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EXISTENCE THEORY FOR THE RADIALLY SYMMETRIC CONTACT LENS EQUATION*

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Abstract. In this paper we present a variational formulation of the problem of determining the elastic stresses in a contact lens on an eye and the induced suction pressure distribution in the tear film between the eye and the lens. This complements the force-balance derivation that we used in earlier work [K. L. Maki and D. S. Ross, *J. Bio. Sys.*, 22 (2014), pp. 235–248]. We investigate the existence of solutions of the relevant boundary value problem for the singular, second-order Euler–Lagrange equation. We prove that, for lenses of constant thickness, solutions exist. We present an example to show that in some cases in which the lens thickness increases with distance from the lens center no solution exists.

Key words. contact lens, ordinary differential equation, existence, suction pressure

AMS subject classifications. 15A15, 15A09, 15A23

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1. Introduction.

About one in ten Americans wears contact lenses [28]. However, for every three persons who wear contact lenses, one person tries them but stops using them; the majority of such dropouts stop using contact lenses because they find them uncomfortable [30]. Improving comfort is a central challenge for the contact lens industry. Improving comfort will expand the market for contact lenses and could bring the benefits of contact lenses to millions. In addition, comfort will be crucial to the success of novel applications for contact lenses, such as metabolic monitoring [29], drug delivery [4], augmented-reality [1] displays, and sensory enhancement [32].

In order to understand what makes contact lenses comfortable or uncomfortable, it is important to understand the solid mechanics and fluid mechanics of the interactions of a lens with an eye. A contact lens is a suction cup; it stays on the eye because any perturbation that might jostle it off the eye induces a negative pressure in the tear film between the lens and the eye, the *postlens tear film*, which holds the lens in place. In equilibrium the pressure in the tear film *integrates* to zero, but the elastic stresses in the lens—which is distorted from its rest shape when it is on the eye—induce a pressure distribution in the tear film. That distribution mediates the eye's feeling the contact lens, and so understanding it is crucial to understanding and improving comfort. We undertook the work whose mathematical aspects we discuss in this paper in order to characterize such pressure distributions and to understand how they depend on the elastic properties of lenses and on the shapes of lenses and eyes.

In the early days of the study of contact lenses it was thought that surface tension held lenses in place [18, 21, 25]. In addition to experimental investigations [10, 11, 24], by the 1980s theoretical researchers had begun to investigate the pressure in the

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postlens tear film treated as a squeeze film [2, 7, 8, 13]. More recently researchers have addressed, computationally, the interaction of the lens with the tear film [5, 12, 16]. We developed the model that we present in the next section in order to identify the essential mechanical contributions to the suction pressure distribution. Our model is mathematically simpler than other approaches; the equation that expresses it is a single, nonlinear, singular, second-order ordinary differential equation (ODE). In this paper, we investigate the mathematical structure of the equation.

2. Suction pressure model. In recent papers we developed [22] and applied [23] a mathematical model of the elastic equilibrium of a radially symmetric, soft, hydrogel, contact lens in the tear film of an eye. The purpose of the model is to predict the suction pressure under a contact lens and to help us understand how that pressure distribution depends on the shapes of the lens and the eye.

A typical contact lens is between 50 and 200 microns thick and has a radius of about 0.7 cm. So it's thin: its thickness is a small fraction of its radius. When it is deformed from its rest shape and placed on the eye, it is submerged in the tear film. (The term *submerged* may be misleading, as the tear film is only a few microns thick.) The lens is separated from the eye by the *postlens tear film*, which is 1 or 2 microns thick; a *prelens tear film* of roughly the same thickness separates the lens from the air [27]. See Figure 1.

In Maki and Ross [22], we presented estimates to establish that bending stresses within the lens are negligible and that shear stresses in the tear film are resolved so quickly that they, too, are negligible. This leaves three stresses that must balance in equilibrium: the radial stress in the lens, the hoop stress in the lens, and the pressure in the postlens tear film which acts on the eye-facing surface of the lens. We follow common usage and refer to this pressure as the *suction pressure*; however, it is important to note that this pressure changes sign; it acts to draw the lens toward the eye at some places, and to push the lens away from the eye at others. Because it is an equilibrium distribution, it must produce no net force on the lens; it must integrate to zero. Thus the pressure distribution in the postlens tear film has the quality of a squeeze film in some regions [17] and that of an adhesive film in others [26].

An interesting quirk of this problem is that although the suction pressure is what matters in the application, it has no role in the mathematical problem; it is computed from the radial tension once that tension has been determined by solving the problem.

We consider a radially symmetric contact lens, with the unstressed shape of the lens given by the graph of a function $z = g(r)$, where r is the radial distance from the center of the lens, on a radially symmetric eye. The lens is deformed so that it conforms to the shape of the eye, which is the graph of a function $z = f(R)$, where R is the radial distance from the center of the corneal surface. In fact, the postlens tear film is between the lens and the eye, but it is so thin—on the order of two percent of the thickness of the lens—that we neglect it. The deformed shape of the contact lens on the eye is characterized by the function $R(r)$: each point $(r, g(r))$ on the undeformed lens moves to a point $(R(r), f(R(r)))$ on the deformed lens.

We take $g(r)$ and $f(R)$ to be C^2 functions on the nonnegative reals, and we take them to have bounded derivatives. This implies that there are positive numbers $m < 1$ and $M > 1$ such that

$$m < \frac{\sqrt{1 + g'(r)^2}}{\sqrt{1 + f'(R)^2}} < M$$

for all nonnegative r and R. We take $\tau(r)$, a positive C^1 function on the nonnegative

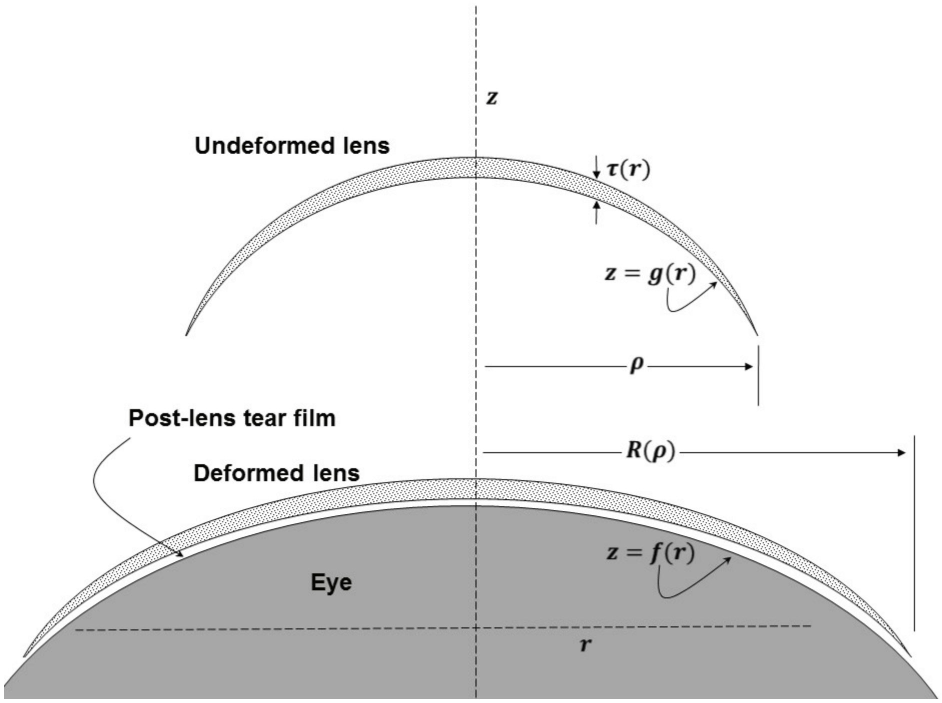

FIG. 1. *A schematic of the eye and the lens. This schematic is not drawn to scale; in particular, the thickness of the lens is generally on the order of* 100 *times that of the post-lens tear film. In the schematic, the lens is shown twice: in its undeformed state and in its deformed state, conformed to the eye. The graphs of the known functions* $z = g(r)$ *and* $z = f(r)$ *specify the shapes of the undeformed lens and the deformed lens, respectively. The unknown function for which we solve is* $R(r)$*, the radial coordinate, in the deformed state, of the point on the lens whose radial coordinate in the undeformed state is* r*. The radial coordinate of the edge of the undeformed lens is* ρ*; the radial coordinate* $R(\rho)$ *of the deformed lens is determined as part of the solution.*

reals, to be the thickness of the lens at a distance r from its center. Because of the symmetry of the problem we must have $g'(0) = f'(0) = \tau'(0) = 0$.

