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# Second Order Quasi-Normal Mode of the Schwarzschild Black Hole

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We formulate and calculate the second order quasi-normal modes (QNMs) of a Schwarzschild black hole (BH). Gravitational wave (GW) from a distorted BH, so called ringdown, is well understood as QNMs in general relativity. Since QNMs from binary BH mergers will be detected with high signal-to-noise ratio by GW detectors, it is also possible to detect the second perturbative order of QNMs, generated by nonlinear gravitational interaction near the BH. In the BH perturbation approach, we derive the master Zerilli equation for the metric perturbation to second order and explicitly regularize it at the horizon and spatial infinity. We numerically solve the second order Zerilli equation by implementing the modified Leaver's continued fraction method. The second order QNM frequencies are found to be twice the first order ones, and the GW amplitude is up to  $\sim 10\%$  that of the first order for the binary BH mergers. Since the second order QNMs always exist, we can use their detections (i) to test the nonlinearity of general relativity, in particular the no-hair theorem, (ii) to remove fake events in the data analysis of QNM GWs and (iii) to measure the distance to the BH.

## I. INTRODUCTION

Thanks to the recent technological advance, we have almost come to the stage that gravitational waves are detectable. In the 21 century, the observation of gravitational waves will be absolutely a new window to our universe and also provide a direct experimental test of general relativity.

There are several on-going projects for the gravitational wave detection in the world [1, 2, 3, 4] which are ground-based detectors. Next-generation detectors, such as the Large-scale Cryogenic Gravitational wave Telescope (LCGT) [5] in Japan, are also in progress. In addition, as for space-based interferometric detectors, LISA [6] is now on its R & D stage and DECIGO/BBO [7, 8] is proposed as a future project. Since they are space-based observation, they will be free from the seismic noise and remarkably sensitive to the low frequency gravitational waves below 1 Hz.

One of the most important gravitational wave sources is *ringdown* of black holes [9]. The black hole perturbation, such as in the late stage of a black hole formation, can be described by quasi-normal modes (QNMs) with complex frequencies. Thus the gravitational radiation is expected as a damped sinusoidal waveform. It is important to study QNMs because we can determine the mass and angular momentum of a spinning black hole by observing the QNM frequencies, i.e., the normal-mode frequencies and damping rates.

For ringdown searches using data of gravitational wave detectors, the matched filtering technique is useful since these waveform is well understood. In the paper [10], a data analysis method to search for ringdowns have been discussed by using an efficient tiling method for ringdown filters [11], and an application to the TAMA300 data has been reported. Accuracies in the waveform parameter estimations have been found that [accuracy of black hole mass]  $< 0.9\%$  and [accuracy of Kerr parameter]  $< 24\%$  for events with the signal-to-noise ratio (SNR)  $\geq 10$  [12].

A promising source that excites QNMs is a merger of binary black holes. In these events, we may detect the QNMs with high SNR, e.g., the SNR  $\sim 10^5$  for  $\sim 10^8 M_\odot$  black hole mergers at  $\sim 1\text{Gpc}$  by LISA [13], since a large fraction of energy ( $\sim 1\text{--}5\% \times [\text{total mass}]$ ) is emitted as gravitational waves of the QNMs. Recently, numerical simulations have succeeded in calculating the entire phase of BH mergers [14, 15, 16], and found that the  $\ell = 2$ ,  $m = \pm 2$  mode actually dominates, carrying away  $\sim 1\text{--}5\%$  of the initial rest mass of the system [17]. The merger rate is also estimated to be large enough [18, 19].

Black holes deform appreciably in the merger so that the higher-order QNMs could be prominent. As an order of magnitude estimate, when the gravitational wave energy of ringdown is  $\sim 1\% \times M$ , i.e.,

$$E_{GW} \sim \frac{[\psi^{(1)}]^2}{M} \sim 1\% \times M, \quad (1.1)$$

where  $\psi^{(1)}$  denotes the first order gauge invariant waveform function, the dimensionless amplitude of the metric perturbation is

$$\frac{\psi^{(1)}}{M} \sim 10\%. \quad (1.2)$$

[See the total radiation energy of the first perturbative order in Eq. (7.31) with  $\omega \sim M^{-1}$ .] Hence, the second order amplitude goes up to  $[\psi^{(1)}/M]^2 \sim 1\%$ . This means that the second order amplitude becomes  $\sim 10\%$  of the first order amplitude, which is detectable for high SNR events. Reduced to its simplest terms, the SNR of the second order QNMs is  $\sim 10$  if the SNR is  $\sim 100$  for the first order QNMs.

The essence of higher order QNMs has already been discussed in “Mechanics” by Landau & Lifshitz [20] as an anharmonic oscillation. In general, an oscillation with small amplitude  $x$  is described by an equation,

$$\ddot{x} + \omega^2 x = 0 \quad (1.3)$$

with a frequency  $\omega$ . This gives a solution,

$$x = a \cos(\omega t + \phi), \quad (1.4)$$

where  $a$  and  $\phi$  denote integration constants. Including the second perturbative order with respect to the amplitude, the equation has a correction,

$$\ddot{x} + \omega^2 x = -\alpha x^2, \quad (1.5)$$

where  $\alpha$  is a constant, and so the solution is  $x = a \cos(\omega t + \phi) + x^{(2)}$ , where

$$x^{(2)} = -\frac{\alpha a^2}{2\omega^2} + \frac{\alpha a^2}{6\omega^2} \cos 2\omega t \propto a^2. \quad (1.6)$$

Here the first term and second term in the right hand side of the above equation have a frequency  $(\omega - \omega)$  and  $(\omega + \omega)$ , respectively. The important point is that the second order oscillation always exists and is completely determined by the first order one. We also note that the frequency does not have a correction at this order.

In a recent paper [21], we have proposed that higher perturbative order of QNMs, generated by nonlinear gravitational interaction near the BH, are detectable and worth searching in observations and simulations of binary BH mergers. In this paper, we give full details to calculate the second order QNMs of a black hole. Since the previous paper just put forward an order-of-magnitude estimate, we also perform numerical calculations of the second order QNMs. Considering a Schwarzschild black hole as a first step, we show that the second order QNMs appear at the frequency  $\omega^{(2)} = 2\omega^{(1)}$  and numerically determine their amplitude by the modified Leaver’s continued fraction method [22]. Since the second order QNMs always exist, their detections can be used as a new test of general relativity. Although this paper and the previous paper [21] have been the first to study second-order QNMs, the second-order analysis of general relativity is pioneered by Tomita [23], and the  $\ell = 2$ ,  $m = 0$  case is studied by Gleiser *et al.* [24, 25, 26, 27]. It is also extended to cosmology [28, 29, 30, 31].

This paper is organized as follows. In Sec. II, we consider the second order metric perturbation and equations to be satisfied, i.e., the perturbed Einstein equation. Our strategy to solve this equation will be given in this section. In Sec. III, we summarize the tensor harmonics expansion of the first and second order perturbation. In Sec. IV, we review the Regge-Wheeler-Zerilli formalism [32, 33] in the black hole perturbation approach. Here, the first order QNMs are also discussed for later use. In Sec. V, we derive the second order Zerilli equation with a source term. The source term consists of quadratic terms of the first order wave-function. In Sec. VI, we regularize the second order source term so that it is regular at the horizon and spatial infinity. In Sec. VII, we discuss how to extract physical information from the second order Zerilli function when we obtain this function. In practice, the gauge transformation from the Regge-Wheeler gauge to an asymptotic flat gauge is considered. In Sec. VIII, we numerically solve the second order Zerilli equation and calculate the QNM amplitude by implementing a modified Leaver’s continued fraction method. In Sec. IX, we summarize this paper and discuss some remaining problems. Some discussions on the first QNMs are given in Appendix A. We clarify the complex nature of QNMs and the relation between  $m$  and  $-m$  modes in the spherical harmonics expansion in this appendix. In this paper, we use units in which  $c = G = 1$  and follow the conventions of Misner *et al.* [34] with the signature  $- + + +$  for the metric.

