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Eventually Periodic Solutions of Single Neuron Model

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Abstract

In this paper, we consider a non-autonomous piecewise linear difference equation that describes a discrete version of a single neuron model with a periodic (period two and period three) internal decay rate. We investigated the periodic behavior of solutions relative to the periodic internal decay rate in our previous papers. Our goal is to prove that this model contains a large quantity of initial conditions that generate eventually periodic solutions. We will show that only periodic solutions and eventually periodic solutions exist in several cases.

Keywords: neuron model, difference equation, periodic solution, eventually periodic solution.

1 Introduction

In [18], the authors investigated the delayed differential equation

$$x'(t) = -g(x(t-\tau)), \tag{1}$$

that is used to model a single neuron with no internal decay, where $g\colon \mathbf{R} \to \mathbf{R}$ is either a sigmoid function or a piecewise linear signal function and $\tau \leq 0$ is a synaptic transmission delay. From (1) the corresponding difference equation was obtained as a discrete-time network of a single neuron model ([8]):

$$x_{n+1} = \beta x_n - g(x_n),\ n = 0, 1, 2, ..., \tag{2}$$

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where $\beta > 0$ is an internal decay rate and g is a signal function. Several authors investigated equation (2) (e.g., [5, 8, 17, 24, 23, 22, 25, 21, 20]). In addition, equation (2) have been investigated as a single neuron model where the signal function g is the following piecewise constant function with McCulloch-Pitts nonlinearity:

$$g(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases} \tag{3}$$

In [2, 3], the authors studied models by applying a different signal function (with more than one threshold). In [14], the authors investigated a discrete neuron model with periodic solutions. Piecewise difference equations have been used as mathematical models for various applications including neurons (see [13]).

Furthermore, in [6, 7], we studied the periodic character of the following non-autonomous piecewise linear difference equation:

$$x_{n+1} = \beta_n x_n - g(x_n), \tag{4}$$

where

$$\beta_n = \begin{cases} \beta_0, & \text{if } n = 2k, \\ \beta_1, & \text{if } n = 2k+1, \end{cases} \quad k = 0, 1, 2, ..., \ \beta_0 \neq \beta_1, \tag{5}$$

and

$$\beta_n = \begin{cases} \beta_0, & \text{if } n = 3k, \\ \beta_1, & \text{if } n = 3k+1, \\ \beta_2, & \text{if } n = 3k+2, \end{cases} \quad k = 0, 1, 2, ..., \ \beta_0 \neq \beta_1 \text{ or } \beta_0 \neq \beta_2, \tag{6}$$

where $\beta_n > 0$ for all $n \geq 0$, and g is in the form (3).

In [6], the coefficient $(\beta_n)_{n=0}^{\infty}$ is a period two sequence (5) and in [7] $(\beta_n)_{n=0}^{\infty}$ is a period three sequence (6). In [6], we showed that periodic cycles can exist only with even periods and investigated the stability character of these cycles. In addition, in [7], we proved that periodic solutions can exist only with period $3k$, $k = 1, 2, 3, ...$ and examined their stability character.

While studying equation (4) with (5) or (6), we observed that cases appear only when periodic and eventually periodic solutions exist. The goal of this paper is to analytically investigate the existence of eventually periodic solutions of (4) together with (5) and (6).

We give the necessary definitions about stable and unstable periodic orbits (see [10] or [11]). Let

$$x_{n+1} = f(x_n), \tag{7}$$

where $f : \mathbf{R} \to \mathbf{R}$. Then the orbit of a point $x_0 \in \mathbf{R}$ is defined to be the set of points

$$\{x_0, \, x_1 = f(x_0), \, x_2 = f(f(x_0)) = f^2(x_0), ..., \, x_n = f^n(x_0), ...\}.$$

Definition 1. *A point x^* is said to be a fixed point of the map f or an equilibrium point of equation (7) if $f(x^*) = x^*$.*

For an equilibrium point x^* the orbit consists of only the point x^*. Closely related to fixed points are the eventually fixed points.

Definition 2. *A point x is said to be an eventually fixed point of the map f if there exists a positive integer r and a fixed point x^* of f such that $f^r(x) = x^*$, but $f^{r-1}(x) \neq x^*$.*

If x is an eventually fixed point, then the orbit is

$$\{x,\, x_1 = f(x),\, ...,\, x_{r-1} = f^{r-1}(x),\, x_r = f^r(x) = x^*,\, x^*,\, x^*,\, ...\}.$$

Definition 3. *The equilibrium point x^* of (7) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies $|f^n(x_0) - x^*| < \varepsilon$ for all $n > 0$. If x^* is not stable, then it is called unstable.*

The stability of an equilibrium x^* means that initial condition x_0 slightly different from x^* generate an orbit that remains close to the equilibrium.

In this paper our goal is not to investigate the stability of equation (4) however the stability is one of the main objectives in the theory of dynamical systems. In many studies on solutions of difference schemes, the stability is established under the assumption that the magnitude of the grid steps τ and h with respect to time and space variables is connected. Of growing interest is the study of absolutely stable difference schemes, in which the stability is established without any assumptions with respect to the grid steps τ and h (see, for example, [19] and [4]).

The concept of periodicity is one of the most important notion in the field of dynamical systems. Its importance follows from the fact that many physical phenomena have certain patterns that repeat themselves (for example, the motion of a pendulum, the motion of planets, the population size of blowflies or other insects at time n, the price of commodity at time n).

Let $\overline{x}$ be in the domain of a mapping f.

Definition 4. *A point $\overline{x}$ is said to be a periodic point of f with period k if $f^k(\overline{x}) = \overline{x}$ for some positive integer k. Note that $\overline{x}$ is a periodic point with period k if it is a fixed point of the map f^k.*

For the periodic point $\overline{x}$ the orbit consists of k points that repeat infinitely many times

$$\{\overline{x},\, x_1 = f(\overline{x}),\, ...,\, x_{k-1} = f^{k-1}(\overline{x})\}.$$

Definition 5. *A point $\overline{x}$ is said to be an eventually periodic point with period k if $\overline{x}$ is not periodic, but there exists $m > 0$ such that $f^{k+i}(\overline{x}) = f^i(\overline{x})$ for all $i \geq m$. That is, $f^i(\overline{x})$ is periodic for $i \geq m$.*

For an eventually periodic point with period k the orbit consists of m points in the beginning and k points which are repeated infinitely many times.

