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Bounds on Some Ramsey Numbers Involving Quadrilateral †

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Abstract. For graphs $G_1, G_2, \cdots, G_m$, the Ramsey number $R(G_1, G_2, \cdots, G_m)$ is defined to be the smallest integer n such that any m-coloring of the edges of the complete graph K_n must include a monochromatic G_i in color i, for some i. In this note we establish several lower and upper bounds for some Ramsey numbers involving quadrilateral C_4, including $R(C_4, K_9) \leq 32$, $19 \leq R(C_4, C_4, K_4) \leq 22$, $31 \leq R(C_4, C_4, C_4, K_4) \leq 50$, $52 \leq R(C_4, K_4, K_4) \leq 72$, $42 \leq R(C_4, C_4, K_3, K_4) \leq 76$, and $87 \leq R(C_4, C_4, K_4, K_4) \leq 179$.

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1 Introduction and Overview

We consider only graphs without multiple edges or loops. If $G = (V, E)$ is a graph, then the set of vertices of G is denoted by $V(G)$, the set of edges by $E(G)$, and their cardinalities by $|V(G)|$ and $|E(G)|$, respectively. We denote by $\delta(G)$ the minimum degree of G, $N(v)$ the neighborhood of vertex v and $N[v] = \{v\} \cup N(v)$. If the degree of each vertex of G is k, then G is called a k-regular graph. The graph C_n is a cycle of length n and K_n a complete graph of order n. Define $t(n)$ to be the maximum number of edges in any graph of order n not containing C_4.

For graphs $G_1, G_2, \cdots, G_m$, the Ramsey number $R(G_1, G_2, \cdots, G_m)$ is defined to be the least positive integer n such that every m-coloring of the edges of the complete graph K_n contains a monochromatic subgraph isomorphic to G_i whose all edges have color i, for some i, $1 \leq i \leq m$. A coloring of the edges of complete graph with m colors is called a $(G_1, G_2, \cdots, G_m)$-coloring (or graph for $m = 2$), if it does not contain a subgraph isomorphic to G_i whose all edges are colored with color i, for each i. A $(G_1, G_2, \cdots, G_m)$-coloring (graph) of K_n is denoted as $(G_1, G_2, \cdots, G_m; n)$. For other graph theory concepts refer to [5] and [4].

In this note we establish several lower and upper bounds for some Ramsey numbers involving quadrilateral C_4. The lower bounds are obtained by an improvement to Bevan's construction [2] or an explicit coloring. Several concrete upper bounds are obtained by using general properties of the function $t(n)$ first derived by Irving [8] and later enhanced by others [6, 14]. Table 1 summarizes all known values and bounds on Ramsey numbers of the form $R(C_4, G_1, G_2)$ and $R(C_4, C_4, G_1, G_2)$ where both of G_1 and G_2 are one of the graphs C_4, $C_3(= K_3)$ or K_4. The bounds in Table 1 without references are obtained in the next two sections. For completeness we note that $R(C_4, C_4) = 6$, $R(C_4, C_3) = R(C_4, K_3) = 7$, and $R(C_4, K_4) = 10$ (cf. [9]). For known values and bounds on $R(C_4, K_n)$ for higher n see [10, 9]. In section 3 we derive a new upper bound $R(C_4, K_9) \leq 32$, which in turn easily implies $R(C_4, K_{10}) \leq 39$.

In the sequel we don't discuss asymptotic problems associated with such types of Ramsey numbers. Let us only mention that several interesting results were obtained by Alon and Rödl [1], including $R(C_4, C_4, K_m) = \Theta(m^2 poly(\log m))$ and $R(C_4, C_4, C_4, K_m) = \Theta(m^2/\log^2 m)$. For the discussion of asymptotics in the two-color case see [10]. For many other references to multicolor cases consult the results and citations in [1, 9].

