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IDENTIFICATION OF A PARAMETER IN FOURTH-ORDER PARTIAL DIFFERENTIAL EQUATIONS BY AN EQUATION ERROR APPROACH

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ABSTRACT. The objective of this short note is to employ an equation error approach to identify a variable parameter in fourth-order partial differential equations. Existence and convergence results are given for the optimization problem emerging from the equation error formulation. Finite element based numerical experiments show the effectiveness of the proposed framework.

1. Introduction

Let Ω be a bounded open domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a sufficiently smooth boundary Γ. Given a function $f \in L^2(\Omega)$, we consider the following fourth-order elliptic boundary value problem (BVP):

$$\Delta(a\Delta u) = f \quad \text{in} \quad \Omega, \tag{1.1a}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma. \tag{1.1b}$$

In this short note, we are interested in the inverse problem of identifying the material parameter a from a measurement z of u. Interesting applications of this

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study are in beam and plate models and car windshield modeling (see [10, 15]). This inverse problem has been solved by the output least-squares (OLS) in which one seeks a minimizer of the functional

$$a \to \frac{1}{2}\|u(a) - z\|^2,$$

defined by an appropriate norm. Here z is the data (the measurement of u) and $u(a)$ is the unique weak solution of (1.1) that corresponds to the coefficient a. See [1]–[15].

One of the main difficulties associated to the OLS approach is the nonconvexity of the OLS functional. In this work, our objective is to use the equation error approach for solving the inverse problem of identifying the parameter a, which in contrast to the OLS based approach, results in solving a convex optimization problem. The equation error approach has been studied in the context of a simpler second-order BVP:

$$-\nabla\cdot(a\nabla u) = f \quad \text{in} \quad \Omega, \tag{1.2a}$$

$$u = 0 \quad \text{on} \quad \Gamma. \tag{1.2b}$$

For (1.2), the equation error approach consists of minimizing the functional

$$a \to \frac{1}{2}\|\nabla\cdot(a\nabla z) + f\|^2_{H^{-1}(\Omega)},$$

where $H^{-1}(\Omega)$ is the topological dual of $H^1_0(\Omega)$ and z is the data. See [1, 9].

In this paper, we extend the equation error approach to identify the variable coefficient a in the fourth-order boundary value problem (1.1). Our strategy is motivated by the ideas presented originally by Acar [1] and Kärkkäinen [9] for (1.2). In addition to giving an existence theorem and a convergence result for the discretized problem, we also give some numerical examples.

We remark that the equation error approach has two distinct advantages over the OLS approach. Firstly, as mentioned above, it leads to a convex optimization problem and hence it only possesses global minimizers. Secondly, the equation approach is computationally inexpensive as there is no underlying variational problem to be solved. On the other hand, a deficiency of the equation error approach is that it relies on differentiating the data and hence it is quite sensitive to the noise in the data.

The contents of this paper are organized into four sections. In Section 2 we pose a minimization problem and ensure its solvability. The problem is discretized by finite elements and it is shown that the continuous minimization problem can be approximated by the discrete analogue. Computational framework is given in Section 3 whereas two numerical examples are given in Section 4 to show the effectiveness of the approach.

2. Equation error approach

The variational formulation of (1.1) plays an important role in formulating the equation error approach. The space suitable for the weak formulation is given by

$$V := \left\{ v \in H^2(\Omega) \mid u = \tfrac{\partial u}{\partial n} = 0 \text{ on } \Gamma \right\}.$$

The weak formulation of (1.1) is given by: Find $u \in V$ such that

$$\int_\Omega a\Delta u\, \Delta v = \int_\Omega fv, \qquad \text{for all} \quad v \in V. \tag{2.1}$$

For a fixed pair $(a, w) \in L^\infty(\Omega) \times V$, we define the map $E(a, w)\colon V \to \mathbb{R}$ given by

$$E(a, w)(v) = \int_\Omega a\Delta w\, \Delta v - \int_\Omega fv.$$

The map $E(a, w)(\cdot)$ is linear and continuous and hence belongs to the topological dual V^* of V. We denote by $e(a, w) \in V$, the image of $E(a, w)$ under the Riesz map, that is

$$\langle e(a, w), v\rangle_V = \int_\Omega a\Delta w\, \Delta v - \int_\Omega fv, \qquad \text{for all} \quad v \in V,$$

where $\langle \cdot, \cdot\rangle_V$ is the inner product on V.