Definition 6. *The periodic point $\overline{x}$ with period k of f is stable if it is a stable fixed point of f^k. If $\overline{x}$ is an unstable fixed point of f^k, then it is called unstable.*

The goal of dynamical systems is to understand the nature of all orbits and to identify the set of orbits which are periodic, eventually periodic, etc. Generally, this is an impossible task. But for some mappings we can can obtain more precise information about the behavior of solutions than for others. For example, in our case we can find analytically periodic and eventually periodic solutions.

The existence of eventually periodic solutions of (2) was investigated in [8] and [12]. Additional literature about difference equations with eventually periodic solutions is available on max-type difference equations and their periodic character ([1, 9, 16, 15]).

2 Existence of Eventually Periodic Solutions if the Internal Decay Rate is Periodic with Period Two

In this section, we consider a difference equation (4) with a sequence of periodic coefficients $(\beta_n)_{n=0}^{\infty}$ that are periodic with period two.

In [6], we proved that equation (4) with (5) has no periodic orbits of odd period and that there exist solutions only with an even period. More precisely, we showed that if the coefficients $0 < \beta_0 \leq 1$ and $0 < \beta_1 \leq 1$, that is, coefficients are in the region I (see Fig.1), then there exist solutions only with period two. If coefficients belong to the the region II, then exist solutions only with period four. If the coefficients belong to the the region III, then exist solutions with period two but in this case also exist solutions with an arbitrary even period. The surprising situation is in the case when $\beta_1 = \frac{1}{\beta_0}$ (except for $\beta_1 = \beta_0 = 1$). In this situation, there exist segments of initial conditions from which period four solutions arise. In [6], it has not been proved that for all other initial conditions solutions are eventually periodic with period four.

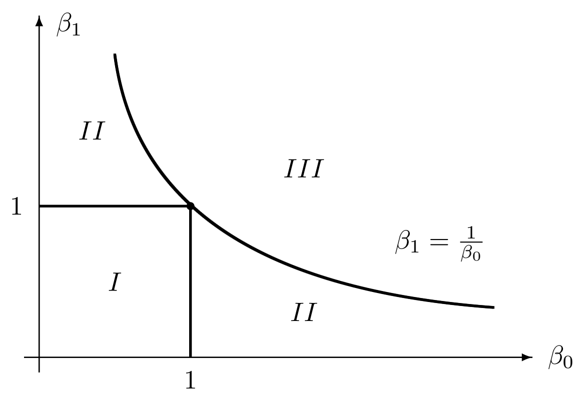

Figure 1: Existence of cycles depending on coefficients $\beta_0 > 0$ and $\beta_1 > 0$.

The first result in [6] is the following theorem.

Theorem 1. *([6]) If $0 < \beta_0 < 1$ and $0 < \beta_1 < 1$ (one of two coefficients is possible to be 1), then the periodic orbits*

$$\left\{\frac{1-\beta_1}{1-\beta_0\beta_1}, \frac{\beta_0-1}{1-\beta_0\beta_1}\right\} \text{ and } \left\{\frac{\beta_1-1}{1-\beta_0\beta_1}, \frac{1-\beta_0}{1-\beta_0\beta_1}\right\}$$

are stable periodic orbits with period two.

Now the following theorem will address the question regarding the existence of eventually periodic solutions.

Theorem 2. *If $0 < \beta_0 < 1$ and $0 < \beta_1 < 1$, then the initial conditions*

$$x_0 = \frac{2-\beta_0^k\beta_1^k(1+\beta_1)}{\beta_0^k\beta_1^k(1-\beta_0\beta_1)} > 0,\ k = 1, 2, ...,$$
$$x_0 = \frac{\beta_0^k\beta_1^k(1+\beta_1)-2\beta_1}{\beta_0^k\beta_1^k(1-\beta_0\beta_1)} < 0,\ k = 1, 2, ...,$$

produce eventually periodic solutions; precisely, $x_{2k} = \frac{1-\beta_1}{1-\beta_0\beta_1}$.

Also the initial conditions

$$x_0 = \frac{2\beta_1-\beta_0^k\beta_1^k(1+\beta_1)}{\beta_0^k\beta_1^k(1-\beta_0\beta_1)} > 0,\ k = 1, 2, ...,$$
$$x_0 = \frac{\beta_0^k\beta_1^k(1+\beta_1)-2}{\beta_0^k\beta_1^k(1-\beta_0\beta_1)} < 0,\ k = 1, 2, ...,$$

produce eventually periodic solutions; precisely, $x_{2k} = \frac{\beta_1-1}{1-\beta_0\beta_1}$.

Proof. We will only prove the first case, when $x_0 = \frac{2-\beta_0^k\beta_1^k(1+\beta_1)}{\beta_0^k\beta_1^k(1-\beta_0\beta_1)}$. The second case is symmetric as g is an odd function.

Let $k = 1$. Then $x_0 = \frac{2-\beta_0\beta_1(1+\beta_1)}{\beta_0\beta_1(1-\beta_0\beta_1)}$. Therefore we get

$$x_1 = \beta_0 x_0 - 1 = \frac{2-\beta_0\beta_1(1+\beta_1)}{\beta_1(1-\beta_0\beta_1)} - 1 = \frac{2-\beta_0\beta_1-\beta_1}{\beta_1(1-\beta_0\beta_1)} > 0,$$
$$x_2 = \beta_1 x_1 - 1 = \frac{2-\beta_0\beta_1-\beta_1}{1-\beta_0\beta_1} - 1 = \frac{1-\beta_1}{1-\beta_0\beta_1}.$$

We will assume that the initial condition $x_0 = \frac{2-\beta_0^k\beta_1^k(1+\beta_1)}{\beta_0^k\beta_1^k(1-\beta_0\beta_1)}$ produces an eventually periodic solution and $x_{2k} = \frac{1-\beta_1}{1-\beta_0\beta_1}$.