Ramsey number parameters	value/ bounds	reference
C_4, C_4, C_4	11	[3]
C_4, C_4, C_3	12	[11]
C_4, C_4, K_4	19-22	
C_4, C_3, C_3	17	[7]
C_4, C_3, K_4	25-32	
C_4, K_4, K_4	52-72	
C_4, C_4, C_4, C_4	18	[12]
C_4, C_4, C_4, C_3	21-27	[13]
C_4, C_4, C_4, K_4	31-50	
C_4, C_4, C_3, C_3	28-36	[13]
C_4, C_4, C_3, K_4	42-76	
C_4, C_4, K_4, K_4	87-179	

Table 1. $R(C_4, G_1, G_2)$ and $R(C_4, C_4, G_1, G_2)$, values and bounds for $G_1, G_2 \in \{C_4, C_3, K_4\}$.

2 Constructive Lower Bounds

Several general lower bound constructions for multicolor Ramsey numbers avoiding complete and other graphs are listed in the survey [9]. We first cite as Theorem 1, and then extend it to Theorem 2, a version derived by Bevan in 2002 [2].

Theorem 1 *For arbitrary connected graphs $G_1, G_2, \cdots, G_r$, we have* $R(G_1, \cdots, G_r, K_{k_1}, \cdots, K_{k_s}) \geq (R(G_1, \cdots, G_r) - 1)(R(k_1, \cdots, k_s) - 1) + 1.$

Theorem 1 and lower bounds listed in [9] easily imply the following.

Corollary 1
(1) $R(C_4, K_3, K_4) \geq 25 \quad (= 3 \cdot 8 + 1)$,
(2) $R(C_4, K_4, K_4) \geq 52 \quad (= 3 \cdot 17 + 1)$,
(3) $R(C_4, C_4, C_4, K_4) \geq 31 \quad (= 10 \cdot 3 + 1)$.

We can improve on the Bevan's construction at least in some cases as follows.

Theorem 2
(1) $87 \leq R(C_4, C_4, K_4, K_4)$,
(2) $42 \leq R(C_4, C_4, K_3, K_4)$.

Proof. (1) Let G be the Paley graph of order 17 with the vertex set $V(G) = Z_{17} = \{0, 1, \cdots, 16\}$. The edge between i and j is in color 3 iff either $|i-j|$ or $17 - |i-j|$ is in the set $S = \{1, 2, 4, 8\}$, and other edges are in color 4. Up to renaming of colors, this is the well known self-complementary unique $(4, 4; 17)$-coloring. Note that every triangle in color 3 uses at least one edge yielding value 1 or 4 in S.

Consider a graph H (isomorphic to C_5) with the vertices $V(H) = \{v_j | 1 \leq j \leq 5\}$, the edges $\{(v_j, v_{j+1}) | 1 \leq j \leq 4\} \cup \{(v_1, v_5)\}$ in color 1, and the other edges in color 2. First, we construct the product graph $G[H]$ with vertices $V(G) \times V(H)$, so its order is 85. For different $j_1, j_2 \in \{1, 2, 3, 4, 5\}$ and $i \in \{0, 1, \cdots, 16\}$, the edge between (i, v_{j_1}) and (i, v_{j_2}) is in the same color as that of the edge (v_{j_1}, v_{j_2}) in H. For any $j_1, j_2 \in \{1, 2, 3, 4, 5\}$ and different $i_1, i_2 \in \{0, 1, \cdots, 16\}$, the edge between (i_1, v_{j_1}) and (i_2, v_{j_2}) is in the same color as that of (i_1, i_2) in G. This coloring of K_{85} is the same as one in the proof of Theorem 1, and it is not difficult to see that it is a $(C_4, C_4, K_4, K_4; 85)$-coloring.