We treat the lens as a linearly elastic material [20]. The radial and hoop strains of the deformed lens are

$$\frac{\frac{dR}{dr}\sqrt{1 + f'(R(r))^2}}{\sqrt{1 + g'(r)^2}} - 1$$

and

$$\frac{R(r) - r}{r},$$

respectively. Because the lens is thin and we regard its stresses as uniform across its thickness, we formulate our equations in terms of tensions (force/length) rather than in terms of stresses; the tensions, in this problem, are simply the products of the stresses with the thickness $\tau(r)$. If E and σ are the Young's modulus and the Poisson ratio of the hydrogel, then the radial and hoop tensions are, respectively,

$$S_{rr}(r) = \frac{E}{1-\sigma^2}\left(\left(\frac{\frac{dR}{dr}\sqrt{1+f'(R(r))^2}}{\sqrt{1+g'(r)^2}} - 1\right) + \sigma\left(\frac{R(r)-r}{r}\right)\right)\tau(r),$$

$$S_{\theta\theta}(r) = \frac{E}{1-\sigma^2}\left(\sigma\left(\frac{\frac{dR}{dr}\sqrt{1+f'(R(r))^2}}{\sqrt{1+g'(r)^2}} - 1\right) + \left(\frac{R(r)-r}{r}\right)\right)\tau(r).$$

The Young's modulus E is positive; the Poisson's ratio σ is bounded, $0 < \sigma \leq \frac{1}{2}$. The undeformed lens lies within the cylinder $r \leq \rho$, where $\rho > 0$ is a known, specified parameter. At the edge of the lens, $r = \rho$, the radial tension must be 0 because there is nothing there to exert any force:

$$S_{rr}(\rho) = 0. \tag{2.1}$$

At the center of the lens, by symmetry,

$$R(0) = 0. \tag{2.2}$$

We can characterize equilibrium deformations as those associated with functions $R(r)$ that satisfy the boundary conditions of (2.1) and (2.2) and which minimize the energy functional:

$$\frac{1}{2}\left(\frac{E}{1-\sigma^2}\right)\int_0^\rho \left[\left(\frac{\frac{dR}{dr}\sqrt{1+f'(R(r))^2}}{\sqrt{1+g'(r)^2}} - 1\right)^2 + \left(\frac{R(r)-r}{r}\right)^2 \right.$$
$$\left. + 2\sigma\left(\frac{\frac{dR}{dr}\sqrt{1+f'(R(r))^2}}{\sqrt{1+g'(r)^2}} - 1\right)\left(\frac{R(r)-r}{r}\right)\right]\tau(r)\sqrt{1+g'(r)^2}\,r\,dr.$$

The Euler–Lagrange equation associated with this functional is

$$\frac{d}{dr}(rS_{rr}(r)) = S_{\theta\theta}(r)\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R(r))^2}}. \tag{2.3}$$

The suction pressure $p(r)$ in the tear film must balance the radial stress in the lens:

$$2\pi\int_0^r \xi p(\xi)d\xi + 2\pi r S_{rr}(r)\frac{f'(R(r))}{\sqrt{1+f'(R(r))^2}} = 0.$$

By differentiating, we obtain

$$p(r) = -\frac{1}{r}\frac{d}{dr}\left(rS_{rr}(r)\frac{f'(R(r))}{\sqrt{1+f'(R(r))^2}}\right).$$

For the analyses that we present in this paper it is convenient to introduce a new variable,

$$T(r) = \frac{1-\sigma^2}{E}\frac{rS_{rr}}{\tau(r)},$$

and to write the Euler–Lagrange equation, (2.3), as a first-order system,

$$\frac{dT}{dr} = \left(\sigma T + (1-\sigma^2)(R-r)\right) \frac{\sqrt{1+g'(r)^2}}{r\sqrt{1+f'(R(r))^2}} - \frac{\tau'}{\tau}T, \tag{2.4}$$

$$\frac{dR}{dr} = (T + (1+\sigma)r - \sigma R) \frac{\sqrt{1+g'(r)^2}}{r\sqrt{1+f'(R(r))^2}}. \tag{2.5}$$

There are three boundary conditions for the problem with which we are concerned:

$$R(0) = 0, \tag{2.6}$$

$$T(0) = 0, \tag{2.7}$$

$$T(\rho) = 0. \tag{2.8}$$

The equilibria of a contact lens are characterized by the ODE system that comprises (2.4) and (2.5) and by the boundary conditions defined in (2.6), (2.7), and (2.8). Physically realistic solutions satisfy two other conditions: $\frac{dR}{dr} > 0$ on the entire interval $[0, \rho]$ and $R > 0$ for positive r. (Note that the second condition, $R(r) > 0$ for positive r, is an immediate consequence of the first; if R increases monotonically from 0, it must be positive.) We refer to the condition $\frac{dR}{dr} > 0$ as *the monotonicity condition*. If $\frac{dR}{dr}$ were negative, the lens material would have penetrated itself, which is impossible. In fact, for real materials there will be a positive lower bound on this derivative, as there will be a limit to the degree to which a material can be compressed. In cases in which this derivative is near zero, our model no longer reflects reality; in fact, in such cases, the theory of linear elasticity does not apply. Throughout this paper we will use the notation $\Gamma = \frac{dR}{dr}(0)$, and we will consider only positive values of Γ because only those can be associated with physically realistic solutions. Note that the structure of the equations implies directly that $\frac{dT}{dr}(0) = (1+\sigma)(\Gamma - 1)$. In order to emphasize the dependence of solutions on Γ we will use the notation $R(r, \Gamma)$ and $T(r, \Gamma)$.

This paper has six sections and an appendix. In the appendix we establish the existence of solutions of the singular initial value problem defined by (2.4)–(2.7). This problem is not the essential one in the contact lens application, but we use the fact that it is well-posed in our analyses of the boundary value problems with which we are concerned. Specifically, we use the facts that for every $\Gamma > 0$ there is a unique solution, $R(r, \Gamma)$, $T(r, \Gamma)$, of (2.4)–(2.7) defined on $[0, \infty)$; that this solution depends continuously on Γ; that $R(0, \Gamma) = T(0, \Gamma) = 0$; and that $\frac{dR}{dr}(0, \Gamma) = \Gamma$ and $\frac{dT}{dr}(0, \Gamma) = (1+\sigma)(\Gamma - 1)$. In section 3 we establish the existence of solutions of the boundary value problem defined by (2.4)–(2.8) in cases in which $\tau(r)$ is constant. In section 4 we prove, with an example, that in some cases in which $\tau(r)$ increases with r no solution of this boundary value problem exists. In section 5 we present a closed-form solution for the case in which the lens has constant thickness and the eye is flat; this solution provides both an alternate proof of existence for this case and a proof of uniqueness. In section 6 we review our results in the context of the contact lens problem.

3. Existence of solutions in the case of a lens of constant thickness. We take the lens to be of uniform thickness, $\tau' \equiv 0$, and we introduce two new variables, $P(r, \Gamma) = T(r, \Gamma) + (1-\sigma)R(r, \Gamma)$ and $Q(r, \Gamma) = T(r, \Gamma) - (1+\sigma)R(r, \Gamma)$. In the limit as r approaches 0, $\frac{P(r,\Gamma)}{r}$ and $\frac{Q(r,\Gamma)}{r}$ approach $2\Gamma - (1+\sigma)$ and $-(1+\sigma)$, respectively.

In terms of these variables, the differential equations become

$$\frac{d}{dr}P = \frac{P}{r}\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R)^2}}, \tag{3.1}$$

$$\frac{d}{dr}Q = \left(\frac{-Q-2r(1+\sigma)}{r}\right)\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R)^2}}. \tag{3.2}$$

These equations have a simple structure that allows us to characterize solutions crisply.