## II. SECOND ORDER METRIC PERTURBATION

In the black hole perturbation approach, we consider second order metric perturbations,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}^{(1)} + h_{\mu\nu}^{(2)},$$

with an expansion parameter  $A$  which we can identify as a first order amplitude. Here, superscripts  $(i)$  ( $i = 1, 2$ ) denote the perturbative order, i.e.,  $h_{\mu\nu}^{(1)}$  and  $h_{\mu\nu}^{(2)}$  are called the first and second order metric perturbations, respectively,

and  $g_{\mu\nu}$  is the background metric. In this paper, we consider the Schwarzschild metric as the background and use the usual Schwarzschild coordinates,

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2); \\ f(r) &= 1 - \frac{2M}{r}. \end{aligned} \quad (2.1)$$

Thus, the Greek superscript and subscript indices denote  $\{t, r, \theta, \phi\}$ . In the perturbation calculation, we raise and lower all tensor indices with this background metric.

The Einstein tensor  $G_{\mu\nu}$  up to the second order is formally derived as

$$G_{\mu\nu}[\tilde{g}_{\mu\nu}] = G_{\mu\nu}^{(1)}[h^{(1)}] + G_{\mu\nu}^{(1)}[h^{(2)}] + G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}] + O((h^{(1)})^3, h^{(1)}h^{(2)}, (h^{(2)})^2), \quad (2.2)$$

where we have omitted the spacetime indices  $\mu$  and  $\nu$  of the metric perturbations,  $h_{\mu\nu}^{(1)}$  and  $h_{\mu\nu}^{(2)}$ , and ignored the third perturbative order, i.e.,  $O(A^3)$  in terms of the expansion parameter.  $G_{\mu\nu}^{(1)}$  is well known as the linearized Einstein equation,

$$\begin{aligned} G_{\mu\nu}^{(1)}[H] &= -\frac{1}{2}H_{\mu\nu;\alpha}{}^{;\alpha} + F_{(\mu;\nu)} - R_{\alpha\mu\beta\nu}H^{\alpha\beta} - \frac{1}{2}H_{;\mu\nu} - \frac{1}{2}g_{\mu\nu}(F_{\lambda}{}^{;\lambda} - H_{;\lambda}{}^{;\lambda}); \\ F_{\mu} &= H_{\mu\alpha}{}^{;\alpha}. \end{aligned} \quad (2.3)$$

Here,  $H_{\mu\nu}$  denotes  $h_{\mu\nu}^{(1)}$  or  $h_{\mu\nu}^{(2)}$ , and semicolon ";" in the index indicates the covariant derivative with respect to the background metric.  $G_{\mu\nu}^{(2)}$  consists of quadratic terms in the first order perturbation,

$$\begin{aligned} G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}] &= R_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}] - \frac{1}{2}g_{\mu\nu}R^{(2)}[h^{(1)}, h^{(1)}]; \\ R_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}] &= \frac{1}{4}h_{\alpha\beta;\mu}^{(1)}h^{(1)\alpha\beta}{}_{;\nu} + \frac{1}{2}h^{(1)\alpha\beta}(h_{\alpha\beta;\mu\nu}^{(1)} + h_{\mu\nu;\alpha\beta}^{(1)} - 2h_{\alpha(\mu;\nu)\beta}^{(1)}) \\ &\quad - \frac{1}{2}(h^{(1)\alpha\beta}{}_{;\beta} - \frac{1}{2}h_{\beta}^{(1)\beta;\alpha})(2h_{\alpha(\mu;\nu)}^{(1)} - h_{\mu\nu;\alpha}^{(1)}) + \frac{1}{2}h_{\mu\alpha;\beta}^{(1)}h_{\nu}^{(1)\alpha;\beta} - \frac{1}{2}h_{\mu\alpha;\beta}^{(1)}h_{\nu}^{(1)\beta;\alpha}. \end{aligned} \quad (2.4)$$

Since we consider the vacuum Einstein equation now, we may solve the following equation for the first perturbative order,

$$G_{\mu\nu}^{(1)}[h^{(1)}] = 0. \quad (2.5)$$

For the second perturbative order, once the first order metric perturbation  $h^{(1)}$  is obtained, we may solve the equation with a source term which can be considered as an effective energy momentum tensor.

$$G_{\mu\nu}^{(1)}[h^{(2)}] = -G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}]. \quad (2.6)$$

Thus, expanding the Einstein's vacuum equation, we can obtain basic equations order by order [35, 36, 37].

In the following section, we consider the equations in Eqs. (2.5) and (2.6) by using the tensor harmonics expansion which is summarized in the next section. Then, the Regge-Wheeler-Zerilli formalism [32, 33] is used for coefficients of this expansion.

In this formalism, the first order QNMs of a black hole is derived from the first order Einstein equation (2.5). We discuss the first order QNMs and summarize some formulae in the Regge-Wheeler-Zerilli formalism in Sec. IV. (See also Appendix A.)

For the second order Einstein equation (2.6), we also use the same Regge-Wheeler-Zerilli formalism in Sec. V, while there are some differences. The second order equation has a source term which is written by quadratic terms of the first order wave-function. This source term which is shown in Eq. (5.4), does not behave well at the boundaries. The reason of the behavior simply comes from the gauge choice and this has no physical meaning. Hence, we can regularize this behavior in Sec. VI. Based on this regularized source in Eq. (6.5), we numerically compute the second order wave-function in Sec. VIII.

In order to extract physical information from the first and second order wave-functions, we must consider the metric perturbations under an asymptotic flat gauge condition. The first and second order metric perturbations are obtained under the Regge-Wheeler gauge at first, but this gauge is not an asymptotic flat one. It is necessary to formulate the gauge transformation for both the first and second perturbative order. This is done in Sec. VII.

### III. TENSOR HARMONICS EXPANSION

Since the background spacetime has the spherical symmetry, all perturbative quantities can be expanded by the spherical harmonics  $Y_{\ell m}(\theta, \phi)$  and its angular derivative. In this paper, we consider the following tensor harmonics. Almost of all is the same as that of Zerilli's paper [33]. There are some differences in the notation, therefore we summarize them in this section.