Let K be the set of admissible coefficients which we assume to be a closed and convex subset of $B := H^2(\Omega)$. For given $z \in V$, we consider the following minimization problem: Find $a^* \in K$ by solving

$$\min_{a\in K} J(a) = \frac{1}{2}\|e(a, z)\|_V^2 + \frac{\varepsilon}{2}\|a\|^2_{H^2(\Omega)}, \tag{2.2}$$

where $\varepsilon > 0$ is a regularizing parameter, $z \in V$ is the data, and $\|\cdot\|^2_{H^2(\Omega)}$ is the regularization term.

The following result ensures that the above minimization problem is solvable.

Theorem 2.1. *The minimization problem* (2.2) *is uniquely solvable.*

Proof. The proof is based on standard arguments. Since $J(a) \geq 0$ for every $a \in K$, there exists a minimizing sequence $\{a_n\} \subset K$ such that $\lim\limits_{n\to\infty} J(a_n) = \inf\limits_{a\in K} J(a)$. From

$$\frac{\varepsilon}{2}\|a_n\|^2_{H^2(\Omega)} \leq \frac{1}{2}\|e(a_n, z)\|_V^2 + \frac{\varepsilon}{2}\|a_n\|^2_{H^2(\Omega)},$$

we deduce that the sequence $\{a_n\}$ is bounded in $\|\cdot\|_{H^2(\Omega)}$. Due to the reflexivity of the space $H^2(\Omega)$ and the compact embedding of $H^2(\Omega)$ in $L^\infty(\Omega)$, there exists a subsequence that converges weakly in $H^2(\Omega)$ and strongly in $L^\infty(\Omega)$. Using the same notation for the subsequences as well, we have that $a_n \to \tilde{a} \in K$ in $L^\infty(\Omega)$. From the definition of $e(\cdot,\cdot)$, we have the following two equations

$$\begin{aligned} \langle e(a_n, z), v\rangle_V &= \int_\Omega a_n \Delta z \, \Delta v - \int_\Omega fv, \qquad \text{for all} \quad v \in V, \\ \langle e(\tilde{a}, z), v\rangle_V &= \int_\Omega \tilde{a} \Delta z \, \Delta v - \int_\Omega fv, \qquad \text{for all} \quad v \in V. \end{aligned}$$

By subtracting the above two equations and setting $v = e(a_n, z) - e(\tilde{a}, z)$, we obtain

$$\begin{aligned} \|e(a_n, z) - e(\tilde{a}, z)\|_V^2 &= \int_\Omega (a_n - \tilde{a})\Delta z \, \Delta(e(a_n, z) - e(\tilde{a}, z)) \\ &\le \|a_n - \tilde{a}\|_{L^\infty(\Omega)} \|e(a_n, z) - e(\tilde{a}, z)\|_V \|z\|_V. \end{aligned}$$

This ensures that $e(a_n, z) \to e(\tilde{a}, z)$ in V. By invoking the lower-semicontinuity of the norm $\|\cdot\|_{H^2(\Omega)}$, we obtain

$$\begin{aligned} J(\tilde{a}) &= \frac{1}{2}\|e(\tilde{a}, z)\|_V^2 + \frac{\varepsilon}{2}\|\tilde{a}\|_{H^2(\Omega)}^2 \\ &\le \lim_{n\to\infty} \frac{1}{2}\|e(a_n, z)\|_V^2 + \liminf_{n\to\infty} \frac{\varepsilon}{2}\|a_n\|_{H^2(\Omega)}^2 \\ &= \liminf_{n\to\infty} \left\{ \frac{1}{2}\|e(a_n, z)\|_V^2 + \frac{\varepsilon}{2}\|a_n\|_{H^2(\Omega)}^2 \right\} = \inf_{a\in K} J(a). \end{aligned}$$

This ensures that $\tilde{a} \in K$ is a solution of (2.2) and the proof is complete. □