We first prove several lemmas that will allow us to establish the existence result.

LEMMA 3.1. *The solution* $P(r,\Gamma)$ *of* (3.1) *is identically zero if* $\Gamma = \frac{1+\sigma}{2}$*, it is strictly positive if* $\Gamma > \frac{1+\sigma}{2}$*, and it is strictly negative if* $\Gamma < \frac{1+\sigma}{2}$*. The solution* $Q(r,\Gamma)$ *of* (3.2) *is strictly negative for all* Γ*, and* $Q(r,\Gamma) > -2r(1+\sigma)$ *for* $r > 0$*.*

Proof. We establish in the appendix that $P(r,\Gamma)$ is smooth at $r=0$. For $r>0$, the form of (3.1) implies that $P(r,\Gamma)$ is either positive for all r or negative for all r. This establishes the result for $\Gamma \neq \frac{1+\sigma}{2}$, and the fact that $P(r, \frac{1+\sigma}{2}) \equiv 0$ follows from this function's continuous dependence on Γ.

The function $Q(r,\Gamma)$ is zero at $r=0$ and is smooth at 0, and it has a negative derivative near 0, so it is negative in a small neighborhood of 0. If δ is the right endpoint of such a neighborhood, $Q(\delta,\Gamma) \le 0$, by continuity. But if $Q(\delta,\Gamma)=0$, then $\frac{dQ}{dr}(\delta,\Gamma) < 0$, contradicting the fact that $Q(r,\Gamma) < 0$ in a small one-sided neighborhood to the left of δ. So $Q(r,\Gamma)$ is strictly negative. The smoothness of $Q(r,\Gamma)$ near 0 and the fact that $\frac{Q(r,\Gamma)}{r}$ approaches $-(1+\sigma)$ as r approaches 0 imply that there is a neighborhood of 0 in which $Q(r,\Gamma) > -2r(1+\sigma)$. Because

$$\frac{d}{dr}(-Q(r,\Gamma)-2r(1+\sigma)) = \left(\frac{Q(r,\Gamma)+2r(1+\sigma)}{r}\right)\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R(r,\Gamma))^2}} - 2(1+\sigma),$$

if $-Q(r,\Gamma)-2r(1+\sigma)$ were to approach 0 at some positive r, its derivative would be negative in a neighborhood of that point, which would be a contradiction. ∎

LEMMA 3.2. *If* $\Gamma \ge \frac{1+\sigma}{2}$*, then* $R(r,\Gamma)$ *satisfies the monotonicity condition.*

Proof. Lemma 3.1 established that for $\Gamma \ge \frac{1+\sigma}{2}$, $P(r,\Gamma) \ge 0$ and

$$0 > Q(r,\Gamma) > -2r(1+\sigma).$$

Thus

$$\frac{P(r,\Gamma)+Q(r,\Gamma)}{2} > -r(1+\sigma).$$

By rewriting (2.5) in the form

$$\frac{d}{dr}R(r,\Gamma) = \frac{\frac{P(r,\Gamma)+Q(r,\Gamma)}{2} + r(1+\sigma)}{r}\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R)^2}} > 0,$$

we find that $R(r,\Gamma)$ is strictly monotone in r. ∎

LEMMA 3.3. *There are constants* $B > 1$ *and* C *such that for* $\Gamma > \frac{1+\sigma}{2}$ *and* $r > 0$

$$P(r,\Gamma) \le (2\Gamma - (1+\sigma))r(1+C^2r^2)^{\frac{B-1}{2}}$$

and

$$R(r,\Gamma) \le (2\Gamma)r(1+C^2r^2)^{\frac{B-1}{2}}.$$

Proof. The conditions on the function $g(r)$ ensure that $\sqrt{1+g'(r)^2}$ is bounded by a positive constant B, and that for C sufficiently large,

$$\sqrt{1+g'(r)^2} \leq 1 + (B-1)\frac{C^2r^2}{1+C^2r^2}.$$

Thus it follows from (3.1) that

$$\frac{\frac{d}{dr}P(r,\Gamma)}{r} \leq \frac{P(r,\Gamma)}{r^2}\left(1 + (B-1)\frac{C^2r^2}{1+C^2r^2}\right)$$

or

$$\frac{d}{dr}\log\left(\frac{P(r,\Gamma)}{r}\right) \leq (B-1)\frac{C^2r}{1+C^2r^2} = \frac{(B-1)}{2}\frac{d}{dr}\log(1+C^2r^2).$$

By integrating and applying the initial condition on $\frac{P}{r}$ at $r=0$, we obtain

$$P(r,\Gamma) \leq (2\Gamma - (1+\sigma))r(1+C^2r^2)^{\frac{B-1}{2}}.$$

The inequality

$$R(r,\Gamma) \leq \left(\Gamma - \frac{(1+\sigma)}{2}\right)r(1+C^2r^2)^{\frac{B-1}{2}} + (1+\sigma)r \leq (2\Gamma)r(1+C^2r^2)^{\frac{B-1}{2}}$$

follows from Lemma 3.1 and the bound on $P(r,\Gamma)$ because $R(r,\Gamma) = \frac{P(r,\Gamma)-Q(r,\Gamma)}{2}$. ☐

LEMMA 3.4. *For a fixed $\eta > 0$, as $\Gamma \to \infty$, $R(\eta,\Gamma) \to \infty$.*

Proof. Lemma 3.3 established that

$$R(r,\Gamma) \leq (2\Gamma)r(1+C^2r^2)^{\frac{B-1}{2}}$$

for sufficiently large Γ. Thus, if we let

$$K = 2(1+C^2\eta^2)^{\frac{B-1}{2}},$$

we have $R(r,\Gamma) \leq K\Gamma r$ on $[0,\eta]$. The properties of $f(R)$ and our definition of the bound m imply that for sufficiently large constants V,

$$1 - \frac{(1-m)V^2R^2}{1+V^2R^2} \leq \frac{1}{\sqrt{1+f'(R)^2}}$$

for all R. Thus, on $[0,\eta]$, we have

$$1 - \frac{(1-m)V^2K^2\Gamma^2r^2}{1+V^2K^2\Gamma^2r^2} \leq \frac{1}{\sqrt{1+f'(R)^2}}.$$

Thus it follows from (3.1) that

$$\frac{\frac{d}{dr}P(r,\Gamma)}{r} \geq \frac{P(r,\Gamma)}{r^2}\left(1 - \frac{(1-m)V^2K^2\Gamma^2r^2}{1+V^2K^2\Gamma^2r^2}\right)$$

or

$$\frac{d}{dr}\log\left(\frac{P(r,\Gamma)}{r}\right) \geq -(1-m)\frac{V^2K^2\Gamma^2r}{1+V^2K^2\Gamma^2r^2} = -\frac{(1-m)}{2}\frac{d}{dr}\log(1+V^2K^2\Gamma^2r^2).$$

By integrating and applying the boundary condition on $\frac{P(r,\Gamma)}{r}$ at $r = 0$, we obtain

$$P(r,\Gamma) \geq \frac{2\Gamma - (1+\sigma)}{(1 + V^2K^2\Gamma^2r^2)^{\frac{1-m}{2}}}.$$

Thus

$$P(\eta,\Gamma) \geq \frac{2\Gamma^m}{(VK\eta)^{1-m}} + o(1)$$

as $\Gamma \to \infty$, which, because m is positive and $R(r,\Gamma) = \frac{P(r,\Gamma)-Q(r,\Gamma)}{2}$, establishes the lemma. ∎

Lemmas 3.1–3.4 provide the groundwork we need to prove the existence of a solution of the boundary value problem defined by (2.4)–(2.8).

THEOREM 3.5. *There is a pair of functions, $R(r,\Gamma)$ and $T(r,\Gamma)$, that constitute a physically realistic solution of the boundary value problem defined by* (2.4)–(2.8).