For the first and second order metric perturbations, we expand  $h_{\mu\nu}^{(i)}$  ( $i = 1, 2$ ) by tensor harmonics,

$$\begin{aligned} \mathbf{h}^{(i)} = \sum_{\ell m} & \left[ f(r) H_{0\ell m}^{(i)}(t, r) \mathbf{a}_{0\ell m} - i\sqrt{2} H_{1\ell m}^{(i)}(t, r) \mathbf{a}_{1\ell m} + \frac{1}{f(r)} H_{2\ell m}^{(i)}(t, r) \mathbf{a}_{\ell m} \right. \\ & - \frac{i}{r} \sqrt{2\ell(\ell+1)} h_{0\ell m}^{(e)(i)}(t, r) \mathbf{b}_{0\ell m} + \frac{1}{r} \sqrt{2\ell(\ell+1)} h_{1\ell m}^{(e)(i)}(t, r) \mathbf{b}_{\ell m} \\ & + \sqrt{\frac{1}{2} \ell(\ell+1)(\ell-1)(\ell+2)} G_{\ell m}^{(i)}(t, r) \mathbf{f}_{\ell m} + \left( \sqrt{2} K_{\ell m}^{(i)}(t, r) - \frac{\ell(\ell+1)}{\sqrt{2}} G_{\ell m}^{(i)}(t, r) \right) \mathbf{g}_{\ell m} \\ & - \frac{\sqrt{2\ell(\ell+1)}}{r} h_{0\ell m}^{(i)}(t, r) \mathbf{c}_{0\ell m} + \frac{i\sqrt{2\ell(\ell+1)}}{r} h_{1\ell m}^{(i)}(t, r) \mathbf{c}_{\ell m} \\ & \left. + \frac{\sqrt{2\ell(\ell+1)(\ell-1)(\ell+2)}}{2r^2} h_{2\ell m}^{(i)}(t, r) \mathbf{d}_{\ell m} \right], \end{aligned} \quad (3.1)$$

where we should be careful not to confuse  $G_{\ell m}^{(i)}(t, r)$  in the above equation with the perturbed Einstein tensor  $G_{\mu\nu}^{(i)}$  in the previous section. Here,  $\mathbf{a}_{0\ell m}$ ,  $\mathbf{a}_{\ell m}$ ,  $\dots$  are constructed by the spherical harmonics and its derivative. The ten tensor harmonics are defined as the following.

$$\mathbf{a}_{0\ell m} = \begin{pmatrix} Y_{\ell m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.2)$$

$$\mathbf{a}_{1\ell m} = (i/\sqrt{2}) \begin{pmatrix} 0 & Y_{\ell m} & 0 & 0 \\ Sym & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.3)$$

$$\mathbf{a}_{\ell m} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_{\ell m} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.4)$$

$$\mathbf{b}_{0\ell m} = ir[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & (\partial/\partial\theta)Y_{\ell m} & (\partial/\partial\phi)Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \end{pmatrix}, \quad (3.5)$$

$$\mathbf{b}_{\ell m} = r[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (\partial/\partial\theta)Y_{\ell m} & (\partial/\partial\phi)Y_{\ell m} \\ 0 & Sym & 0 & 0 \\ 0 & Sym & 0 & 0 \end{pmatrix}, \quad (3.6)$$

$$\mathbf{c}_{0\ell m} = r[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & (1/\sin\theta)(\partial/\partial\phi)Y_{\ell m} & -\sin\theta(\partial/\partial\theta)Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \\ Sym & 0 & 0 & 0 \end{pmatrix}, \quad (3.7)$$

$$\mathbf{c}_{\ell m} = ir[2\ell(\ell+1)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & (1/\sin\theta)(\partial/\partial\phi)Y_{\ell m} & -\sin\theta(\partial/\partial\theta)Y_{\ell m} \\ 0 & Sym & 0 & 0 \\ 0 & Sym & 0 & 0 \end{pmatrix}, \quad (3.8)$$

$$\mathbf{d}_{\ell m} = -ir^2[2\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(1/\sin\theta)X_{\ell m} & \sin\theta W_{\ell m} \\ 0 & 0 & Sym & \sin\theta X_{\ell m} \end{pmatrix}, \quad (3.9)$$

$$\mathbf{g}_{\ell m} = (r^2/\sqrt{2}) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{\ell m} & 0 \\ 0 & 0 & 0 & \sin^2\theta Y_{\ell m} \end{pmatrix}, \quad (3.10)$$

$$\mathbf{f}_{\ell m} = r^2[2\ell(\ell+1)(\ell-1)(\ell+2)]^{-1/2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_{\ell m} & X_{\ell m} \\ 0 & 0 & Sym & -\sin^2\theta W_{\ell m} \end{pmatrix}. \quad (3.11)$$

Here the *Sym* denotes components derived from the symmetry of the tensors, and the angular functions  $X_{\ell m}$  and  $W_{\ell m}$  are given by

$$X_{\ell m} = 2\frac{\partial}{\partial\phi} \left( \frac{\partial}{\partial\theta} - \cot\theta \right) Y_{\ell m}, \quad (3.12)$$

$$W_{\ell m} = \left( \frac{\partial^2}{\partial\theta^2} - \cot\theta \frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right) Y_{\ell m}. \quad (3.13)$$

From the above definition, the tensor harmonics can be further classified into even (or polar) and odd (or axial) parities. Even parity modes are defined by the parity  $(-1)^\ell$  under the transformation  $(\theta, \phi) \rightarrow (\pi - \theta, \phi + \pi)$ , while odd parity modes are by the parity  $(-1)^{\ell+1}$ . Thus, the seven coefficients of the metric perturbation,  $H_{0\ell m}^{(i)}$ ,  $H_{1\ell m}^{(i)}$ ,  $H_{2\ell m}^{(i)}$ ,  $h_{0\ell m}^{(e)(i)}$ ,  $h_{1\ell m}^{(e)(i)}$ ,  $G_{\ell m}^{(i)}$  and  $K_{\ell m}^{(i)}$  are called as the even parity part, and the three coefficients,  $h_{0\ell m}^{(i)}$ ,  $h_{1\ell m}^{(i)}$  and  $h_{2\ell m}^{(i)}$  are the odd parity part.

On the other hand, we consider the right hand side of Eq. (2.6) as the energy momentum tensor,

$$\mathcal{T}_{\mu\nu}^{(2)} = -\frac{1}{8\pi} G_{\mu\nu}^{(2)} [h^{(1)}, h^{(1)}]. \quad (3.14)$$

Then we define the following tensor harmonics expansion,

$$\begin{aligned} \mathcal{T}^{(2)} = \sum_{\ell m} [ & \mathcal{A}_{0\ell m}(t, r) \mathbf{a}_{0\ell m} + \mathcal{A}_{1\ell m}(t, r) \mathbf{a}_{1\ell m} + \mathcal{A}_{\ell m}(t, r) \mathbf{a}_{\ell m} + \mathcal{B}_{0\ell m}(t, r) \mathbf{b}_{0\ell m} + \mathcal{B}_{\ell m}(t, r) \mathbf{b}_{\ell m} \\ & + \mathcal{Q}_{0\ell m}(t, r) \mathbf{c}_{0\ell m} + \mathcal{Q}_{\ell m}(t, r) \mathbf{c}_{\ell m} + \mathcal{D}_{\ell m}(t, r) \mathbf{d}_{\ell m} + \mathcal{G}_{\ell m}(t, r) \mathbf{g}_{\ell m} + \mathcal{F}_{\ell m}(t, r) \mathbf{f}_{\ell m} ] . \end{aligned} \quad (3.15)$$

Here the seven coefficients,  $\mathcal{A}_{0\ell m}$ ,  $\mathcal{A}_{1\ell m}$ ,  $\mathcal{A}_{\ell m}$ ,  $\mathcal{B}_{0\ell m}$ ,  $\mathcal{B}_{\ell m}$ ,  $\mathcal{G}_{\ell m}$  and  $\mathcal{F}_{\ell m}$  are the even parity part, and the three coefficients,  $\mathcal{Q}_{0\ell m}$ ,  $\mathcal{Q}_{\ell m}$  and  $\mathcal{D}_{\ell m}$  are the odd parity part.