Now we consider the initial condition $x_0 = \frac{2-\beta_0^{k+1}\beta_1^{k+1}(1+\beta_1)}{\beta_0^{k+1}\beta_1^{k+1}(1-\beta_0\beta_1)} > 0$. We see that

$$x_1 = \beta_0 x_0 - 1 = \frac{2-\beta_0^{k+1}\beta_1^{k+1}(1+\beta_1)}{\beta_0^k\beta_1^{k+1}(1-\beta_0\beta_1)} - 1 = \frac{2-\beta_0^{k+1}\beta_1^{k+1}-\beta_0^k\beta_1^{k+1}}{\beta_0^k\beta_1^{k+1}(1-\beta_0\beta_1)} > 0,$$
$$x_2 = \beta_1 x_1 - 1 = \frac{2-\beta_0^{k+1}\beta_1^{k+1}-\beta_0^k\beta_1^k}{\beta_0^k\beta_1^k(1-\beta_0\beta_1)} - 1 = \frac{2-\beta_0^k\beta_1^k(1+\beta_1)}{\beta_0^k\beta_1^k(1-\beta_0\beta_1)}$$

and hence by induction we see that it is an eventually periodic solution and therefore $x_{2k+2} = \frac{1-\beta_1}{1-\beta_0\beta_1}$.

The proof for other cases is similar and is omitted. □

In [6], we proved the theorem, which shows that there exist segments of initial conditions from which period four solutions arise.

Theorem 3. *([6]) Suppose that* $0 < \beta_0 < 1$, $\beta_1 > 1$ *and* $\beta_0\beta_1 = 1$, *then every initial condition in the following two intervals generates a period 4 cycle. In fact, if* $x_0 \in [0, \frac{1}{\beta_0} - 1[$, *then we get the following period 4 cycle*

$$\{x_0, \beta_0 x_0 - 1, x_0 - \beta_1 + 1, \beta_0 x_0 + \beta_0\} \ .$$

If $x_0 \in [-\frac{1}{\beta_0} + 1, 0[$, *then we get the following period 4 cycle*

$$\{x_0, \beta_0 x_0 + 1, x_0 + \beta_1 - 1, \beta_0 x_0 - \beta_0\} \ .$$

In both cases the periodic orbits are stable except when $x_0 = 0$ *and* $x_0 = -\frac{1}{\beta_0} + 1$.

The case where $\beta_0 > 1$, $0 < \beta_1 < 1$ and $\beta_0\beta_1 = 1$ is formulated in a similar result. Now we will show that all initial conditions that are not in the segments that are considered in Theorem 3 produce eventually periodic solutions.

Theorem 4. *Suppose that* $0 < \beta_0 < 1$ *and* $\beta_1 = \frac{1}{\beta_0}$, *then every initial condition*

$$x_0 \notin [-\frac{1}{\beta_0} + 1, \frac{1}{\beta_0} - 1[$$

produces eventually periodic solution with period four.

Proof. We denote the following interval $I = [-\frac{1}{\beta_0} + 1, \frac{1}{\beta_0} - 1[$.

First we consider case where $0 < \beta_0 \leq \frac{1}{2}$. Then it follows that $\beta_1 = \frac{1}{\beta_0} \geq 2$ and $[-1, 1[\in I$.

Our objective is to show that there exists $k \in \mathbf{N}$ such that $x_{2k} \in I$.

We assume that $x_0 \geq \frac{1}{\beta_0} - 1$. We devide the segment $[-\frac{1}{\beta_0} + 1, +\infty[$ into smaller segments

$$[\frac{1}{\beta_0} - 1, \frac{1}{\beta_0}[, [\frac{1}{\beta_0}, \frac{2}{\beta_0}[, ..., , [n + \frac{n+1}{\beta_0}, n + \frac{n+2}{\beta_0}[, [n + \frac{n+2}{\beta_0}, n + 1 + \frac{n+2}{\beta_0}[, ...$$

(see Fig.2). Then there exists a segment such that x_0 belongs to this segment.

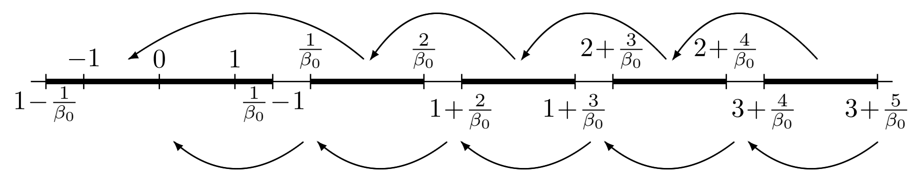

Figure 2: Partition of segment $[1 - \frac{1}{\beta_0}, +\infty[$ if $0 < \beta_0 \leq \frac{1}{2}$.

Now we show that

1) if $x_0 \in [n + \frac{n+1}{\beta_0}, n + \frac{n+2}{\beta_0}[$, then $x_2 \in [n - 1 + \frac{n}{\beta_0}, n - 1 + \frac{n+1}{\beta_0}[$, $n = 1, 2, ...$, and

$x_{2(n+1)} \in I$,
2) if $x_0 \in [n + \frac{n+2}{\beta_0},\, n + 1 + \frac{n+2}{\beta_0}[$, then $x_2 \in [n - 1 + \frac{n+1}{\beta_0},\, n + \frac{n+1}{\beta_0}[$, $n = 0, 1, 2, ...$, and $x_{2(n+1)} \in I$.