Proof. Lemma 3.1 implies that

$$\frac{P\big(r, \frac{(1+\sigma)}{2}\big) + Q\big(r, \frac{(1+\sigma)}{2}\big)}{2} < 0$$

and

$$P\left(r, \frac{(1+\sigma)}{2}\right) = 0$$

for all $r > 0$. Because

$$\frac{P\big(r, \frac{(1+\sigma)}{2}\big) + Q\big(r, \frac{(1+\sigma)}{2}\big)}{2} = T\left(r, \frac{(1+\sigma)}{2}\right) - \sigma R\left(r, \frac{(1+\sigma)}{2}\right)$$

and

$$P\left(r, \frac{(1+\sigma)}{2}\right) = T\left(r, \frac{(1+\sigma)}{2}\right) + (1-\sigma)R\left(r, \frac{(1+\sigma)}{2}\right),$$

this further implies that $R\big(r, \frac{(1+\sigma)}{2}\big) > 0$ and $T\big(r, \frac{(1+\sigma)}{2}\big) < 0$. Lemma 3.1 established that $Q(r,\Gamma)$ is bounded below independent of Γ, and Lemma 3.4 established that $R(r,\Gamma) \to \infty$ as $\Gamma \to \infty$, so for Γ sufficiently large the function

$$T(r,\Gamma) = (1+\sigma)R(r,\Gamma) + Q(r,\Gamma) > 0.$$

Thus, by continuous dependence on Γ, there is a $\Gamma > \frac{1+\sigma}{2}$ at which $T(r,\Gamma) = 0$. Lemma 3.2 established that for this value of Γ, $R(r,\Gamma)$ satisfies the monotonicity condition. ∎

4. Nonexistence of a solution in a case in which the lens thickness increases as a function of distance from the lens center. Numerical experiments suggested that no solution of the boundary value problem exists in certain cases in which the lens thickness $\tau(r)$ increases rapidly as a function of r. Here we simply establish that in some such cases no solution exists.

We consider cases in which the eye is flat and the lens has radius 1; that is, $f'(R) \equiv 0$ and $\rho = 1$. We consider the particular lens shape for which

$$\sqrt{1 + g'(r)^2} = 1 + r^2,$$

and we consider exponentially increasing thickness profiles: $\tau(r) = e^{\lambda r}$, with λ a positive real number. We show that for λ sufficiently large no physically realistic solution of the boundary value problem defined by (2.4)–(2.8) exists. In such cases (2.4) and (2.5) become

$$\frac{d}{dr}T(r,\Gamma) = \big(\sigma T + (1-\sigma^2)(R(r,\Gamma) - r)\big)\,\frac{1+r^2}{r} - \lambda T(r,\Gamma), \tag{4.1}$$

$$\frac{d}{dr}R(r,\Gamma) = (T + (1+\sigma)r - \sigma R(r,\Gamma))\,\frac{1+r^2}{r}. \tag{4.2}$$

We prove this result via two lemmas. We first establish in Lemma 4.1 a negative, $\mathcal{O}(\frac{1}{\lambda})$ lower bound on T. Then in Lemma 4.2, we use this bound to prove that $R - r$ is larger than a positive multiple of r^3 in the limit of large λ. We then use both of these bounds to prove, in Theorem 4.3, that $T(1,\Gamma)$ is strictly positive, for all Γ, for sufficiently large λ. Because $T(1,\Gamma)$ must equal 0 for a physically realistic solution, this establishes that no such solution exists.

Lemma 4.1. *If $T(r,\Gamma)$ and $R(r,\Gamma)$ constitute a physically realistic solution of* (4.1) *and* (4.2)*, then*

$$T(r,\Gamma) \geq -2(1+\sigma)e^{\frac{\sigma}{2}}\frac{(1-e^{-\lambda r})}{\lambda}.$$

Proof. A physically realistic solution has $R(r,\Gamma) \geq 0$, so (4.1) implies that

$$\frac{dT}{dr} \geq \left(\sigma\left(\frac{1}{r} + r\right) - \lambda\right)T - (1-\sigma^2)(1+r^2).$$

By using the integrating factor $r^{-\sigma}e^{\lambda r - \frac{\sigma}{2}r^2}$, we obtain

$$T \geq -(1-\sigma^2)r^{\sigma}e^{-\lambda r + \frac{\sigma}{2}r^2}\int_0^r e^{\lambda\xi - \frac{\sigma}{2}\xi^2}(\xi^{2-\sigma} + \xi^{-\sigma})d\xi.$$

The simple bounds $1 \leq e^{\frac{\sigma}{2}r^2} \leq e^{\frac{\sigma}{2}}$ on $[0,1]$ yield

$$T \geq -(1-\sigma^2)r^{\sigma}e^{-\lambda r + \frac{\sigma}{2}}\int_0^r e^{\lambda\xi}(\xi^{2-\sigma} + \xi^{-\sigma})d\xi \geq -2(1-\sigma^2)r^{\sigma}e^{-\lambda r + \frac{\sigma}{2}}\int_0^r e^{\lambda\xi}\xi^{-\sigma}d\xi.$$

Because $0 < \sigma \leq \frac{1}{2}$,

$$\int_0^r e^{\lambda\xi}\xi^{-\sigma}d\xi = \int_0^r \sum_0^\infty \frac{\lambda^j\xi^{j-\sigma}}{j!}d\xi = \sum_0^\infty \frac{\lambda^j r^{j+1-\sigma}}{j!(j+1-\sigma)}$$
$$\leq \frac{r^{-\sigma}}{1-\sigma}\sum_0^\infty \frac{\lambda^j r^{j+1}}{(j+1)!} = \frac{r^{-\sigma}}{(1-\sigma)}\frac{(e^{\lambda r}-1)}{\lambda}.$$

So,

$$T(r,\Gamma) \geq -2(1+\sigma)e^{\frac{\sigma}{2}}\frac{(1-e^{-\lambda r})}{\lambda}. \qquad \square$$

LEMMA 4.2. *If $T(r,\Gamma)$ and $R(r,\Gamma)$ constitute a physically realistic solution of* (4.1) *and* (4.2)*, then*

$$R(r,\Gamma) \geq -\frac{e^{\sigma}}{\lambda}\frac{4(1+\sigma)^2}{\sigma(2+\sigma)} + r + e^{-\frac{\sigma}{2}}\left(\frac{1-\sigma-\sigma^2}{3+\sigma}\right)r^3.$$

Proof. By incorporating into (4.2) the lower bound on $T(r,\Gamma)$ that we established in Lemma 4.1, we obtain

$$\frac{dR}{dr} + \sigma\left(\frac{1}{r}+r\right)R \geq \left(-2(1+\sigma)e^{\frac{\sigma}{2}}\frac{(1-e^{-\lambda r})}{\lambda}\left(\frac{1}{r}+r\right) + (1+\sigma)(1+r^2)\right).$$

By multiplying by the integrating factor $r^{\sigma}e^{\frac{\sigma}{2}r^2}$ and integrating, we obtain

$$R \geq r^{-\sigma}e^{\frac{-\sigma}{2}r^2}\left(-2(1+\sigma)e^{\frac{\sigma}{2}}\int_0^r \frac{(1-e^{-\lambda\xi})}{\lambda}e^{\frac{\sigma}{2}\xi^2}(\xi^{\sigma-1}+\xi^{\sigma+1})d\xi \right.$$
$$\left. + (1+\sigma)\int_0^r e^{\frac{\sigma}{2}\xi^2}(\xi^{\sigma}+\xi^{\sigma+2})d\xi\right).$$

By using the bounds $1-e^{-\lambda r} \leq 1$ and $1 \leq e^{\frac{\sigma}{2}r^2} \leq e^{\frac{\sigma}{2}}$, we obtain

$$\begin{aligned}
R &\geq r^{-\sigma}e^{\frac{-\sigma}{2}r^2}\left(-\frac{2(1+\sigma)e^{\sigma}}{\lambda}\int_0^r(\xi^{\sigma-1}+\xi^{\sigma+1})d\xi + (1+\sigma)\int_0^r(\xi^{\sigma}+\xi^{\sigma+2})d\xi\right)\\
&\geq -\frac{2(1+\sigma)e^{\sigma}}{\lambda}\left(\frac{1}{\sigma}+\frac{r^2}{2+\sigma}\right) + r\left(\frac{1+\frac{1+\sigma}{3+\sigma}r^2}{e^{\frac{\sigma}{2}r^2}}\right)\\
&\geq -\frac{e^{\sigma}}{\lambda}\frac{4(1+\sigma)^2}{\sigma(2+\sigma)} + r\left(\frac{1+\frac{1+\sigma}{3+\sigma}r^2}{e^{\frac{\sigma}{2}r^2}}\right).
\end{aligned}$$