The continuous problem (2.2) has to be discretized for a numerical solution. We assume that we are given a parameter h converging to 0 and a family $\{V_h\}$ of finite dimensional subspaces of V. As usual, we define a projection operator $P_h \colon V \to V_h$ by $\|P_h v - v\|_V \to 0$, for every $v \in V$. Analogously, we assume that $\{B_h\}$ is a family of finite-dimensional subspaces of B. We define $K_h \subset B_h \bigcap K$ to be the discrete set of admissible coefficients. We assume that K_h is nonempty and uniformly bounded. Furthermore, we assume that for every $a \in K$ there exits a sequence $\{\hat{a}_h\}$ with $\hat{a}_h \in K_h$ such that $\hat{a}_h \to a$ in $\|\cdot\|_{H^2(\Omega)}$ norm.

For any $(a_h, v_h) \in K_h \times V_h$, the element $e_h(a_h, v_h) \in V_h$ is given by the condition that

$$\langle e_h(a_h, v_h), w_h\rangle_V = \int_\Omega a_h \Delta v_h \, \Delta w_h - \int_\Omega f w_h, \qquad \text{for all} \quad w_h \in V_h. \tag{2.3}$$

We consider the following discrete minimization problem: Find $a_h \in K_h$ by solving

$$\min_{a\in K_h} J_h(a) = \frac{1}{2}\|e_h(a,z)\|^2_{H^2(\Omega)} + \frac{\varepsilon}{2}\|a\|^2_{H^2(\Omega)}. \tag{2.4}$$

The following result ensures that the continuous problem can be approached by its discrete analogue.

Theorem 2.2. *The discrete minimization problem* (2.4) *is solvable. If* $\{\tilde{a}_h\}_{h>0}$ *is a sequence of minimizers of* (2.4)*, then there is a subsequence which converges to a minimizer of the continuous problem* (2.2)*.*

Proof. The existence of minimizers of (2.4) can be proved by using same arguments as employed in the proof of Theorem 2.1. Let $\{\tilde{a}_h\}$ be a sequence of minimizers of J_h. Then $\{\tilde{a}_h\}$ remains bounded in $B = H^2(\Omega)$ norm. This further ensures the existence of a subsequence, still denoted by $\{\tilde{a}_h\}$, which converges to some $\tilde{a} \in K$ in the $L^\infty(\Omega)$ norm.

We claim that $e_h(\tilde{a}_h, z) \to e(\tilde{a}, z)$ weakly in V. In fact, for any $w \in V$, we have

$$\begin{aligned}&\langle e_h(\tilde{a}_h,z) - e(\tilde{a},z), w\rangle \\ &= \int_\Omega \tilde{a}_h \Delta z \Delta(P_h w) - \int_\Omega f P_h w + \langle e_h(\tilde{a}_h,z), w - P_h w\rangle - \int_\Omega \tilde{a}\Delta z\Delta w + \int_\Omega f w,\end{aligned}$$

which ensures that $e_h(\tilde{a}_h, z) \to e(\tilde{a}, z)$, weakly in V. In fact, the above expression can be further manipulated to ensure that $e_h(\tilde{a}_h, z) \to e(\tilde{a}, z)$, strongly in V.

Let $a \in K$ be arbitrary. Then, there exists a sequence $\{\hat{a}_h\}$ with $\hat{a}_h \in K_h$ such that $\hat{a}_h \to a$ in $\|\cdot\|_V$. Therefore,

$$\begin{aligned} J(\tilde{a}) &= \frac{1}{2}\|e(\tilde{a},z)\|^2_{H^2(\Omega)} + \frac{\varepsilon}{2}\|\tilde{a}\|^2_{H^2(\Omega)} \\ &\leq \liminf_{h\to 0} \frac{1}{2}\|e_h(\tilde{a}_h,z)\|^2_{H^2(\Omega)} + \liminf_{h\to 0}\frac{\varepsilon}{2}\|\tilde{a}_h\|^2 \\ &\leq \liminf_{h\to 0}\left\{\frac{1}{2}\|e_h(\tilde{a}_h,z)\|^2_{H^2(\Omega)} + \frac{\varepsilon}{2}\|\tilde{a}_h\|^2_{H^2(\Omega)}\right\} \\ &\leq \liminf_{h\to 0}\left\{\frac{1}{2}\|e_h(\hat{a}_h,z)\|^2_{H^2(\Omega)} + \frac{\varepsilon}{2}\|\hat{a}_h\|^2_{H^2(\Omega)}\right\} \\ &= \frac{1}{2}\|e(a,z)\|^2_{H^2(\Omega)} + \frac{\varepsilon}{2}\|a\|^2_{H^2(\Omega)}. \end{aligned}$$

