                LineTo(spots[i].h + 5, spots[i].v + 5);
                MoveTo(spots[i].h - 5, spots[i].v + 5);
                LineTo(spots[i].h + 5, spots[i].v - 5);
            END;
        END;
    END; { inGeometry }

    PROCEDURE TMesh.MakeNewTriangles(aTriangle: TTriangle);
        VAR
            newTriangle: TTriangle;
            i : INTEGER;
            makeOne: BOOLEAN;

    BEGIN
        With aTriangle Do
        BEGIN
            for i := 1 to 3 Do
            BEGIN
                makeOne := FALSE;
                if (i = 1) then
                BEGIN
                    if E12 then
                    BEGIN
                        makeOne := TRUE;
                        New(newTriangle);
                        newTriangle.V1 := V1;
                        newTriangle.V2 := V2;
                        newTriangle.V3 := fNewNode;
                    END;
                END;
                if (i = 2) then
                BEGIN
                    if E23 then
                    BEGIN
                        makeOne := TRUE;
                        New(newTriangle);
                        newTriangle.V1 := fNewNode;
                        newTriangle.V2 := V2;
                        newTriangle.V3 := V3;
                    END;
                END;
                if (i = 3) then
                BEGIN
                    if E13 then
                    BEGIN
                        makeOne := TRUE;
                        New(newTriangle);
                        newTriangle.V1 := V1;
                        newTriangle.V2 := fNewNode;
                        newTriangle.V3 := V3;
                    END;
                END;
                if makeOne then
                BEGIN
                    if (Calculate(newTriangle)) then
                    BEGIN
                        fElementList.InsertLast(newTriangle);
```

```
                    END
                    else
                    BEGIN
                        With newTriangle Do
                        BEGIN
                            V1 := NIL;
                            V2 := NIL;
                            V3 := NIL;
                        END;
                        newTriangle.Free;
                        newTriangle := NIL;
                    END;
                END;
            END; { For Loop }
        END; { With aTriangle }
    END;    { MakeNewTriangles }


    FUNCTION TMesh.ScrutinizeTriangle(theTriangle: TTriangle): BOOLEAN;
    { Boolean Function tells whether the node should be kept or not. }
        Var Dist : Extended;

        FUNCTION OOB(P1,UB,LB: XYZExtPoint): BOOLEAN;
        BEGIN
            OOB := FALSE;
            if ((P1.x > UB.x + 5) OR (P1.y > UB.y + 5) OR (P1.z > UB.z +
5)) then
                OOB := TRUE;
            if ((P1.x < LB.x - 5) OR (P1.y < LB.y - 5) OR (P1.z < LB.z -
5)) then
                OOB := TRUE;
        END; { OOB : Out Of Bounds }
    BEGIN
        if (OOB(theTriangle.V1.fPos,fGeometry.fMaxCoords,
fGeometry.fMinCoords) OR
            OOB(theTriangle.V2.fPos,fGeometry.fMaxCoords,
fGeometry.fMinCoords) OR
            OOB(theTriangle.V3.fPos,fGeometry.fMaxCoords,
fGeometry.fMinCoords) ) then
            ScrutinizeTriangle := TRUE  {means Throw it away}
        else
        BEGIN
            ScrutinizeTriangle := FALSE;    {means keep it}
        END;
    END;    { ScrutinizeTriangle }

    {Adjacent2Node takes centerNode as input, and returns a list of
     triangles and nodes which are adjacent to that node.}

    PROCEDURE TMesh.Adjacent2aNode(centerNode: TNode;Var aTriList: TList;
                                                    Var Nodelist: TList);

    Var
        i, ListSize: integer;
        testTriangle: TTriangle;

    BEGIN
        NodeList := NewList;
        NodeList.InsertFirst(centerNode);

        fElementList.RemoveDeletions;
        ListSize := fElementList.fSize;
        for i := ListSize Downto 1 Do
        BEGIN
            testTriangle := TTriangle(fElementList.At(i));
            if (NodeInTriangle(centerNode, testTriangle)) then
            BEGIN
```

```
                    if (NOT NodeInList(testTriangle.V1, NodeList)) then
                        NodeList.InsertLast(testTriangle.V1);
                    if (NOT NodeInList(testTriangle.V2, NodeList)) then
                        NodeList.InsertLast(testTriangle.V2);
                    if (NOT NodeInList(testTriangle.V3, NodeList)) then
                        NodeList.InsertLast(testTriangle.V3);
                END;
            END;
            NodeList.Delete(centerNode);
            NodeList.RemoveDeletions;
        END;

        {ShiftNode takes a node in a mesh, and finds the triangles
         containing that node.  the position of that node is then
         set to the average value of the other nodes found in those
         triangles.}

        PROCEDURE TMesh.ShiftNode(theNode: TNode);
        Var
            aTriangleList, aNodeList: TList;
            averagePoint: XYZExtPoint;
            index, length: Integer;
            thisNode: TNode;
        BEGIN
            {Don't initialize the lists, this is done in Adjacent2aNode}
            Adjacent2aNode(theNode, aTriangleList, aNodeList);
            aTriangleList.DeleteAll;
            aTriangleList.RemoveDeletions;
            aTriangleList.Free; {Deallocate TList allocated in Adjacent2aNode
Routine}
            length := aNodeList.fSize;
            if (length > 0) then
            BEGIN
                thisNode := TNode(aNodeList.At(1));
                if (thisNode <> NIL) then
                    averagePoint := thisNode.fPos;
            END;
            for index := 2 to length Do
            BEGIN
                thisNode := TNode(aNodeList.At(index));
                averagePoint.x := averagePoint.x + thisNode.fPos.x;
                averagePoint.y := averagePoint.y + thisNode.fPos.y;
                averagePoint.z := averagePoint.z + thisNode.fPos.z;
            END;
            theNode.fPos.x := averagePoint.x / length ;
            theNode.fPos.y := averagePoint.y / length ;
            theNode.fPos.z := averagePoint.z / length ;
            aNodeList.DeleteAll;
            aNodeList.RemoveDeletions;
            aNodeList.Free; {Deallocate TList allocated in Adjacent2aNode
Routine}
        END;
```