Because $(1+\frac{1+\sigma}{3+\sigma}r^2)/e^{\frac{\sigma}{2}r^2} - 1 - e^{-\frac{\sigma}{2}}(\frac{1-\sigma-\sigma^2}{3+\sigma})r^2 = 0$ when $r=0$, and

$$\begin{aligned}
&\frac{d}{dr}\left[\left(\frac{1+\frac{1+\sigma}{3+\sigma}r^2}{e^{\frac{\sigma}{2}r^2}}\right) - 1 - e^{-\frac{\sigma}{2}}\left(\frac{1-\sigma-\sigma^2}{3+\sigma}\right)r^2\right]\\
&= 2r\left[\frac{\frac{1+\sigma}{3+\sigma} - \frac{\sigma}{2}(1+\frac{1+\sigma}{3+\sigma}r^2)}{e^{\frac{\sigma}{2}r^2}} - e^{-\frac{\sigma}{2}}\left(\frac{1-\sigma-\sigma^2}{3+\sigma}\right)\right]\\
&\geq 2r\left[\frac{\frac{1+\sigma}{3+\sigma} - \frac{\sigma}{2}(1+\frac{1+\sigma}{3+\sigma})}{e^{\frac{\sigma}{2}r^2}} - e^{-\frac{\sigma}{2}}\left(\frac{1-\sigma-\sigma^2}{3+\sigma}\right)\right]\\
&= 2r\left[\frac{\frac{1-\sigma-\sigma^2}{3+\sigma}}{e^{\frac{\sigma}{2}r^2}} - e^{-\frac{\sigma}{2}}\left(\frac{1-\sigma-\sigma^2}{3+\sigma}\right)\right] = 2r\frac{(\frac{1-\sigma-\sigma^2}{3+\sigma})}{e^{\frac{\sigma}{2}r^2}}\left(1-e^{\frac{\sigma}{2}(r^2-1)}\right) \geq 0
\end{aligned}$$

on $[0,1]$, the result follows. ∎

THEOREM 4.3. *For λ sufficiently large there is no physically realistic solution of* (4.1) *and* (4.2).

Proof. The bound on $T(r,\Gamma)$ that we established in Lemma 4.1 and the bound on $R(r,\Gamma)$ that we established in Lemma 4.2 imply that

$$\begin{aligned}
\frac{dT}{dr} + \lambda T &\geq \sigma T + (1-\sigma^2)(R-r)\\
&\geq -\frac{1}{\lambda}\left(e^{\sigma}\left(2\sigma(1+\sigma) + \frac{4(1-\sigma^2)(1+\sigma)^2}{\sigma(2+\sigma)}\right)\right) + (1-\sigma^2)e^{-\frac{\sigma}{2}}\frac{1-\sigma-\sigma^2}{3+\sigma}r^3.
\end{aligned}$$

Thus,

$$T \geq -\frac{1-e^{-\lambda r}}{\lambda^2}\left(e^{\sigma}\left(2\sigma(1+\sigma)+\frac{4(1-\sigma^2)(1+\sigma)^2}{\sigma(2+\sigma)}\right)\right)$$
$$+(1-\sigma^2)e^{-\frac{\sigma}{2}}\frac{1-\sigma-\sigma^2}{3+\sigma}\left(\frac{r^3}{\lambda}-3\frac{r^2}{\lambda^2}+6\frac{r}{\lambda^3}-6\frac{(1-e^{-\lambda r})}{\lambda^4}\right).$$

So

$$T(1,\Gamma) \geq (1-\sigma^2)e^{-\frac{\sigma}{2}}\frac{1-\sigma-\sigma^2}{3+\sigma}\left(\frac{1}{\lambda}\right)$$
$$-\left(e^{\sigma}\left(2\sigma(1+\sigma)+\frac{4(1-\sigma^2)(1+\sigma)^2}{\sigma(2+\sigma)}\right)+3(1-\sigma^2)e^{-\frac{\sigma}{2}}\frac{1-\sigma-\sigma^2}{3+\sigma}\right)\left(\frac{1}{\lambda^2}\right)$$
$$-(1-\sigma^2)e^{-\frac{\sigma}{2}}\frac{1-\sigma-\sigma^2}{3+\sigma}\left(\frac{6}{\lambda^4}\right).$$

The right-hand side of this inequality is positive for λ sufficiently large, so $T(1,\Gamma)$ is also positive for such values of λ, so no physically realistic solution exists in such cases. ∎

5. Closed-form solutions, existence, and uniqueness for cases in which the eye is flat and the lens thickness is constant. If the eye is flat, i.e., if $f'(R) \equiv 0$, the system is linear, and if $\tau' \equiv 0$, the equations have this form:

$$\frac{dT}{dr} = \left(\sigma T + (1-\sigma^2)(R-r)\right)\frac{\sqrt{1+g'(r)^2}}{r}, \tag{5.1}$$

$$\frac{dR}{dr} = \left(T + (1+\sigma)r - \sigma R\right)\frac{\sqrt{1+g'(r)^2}}{r}. \tag{5.2}$$

In such cases we have a closed-form solution in terms of the function

$$\phi(r) = e^{\int_0^r \frac{\sqrt{1+g'(\xi)^2}-1}{\xi}d\xi}.$$

That solution is

$$T(r,\Gamma) = \left(\Gamma - \frac{(1+\sigma)}{2}\right)(1+\sigma)r\phi(r) - \frac{(1-\sigma^2)}{r\phi(r)}\int_0^r \xi\sqrt{1+g'(\xi)^2}\phi(\xi)d\xi, \tag{5.3}$$

$$R(r,\Gamma) = \left(\Gamma - \frac{(1+\sigma)}{2}\right)r\phi(r) + \frac{(1+\sigma)}{r\phi(r)}\int_0^r \xi\sqrt{1+g'(\xi)^2}\phi(\xi)d\xi. \tag{5.4}$$

It follows directly from these expressions that both $T(r,\Gamma)$ and $R(r,\Gamma)$ increase monotonically with Γ; that $T(r,\Gamma)$ is strictly negative for positive r if $\Gamma \leq \frac{1+\sigma}{2}$; that the solution is physically realistic if $\Gamma > \frac{1+\sigma}{2}$; and that, at any fixed positive r, $T(r,\Gamma) > 0$ for Γ sufficiently large. These facts establish both the existence and the uniqueness of solutions in such cases for any $\rho > 0$. The more general existence result, Theorem 3.5, applies in this case, so here we simply formulate the uniqueness of solutions in a theorem, as follows.

THEOREM 5.1. *The physically realistic solution of the boundary value problem defined by* (5.1)*,* (5.2)*,* (2.6)*,* (2.7)*, and* (2.8)*, whose existence is established in Theorem* 3.5*, is unique.*

Proof. It follows from (5.3) that

$$\frac{\partial}{\partial\Gamma}T(\rho,\Gamma) = (1+\sigma)\rho\phi(\rho) > 0$$

if $\rho > 0$. So $T(\rho,\Gamma)$ has at most one root as a function of Γ for fixed positive ρ. ◻

6. Conclusions. The analyses that we have presented here are a foundation, but just a foundation, for a full theory of well-posedness of the boundary value problem for the contact lens equation. Taken together, the results of sections 3 and 4 suggest that the existence of solutions depends in a subtle way on thickness variations of the lens. Given the nonlinearity of the problem, it seems likely that it will also depend on the shapes of the eye and the lens.

After we discovered—by solving the equation numerically—the nonexistence result that is Theorem 4.3, we investigated other classes of shapes numerically. In cases in which the thickness of the lens decreases monotonically as a function of distance from the center, our numerical experiments have always yielded unique solutions of the boundary value problem. Our current, tentative, conjecture is that if $\frac{d\tau}{dr} \le 0$ for all r, a solution of the boundary value problem exists.