By using the orthogonality of the above tensor harmonics, we can derive the coefficient of the tensor harmonics expansion. For example, the coefficient of the energy momentum tensor is calculated by

$$\begin{aligned} \mathcal{A}_{0\ell m}(t, r) &= \int \mathcal{T}^{(2)} \cdot \mathbf{a}_{0\ell m}^* d\Omega \\ &= \int \delta^{\mu\alpha} \delta^{\nu\beta} \mathcal{T}_{\mu\nu}^{(2)} a_{0\ell m \alpha\beta}^* d\Omega, \end{aligned} \quad (3.16)$$

where  $*$  denotes the complex conjugate,  $d\Omega = \sin\theta d\theta d\phi$  and  $\delta^{\mu\alpha}$  has the component,  $diag(1, 1, 1/r^2, 1/(r^2 \sin^2\theta))$ .

#### IV. FIRST ORDER ZERILLI EQUATION

We use the Regge-Wheeler-Zerilli formalism [32, 33] for the first order metric perturbation in the Schwarzschild spacetime. There are some reviews about this formalism in [38]. Separating angular variables with tensor harmonics of indices  $(\ell, m)$  as mentioned in the above section, the equations are divided into the even and odd parity parts. In this formalism, the master equation for the odd or even parity part arises as the Regge-Wheeler or Zerilli equation, respectively.

Here, we consider only the even parity mode in the first order calculation. The reason is the following. In the case of a head-on collision, we have already known that the odd parity perturbation does not arise due to symmetry. When we treat black hole binaries, the radial motion always exists in order to merge through the potential barrier of the system. Actually the numerical simulations show that the even parity mode dominates. Thus, the above assumption for the first perturbative order is adequate as a first step, though it may not be the best approximation. Furthermore, we should note that it is also sufficient to discuss the even parity part for the second order calculation under the above assumption.

In order to discuss the metric perturbations, it is necessary to fix the gauge. For the first order metric perturbation, we impose the Regge-Wheeler (RW) gauge conditions, the vanishing of some coefficients of the first order metric perturbation:

$$h_{0\ell m}^{(e)(1)\text{RW}} = h_{1\ell m}^{(e)(1)\text{RW}} = G_{\ell m}^{(1)\text{RW}} = 0.$$

Here, the suffix RW stands for the RW gauge. It is noted that in the RW gauge the gauge freedom is completely fixed. Although there are seven equations for the even parity part, introducing the following wave-function,

$$\begin{aligned} \psi_{\ell m}^{(1)}(t, r) &= \frac{r}{\lambda + 1} \left[ K_{\ell m}^{(1)\text{RW}}(t, r) + \frac{r - 2M}{\lambda r + 3M} \left( H_{2\ell m}^{(1)\text{RW}}(t, r) - r \frac{\partial}{\partial r} K_{\ell m}^{(1)\text{RW}}(t, r) \right) \right]; \\ \lambda &= \frac{(\ell - 1)(\ell + 2)}{2}, \end{aligned} \quad (4.1)$$

we can reduce the seven equations to a single equation for the function  $\psi_{\ell m}^{(1)}$ . This function  $\psi_{\ell m}^{(1)}$  obeys the following Zerilli equation.

$$\begin{aligned} \left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_Z(r) \right] \psi_{\ell m}^{(1)}(t, r) &= 0; \\ V_Z(r) &= \left( 1 - \frac{2M}{r} \right) \frac{2\lambda^2(\lambda + 1)r^3 + 6\lambda^2Mr^2 + 18\lambda M^2r + 18M^3}{r^3(\lambda r + 3M)^2}, \end{aligned} \quad (4.2)$$

where

$$r_* = r + 2M \ln \left( \frac{r}{2M} - 1 \right).$$

All the first order metric perturbations can be reconstructed from the wave-function  $\psi_{\ell m}^{(1)}$ . The detailed method of the first order  $\ell = 2$  metric reconstruction under the RW gauge condition is summarized in the end of this section.

If  $\psi_{\ell m}^{(1)}$  is Fourier analyzed,

$$\psi_{\ell m}^{(1)}(t, r) = \int e^{-i\omega t} \psi_{\ell m \omega}^{(1)}(r) d\omega, \quad (4.3)$$

the Zerilli equation gives a one-dimensional scattering problem with a potential,

$$\left[ \frac{\partial^2}{\partial r_*^2} + \omega^2 - V_Z(r) \right] \psi_{\ell m \omega}^{(1)}(r) = 0. \quad (4.4)$$

Then, the QNMs are obtained by imposing the boundary conditions with purely ingoing waves,

$$\psi_{\ell m \omega}^{(1)}(r) e^{-i\omega t} \sim e^{-i\omega(t+r_*)},$$

at the horizon of a black hole and purely outgoing waves

$$\psi_{\ell m \omega}^{(1)}(r) e^{-i\omega t} \sim e^{-i\omega(t-r_*)},$$

at infinity. Such boundary conditions are satisfied at discrete QNM frequencies. These frequencies are complex with the real part representing the actual frequency of the oscillation, i.e., the normal mode frequency, and the imaginary part representing the damping. There is an infinite number of QNMs for each harmonic index  $(\ell, m)$  which are labeled by  $n$ . Thus, the QNM frequency has three indices,  $(\ell, m, n)$ . We note that the QNM frequencies have some symmetry shown in Eq. (A3).

In the following, we consider only the  $\ell = 2, m = \pm 2$  modes as the first order perturbations. This is because these modes dominate for binary black hole mergers. Although we can derive all QNMs without fixing the  $m$  mode in the case of a Schwarzschild black hole, we need to specify  $m$  mode when we consider the second order perturbations. (See Appendix A.)

Furthermore, in this paper, we mainly concentrate on the most long-lived QNM in the first perturbative order. This mode is characterized by the fundamental ( $n = 0$ ) QNM frequency. Just for comparison, we will calculate the second order QNM in the cases that only the  $n = 1$  or 2 QNM is excited in the first perturbative order. Although the QNM frequencies for  $\ell = 2$  are given in [22], we calculate them with higher-precision in Table I that is necessary to calculate the second perturbative order.

For later use, we show that  $\psi_{2-2}^{(1)}$  must be the complex conjugate of  $\psi_{22}^{(1)}$  in order to assure that the metric perturbation has a real value. For example, the first order metric component  $h_{tt}^{(1)RW}$  is written by

$$\begin{aligned} h_{tt}^{(1)RW} &= f(r) \left( H_{022}^{(1)RW}(t, r) Y_{22}(\theta, \phi) + H_{02-2}^{(1)RW}(t, r) Y_{2-2}(\theta, \phi) \right) \\ &= f(r) \left( \hat{H}_{022}^{(1)RW}[\psi_{22}^{(1)}(t, r)] Y_{22}(\theta, \phi) + \hat{H}_{02-2}^{(1)RW}[\psi_{2-2}^{(1)}(t, r)] Y_{2-2}(\theta, \phi) \right) \\ &= f(r) \left( \hat{H}_{022}^{(1)RW}[\psi_{22}^{(1)}(t, r) Y_{22}(\theta, \phi) + \psi_{2-2}^{(1)}(t, r) Y_{2-2}(\theta, \phi)] \right), \end{aligned} \quad (4.5)$$

where we have used Eqs. (3.1) and (3.2) in the first line, and  $\hat{H}_{022}^{(1)RW} = \hat{H}_{02-2}^{(1)RW}$  is a real differential operator with respect to  $t$  and  $r$  defined by Eqs. (4.8). By using

$$\begin{aligned} Y_{22}(\theta, \phi) &= \frac{1}{8} \sqrt{\frac{30}{\pi}} \sin^2 \theta e^{2i\phi}, \\ Y_{2-2}(\theta, \phi) &= \frac{1}{8} \sqrt{\frac{30}{\pi}} \sin^2 \theta e^{-2i\phi}, \end{aligned} \quad (4.6)$$

we can show

$$\begin{aligned} \psi_{22}^{(1)}(t, r) &= \psi_{2-2}^{(1)*}(t, r) \\ &\sim A e^{-i\omega_{22}^{(1)}(t-r_*)} \quad \text{for } r \rightarrow \infty, \end{aligned} \quad (4.7)$$

where the second line represents the most dominant QNM (see Appendix A) and  $A$  is some complex number. Berti *et al.* [39] have discussed that the quasi-normal waveform for each  $(\ell, m)$  mode has four degrees of freedom, i.e., two amplitudes and two phases. In this paper, we consider only the even parity mode. This means that we need to consider two degrees of freedom in our calculation, i.e., the complex number  $A$ .