Since $a \in K$ was chosen arbitrarily, we have shown that $\tilde{a} \in K$ is a minimizer. □

3. Computational framework

In this section, we develop a computational framework for the equation error approach in the context of the following one-dimensional analogue of (1.1):

$$(a(x)u'')'' = f(x), \qquad \text{for all} \quad x \in \Omega, \tag{3.1a}$$

$$u(0) = u'(0) = 0, \tag{3.1b}$$

$$u(1) = u'(1) = 0, \tag{3.1c}$$

where $\Omega = (0,1)$, $a(x)$ is a variable coefficient and f is a suitable function.

The weak form of (3.1) reads: Find $u \in V$ such that

$$\langle a(x)u'', v''\rangle = \langle f, v\rangle, \qquad \text{for all} \quad v \in V. \tag{3.2}$$

To introduce the finite element space, we define the following partition of Ω:

$$0 = x_0 < x_1 < \cdots < x_j < \cdots < x_n < x_{n+1} = 1,$$

and set $I_j =]x_{j-1}, x_j[$, for $j = 1, \ldots, n+1$. For simplicity, we take $h_j = x_j - x_{j-1}$.

We define a finite dimensional space V_h, consisting of elements v that satisfy the following condition:

- v and v' are continuous on $[0,1]$.
- v is a polynomial of degree 3 on each subinterval I_j.
- The boundary conditions (3.1a) and (3.1b) hold for v.

We consider the following discretized weak form: Find $u_h \in V_h$ such that

$$\langle a(x)u_h'', v''\rangle = \langle f, v\rangle, \qquad \text{for all} \quad v \in V_h. \tag{3.3}$$

Since a degree three polynomial has four degrees of freedom, an element $v \in V_h$ on any interval I_j can be uniquely determined by the four values, namely, $v(x_{j-1})$, $v(x_j)$, $v'(x_{j-1})$ and $v'(x_j)$. Therefore, at every point of the mesh, any $v \in V_h$ has two degrees of freedom, namely, the function value v and its derivative value v'. To define a bases for V_h we will define two basis functions for every node, namely Φ_j that corresponds to v and Ψ_j for v'. By using standard arguments, we obtain that for $x \in [0,1]$ and for $j = 1, \ldots, n$, the basis function Φ_j that corresponds to $v(x_j)$ is given by

$$\Phi_j(x) = \begin{cases} \frac{1}{h_j^3}\left[-2x^3 + 3(x_{j-1} + x_j)x^2 - 6x_{j-1}x_j x + (3x_j - x_{j-1})x_{j-1}^2\right] & x \in I_j \\ \frac{1}{h_{j+1}^3}\left[2x^3 - 3(x_j + x_{j+1})x^2 + 6x_j x_{j+1} x + (x_{j+1} - 3x_j)x_{j+1}^2\right] & x \in I_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the basis function Ψ_j that corresponds to $v'(x_j)$ is given by:

$$\Psi_j(x) = \begin{cases} \frac{1}{h_j^2}\left[x^3 - (2x_{j-1} + x_j)x^2 + x_{j-1}(x_{j-1} + 2x_j)x - x_{j-1}^2 x_j\right] & x \in I_j \\ \frac{1}{h_{j+1}^2}\left[x^3 - (x_j + 2x_{j+1})x^2 + x_{j+1}(2x_j + x_{j+1})x - x_j x_{j+1}^2\right] & x \in I_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

We have now constructed a set of basis functions $\{\Phi_1, \ldots, \Phi_n, \Psi_1, \ldots, \Psi_n\}$ or $\{\Phi_j, \Psi_j\}_{j=1}^n$ for V_h. By the definition of V_h, any element $v \in V_h$ can be uniquely written as

$$v = \sum_{j=1}^{n} [v_j \Phi_j + \hat{v}_j \Psi_j], \tag{3.4}$$

where $v_j = v(x_j)$ and $\hat{v}_j = v'(x_j)$.