Real lenses are generally tapered toward their edges. This is true of most lenses produced by major manufacturers and in use today [3, 19, 31], and of many designs for cutting-edge applications [9], too. However, there are some common hydrogel lenses whose thicknesses increase, in some intervals, with distance from the lens center [14].

Regarding uniqueness of solutions we have no rigorous results and no conjecture. On one hand, the strongly nonlinear structure that our mild conditions on $f(R)$ allow suggests the possibility of nonuniqueness; to put the matter colloquially, if the eye shape has lots of bumps, it seems possible that there will be two or more equilibria of the lens. On the other hand, we have not found any cases of nonuniqueness in our numerical experiments.

Our theoretical results have implications for lens design; they indicate the importance of thickness variations to the mechanics of contact lenses. The contact lens industry is exploring the tailoring of thickness profiles as a way of improving patient comfort. The fundamental question is what thickness profiles are associated with comfortable lenses; this work establishes a basic mathematical framework for addressing that question.

In addition, the model that we have analyzed in this paper is an essential step in understanding the problem of the centering of a contact lens, because it is an account of how the shapes of the eye and the lens determine the suction pressure distribution. Gradients in this pressure distribution produce flow in the postlens tear film [23], and the consequent drag on the lens is the driving force of lens centration.

Appendix. In this appendix we prove the existence and uniqueness of solutions of the initial value problem defined by (2.4)–(2.7). The well-posedness of this problem is a foundation for the other proofs in this paper.

In this section we will use slightly different variables. We let $S(r,\Gamma) = \frac{\tau(r)T(r,\Gamma)}{r}$ $(= S_{rr})$, and we rewrite the ODE as a system:

$$\text{(A.1)}\quad \frac{d}{dr}(rS(r,\Gamma)) = \left(\sigma S(r,\Gamma) + (1-\sigma^2)\left(\frac{R(r,\Gamma)-r}{r}\right)\tau(r)\right)\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R(r,\Gamma))^2}},$$

$$\text{(A.2)}\quad \frac{d}{dr}R(r,\Gamma) = \left(\frac{S(r,\Gamma)}{\tau(r)} + 1 - \sigma\frac{R(r,\Gamma)-r}{r}\right)\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R(r,\Gamma))^2}}.$$

This system (like the ODE in (2.3)) is singular at $r = 0$; this is why the existence and uniqueness of solutions doesn't follow from standard ODE theory. However, the initial value problem for this equation at any point $r > 0$ is well-posed by the standard theory. So, here we need only establish existence and uniqueness in some arbitrarily small interval with 0 as its left endpoint. With that established, we can invoke standard results [6, 15] to show that unique solutions exist on $r \geq 0$.

We re-express the system given by (A.1) and (A.2) as a first-order system of integral equations:

$$\begin{aligned} S(r,\Gamma) = \frac{1}{r}\int_0^r \Bigg(&\sigma S(\xi,\Gamma) \\ &+ (1-\sigma^2)\left(\frac{R(\xi,\Gamma)-\xi}{\xi}\right)\tau(\xi)\Bigg)\frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R(\xi,\Gamma))^2}}d\xi, \end{aligned} \tag{A.3}$$

$$R(r,\Gamma) = \int_0^r \left(\frac{S(\xi,\Gamma)}{\tau(\xi)} + 1 - \sigma\left(\frac{R(\xi,\Gamma)-\xi}{\xi}\right)\right)\frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R(\xi,\Gamma))^2}}d\xi. \tag{A.4}$$

Lemma A.1. *If $S(r,\Gamma)$ and $R(r,\Gamma)$ are smooth functions on $[0,\rho]$ and $S(0,\Gamma) = (\Gamma-1)(\sigma+1)\tau(0)$ and $R(0,\Gamma) = 0$ and $R'(0,\Gamma) = \Gamma$, then if*

$$\begin{aligned} \tilde{S}(r,\Gamma) = \frac{1}{r}\int_0^r \Bigg(&\sigma S(\xi,\Gamma) \\ &+ (1-\sigma^2)\left(\frac{R(\xi,\Gamma)-\xi}{\xi}\right)\tau(\xi)\Bigg)\frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R(\xi,\Gamma))^2}}d\xi, \end{aligned} \tag{A.5}$$

$$\tilde{R}(r,\Gamma) = \int_0^r \left(\frac{S(\xi,\Gamma)}{\tau(\xi)} + 1 - \sigma\left(\frac{R(\xi,\Gamma)-\xi}{\xi}\right)\right)\frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R(\xi,\Gamma))^2}}d\xi, \tag{A.6}$$

$\tilde{S}(r,\Gamma)$ and $\tilde{R}(r,\Gamma)$ are smooth on $[0,\rho]$ and $\tilde{S}(0,\Gamma) = (\Gamma-1)(\sigma+1)\tau(0)$ and $\tilde{R}(0,\Gamma) = 0$ and $\tilde{R}'(0,\Gamma) = \Gamma$.

Proof. The smoothness of the integrands ensures that $\tilde{S}(r,\Gamma)$ and $\tilde{R}(r,\Gamma)$ are smooth (the apparent singularity in the term $\frac{R(\xi,\Gamma)-\xi}{\xi}$ is removable), and it ensures that

$$\tilde{S}(r,\Gamma) = \frac{1}{r}\int_0^r (\Gamma-1)(\sigma+1)\tau(0) + O(\xi)d\xi = (\Gamma-1)(\sigma+1)\tau(0) + O(r), \tag{A.7}$$

$$\tilde{R}(r,\Gamma) = \int_0^r \Gamma + O(\xi)d\xi = \Gamma r + O(r^2). \tag{A.8}$$

∎

Lemma A.2. *If $\Gamma > 0$, there are positive numbers γ and ρ such that if $\left|\frac{S(r,\Gamma)}{\tau(0)} - (\Gamma-1)(1+\sigma)\right| \leq \gamma r$ and $|R(r,\Gamma) - \Gamma r| \leq \gamma r^2$ on $[0,\rho)$, and $\tilde{S}(r,\Gamma)$ and $\tilde{R}(r,\Gamma)$ are defined by* (A.5) *and* (A.6)*, then $\tilde{S}(r,\Gamma)$ and $\tilde{R}(r,\Gamma)$ satisfy these same inequalities on the same interval. Further, γ can be chosen to depend continuously on Γ.*

Proof. Because $\tau(r)$ is positive and bounded away from 0 and is smooth with a bounded derivative, and because $g(r)$ is C^2 with bounded first and second derivatives and $g'(0) = 0$, there is a positive constant β such that $\sqrt{1+g'(r)^2}$, $\sqrt{1+g'(r)^2}\frac{\tau(r)}{\tau(0)}$, and $\sqrt{1+g'(r)^2}\frac{\tau(0)}{\tau(r)}$ are all bounded above by $1+\beta r$ and below by $1-\beta r$. We restrict ρ so that $1-\beta\rho > 0$, and we will restrict ρ further. The smoothness conditions on

$f(R)$ ensure that there is a positive constant K such that $\frac{1}{\sqrt{1+f'(R)^2}} \geq 1 - KR$. So, if $R(r,\Gamma)$ satisfies the bound given in the statement of the lemma, then

$$\frac{1}{\sqrt{1+f'(R)^2}} \geq 1 - K\Gamma r - K\gamma r^2 \geq 1 - (K\Gamma + K\gamma\rho)r.$$

Taken together, these bounds imply that $\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R)^2}}$, $\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R)^2}}\frac{\tau(r)}{\tau(0)}$, and $\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R)^2}}\frac{\tau(0)}{\tau(r)}$ are all bounded above by $1 + \beta r$ and below by $1 - (K\Gamma + \beta + K\gamma\rho)r$. We let $b = K(\Gamma+1)+\beta$, and we restrict ρ so that $\rho\gamma < 1$. (Thus ρ will depend on our choice of γ.) Thus we have that all of the expressions we have been considering are bounded above and below by $1+br$ and $1-br$, respectively. Of course, we now further restrict ρ so that $b\rho < 1$.