From these wave-functions, the reconstruction of the first order metric perturbation under the RW gauge is given as the following,

$$\begin{aligned} H_{02\pm 2}^{(1)RW}(t, r) &= \hat{H}_{02\pm 2}^{(1)RW}[\psi_{2\pm 2}^{(1)}(t, r)] \\ &= \frac{2r^2 - 2rM + 3M^2}{r(2r + 3M)} \frac{\partial}{\partial r} \psi_{2\pm 2}^{(1)}(t, r) - 3 \frac{4r^3 + 4r^2M + 6rM^2 + 3M^3}{r^2(2r + 3M)^2} \psi_{2\pm 2}^{(1)}(t, r) \\ &\quad + (r - 2M) \frac{\partial^2}{\partial r^2} \psi_{2\pm 2}^{(1)}(t, r), \\ H_{12\pm 2}^{(1)RW}(t, r) &= \hat{H}_{12\pm 2}^{(1)RW}[\psi_{2\pm 2}^{(1)}(t, r)] \\ &= \frac{2r^2 - 6rM - 3M^2}{(2r + 3M)(r - 2M)} \frac{\partial}{\partial t} \psi_{2\pm 2}^{(1)}(t, r) + r \frac{\partial^2}{\partial t \partial r} \psi_{2\pm 2}^{(1)}(t, r), \\ H_{22\pm 2}^{(1)RW}(t, r) &= \hat{H}_{02\pm 2}^{(1)RW}[\psi_{2\pm 2}^{(1)}(t, r)] \\ &= H_{02\pm 2}^{(1)RW}(t, r), \\ K_{2\pm 2}^{(1)RW}(t, r) &= \hat{K}_{2\pm 2}^{(1)RW}[\psi_{2\pm 2}^{(1)}(t, r)] \\ &= \frac{r - 2M}{r} \frac{\partial}{\partial r} \psi_{2\pm 2}^{(1)}(t, r) + 6 \frac{r^2 + rM + M^2}{(2r + 3M)r^2} \psi_{2\pm 2}^{(1)}(t, r). \end{aligned} \quad (4.8)$$

In the above equations, we can derive the first order metric perturbation only from the wave-functions in simple differential forms. This is because there is no source term in the vacuum Einstein equation. On the contrary, when



we construct a second order metric perturbation, we need both the second order wave-functions and the second order source which have quadratic terms of the first order perturbation.

We use the above metric perturbation under the RW gauge to derive the source term of the second order Zerilli equation. But, when we extract physical information, it is necessary to consider a gauge transformation from the RW gauge to an asymptotic flat (AF) gauge. This will be discussed in Sec. VII.

## V. SECOND ORDER ZERILLI EQUATION

For the second order perturbations, we also separate angular variables by tensor harmonics and choose the RW gauge condition. First of all, we have considered the first order metric perturbation only for the even parity part. The second order source term, which is derived from  $G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}]$  in Eq. (2.6), has also the even parity. Therefore, we may discuss the second order metric perturbation only for the even parity part, i.e., the Zerilli equation.

Here if the dominant first order perturbations are the  $\ell = 2, m = \pm 2$  even parity part, we are sufficient to consider the  $\ell = 4, m = \pm 4$  even parity mode for the second order perturbations as shown below. The second order source term arises from  $G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}]$ . This contains products,  $(\ell = 2, m = 2) \times (\ell = 2, m = 2)$ ,  $(\ell = 2, m = -2) \times (\ell = 2, m = -2)$ ,  $(\ell = 2, m = -2) \times (\ell = 2, m = 2)$  and  $(\ell = 2, m = 2) \times (\ell = 2, m = -2)$  of the first order metric perturbation. Then the source term has  $m = 0$  and  $\pm 4$  modes as a result of the product of the spherical harmonics. We can find that the  $m = 0$  source term does not oscillate as a function of time, because of the symmetry of the QNM frequencies in Eq. (A3), and hence this  $m = 0$  mode can not be observed as a QNM wave. Therefore, we consider only  $\ell = 4, m = \pm 4$  modes in the following calculation.

In [24, 26, 27], the second order source term of the  $\ell = 2, m = 0$  mode which arises from the  $\ell = 2, m = 0$  first order perturbations, has been discussed. These discussion is for example, for the treatment in the "close limit approximation" of the collision of black holes. Their situation is restricted to the axisymmetric case, i.e., the  $m = 0$  mode. Therefore, the second order calculation is also only for the  $m = 0$  mode. In our case, the harmonics of the second order metric perturbation changes in the  $m$  mode as well as in the  $\ell$  mode as discussed above.

Now, we introduce a function for the second perturbative order,

$$\chi_{4\pm 4}^{(2)}(t, r) = \frac{r - 2M}{3(3r + M)} \left[ \frac{r^2}{r - 2M} \frac{\partial K_{4\pm 4}^{(2)RW}(t, r)}{\partial t} - H_{14\pm 4}^{(2)RW}(t, r) \right], \quad (5.1)$$

where the functions  $K_{4\pm 4}^{(2)RW}$  and  $H_{14\pm 4}^{(2)RW}$  are coefficients in Eq. (3.1), i.e., derived from the expansion of the second order metric perturbation by tensor harmonics as in the first order case. We note that the first-order counterpart exactly satisfies  $\chi_{\ell m}^{(1)} = \partial \psi_{\ell m}^{(1)} / \partial t$ . Hence, the dimensions are  $\psi_{\ell m}^{(1)}(t, r) \sim O(M)$ ,  $\chi_{\ell m}^{(1)}(t, r) \sim O(M^0)$  and  $\chi_{\ell m}^{(2)}(t, r) \sim O(M^0)$ .