Let $u_h \in V_h$ be the solution of the finite-dimensional weak form. Using (3.4), we obtain

$$u_h = \sum_{j=1}^{n} [u_j \Phi_j + \hat{u}_j \Psi_j]. \tag{3.5}$$

The matrix form of the discretized weak form reads

$$KU = F$$

where

$$U = (u_1, \ldots, u_n, \hat{u}_1, \ldots, \hat{u}_n)^T,$$

K is called the stiffness matrix and has the form

$$K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

and the load vector F is given by

$$F = \begin{pmatrix} F_\Phi \\ F_\Psi \end{pmatrix}.$$

Here, for $i, j = 1, \ldots, n$, the $n \times n$ matrices A, B, and C are given by

$$A_{ij} = \left\langle a(x)\Phi_j'', \Phi_i'' \right\rangle,$$
$$B_{ij} = \left\langle a(x)\Psi_j'', \Phi_i'' \right\rangle,$$
$$C_{ij} = \left\langle a(x)\Psi_j'', \Psi_i'' \right\rangle.$$

and the vectors $F_\Phi, F_\Psi \in R^n$ by

$$F_\Phi = (\langle f, \Phi_1 \rangle, \ldots, \langle f, \Phi_n \rangle)^T$$
$$F_\Psi = (\langle f, \Psi_1 \rangle, \ldots, \langle f, \Psi_n \rangle)^T.$$

An important step is the computation of the so-called adjoint-stiffness matrix defined by the condition

$$\big(\forall \tilde{A} \in R^{n+2}\big)\big(\forall \tilde{V} \in R^{2n}\big)\ \big[L(\tilde{V})\tilde{A} = K(\tilde{A})\tilde{V}\big], \tag{3.6}$$

where $\tilde{V} = (V, \hat{V})^T$.

By a simple but lengthy computation, it can be shown that the matrix $L(\tilde{V})$ is given by

$$L = \begin{pmatrix} A + B \\ D + C \end{pmatrix}$$

where

$$\begin{aligned}
A_{ik} &= \sum_{j=1}^{n} \Big(\int_0^1 a_k \Phi_j'' \Phi_i''\, \mathrm{d}x \Big) V_j \\
B_{ik} &= \sum_{j=1}^{n} \Big(\int_0^1 a_k \Psi_j'' \Phi_i''\, \mathrm{d}x \Big) \hat{V}_j \\
D_{ik} &= \sum_{j=1}^{n} \Big(\int_0^1 a_k \Phi_j'' \Psi_i''\, \mathrm{d}x \Big) V_j \\
C_{ik} &= \sum_{j=1}^{n} \Big(\int_0^1 a_k \Psi_j'' \Psi_i''\, \mathrm{d}x \Big) \hat{V}_j.
\end{aligned}$$

By using the specific forms of the basis functions Φ and Ψ, we can show that

$$A = \frac{6}{h^3} \left(\begin{matrix} V_1 & 2V_1 - V_2 & V_1 - V_2 & 0 \\ 0 & -V_1 + V_2 & -V_1 + 2V_2 - V_3 & V_2 - V_3 \\ 0 & \dots & -V_2 + V_3 & -V_2 + 2V_3 - V_4 \\ & & & \vdots \\ 0 & \dots & 0 & -V_{n-2} + V_{n-1} \\ 0 & \dots & 0 & 0 \end{matrix} \right.$$

$$\left. \begin{matrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ V_3 - V_4 & \dots & 0 \\ & & \\ -V_{n-2} + 2V_{n-1} - V_n & V_{n-1} - V_n & 0 \\ -V_{n-1} + V_n & -V_{n-1} + 2V_n & V_n \end{matrix} \right)$$