By using these bounds and the bounds on $\frac{S(r,\Gamma)}{\tau(0)}$ and $R(r,\Gamma)$ given in the statement of the lemma, in (A.5) and (A.6), we obtain bounds on $\tilde{S}(r,\Gamma)$ and $\tilde{R}(r,\Gamma)$.

If $\Gamma \geq 1$, we obtain

$$(1+\sigma)(\Gamma-1) - \left(\frac{\gamma(1+\sigma-\sigma^2)}{2} + \frac{b(1+\sigma)(\Gamma-1)}{2}\right) r \leq \frac{\tilde{S}(r,\Gamma)}{\tau(0)}$$
$$\leq (1+\sigma)(\Gamma-1) + \left(\frac{(1+\sigma-\sigma^2)\gamma}{2} + \frac{b(1+\sigma)(\Gamma-1)}{2} + \frac{(1+\sigma-\sigma^2)\gamma b\rho}{3}\right) r$$

and

$$\Gamma r - \left(\frac{(1+\sigma)\gamma}{2} + \frac{b((\sigma+1)\Gamma-1)}{2}\right) r^2 \leq \tilde{R}(r,\Gamma)$$
$$\leq \Gamma r + \left(\frac{(1+\sigma)\gamma}{2} + \frac{b((\sigma+1)\Gamma-1)}{2} + \frac{(1-\sigma)\gamma b\rho}{3}\right) r^2.$$

So if we choose $\gamma = 3\Gamma b + 1$ and we further restrict ρ so that $\rho < \frac{1}{\gamma b}$, these constants have the required properties.

If $\Gamma < 1$, we obtain

$$(1+\sigma)(\Gamma-1) - \left(\frac{\gamma(1+\sigma-\sigma^2)}{2} - \frac{b(1+\sigma)(\Gamma-1)}{2} + \frac{(1+\sigma-\sigma^2)\gamma b\rho}{3}\right) r \leq \frac{\tilde{S}(r,\Gamma)}{\tau(0)}$$
$$\leq (1+\sigma)(\Gamma-1) + \left(\frac{(1+\sigma-\sigma^2)\gamma}{2} - \frac{b(1+\sigma)(\Gamma-1)}{2}\right) r$$

and

$$\Gamma r - \left(\frac{(1+\sigma)\gamma}{2} - \frac{b((\sigma+1)\Gamma-1)}{2} + \frac{(1-\sigma)\gamma b\rho}{3}\right) r^2 \leq \tilde{R}(r,\Gamma)$$
$$\leq \Gamma r + \left(\frac{(1+\sigma)\gamma}{2} - \frac{b((\sigma+1)\Gamma-1)}{2}\right) r^2.$$

So if we choose $\gamma = 3b+1$ and we further restrict ρ so that $\rho < \frac{1}{\gamma b}$, these constants have the required properties.

Our choices of γ in the two cases make it a continuous function of Γ. ∎

We prove existence and uniqueness by a modified Picard iteration. We define $R_0(r,\Gamma) \equiv \Gamma r$ and $S_0(r,\Gamma) \equiv (\Gamma - 1)(\sigma + 1)\tau(0)$, and

$$S_{j+1}(r,\Gamma) = \frac{1}{r}\int_0^r \left(\sigma S_j(\xi,\Gamma) + (1-\sigma^2)\left(\frac{R_j(\xi,\Gamma)-\xi}{\xi}\right)\tau(\xi)\right)\frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R_j(\xi,\Gamma))^2}}d\xi, \tag{A.9}$$

$$R_{j+1}(r,\Gamma) = \int_0^r \left(\frac{S_j(\xi,\Gamma)}{\tau(\xi)} + 1 - \sigma\left(\frac{R_j(\xi,\Gamma)-\xi}{\xi}\right)\right)\frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R_j(\xi,\Gamma))^2}}d\xi. \tag{A.10}$$

THEOREM A.3. *If ρ is the positive constant whose existence is guaranteed by Lemma* A.2*, then there is a subinterval $[0,\rho')$ of $[0,\rho)$ on which $S_j(r,\Gamma)$ and $R_j(r,\Gamma)$ converge uniformly.*

Proof. By Lemma A.2 we know that the expressions

$$\left|\sigma S_j(r,\Gamma) + (1-\sigma^2)\left(\frac{R_j(r,\Gamma)-r}{r}\right)\tau(r)\right| \tag{A.11}$$

and

$$\left|\frac{S_j(r,\Gamma)}{\tau(r)} + 1 - \sigma\left(\frac{R_j(r,\Gamma)-r)}{r}\right)\right| \tag{A.12}$$

are bounded on $[0,\rho)$ independent of j. The conditions on $\tau(r)$, $g(r)$, and $f(R)$, along with Lemma A.2, establish that there is a positive constant c such that $\frac{\tau(r)}{\tau(0)}$, $\frac{\tau(0)}{\tau(r)}$, and $\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R_j)^2}}$ are all bounded by $1+cr^2$. The lemma and the conditions on these functions also imply that there is a positive constant W such that

$$\left|\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R_j)^2}} - \frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R_{j-1})^2}}\right| < Wr$$

on $[0,\rho)$.

With these bounds, from the definitions

$$\begin{aligned}\Delta S_{j+1}(r,\Gamma) = \frac{1}{r}\int_0^r &\left(\sigma\Delta S_j(\xi,\Gamma) + (1-\sigma^2)\left(\frac{\Delta R_j(\xi,\Gamma)}{\xi}\right)\tau(\xi)\right)\frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R_j(\xi,\Gamma))^2}}\\ &+ \left(\sigma S_{j-1}(\xi,\Gamma) + (1-\sigma^2)\left(\frac{R_{j-1}(\xi,\Gamma)-\xi}{\xi}\right)\tau(\xi)\right)\\ &\cdot\left(\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R_j(\xi,\Gamma))^2}} - \frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R_{j-1}(\xi,\Gamma))^2}}\right)d\xi,\end{aligned}$$

$$\begin{aligned}\Delta R_{j+1}(r,\Gamma) = \int_0^r &\left(\frac{\Delta S_j(\xi,\Gamma)}{\tau(\xi)} - \sigma\left(\frac{\Delta R_j(\xi,\Gamma)}{\xi}\right)\right)\frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R_j(\xi,\Gamma))^2}}\\ &+ \left(\frac{S_{j-1}(\xi,\Gamma)}{\tau(\xi)} + 1 - \sigma\left(\frac{R_{j-1}(\xi,\Gamma)-\xi}{\xi}\right)\right)\\ &\cdot\left(\frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R_j(\xi,\Gamma))^2}} - \frac{\sqrt{1+g'(\xi)^2}}{\sqrt{1+f'(R_{j-1}(\xi,\Gamma))^2}}\right)d\xi,\end{aligned}$$

we have

$$|\Delta S_{j+1}(r,\Gamma)| \le \frac{1}{r}\int_0^r \left(\sigma|\Delta S_j(\xi,\Gamma)| + (1-\sigma^2)\left(\frac{|\Delta R_j(\xi,\Gamma)|}{\xi}\right)(1+c\xi^2)\right)(1+c\xi^2) + MW\xi|\Delta R_j(\xi,\Gamma)|d\xi,$$

$$|\Delta R_{j+1}(r,\Gamma)| \le \int_0^r \left(\frac{|\Delta S_j(\xi,\Gamma)|}{\tau(0)}(1+c\xi^2) + \sigma\left(\frac{\Delta R_j(\xi,\Gamma)}{\xi}\right)\right)(1+c\xi^2) + MW\xi|\Delta R_j(\xi,\Gamma)|d\xi.$$