Using the function of Eq. (5.1), the second order equations for the even parity mode is reduced to the Zerilli equation with a second order source term,

$$\left[ -\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_Z(r) \right] \chi_{4\pm 4}^{(2)}(t, r) = S_{4\pm 4}(t, r), \quad (5.2)$$

where the potential  $V_Z$  is the same function defined in Eq. (4.2) with  $\lambda = 9$ . The source term  $S_{4\pm 4}$  is derived as

$$\begin{aligned} S_{4\pm 4}(t, r) = & \frac{4\pi\sqrt{10}(r-2M)^2}{15(3r+M)} \frac{\partial}{\partial t} \mathcal{B}_{4\pm 4}(t, r) + \frac{8\pi(r-2M)^2}{3(3r+M)} \frac{\partial}{\partial t} \mathcal{A}_{4\pm 4}(t, r) \\ & - \frac{4\sqrt{2}i\pi(r-2M)^2}{3(3r+M)} \frac{\partial}{\partial r} \mathcal{A}_{14\pm 4}(t, r) - \frac{8\sqrt{2}i\pi(r-2M)(4r-M)M}{3r(3r+M)^2} \mathcal{A}_{14\pm 4}(t, r) \\ & - \frac{4\sqrt{10}i\pi(r-2M)^2}{15(3r+M)} \frac{\partial}{\partial r} \mathcal{B}_{04\pm 4}(t, r) - \frac{8\sqrt{5}\pi(r-2M)}{15} \frac{\partial}{\partial t} \mathcal{F}_{4\pm 4}(t, r) \\ & + \frac{4\sqrt{10}i\pi(4M^2 + 7Mr + 27r^2)(r-2M)}{15r(3r+M)^2} \mathcal{B}_{04\pm 4}(t, r), \end{aligned} \quad (5.3)$$

where the functions  $\mathcal{B}_{4\pm 4}$  etc. are coefficients of the tensor harmonics expansion of  $G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}]$  in Eqs. (3.14) and (3.15). Since these coefficients are written by quadratic terms of the first order metric perturbation, we can rewrite

the second order source term  $S_{4\pm 4}$  in terms of  $\psi_{2\pm 2}^{(1)}$  by using Eq. (4.8) as

$$\begin{aligned}
S_{4\pm 4}(t, r) = & \frac{r-2M}{42} \frac{\sqrt{70}}{\sqrt{\pi}} \left\{ -\frac{3}{r^5(3r+M)^2(2r+3M)^3} (24r^7 + 1120r^6M + 1052r^5M^2 - 798r^4M^3 \right. \\
& - 2586r^3M^4 - 2396r^2M^5 - 270rM^6 - 9M^7) \left( \frac{\partial}{\partial t} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial}{\partial r} \psi_{2\pm 2}^{(1)}(t, r) \\
& + \frac{1}{r^4(3r+M)^2(2r+3M)^2} (132r^6 - 136Mr^5 + 994r^4M^2 - 378r^3M^3 - 2306r^2M^4 \\
& - 270rM^5 - 9M^6) \left( \frac{\partial^2}{\partial r \partial t} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial}{\partial r} \psi_{2\pm 2}^{(1)}(t, r) \\
& + \frac{(78r^4 - 128r^3M - 264r^2M^2 - 100rM^3 + 3M^4)}{r(2r+3M)(3r+M)^2(r-2M)} \left( \frac{\partial^3}{\partial t^3} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial}{\partial r} \psi_{2\pm 2}^{(1)}(t, r) \\
& + \frac{(84r^5 - 122r^4M - 281r^3M^2 - 6r^2M^3 + 237rM^4 + 33M^5)}{r^2(2r+3M)(r-2M)^2(3r+M)^2} \left( \frac{\partial^2}{\partial t^2} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial}{\partial t} \psi_{2\pm 2}^{(1)}(t, r) \\
& - \frac{(66r^4 - 106r^3M - 220r^2M^2 - 156rM^3 - 45M^4)}{r(2r+3M)(3r+M)^2(r-2M)} \left( \frac{\partial^3}{\partial t^2 \partial r} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial}{\partial t} \psi_{2\pm 2}^{(1)}(t, r) \\
& - \frac{9}{r^6(3r+M)^2(2r+3M)^4} (336r^8 + 3664r^7M + 10144r^6M^2 + 15052r^5M^3 \\
& + 13444r^4M^4 + 7386r^3M^5 + 2648r^2M^6 + 270rM^7 + 9M^8) \psi_{2\pm 2}^{(1)}(t, r) \frac{\partial}{\partial t} \psi_{2\pm 2}^{(1)}(t, r) \\
& + 2 \frac{(r-2M)(3r+2M)}{r(3r+M)^2} \left( \frac{\partial^2}{\partial t^2} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial^2}{\partial r \partial t} \psi_{2\pm 2}^{(1)}(t, r) - 3 \left( \frac{\partial^3}{\partial t^2 \partial r} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial^2}{\partial r \partial t} \psi_{2\pm 2}^{(1)}(t, r) \\
& - \frac{3}{r^5(3r+M)^2(2r+3M)^3} (72r^7 + 1240r^6M + 944r^5M^2 - 1284r^4M^3 \\
& - 2910r^3M^4 - 2396r^2M^5 - 270rM^6 - 9M^7) \psi_{2\pm 2}^{(1)}(t, r) \frac{\partial^2}{\partial r \partial t} \psi_{2\pm 2}^{(1)}(t, r) \\
& + 3 \frac{r^2}{(r-2M)^2} \left( \frac{\partial^3}{\partial t^3} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial^2}{\partial t^2} \psi_{2\pm 2}^{(1)}(t, r) \\
& \left. + 3 \frac{(132r^5 + 196r^4M + 174r^3M^2 + 114r^2M^3 + 28rM^4 - 3M^5)}{r^2(2r+3M)^2(3r+M)^2(r-2M)} \psi_{2\pm 2}^{(1)}(t, r) \frac{\partial^3}{\partial t^3} \psi_{2\pm 2}^{(1)}(t, r) \right\}. \quad (5.4)
\end{aligned}$$

The second order metric perturbations can be reconstructed from  $\chi_{4\pm 4}^{(2)}$  under the RW gauge as

$$\begin{aligned}
\frac{\partial}{\partial t} K_{4\pm 4}^{(2)RW}(t, r) &= \frac{30r^2 + 9rM + 2M^2}{r^2(3r+M)} \chi_{4\pm 4}^{(2)}(t, r) + \frac{r-2M}{r} \frac{\partial}{\partial r} \chi_{4\pm 4}^{(2)}(t, r) \\
&+ \frac{4\sqrt{2}}{3} \frac{i\pi r(r-2M)}{3r+M} \mathcal{A}_{14\pm 4}(t, r) + \frac{4\sqrt{5}}{15} \frac{i\pi r(r-2M)}{3r+M} \mathcal{B}_{04\pm 4}(t, r), \\
\frac{\partial}{\partial t} H_{24\pm 4}^{(2)RW}(t, r) &= r \frac{\partial^2}{\partial r \partial t} K_{4\pm 4}^{(2)RW}(t, r) + 3 \frac{M}{r^2} \chi_{4\pm 4}^{(2)}(t, r) - 3 \frac{3r+M}{r} \frac{\partial}{\partial r} \chi_{4\pm 4}^{(2)}(t, r) - \frac{4\sqrt{5}}{5} i\pi r \mathcal{B}_{04\pm 4}(t, r), \\
H_{14\pm 4}^{(2)RW}(t, r) &= -3 \frac{3r+M}{r-2M} \chi_{4\pm 4}^{(2)}(t, r) + \frac{r^2}{r-2M} \frac{\partial}{\partial t} K_{4\pm 4}^{(2)RW}(t, r), \\
H_{04\pm 4}^{(2)RW}(t, r) &= H_{24\pm 4}^{(2)RW}(t, r) + \frac{8\sqrt{5}}{15} \pi r^2 \mathcal{F}_{4\pm 4}(t, r). \quad (5.5)
\end{aligned}$$

Unlike the first order metric reconstruction, we need the information derived from  $G_{\mu\nu}^{(2)}[h^{(1)}, h^{(1)}]$ , such as  $\mathcal{A}_{14\pm 4}$  and  $\mathcal{B}_{04\pm 4}$  in Eq. (3.15).