In the first case, we let $x_0 \in [n + \frac{n+1}{\beta_0},\, n + \frac{n+2}{\beta_0}[$. Then we see that

$$x_1 = \beta_0 x_0 - 1 \geq \beta_0(n + \frac{n+1}{\beta_0}) - 1 = n\beta_0 + n > 0,$$

$x_2 = \frac{1}{\beta_0}(\beta_0 x_0 - 1) - 1 = x_0 - \frac{1}{\beta_0} - 1$ and therefore

$$n - 1 + \frac{n}{\beta_0} = n + \frac{n+1}{\beta_0} - \frac{1}{\beta_0} - 1 \leq x_2 < n + \frac{n+2}{\beta_0} - \frac{1}{\beta_0} - 1 = n - 1 + \frac{n+1}{\beta_0}.$$

Now we assume that $x_0 \in [\frac{1}{\beta_0},\, \frac{2}{\beta_0}[$. Then we see that $x_1 = \beta_0 x_0 - 1 \geq \beta_0 \frac{1}{\beta_0} - 1 = 0$. Thus we get

$$-1 = \frac{1}{\beta_0} - \frac{1}{\beta_0} - 1 \leq x_2 = \frac{1}{\beta_0}(\beta_0 x_0 - 1) - 1 = x_0 - \frac{1}{\beta_0} - 1 < \frac{2}{\beta_0} - \frac{1}{\beta_0} - 1 = \frac{1}{\beta_0} - 1$$

and consequently $x_2 \in [-1,\, \frac{1}{\beta_0} - 1[\subset I$. This means that if we start with $x_0 \in [n + \frac{n+1}{\beta_0},\, n + \frac{n+2}{\beta_0}[$, then $x_{2(n+1)} \in I$.

Now in the second case, we let $x_0 \in [n + \frac{n+2}{\beta_0},\, n + 1 + \frac{n+2}{\beta_0}[$. Then we see that

$$x_1 = \beta_0 x_0 - 1 \geq \beta_0(n + \frac{n+2}{\beta_0}) - 1 = n\beta_0 + n + 1 > 0,$$

$x_2 = \frac{1}{\beta_0}(\beta_0 x_0 - 1) - 1 = x_0 - \frac{1}{\beta_0} - 1$ and therefore

$$n - 1 + \frac{n+1}{\beta_0} = n + \frac{n+2}{\beta_0} - \frac{1}{\beta_0} - 1 \leq x_2 < n + 1 + \frac{n+2}{\beta_0} - \frac{1}{\beta_0} - 1 = n + \frac{n+1}{\beta_0}.$$

Now we assume that $x_0 \in [-1 + \frac{1}{\beta_0},\, \frac{1}{\beta_0}[$. Then it follows that $x_1 = \beta_0 x_0 - 1$ and

$$-\beta_0 = \beta_0(-1 + \frac{1}{\beta_0}) - 1 \leq x_1 < \beta_0 \frac{1}{\beta_0} - 1 = 0,$$

$x_1 < 0$, $x_2 = \frac{1}{\beta_0}(\beta_0 x_0 - 1) + 1 = x_0 - \frac{1}{\beta_0} + 1$ and hence

$$0 = -1 + \frac{1}{\beta_0} - \frac{1}{\beta_0} + 1 \leq x_2 < \frac{1}{\beta_0} - \frac{1}{\beta_0} + 1 = 1.$$

Consequently, we get $x_2 \in [0,\, 1[\subset I$. This implies that if $x_0 \in [n + \frac{n+2}{\beta_0},\, n + 1 + \frac{n+2}{\beta_0}[$, then $x_{2(n+1)} \in I$.

The case where $x_0 < 1 - \frac{1}{\beta_0}$ is similar and will be omitted.

Now we consider the case where $\frac{1}{2} < \beta_0 < 1$. Then $1 < \beta_1 = \frac{1}{\beta_0} < 2$ and $0 < \frac{1}{\beta_0} - 1 < 1$ (therefore this situation is different from the previous where $0 < \beta \leq \frac{1}{2}$). In this case we have $I = [-\frac{1}{\beta_0} + 1,\, \frac{1}{\beta_0} - 1[\subset [-1,\, 1[$.

Now we let $x_0 \in I_{1+} = [\frac{1}{\beta_0} - 1,\ \frac{2}{\beta_0} - 2[$, then

$$x_1 = \beta_0 x_0 - 1 < \beta_0(\frac{2}{\beta_0} - 2) - 1 = 1 - 2\beta_0 < 0 \quad (\text{since } \beta_0 > \frac{1}{2}).$$

In addition, we acquire $x_2 = \frac{1}{\beta_0}(\beta_0 x_0 - 1) + 1 = x_0 - \frac{1}{\beta_0} + 1$ and it follows that

$$0 = \frac{1}{\beta_0} - 1 - \frac{1}{\beta_0} + 1 \leq x_0 - \frac{1}{\beta_0} + 1 < \frac{2}{\beta_0} - 2 - \frac{1}{\beta_0} + 1 = \frac{1}{\beta_0} - 1,$$

and, consequently, we see that $x_2 \in I$. Furthermore, if $x_0 \in I_{2+} = [2, \frac{2}{\beta_0}[$ then $x_2 \in I$. Moreover, in similar symmetric cases where $x_0 \in [2 - \frac{2}{\beta_0},\ 1 - \frac{1}{\beta_0}[= I_{1-}$ or $x_0 \in [-\frac{2}{\beta_0},\ -2[= I_{2-}$, then $x_2 \in I$ (see Fig. 3).

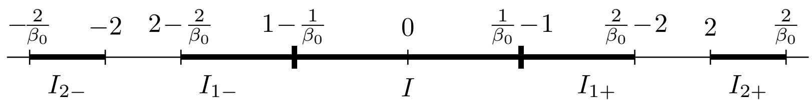

Figure 3: If $x_0 \in I_{2-} \cup I_{1-} \cup I_{1+} \cup I_{2+}$, then $x_2 \in I$ ($\frac{1}{2} < \beta_0 < 1$).