We extend $G[H]$ by a new vertex w. For each $i \in Z_{17}$, we color the edge between w and (i, v_1) with color 3. For each $i \in Z_{17}$ and $j \in \{2, 3, 4, 5\}$, we color the edge between w and (i, v_j) with one of the colors 1 or 2, which is different from the color of (v_1, v_j) in H. Note that no C_4's in color 1 or 2 are created, however now some monochromatic K_4's in color 3 containing w are present. Thus, finally, for all $i \in Z_{17}$, we recolor the edges between (i, v_1) and $(i+1, v_1)$ with color 1, and recolor the edges between (i, v_1) and $(i+4, v_1)$ with color 2. One can easily see that this does not create C_4's in colors 1 or 2, but eliminates monochromatic K_4's in color 3 containing w (since, as observed above, every triangle in color 3 in G has an edge yielding difference 1 or 4 in S). Hence $R(C_4, C_4, K_4, K_4) \geq 87$.

(2) We use the same method as in part (1). Graph G is the cyclic $(3, 4; 8)$-graph with the arc set $S = \{1, 4\}$, while H is the same as before. This leads to the construction of a $(C_4, C_4, K_3, K_4; 40)$-coloring. One can easily check that no forbidden monochromatic subgraphs are created after adding new vertex w and simple recoloring of the edges. Thus we have $R(C_4, C_4, K_3, K_4) \geq 42$. □

Theorem 3 $19 \leq R(C_4, C_4, K_4)$.

Proof. The unique graph G of order 18 containing no C_4 with the maximum of 39 edges was found in [6]. Consider the edges of G to be of color 1. Using the computer we colored the complement of G using colors 2 and 3, so that there is no C_4 in color 2 and no K_4 in color 3. The resulting $(C_4, C_4, K_4; 18)$-coloring, proving that $19 \leq R(C_4, C_4, K_4)$, is shown in Figure 1.

```
1     0 2 2 2 2 3 3 3 3 1 3 2 1 1 3 3 3 3
2     2 0 2 3 3 3 2 2 3 2 1 3 3 3 1 3 1 3
3     2 2 0 3 3 2 3 3 2 3 2 1 3 3 3 1 3 1
4     2 3 3 0 1 3 3 3 3 1 3 2 3 3 1 1 2 2
5     2 3 3 1 0 3 3 3 3 1 3 3 3 3 2 2 1 1
6     3 3 2 3 3 0 3 3 1 3 2 1 1 2 1 3 2 3
7     3 2 3 3 3 3 0 1 3 2 1 3 2 1 3 1 3 2
8     3 2 3 3 3 3 1 0 3 3 1 3 1 2 3 2 3 1
9     3 3 2 3 3 1 3 3 0 3 3 1 2 1 2 3 1 3
10    1 2 3 1 1 3 2 3 3 0 1 1 3 3 2 3 2 3
11    3 1 2 3 3 2 1 1 3 1 0 1 3 3 3 2 3 2
12    2 3 1 2 3 1 3 3 1 1 1 0 2 2 3 3 3 3
13    1 3 3 3 3 1 2 1 2 3 3 2 0 2 1 3 3 1
14    1 3 3 3 3 2 1 2 1 3 3 2 2 0 3 1 1 3
15    3 1 3 1 2 1 3 3 2 2 3 3 1 3 0 1 2 3
16    3 3 1 1 2 3 1 2 3 3 2 3 3 1 1 0 3 2
17    3 1 3 2 1 2 3 3 1 2 3 3 3 1 2 3 0 1
18    3 3 1 2 1 3 2 1 3 3 2 3 1 3 3 2 1 0
```

Figure 1. Matrix of a $(C_4, C_4, K_4; 18)$-coloring.

□

3 Upper Bounds

The main tool we use in this section for establishing some upper bounds on Ramsey numbers involving C_4 are the known values and upper bounds on $t(n)$ (the maximum number of edges in a C_4-free graph on n vertices). Lemma 1 below summarizes the facts about $t(n)$ studied by various authors [8, 4, 6, 14] which we will need. The exact values of $t(n)$ up to 21 were obtained in [6], and the list was extended to up to 31 in [14].

Lemma 1

(1) *For any n-vertex C_4-free graph G, $n > 3$,*
$|E(G)| \leq t(n) < \frac{1}{4}n(1+\sqrt{4n-3})$ *and* $\delta(G) < \frac{1}{2}(1+\sqrt{4n-3})$.