## VI. REGULARIZATION OF SOURCE TERM

Although we show the second order  $\ell = 4$ ,  $m = \pm 4$  source term  $S_{4\pm 4}$  in terms of  $\psi_{2\pm 2}^{(1)}$  in Eq. (5.4), the raw expression  $S_{4\pm 4}$  does not behave well at infinity. This is not suitable for numerical calculations. We can find  $S_{4\pm 4} \sim O(r^0)$  at infinity because at large  $r$  the first order wave-function behaves as

$$\psi_{2\pm 2}^{(1)}(t, r) = \frac{1}{3}F_I''(t - r_*) + \frac{1}{r}F_I'(t - r_*) + \frac{1}{r^2}[F_I(t - r_*) - MF_I'(t - r_*)] + O(r^{-3}), \quad (6.1)$$

where  $F_I$  is some function of  $(t - r_*)$  and  $F_I'(x)$  denotes  $dF_I(x)/dx$ . In order to obtain a finite solution  $\chi_{4\pm 4}^{(2)}$ , we have to make at least the second order source term  $\sim O(r^{-2})$  by some regularization, which is the same order of the potential  $V_Z \sim O(r^{-2})$  in Eqs. (5.2) and (4.2). On the other hand, the second order source behaves well at the horizon, i.e.,  $\sim O(r - 2M)$ , with

$$\psi_{2\pm 2}^{(1)}(t, r) = F_H'(t + r_*) + \frac{1}{4}\frac{F_H(t + r_*)}{M} + \frac{27}{56}\frac{F_H(t + r_*)}{M^2}(r - 2M) + O[(r - 2M)^2], \quad (6.2)$$

where  $F_H$  is some function of  $(t + r_*)$ .

We can regularize the second order source term by introducing the following regularized function,

$$\begin{aligned} \chi_{4\pm 4}^{(2)reg}(t, r) &= \chi_{4\pm 4}^{(2)}(t, r) - \zeta_{4\pm 4}^{(2)}(t, r); \\ \zeta_{4\pm 4}^{(2)}(t, r) &= \frac{\sqrt{70}}{126\sqrt{\pi}}\frac{(r - 2M)^2}{r}\left(\frac{\partial}{\partial r}\psi_{2\pm 2}^{(1)}(t, r)\right)\frac{\partial^2}{\partial r\partial t}\psi_{2\pm 2}^{(1)}(t, r). \end{aligned} \quad (6.3)$$

The regularized function  $\chi_{4\pm 4}^{(2)reg}$  satisfies the Zerilli equation (5.2) with a well-behaved source term, i.e.,

$$\begin{aligned} \left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_Z(r)\right]\chi_{4\pm 4}^{(2)reg}(t, r) &= S_{4\pm 4}(t, r) - \left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_*^2} - V_Z(r)\right]\zeta_{4\pm 4}^{(2)}(t, r) \\ &= S_{4\pm 4}^{reg}(t, r), \end{aligned} \quad (6.4)$$

where  $S_{4\pm 4}^{reg} \sim O(r^{-2})$  at infinity and  $S_{4\pm 4}^{reg} \sim O(r - 2M)$  at the horizon. Thus we can remove an unphysical gauge-dependent divergence.

It should be noted that such a regularization is not unique, and for example, we can replace  $\partial/\partial r$  with  $-\partial/\partial t$  in Eq. (6.3). However the observed quantities do not depend on the choice of the regularization method. This is because the regularization is equivalent to adding quadratic terms in the first order gauge invariant function  $\psi_{2\pm 2}^{(1)}$  to the second order gauge invariant function  $\zeta_{4\pm 4}^{(2)}$ , so that it preserves the gauge invariance [40].

The explicit expression for the regularized second order source term is given by

$$\begin{aligned} S_{4\pm 4}^{reg}(t, r) &= \frac{r - 2M}{42}\frac{\sqrt{70}}{\sqrt{\pi}}\left\{-\frac{1}{r^6(3r + M)^2(2r + 3M)^3}(72r^8 + 3936r^7M + 2316r^6M^2 - 2030r^5M^3 \right. \\ &\quad \left. - 7744r^4M^4 - 9512r^3M^5 - 3540r^2M^6 - 1119rM^7 - 144M^8)\left(\frac{\partial}{\partial r}\psi_{2\pm 2}^{(1)}(t, r)\right)\frac{\partial}{\partial t}\psi_{2\pm 2}^{(1)}(t, r) \right. \\ &\quad \left. + \frac{1}{r^3(3r + M)^2(2r + 3M)^2(r - 2M)^2}(24r^7 + 344r^6M - 872r^5M^2 - 771r^4M^3 \right. \\ &\quad \left. + 120r^3M^4 + 77r^2M^5 - 237rM^6 - 48M^7)\left(\frac{\partial^2}{\partial t^2}\psi_{2\pm 2}^{(1)}(t, r)\right)\frac{\partial}{\partial t}\psi_{2\pm 2}^{(1)}(t, r) \right. \\ &\quad \left. - \frac{66r^4 - 106r^3M - 220r^2M^2 - 156rM^3 - 45M^4}{r(M + 3r)^2(2r + 3M)(r - 2M)}\left(\frac{\partial^3}{\partial r\partial t^2}\psi_{2\pm 2}^{(1)}(t, r)\right)\frac{\partial}{\partial t}\psi_{2\pm 2}^{(1)}(t, r) \right. \\ &\quad \left. - \frac{3}{r^7(2r + 3M)^4(3r + M)^2}(2160r^9 + 11760r^8M + 30560r^7M^2 + 41124r^6M^3 + 31596r^5M^4 \right. \\ &\quad \left. + 11630r^4M^5 - 1296r^3M^6 - 4182r^2M^7 - 1341rM^8 - 144M^9)\psi_{2\pm 2}^{(1)}(t, r)\frac{\partial}{\partial t}\psi_{2\pm 2}^{(1)}(t, r) \right. \\ &\quad \left. + \frac{1}{r^5(3r + M)^2(2r + 3M)^2}(228r^7 + 8r^6M - 370r^5M^2 + 142r^4M^3 - 384r^3M^4 \right. \end{aligned}$$

$$\begin{aligned}
& -514r^2M^5 - 273rM^6 - 48M^7 \left( \frac{\partial^2}{\partial r \partial t} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial}{\partial r} \psi_{2\pm 2}^{(1)}(t, r) \\
& + \frac{(198r^5 - 318r^4M - 664r^3M^2 - 458r^2M^3 - 127rM^4 - 24M^5)}{3r^2(3r+M)^2(2r+3M)(r-2M)} \left( \frac{\partial^3}{\partial t^3} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial}{\partial r} \psi_{2\pm 2}^{(1)}(t, r) \\
& - \frac{2(r-2M)M^2}{3(3r+M)^2r^2} \left( \frac{\partial^2}{\partial t^2} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial^2}{\partial r \partial t} \psi_{2\pm 2}^{(1)}(t, r) - \frac{7r+4M}{3r} \left( \frac{\partial^3}{\partial r \partial t^2} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial^2}{\partial r \partial t} \psi_{2\pm 2}^{(1)}(t, r) \\
& - \frac{1}{r^6(3r+M)^2(2r+3M)^3} (216r^8 + 4296r^7M + 1992r^6M^2 - 3488r^5M^3 - 8716r^4M^4 \\
& - 9512r^3M^5 - 3540r^2M^6 - 1119rM^7 - 144M^8) \psi_{2\pm 2}^{(1)}(t, r) \frac{\partial^2}{\partial r \partial t} \psi_{2\pm 2}^{(1)}(t, r) \\
& + \frac{1}{r^3(3r+M)^2(2r+3M)^2(r-2M)} (252r^6 + 636r^5M + 674r^4M^2 + 730r^3M^3 + 524r^2M^4 \\
& + 171rM^5 + 24M^6) \psi_{2\pm 2}^{(1)}(t, r) \frac{\partial^3}{\partial t^3} \psi_{2\pm 2}^{(1)}(t, r) + \frac{(7r+4M)r}{3(r-2M)^2} \left( \frac{\partial^3}{\partial t^3} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial^2}{\partial t^2} \psi_{2\pm 2}^{(1)}(t, r) \Big\}. \quad (6.5)
\end{aligned}$$