Now let $\frac{2}{\beta_0} - 2 \leq x_0 < 2$. Then

$$1 - 2\beta_0 = \beta_0(\frac{2}{\beta_0} - 2) - 1 \leq x_1 = \beta_0 x_0 - 1 < 2\beta_0 - 1.$$

If $\frac{2}{\beta_0} - 2 \leq x_0 < \frac{1}{\beta_0}$, then $x_1 < 0$. Hence we see that $x_2 = x_0 - \frac{1}{\beta_0} + 1$ and it follows that

$$\frac{1}{\beta_0} - 1 = \frac{2}{\beta_0} - 2 - \frac{1}{\beta_0} + 1 \leq x_2 < \frac{1}{\beta_0} - \frac{1}{\beta_0} + 1 = 1.$$

Since $I_{1+} = \left[\frac{1}{\beta_0} - 1,\ \frac{2}{\beta_0} - 2\right[$ and $\frac{1}{\beta_0} - 1 \leq x_2 < 1$, then $x_2 \in I_{1+}$ only if $1 \leq \frac{2}{\beta_0} - 2$ or $\beta_0 \leq \frac{2}{3}$. In this case $x_4 \in I$. If, on the other hand $\beta_0 > \frac{2}{3}$, then for all $x_0 \in [\frac{2}{\beta_0} - 2,\ \frac{3}{\beta_0} - 3[= I_{3+}$ corresponding $x_2 \in I_{1+}$ (consequently $x_4 \in I$). However, if we continue further with $\beta_0 > \frac{2}{3}$ and

$$\frac{3}{\beta_0} - 3 \leq x_0 < \frac{1}{\beta_0},$$

then we obtain

$$x_3 = \beta_0(x_0 - \tfrac{1}{\beta_0} + 1) - 1 = \beta_0(x_0 + 1) - 2 < \beta_0(\tfrac{1}{\beta_0} + 1) - 2 = \beta_0 - 1 < 0,$$
$$x_4 = \tfrac{1}{\beta_0}(\beta_0(x_0 + 1) - 2) + 1 = x_0 + 2 - \tfrac{2}{\beta_0}$$

and we get

$$\frac{1}{\beta_0} - 1 = \frac{3}{\beta_0} - 3 + 2 - \frac{2}{\beta_0} \leq x_4 < \frac{1}{\beta_0} + 2 - \frac{2}{\beta_0} = 2 - \frac{1}{\beta_0}.$$

Now note that if

$$2 - \frac{1}{\beta_0} \le \frac{3}{\beta_0} - 3 \text{ or equivalent } \ \beta_0 \le \frac{4}{5},$$

then $x_4 \in I_{1+} \cup I_{3+}$ and all the initial conditions from the segment $[\frac{3}{\beta_0} - 3,\ \frac{1}{\beta_0}[$ produce eventually periodic solutions. However, if $\beta_0 > \frac{4}{5}$, then only the initial condition $x_0 \in [\frac{3}{\beta_0} - 3,\ \frac{4}{\beta_0} - 4[= I_{4+}$ produces eventually periodic solutions and we can continue further with $\beta_0 > \frac{4}{5}$ and

$$\frac{4}{\beta_0} - 4 \le x_0 < \frac{1}{\beta_0}.$$

Since β_0 is fixed, then $\exists n \in \{2, 3, 4, ...\}$ such that $(n-1) - \frac{n-2}{\beta_0} \le \frac{n}{\beta_0} - n$ holds or equivalent form $\beta_0 \le \frac{2n-2}{2n-1}$. This implies that all the initial conditions x_0 from the segment $[\frac{1}{\beta_0} - 1,\ \frac{1}{\beta_0}[$ produce eventually periodic solutions.

The case where $x_1 \ge 0$ (that is, $\frac{1}{\beta_0} \le x_0 < 2$) is similar and is omitted.

We can obtain very similar results if we start with $x_0 \in [-2,\ 2 - \frac{2}{\beta_0}[$ and we can conclude that all the solutions with x_0 from the previously mentioned segment become eventually periodic.

Moreover, we remark that if $x_{2n} - x_{2n+2} = 1 + \frac{1}{\beta_0}$, $x_{2n} > \frac{2}{\beta_0}$, $x_{2n+2} > \frac{2}{\beta_0}$, then the sequence $(x_{2n})_{N \in \mathbf{N}}$ is strictly decreasing and there exists k such that $x_{2k} \in [-\frac{2}{\beta_0},\ \frac{2}{\beta_0}[$ and therefore for all the initial conditions $x_0 \notin [-\frac{2}{\beta_0},\ \frac{2}{\beta_0}[$ the corresponding solutions are eventually periodic. □

If both periodic coefficients β_0 and β_1 are greater than 1, then there exist periodic solutions with period two and other periodic solutions with even periods. For example, in [6] the authors proved the following result.

Theorem 5. *([6]) If $\beta_0 > 1$ and $\beta_1 \ge 1$, then the periodic orbit*

$$\left\{ \frac{\beta_1 - 1}{\beta_0\beta_1 - 1}, \frac{1 - \beta_0}{\beta_0\beta_1 - 1} \right\}$$

of equation (4) with (5) is an unstable periodic orbit with period two.

Now we formulate the corresponding result about the eventually periodic solutions.

Theorem 6. *If $\beta_0 > 1$ and $\beta_1 \ge 1$, then the initial conditions*

$$x_0 = \frac{\beta_0^k \beta_1^k (1 + \beta_1) - 2}{\beta_0^k \beta_1^k (\beta_0\beta_1 - 1)} > 0,\ k = 1, 2, ...,$$

produce eventually periodic solutions; precisely, $x_{2k} = \frac{\beta_1 - 1}{\beta_0\beta_1 - 1}$.

Proof. The proof is similar as in Theorem 2. □

Fig. 4 is an illustration of Theorem 6 with $k = 4$. If $\beta_0 = 1.4$ and $\beta_1 = 3$, then $x_0 \approx 1.247991449$ and $x_8 = 0.625$ that is the first point of cycle $\{0.625,\ -0.125\}$.

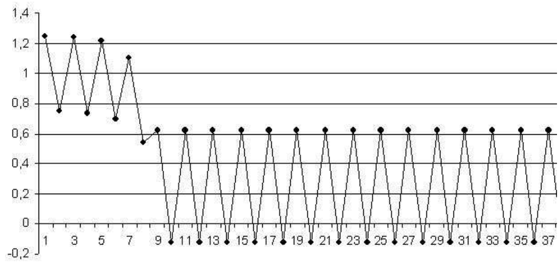

Figure 4: Solution of difference equation (4), if $\beta_0 = 1.4$, $\beta_1 = 3$ and $x_0 \approx 1.247991449$.