$$B = \frac{2}{h^2} \begin{pmatrix} -\hat{V}_1 & \hat{V}_2 & \hat{V}_1 + 2\hat{V}_2 & 0 & 0 & \dots & 0 \\ 0 & -2\hat{V}_1 - \hat{V}_2 & -\hat{V}_1 + \hat{V}_3 & \hat{V}_2 + 2\hat{V}_3 & 0 & \dots & 0 \\ 0 & 0 & -2\hat{V}_2 - \hat{V}_3 & -\hat{V}_2 + \hat{V}_4 & \hat{V}_3 + 2\hat{V}_4 & \dots & 0 \\ & & & \vdots & & & \\ 0 & \dots & 0 & -2\hat{V}_{n-2} - \hat{V}_{n-1} & -\hat{V}_{n-2} + \hat{V}_n & \hat{V}_{n-1} + 2\hat{V}_n & 0 \\ 0 & \dots & 0 & 0 & -2\hat{V}_{n-1} - \hat{V}_n & -\hat{V}_{n-1} & \hat{V}_n \end{pmatrix}$$

$$C = \frac{1}{h} \begin{pmatrix} \hat{V}_1 & 6\hat{V}_1 + \hat{V}_2 & \hat{V}_1 + \hat{V}_2 & 0 & 0 & \dots & 0 \\ 0 & \hat{V}_1 + \hat{V}_2 & \hat{V}_1 + 6\hat{V}_2 + \hat{V}_3 & \hat{V}_2 + \hat{V}_3 & 0 & \dots & 0 \\ 0 & \dots & \hat{V}_2 + \hat{V}_3 & \hat{V}_2 + 6\hat{V}_3 + \hat{V}_4 & \hat{V}_3 + \hat{V}_4 & \dots & 0 \\ & & & \vdots & & & \\ 0 & \dots & 0 & \hat{V}_{n-2} + \hat{V}_{n-1} & \hat{V}_{n-2} + 6\hat{V}_{n-1} + \hat{V}_n & \hat{V}_{n-1} + \hat{V}_n & 0 \\ 0 & \dots & 0 & 0 & \hat{V}_{n-1} + \hat{V}_n & \hat{V}_{n-1} + 6\hat{V}_n & \hat{V}_n \end{pmatrix}$$

$$D = \frac{2}{h^2}\begin{pmatrix} -V_1 & -2V_2 & V_1 - V_2 & 0 & 0 & \dots & 0 \\ 0 & V_1 - V_2 & 2V_1 - 2V_3 & V_2 - V_3 & 0 & \dots & 0 \\ 0 & \dots & V_2 - V_3 & -V_2 - 2V_4 & V_3 - V_4 & \dots & 0 \\ & & & \vdots & & & \\ 0 & \dots & 0 & V_{n-2} - V_{n-1} & 2V_{n-2} - 2V_n & V_{n-1} - V_n & 0 \\ 0 & \dots & 0 & 0 & V_{n-1} - V_n & 2V_{n-1} & V_n \end{pmatrix}.$$

We recall that for a fixed pair $(a, z) \in K_h \times V_h$, the element $e_h(\cdot, \cdot)$ is defined by

$$\langle e_h(a, z), v\rangle_V = \int_0^1 az'' v'' - \int_0^1 fv, \qquad \text{for all} \quad v \in V_h. \tag{3.7}$$

Therefore, for $e_h \in V_h$, the corresponding vector of the nodal values $E \in \mathbb{R}^{2n}$ is given by

$$KE = K(A)Z - F,$$

where K is the stiffness matrix from the $H^2(\Omega)$ inner product and $Z \in \mathbb{R}^{2n}$ corresponds to the data z. Consequently,

$$E(A, Z) = K^{-1}(L(Z)A - F).$$

The above calculation then leads to

$$J(A) = \frac{1}{2}(L(Z)A - F)^T K^{-1}(L(Z)A - F).$$

Let us now compute the gradient and the Hessian of the objective functional. For $\delta A \in \mathbb{R}^m$, we have

$$\begin{aligned} DJ(A)(\delta A) &= \frac{1}{2}\langle L(Z)\delta A, K^{-1}(L(Z)A - F)\rangle_{\mathbf{R}^{2n}} \\ &\quad + \frac{1}{2}\langle L(Z)A - F, K^{-1}L(Z)\delta A\rangle_{\mathbf{R}^{2n}} \\ &= \langle \delta A, L(Z)^T K^{-1}(L(Z)A - F)\rangle_{\mathbf{R}^{2n}}, \\ D^2 J(A)(\delta A, \delta A) &= \langle L(Z)\delta A, K^{-1}(L(Z)\delta A\rangle_{\mathbf{R}^{2n}} \\ &= \langle L(Z)^T K^{-1} L(Z)\delta A, \delta A\rangle_{\mathbf{R}^{2n}}. \end{aligned} \tag{3.8}$$