Now suppose that $\frac{|\Delta S_j(r,\Gamma)|}{\tau(0)} \le A_j r$ and $|\Delta R_j(r,\Gamma)| \le B_j r^2$. Our inequalities imply that

$$\begin{aligned}\frac{|\Delta S_{j+1}(r,\Gamma)|}{\tau(0)} &\le \frac{1}{r}\int_0^r (\sigma A_j\xi + (1-\sigma^2)B_j\xi)(1+c\xi^2)(1+c\xi^2) + MWB_j\xi^3 d\xi\\ &= \frac{\sigma}{2}\left(r+\frac{c}{2}r^2\right)A_j + (1-\sigma^2)\left(\frac{r}{2}+\frac{2cr^3}{4}+\frac{c^2r^5}{6}+MW\frac{r^3}{4}\right)B_j,\end{aligned}$$

$$\begin{aligned}|\Delta R_{j+1}(r,\Gamma)| &\le \int_0^r A_j\xi(1+c\xi^2)^2 + \sigma B_j\xi(1+c\xi^2) + MWB_j\xi^3 d\xi\\ &= \left(\frac{r^2}{2}+\frac{cr^4}{2}+\frac{c^2r^6}{6}\right)A_j + \sigma\left(\frac{r^2}{2}+\frac{cr^4}{4}+MW\frac{r^4}{4}\right)B_j.\end{aligned}$$

Because we're on the interval $[0,\rho')$, these inequalities imply

$$\frac{|\Delta S_{j+1}(r,\Gamma)|}{\tau(0)} \le r\left(\frac{\sigma}{2}\left(1+\frac{c}{2}\rho'\right)A_j + (1-\sigma^2)\left(\frac{1}{2}+\frac{2c\rho'^2}{4}+\frac{c^2\rho'^4}{6}+MW\frac{\rho'^2}{4}\right)B_j\right),$$

$$|\Delta R_{j+1}(r,\Gamma)| \le r^2\left(\left(\frac{1}{2}+\frac{c\rho'^2}{2}+\frac{c^2\rho'^4}{6}\right)A_j + \sigma\left(\frac{1}{2}+\frac{c\rho'^2}{4}+MW\frac{\rho'^2}{4}\right)B_j\right).$$

Given $\epsilon > 0$, we can choose $\rho' > 0$ sufficiently small that these inequalities imply

$$\begin{aligned}\frac{|\Delta S_{j+1}(r,\Gamma)|}{\tau(0)} &\le r\left(\left(\frac{\sigma}{2}+\epsilon\right)A_j + \left(\frac{1}{2}+\epsilon\right)B_j\right),\\ |\Delta R_{j+1}(r,\Gamma)| &\le r^2\left(\left(\frac{1}{2}+\epsilon\right)A_j + \left(\frac{\sigma}{2}+\epsilon\right)B_j\right).\end{aligned}$$

If we take $A_1 = B_1 = 2\gamma$, where γ is the constant defined in Lemma A.2, and we define

$$\begin{aligned}A_{j+1} &= \left(\frac{\sigma}{2}+\epsilon\right)A_j + \left(\frac{1}{2}+\epsilon\right)B_j,\\ B_{j+1} &= \left(\frac{1}{2}+\epsilon\right)A_j + \left(\frac{\sigma}{2}+\epsilon\right)B_j,\end{aligned}$$

then we find that

$$\begin{aligned}\frac{|\Delta S_{j+1}(r,\Gamma)|}{\tau(0)} &\le rA_{j+1},\\ |\Delta R_{j+1}(r,\Gamma)| &\le r^2B_{j+1},\end{aligned}$$

$A_j = B_j = 2\gamma(\frac{1+\sigma+2\epsilon}{2})^{j-1}$, and $(\frac{1+\sigma+2\epsilon}{2}) < 1$ for $\epsilon < \frac{1}{4}$, because the Poisson's ratio, σ, is between 0 and $\frac{1}{2}$.

This means that the sums $S_0(r,\Gamma)+\sum_{j=1}^{\infty} \Delta S_j(r,\Gamma)$ and $R_0(r,\Gamma)+\sum_{j=1}^{\infty} \Delta R_j(r,\Gamma)$ converge absolutely and uniformly on $[0,\rho')$, because each is dominated by a multiple of the geometric series $\sum_{j=1}^{\infty}(\frac{1+\sigma+2\epsilon}{2})^{j-1}$. ∎

COROLLARY A.4. *If both pairs* $(S(r,\Gamma), R(r,\Gamma))$ *and* $(\hat{S}(r,\Gamma), \hat{R}(r,\Gamma))$ *satisfy* (A.1) *and* (A.2)*, and if both satisfy* $|\frac{S(r,\Gamma)}{\tau(0)} - (\Gamma-1)(1+\sigma)| \le \gamma r$ *and* $|R(r,\Gamma) - \Gamma r| \le \gamma r^2$ *on an interval* $[0,\rho)$*, then* $S(r,\Gamma) \equiv \hat{S}(r,\Gamma)$ *and* $R(r,\Gamma) \equiv \hat{R}(r,\Gamma)$ *on that interval.*

Proof. The contraction argument that established Theorem A.4 establishes that for any $\epsilon > 0$ (we're concerned with small values of ϵ), on some subinterval $[0,\rho')$, both $\frac{|S(r,\Gamma)-\hat{S}(r,\Gamma)|}{\tau(0)}$ and $|R(r,\Gamma) - \hat{R}(r,\Gamma)|$ are less than or equal to $2\gamma(\frac{1+\sigma+2\epsilon}{2})^{j-1}$ for every j, and thus $S(r,\Gamma) \equiv \hat{S}(r,\Gamma)$ and $R(r,\Gamma) \equiv \hat{R}(r,\Gamma)$ on that interval. The ODE system defined by (A.1) and (A.2) is not singular for $r > 0$, and the requisite Lipschitz condition applies, so by choosing any point interior to $[0,\rho')$ as an initial point, we can invoke standard ODE theory to establish that $S(r,\Gamma) \equiv \hat{S}(r,\Gamma)$ and $R(r,\Gamma) \equiv \hat{R}(r,\Gamma)$ on $[0,\rho)$. ∎

THEOREM A.5. *For each* $\Gamma > 0$ *there is a unique solution* $S(r,\Gamma)$ *and* $R(r,\Gamma)$ *of* (A.1) *and* (A.2) *on* $[0,\infty)$ *that satisfies bounds of the form* $\left|\frac{S(r,\Gamma)}{\tau(0)} - (\Gamma-1)(1+\sigma)\right| \le \gamma r$ *and* $|R(r,\Gamma) - \Gamma r| \le \delta r^2$ *in some one-sided neighborhood of* 0*. These solutions depend continuously on* Γ*.*

Proof. The fact that $S(r,\Gamma)$ and $R(r,\Gamma)$ are uniform limits of continuous functions, as we established in the proof of Theorem A.4, ensures that they are continuous and that they satisfy (A.3) and (A.4). Their satisfying this system of integral equations ensures that they are differentiable and that they satisfy the ODE systems of (A.1) and (A.2) on the small interval $[0,\rho')$ on which they are defined by the construction. On that interval they satisfy the bounds $|\frac{S(r,\Gamma)}{\tau(0)} - (\Gamma-1)(1+\sigma)| \le \gamma r$ and $|R(r,\Gamma) - \Gamma r| \le \gamma r^2$ because they are constructed as limits of functions that satisfy these bounds. Because, by Lemma A.2, γ depends continuously on Γ, the functions $S(r,\Gamma)$ and $R(r,\Gamma)$ are continuous as functions of Γ on $[0,\rho')$.

By Corollary A.4, $S(r,\Gamma)$ and $R(r,\Gamma)$ constitute the unique such solution of (A.1) and (A.2) that satisfies the bounds in the statement of this theorem.

If ρ'' is in $(0,\rho')$, we can invoke standard existence theory [6, 15] to extend the functions $S(r,\Gamma)$ and $R(r,\Gamma)$. Consider the initial value problem for (A.1) and (A.2) with the initial values being $S(\rho'',\Gamma)$ and $R(\rho'',\Gamma)$. In a neighborhood of ρ'' and for $r \ge \rho''$, the system satisfies the necessary Lipschitz condition to ensure existence and uniqueness (Theorems 1.21 and 1.41 in Hu and Li [15]). Moreover, on any interval with ρ'' as its left endpoint, $\frac{\sqrt{1+g'(r)^2}}{\sqrt{1+f'(R)^2}}$, r, $\frac{1}{r}$, and $\tau(r)$ are all bounded above and below, so by Theorem 1.7.1 in Hu and Li [15], the solution exists for $r \ge 0$. ∎

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