(2) $t(22) = 52$, $t(29) = 80$, $t(31) = 90$.

Theorem 4

(1) $R(C_4, K_9) \leq 32$, (2) $R(C_4, K_{10}) \leq 39$.

Proof. (1) Suppose G is a $(C_4, K_9; 32)$-graph. From $R(C_4, K_8) = 26$ [10] we see that $\delta(G) \geq 6$. By Lemma 1.1 $\delta(G) \leq 6$, it follows $\delta(G) = 6$. And since $t(31) = 90$, we have $|E(G)| \leq 96$. Hence $|E(G)| = 96$ and G is a 6-regular graph. If G has a triangle xyz, then one can easily count that $V(G) - \{x, y, z\}$ induces in G a graph on $96 - 3 - 12 = 81$ edges, which by Lemma 1.2 is impossible since $t(29) = 80$. Thus G has no triangles. For any vertex x, there are $30 = 6 \times 5$ edges between $X = N[x]$ and $Y = V(G) - X$. Note that Y has 25 vertices, hence for some y in Y, y is connected to at least two vertices s, t in $N(x)$. Thus $xsyt$ forms a C_4, a contradiction, and we have $R(C_4, K_9) \leq 32$.

(2) Suppose G is a $(C_4, K_{10}; 39)$-graph. Now $R(C_4, K_9) \leq 32$ implies that $\delta(G) \geq 7$ which contradicts Lemma 1.1. Therefore, $R(C_4, K_{10}) \leq 39$. □

The best known lower bounds for the cases considered in Theorem 4 are $30 \leq R(C_4, K_9)$ and $34 \leq R(C_4, K_{10})$ [10].

Theorem 5

(1) $R(C_4, C_4, K_4) \leq 22$,
(2) $R(C_4, K_3, K_4) \leq 32$,
(3) $R(C_4, K_4, K_4) \leq 72$,
(4) $R(C_4, C_4, C_4, K_4) \leq 50$,
(5) $R(C_4, C_4, K_3, K_4) \leq 76$,
(6) $R(C_4, C_4, K_4, K_4) \leq 179$.

Proof.
(1) Suppose G is a $(C_4, C_4, K_4; 22)$-graph. Since $R(C_4, C_4, K_3) = 12$ [11] then for each $v \in V(G)$, the number of edges of color 3 incident to v is at most 11, and their total number is at most $11 \cdot 22/2 = 121$. Since $t(22) = 52$, the number of colored edges in G is at most $2t(22) + 121 = 225$, which is less than $|E(K_{22})| = 231$, a contradiction.

(2) $R(C_4, K_3, K_4) \leq R(C_4, K_9) \leq 32$ since $R(K_3, K_4) = 9$.

(3) Suppose G is a $(C_4, K_4, K_4; 72)$-graph. $R(C_4, K_3, K_4) \leq 32$ implies that for each vertex $v \in G$ the number of edges of colors 2 and 3 incident to v are both at most 31. By Lemma 1.1 $t(72) \leq 321$, hence the number of colored edges in G is at most $321 + 72 \cdot 31 = 2553$, which is less than $|E(K_{72})| = 2556$, a contradiction.

(4) In order to obtain $R(C_4, C_4, C_4, K_4) \leq 50$, proceed similarly as in (1) and (3) using $R(C_4, C_4, C_4, K_3) \leq 27$ [13] and $t(50) \leq 187$.

(5) In order to obtain $R(C_4, C_4, K_3, K_4) \leq 76$, proceed similarly using $R(C_4, C_4, K_3, K_3) \leq 36$ [13], $R(C_4, C_4, K_4) \leq 22$, and $t(76) \leq 348$.

(6) In order to obtain $R(C_4, C_4, K_4, K_4) \leq 179$, proceed similarly using $R(C_4, C_4, K_3, K_4) \leq 76$ and $t(179) \leq 1239$.

$\square$

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