3 Existence of Eventually Periodic Solutions if the Internal Decay Rate is Periodic with Period Three

In this section, we consider a difference equation (4) with a sequence of periodic coefficients $(\beta_n)_{n=0}^{\infty}$ that are periodic with period three.

If the internal decay rate $(\beta_n)_{n\in\mathbf{N}}$ is a periodic with period three, then we obtain some similar properties of solutions as in Sect. 2. However, different properties emerge as well.

First of all, if all three periodic coefficients are less than 1, then there are no periodic solutions with period three, and we acquire periodic solutions with period six instead. However, if $\beta_0\beta_1\beta_2 > 1$, then we obtain a different result.

Theorem 7. *([7]) If $\beta_0\beta_1\beta_2 > 1$, then initial conditions*

$$x_0 = \frac{\beta_1\beta_2 + \beta_2 + 1}{\beta_0\beta_1\beta_2 - 1} \text{ and } x_0 = -\frac{\beta_1\beta_2 + \beta_2 + 1}{\beta_0\beta_1\beta_2 - 1}$$

form periodic solutions of equation (4) with period three; in fact, all points of the orbit are positive in first case, are negative in the second case and both orbits are unstable.

In [7], it is shown that if $\beta_0\beta_1\beta_2 > 1$ and $x_0 > \frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}$, then the solution is unbounded - going to $+\infty$ (in negative case similar).

This means that in Theorem 7 we cannot find an initial condition, which is greater than the first point of cycle which forms an eventually periodic solution. Furthermore, in this situation eventually periodic solutions exist. For instance, see Fig. 5. In fact, in this case we have $\beta_0 = 1.5$, $\beta_1 = 4$, $\beta_2 = 3$ and x_0 is determined by the following formula

$$x_0 = \frac{2 + 2\beta_2 + \beta_0\beta_1\beta_2(\beta_1\beta_2 - \beta_2 - 1)}{\beta_0\beta_1\beta_2(\beta_0\beta_1\beta_2 - 1)}.$$

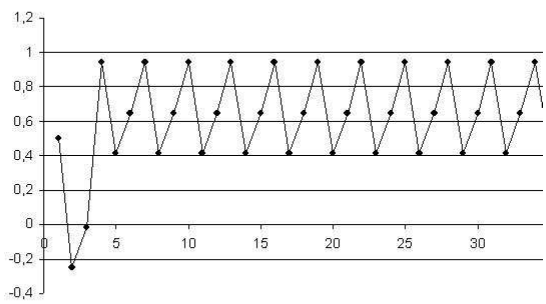

Figure 5: Solution of difference equation (4) with (6), if $\beta_0 = 1.5$, $\beta_1 = 4$, $\beta_2 = 3$ and $x_0 \approx 0,496732026$.

Then x_3 is the starting point of the period three cycle. This formula, however, does not always work. In fact, the coefficients β_0, β_1 and $\beta_2 > 1$ must satisfy particular conditions.

Now we will focus our attention on the case when there exists a segment of initial points such that all points are periodic points with period three.

Theorem 8. *([7]) Let $\beta_0\beta_1\beta_2 = 1$. Then the following statements are true*
1) if $1-\beta_1\beta_2-\beta_2 = 0$ (this equality holds when $\beta_0 > 1$, $\beta_1 = \frac{1}{\beta_0-1}$ and $\beta_2 = \frac{\beta_0-1}{\beta_0}$), then every initial condition $x_0 \in [-1, -\frac{1}{\beta_0}[\cup[\frac{1}{\beta_0}, 1[$ produces cycles with period three which are stable periodic orbits except when $x_0 = \frac{1}{\beta_0}$ and $x_0 = -1$;
2) if $\beta_2-\beta_1\beta_2-1 = 0$ (this equality holds when $\beta_0 > 0$, $\beta_1 = \frac{1}{\beta_0+1}$ and $\beta_2 = \frac{\beta_0+1}{\beta_0}$), then every initial condition $x_0 \in [-\frac{1}{\beta_0}, \frac{1}{\beta_0}[$ produces cycles with period three which are stable periodic orbits except when $x_0 = 0$ and $x_0 = -\frac{1}{\beta_0}$;
3) if $1 + \beta_2 - \beta_1\beta_2 = 0$ (this equality holds when $0 < \beta_0 < 1$, $\beta_1 = \frac{1}{1-\beta_0}$ and $\beta_2 = \frac{1-\beta_0}{\beta_0}$), then every initial condition $x_0 \in [-1, 1[$ produces cycles with period three which are stable periodic orbits except $x_0 = 0$ and $x_0 = -1$.

The vital question to address: what will occur with the solution when x_0 does not belong to the designated segment of Theorem 8? Here we will analyze only case 1) of Theorem 8 and show that all solutions that start outside the segment $[-1, -\frac{1}{\beta_0}[\cup[\frac{1}{\beta_0}, 1[$ become eventually periodic with period three.

Theorem 9. *Let $\beta_0\beta_1\beta_2 = 1$. If $\beta_0 > 1$, $\beta_1 = \frac{1}{\beta_0-1}$ and $\beta_2 = \frac{\beta_0-1}{\beta_0}$, then every initial condition*

$$x_0 \notin [-1, -\frac{1}{\beta_0}[\cup[\frac{1}{\beta_0}, 1[$$

produces eventually periodic solution with period three.