Summarizing,

$$\begin{aligned}\nabla J(A) &= L(Z)^T K^{-1}(L(Z)A - F)\\ \nabla^2 J(A) &= L(Z)^T K^{-1} L(Z).\end{aligned}$$

4. Numerical examples

In this section, we give two numerical examples to show the feasibility of the proposed equation error approach.

***Example* 1.** In this example, we identify a smooth coefficient $a(x) = 1 + x$. The exact solution is $u(x) = -\cos(2\pi x) + 1$ whereas $f = -16\pi^3 \sin(2\pi x) - 16\pi^4 \cos(2\pi x)(x + 1)$.

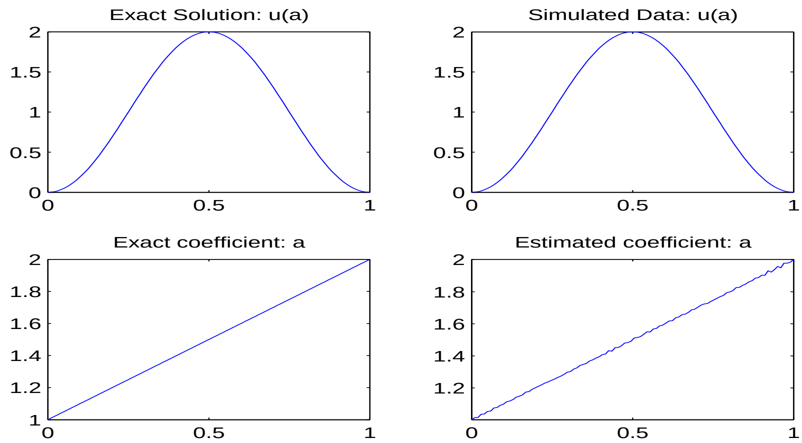

FIGURE 1. Identification by the Equation Error Approach

***Example* 2.** In this example, we identify a smooth coefficient $a(x) = 2 + 5(x-1)x^2$. The exact solution is $u(a) = \cos(2\pi x) - 1$ whereas $f = 16\pi^3 \sin(2\pi x)(2x(5x - 5) + 5x^2) + 16\pi^4 \cos(2\pi x)(x^2(5x - 5) + 2) - 4\pi^2 \cos(2\pi x)(30x - 10)$.

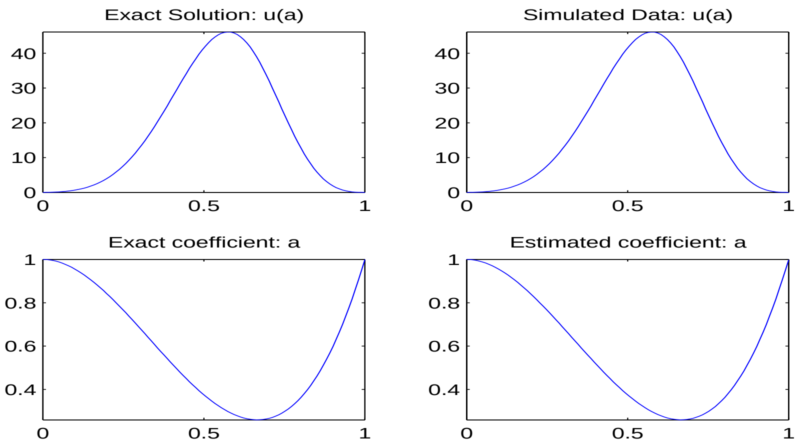

Figure 2. Identification by the Equation Error Approach

Our preliminary numerical results are encouraging. In a future work, we would like to extend the computational framework to higher dimensional setting. We also aim to investigate the impact of using noisy data.

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