Proof. First, we start with $x_0 \geq 1$. We denote $I_1 = [-1, -\frac{1}{\beta_0}[$, $I_2 = [\frac{1}{\beta_0}, 1[$ and $I = I_1 \cup I_2$. Our goal is to show that there exists $k \in \mathbf{N}$ such that $x_{3k} \in I$. Since

$\beta_0 > 1$ and $x_0 \geq 1$, then it follows that

$$\begin{aligned} x_1 &= \beta_0 x_0 - 1 > 0, \\ x_2 &= \beta_1 x_1 - 1 = \tfrac{1}{\beta_0 - 1}(\beta_0 x_0 - 1) - 1 = \tfrac{\beta_0(x_0-1)}{\beta_0-1} \geq 0, \\ x_3 &= \beta_2 x_2 - 1 = \tfrac{\beta_0-1}{\beta_0}\tfrac{\beta_0(x_0-1)}{\beta_0-1} - 1 = x_0 - 2. \end{aligned}$$

Note that if $x_0 \in [1, 2 - \frac{1}{\beta_0}[\cup[2 + \frac{1}{\beta_0}, 3[$, then $x_3 \in I$. Also if $x_0 \in [2 - \frac{1}{\beta_0}, 2 + \frac{1}{\beta_0}[$, then $x_3 \in [-\frac{1}{\beta_0}, \frac{1}{\beta_0}[$.

Now observe that if $x_0 \geq 3$, then we determine the next iterations of our solution

$$\begin{aligned} x_4 &= \beta_0(x_0 - 2) - 1 > 0, \\ x_5 &= \tfrac{1}{\beta_0-1}(\beta_0(x_0 - 2) - 1) - 1 = \tfrac{\beta_0(x_0-3)}{\beta_0-1} \geq 0, \\ x_6 &= \tfrac{\beta_0-1}{\beta_0}\tfrac{\beta_0(x_0-3)}{\beta_0-1} - 1 = x_0 - 4. \end{aligned}$$

Hence we conclude that if $x_0 \in [3, 4 - \frac{1}{\beta_0}[\cup[4 + \frac{1}{\beta_0}, 5[$, then $x_6 \in I$; furthermore, if $x_0 \in [4 - \frac{1}{\beta_0}, 4 + \frac{1}{\beta_0}[$, then $x_6 \in [-\frac{1}{\beta_0}, \frac{1}{\beta_0}[$. Now note that it is possible that $x_0 \geq 5$.

Inductively, we conclude that there exists $k \in \mathbf{N}$ such that

$$x_0 \in [2k-1, 2k - \frac{1}{\beta_0}[\cup[2k + \frac{1}{\beta_0}, 2k+1[\text{ and then } x_{3k} \in I \text{ or}$$

$$x_0 \in [2k - \frac{1}{\beta_0}, 2k + \frac{1}{\beta_0}[\text{ and then } x_{3k} \in [-\frac{1}{\beta_0}, \frac{1}{\beta_0}[,\ k = 1, 2, 3, ...$$

Similarly if we start with $x_0 < -1$, we conclude that there exists $k \in \mathbf{N}$ such that $x_0 \in [-2k-1, -2k - \frac{1}{\beta_0}[\cup[-2k + \frac{1}{\beta_0}, -2k+1[$ and then $x_{3k} \in I$ or $x_0 \in [-2k - \frac{1}{\beta_0}, -2k + \frac{1}{\beta_0}[$ and then $x_{3k} \in [-\frac{1}{\beta_0}, \frac{1}{\beta_0}[$, $k = 1, 2, 3, \ldots$.

This means that all initial conditions

$$x_0 \in \bigcup_{i=-1}^{-\infty}[2i + \frac{1}{\beta_0}, 2i+2-\frac{1}{\beta_0}[\cup\bigcup_{i=1}^{+\infty}[2i + \frac{1}{\beta_0}, 2i+2-\frac{1}{\beta_0}[\cup[-2 + \frac{1}{\beta_0}, -1[\cup[1, 2 - \frac{1}{\beta_0}[$$

produce eventually periodic solutions.

Now our problem is with the initial conditions that are in the segment $[-\frac{1}{\beta_0}, \frac{1}{\beta_0}[$.

So let $0 \leq x_0 < \frac{1}{\beta_0}$. Then we see that

$$\begin{aligned} x_1 &= \beta_0 x_0 - 1 < 0, \\ x_2 &= \tfrac{1}{\beta_0-1}(\beta_0 x_0 - 1) + 1 = \tfrac{\beta_0(x_0+1)-2}{\beta_0-1}. \end{aligned}$$

Notice that the inequality $\beta_0(x_0 + 1) - 2 \geq 0$ holds if $x_0 \geq \frac{2}{\beta_0} - 1$. Also if $\beta_0 \geq 2$, then $0 \geq \frac{2}{\beta_0} - 1$ and the last inequality is always true. Therefore, if $\frac{2}{\beta_0} - 1 \leq x_0 < \frac{1}{\beta_0}$, then $x_2 \geq 0$. In addition, we see that

$$x_3 = \frac{\beta_0 - 1}{\beta_0}\frac{\beta_0(x_0 + 1) - 2}{\beta_0 - 1} - 1 = x_0 - \frac{2}{\beta_0},$$

$-1 < x_3 = x_0 - \frac{2}{\beta_0} < -\frac{1}{\beta_0}$ and therefore $x_3 \in I_1$.

Furthermore, if $0 \leq x_0 < \frac{2}{\beta_0} - 1$ (it is possible only if $1 < \beta_0 < 2$!), then $x_2 < 0$ and

$$x_3 = \frac{\beta_0 - 1}{\beta_0} \frac{\beta_0(x_0 + 1) - 2}{\beta_0 - 1} + 1 = x_0 + 2 - \frac{2}{\beta_0}.$$

We now conclude that

$$0 < 2 - \frac{2}{\beta_0} \leq x_3 = x_0 + 2 - \frac{2}{\beta_0} < \frac{2}{\beta_0} - 1 + 2 - \frac{2}{\beta_0} = 1.$$

Moreover, if $x_0 + 2 - \frac{2}{\beta_0} \geq \frac{1}{\beta_0}$, then $x_3 \in I_2$. Otherwise if $0 \leq x_0 < \frac{3}{\beta_0} - 2$ (it is possible only when $1 < \beta_0 < \frac{3}{2}$!) then $x_3 \notin I$. In this case we continue with this iterative and inductive process by determining the next three iterations of the solution

$$\begin{aligned} x_4 &= \beta_0 \tfrac{\beta_0(x_0+2)-2}{\beta_0} - 1 = \beta_0(x_0 + 2) - 3 < \beta_0(\tfrac{3}{\beta_0} - 2 + 2) - 3 = 0, \\ x_5 &= \tfrac{1}{\beta_0 - 1}(\beta_0(x_0 + 2) - 3) + 1 = \tfrac{\beta_0(x_0+3)-4}{\beta_0-1} < \tfrac{\beta_0(\frac{3}{\beta_0}-2+3)-4}{\beta_0-1} = 1. \end{aligned}$$

Now we see that only two cases are possible. In the first case if we let $\frac{4}{\beta_0} - 3 \leq x_0 < \frac{3}{\beta_0} - 2$, then $x_5 \geq 0$ and

$$x_6 = \frac{\beta_0 - 1}{\beta_0} \frac{\beta_0(x_0 + 3) - 4}{\beta_0 - 1} - 1 = x_0 + 2 - \frac{4}{\beta_0}.$$

Therefore $-1 = \frac{4}{\beta_0} - 3 + 2 - \frac{4}{\beta_0} \leq x_0 + 2 - \frac{4}{\beta_0} < \frac{3}{\beta_0} - 2 + 2 - \frac{4}{\beta_0} = -\frac{1}{\beta_0}$ and $x_6 \in I_1$.

In the second case, $0 \leq x_0 < \frac{4}{\beta_0} - 3$ (it is possible only when $1 < \beta_0 < \frac{4}{3}$!) and $x_5 < 0$. Therefore

$$x_6 = \frac{\beta_0 - 1}{\beta_0} \frac{\beta_0(x_0 + 3) - 4}{\beta_0 - 1} + 1 = x_0 + 4 - \frac{4}{\beta_0}.$$

From the restriction of x_0, we obtain that $0 \leq x_6 < 1$. This means that if $x_6 \geq \frac{1}{\beta_0}$ or $x_0 \geq \frac{5}{\beta_0} - 4$, then $x_6 \in I_2$. However, if $x_6 < \frac{1}{\beta_0}$ or $0 \leq x_0 < \frac{5}{\beta_0} - 4$ (it is possible only when $1 < \beta_0 < \frac{5}{4}$!), then $x_6 \notin I$.

Inductively, we conclude that for every fixed $1 < \beta_0 < 2$ and every fixed $0 \leq x_0 < \frac{1}{\beta_0}$ there exists $M \in \mathbf{N},\ M \geq 2$ such that

$$\frac{2M + 1}{2M} \leq \beta_0 < \frac{2M}{2M - 1} \quad \text{or} \quad \frac{2M}{2M - 1} \leq \beta_0 < \frac{2M - 1}{2M - 2}$$

(this means that $\frac{2M+1}{\beta_0} - 2M \leq 0$ or $\frac{2M}{\beta_0} - (2M - 1) \leq 0$) and there exists $n \in \{1, 2, ..., M\}$ such that

$$\begin{aligned} &\tfrac{2n}{\beta_0} - (2n - 1) \leq x_0 < \tfrac{2n-1}{\beta_0} - (2n - 2) \text{ and then } \ x_{3n} \in I_1 \ \text{ or} \\ &\tfrac{2n+1}{\beta_0} - 2n \leq x_0 < \tfrac{2n}{\beta_0} - (2n - 1) \text{ and then } \ x_{3n} \in I_2. \end{aligned}$$

The case where $-\frac{1}{\beta_0} \leq x_0 < 0$ is similar. □

The last part of proof shows that if β_0 is very close to 1 and the corresponding x_0 is close to 0, then it will require many more iterations until x_n belongs to I as β_0 is not so close to 1.

Example 1. For instance, if we let $M = 10$, then $\frac{2M+1}{2M} = \frac{21}{20} = 1.05$ and $\frac{2M}{2M-1} = \frac{20}{19} \approx 1.0526$. Let $\beta_0 = 1.05$. Then $\frac{2M-1}{\beta_0} - (2M-2) = \frac{19}{1.05} - 18 \approx 0.0952$ and $\frac{2M}{\beta_0} - (2M-1) = \frac{20}{1.05} - 19 \approx 0.0476$. If $x_0 = 0.05$, then $x_{30} \approx -0.99761 \in I_1$, and if $x_0 = 0.03$, then $x_{30} \approx 0.98238 \in I_2$. See Fig.6 with $x_0 = 0.05$.

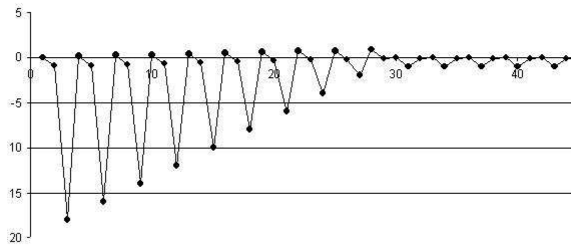

Figure 6: Solution of difference equation (4), if $\beta_0 = 1.05$ and $x_0 = 0.05$.

4 Conclusion

Our main goal of this paper was to show the existence of eventually periodic solutions for the single neuron model (4). However, we did not consider all the possible cases. In fact, the most challenging cases emerge in Theorems 4 and 9 where the solutions of (4) are either periodic or eventually periodic.

In [18], x denotes the activation level of a neuron. First of all, if one neuron works as the proposed model suggests, we can then interpret a stationary state as an equilibrium state where the activation level is constant. Second of all, the periodic orbit indicates the periodic changes of the activation level. On one hand, a chaotic orbit implies unpredictable changes of the activation level. On the other hand, we cannot provide an accurate interpretation of the unstable orbit that gradually diverges to infinity where the activation level increases without restriction. In this paper, we studied the existence and patterns of eventually periodic solutions; in particular, we examined the stability character of periodic orbits where the activation level is bounded.

Finally, we conclude that our model (4) with the signal function (3) and an internal periodic decay rate (with period two and period three), describe a substantially different situation in comparison to [5], [8], [17], [20], [21], [22], [23], [24], [25] and [18]. In the mentioned papers the model has not been studied with a periodic coefficient.

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