In the above equation, we note that the second order source term is quadratic in the first order wave-function  $\psi_{2\pm 2}^{(1)}$  and hence the second order QNMs have a frequency at  $\omega_{44}^{(2)} = 2\omega_{22}^{(1)}$ . Since the second order frequencies are different from the first order ones, we can in principle identify gravitational waves from the second order QNMs.

## VII. GAUGE TRANSFORMATION TO ASYMPTOTIC FLAT GAUGE

In this section, in order to obtain physical information from the wave-functions,  $\psi_{2\pm 2}^{(1)}$  and  $\chi_{4\pm 4}^{(2)reg}$ , we treat a gauge transformation in the first and second order calculation. To obtain the gravitational waveform, it is necessary to go to an asymptotic flat (AF) gauge from the RW gauge. This is because the Regge-Wheeler-Zerilli formalism that we have employed is under the RW gauge and this gauge is not asymptotically flat.

Here, we consider the following gauge transformation [35, 41],

$$x_{RW}^\mu \rightarrow x_{AF}^\mu = x_{RW}^\mu + \xi^{(1)\mu}(x^\alpha) + \frac{1}{2} \left[ \xi^{(2)\mu}(x^\alpha) + \xi^{(1)\nu} \xi^{(1)\mu}{}_{,\nu}(x^\alpha) \right], \quad (7.1)$$

where comma “,” in the index indicates the partial derivative with respect to the background coordinates, and  $\xi^{(1)\mu}$  and  $\xi^{(2)\mu}$  are generators of the first and second order gauge transformation, respectively. By using this gauge transformation, the metric perturbation changes as

$$h_{RW\mu\nu}^{(1)} \rightarrow h_{AF\mu\nu}^{(1)} = h_{RW\mu\nu}^{(1)} - \mathcal{L}_{\xi^{(1)}} g_{\mu\nu}, \quad (7.2)$$

$$h_{RW\mu\nu}^{(2)} \rightarrow h_{AF\mu\nu}^{(2)} = h_{RW\mu\nu}^{(2)} - \frac{1}{2} \mathcal{L}_{\xi^{(2)}} g_{\mu\nu} + \frac{1}{2} \mathcal{L}_{\xi^{(1)}}^2 g_{\mu\nu} - \mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu}^{(1)}, \quad (7.3)$$

where  $\mathcal{L}_{\xi^{(i)}}$  means the Lie derivative in the  $\xi^{(i)}$  direction. We use the following form of a generator for the gauge transformation,

$$\xi^{(i)\mu} = \{V_0^{(i)}(t, r)Y_{\ell m}(\theta, \phi), V_1^{(i)}(t, r)Y_{\ell m}(\theta, \phi), V_2^{(i)}(t, r)\frac{\partial}{\partial \theta}Y_{\ell m}(\theta, \phi), V_2^{(i)}(t, r)\frac{1}{\sin^2 \theta}\frac{\partial}{\partial \phi}Y_{\ell m}(\theta, \phi)\}, \quad (7.4)$$

where we have considered only the even parity mode, and hence we have three degrees of gauge freedom for each order. (Note that the generator for the odd parity part is  $\xi^{(i)\mu} = \{0, 0, -V_3^{(i)}\partial_\phi Y_{\ell m}/\sin \theta, V_3^{(i)}\sin \theta \partial_\theta Y_{\ell m}\}$ .) In this paper, we consider only the  $\ell = 2, m = \pm 2$  modes for the first order gauge transformation, and the  $\ell = 4, m = \pm 4$  modes for the second order one.

The gauge transformation of the metric perturbation is explicitly given as follows. For the first order metric perturbation, we derive

$$\begin{aligned}
H_{02\pm 2}^{(1)AF}(t, r) &= H_{02\pm 2}^{(1)RW}(t, r) + 2\frac{\partial}{\partial t}V_0^{(1)}(t, r) + 2\frac{M}{r(r-2M)}V_1^{(1)}(t, r), \\
H_{12\pm 2}^{(1)AF}(t, r) &= H_{12\pm 2}^{(1)RW}(t, r) + \frac{(r-2M)}{r}\frac{\partial}{\partial r}V_0^{(1)}(t, r) - \frac{r}{r-2M}\frac{\partial}{\partial t}V_1^{(1)}(t, r),
\end{aligned}$$

$$\begin{aligned}
H_{2\pm 2}^{(1)AF}(t, r) &= H_{2\pm 2}^{(1)RW}(t, r) - 2 \frac{\partial}{\partial r} V_1^{(1)}(t, r) + 2 \frac{M}{r(r-2M)} V_1^{(1)}(t, r), \\
K_{2\pm 2}^{(1)AF}(t, r) &= K_{0\pm 2\pm 2}^{(1)RW}(t, r) - 2 \frac{1}{r} V_1^{(1)}(t, r), \\
h_{0\pm 2\pm 2}^{(1)(e)AF}(t, r) &= \frac{(r-2M)}{r} V_0^{(1)}(t, r) - r^2 \frac{\partial}{\partial t} V_2^{(1)}(t, r), \\
h_{1\pm 2\pm 2}^{(1)(e)AF}(t, r) &= -\frac{r}{r-2M} V_1^{(1)}(t, r) - r^2 \frac{\partial}{\partial r} V_2^{(1)}(t, r), \\
G_{2\pm 2}^{(1)AF}(t, r) &= -2 V_2^{(1)}(t, r).
\end{aligned} \tag{7.5}$$

For the second order metric perturbation, we can calculate the gauge transformation straightforward, but obtain very long expressions. For example, they are written formally as

$$H_{0\pm 4\pm 4}^{(2)AF}(t, r) = H_{0\pm 4\pm 4}^{(2)RW}(t, r) + \frac{\partial}{\partial t} V_0^{(2)}(t, r) + \frac{M}{r(r-2M)} V_1^{(2)}(t, r) + \delta H_{0\pm 4\pm 4}^{(2)}(t, r), \tag{7.6}$$

$$K_{4\pm 4}^{(2)AF}(t, r) = K_{4\pm 4}^{(2)RW}(t, r) - \frac{1}{r} V_1^{(2)}(t, r) + \delta K_{4\pm 4}^{(2)}(t, r), \tag{7.7}$$

$$h_{1\pm 4\pm 4}^{(2)(e)AF}(t, r) = -\frac{r}{2(r-2M)} V_1^{(2)}(t, r) - \frac{r^2}{2} \frac{\partial}{\partial r} V_2^{(2)}(t, r) + \delta h_{1\pm 4\pm 4}^{(2)(e)}(t, r). \tag{7.8}$$

$$G_{4\pm 4}^{(2)AF}(t, r) = -V_2^{(2)}(t, r) + \delta G_{4\pm 4}^{(2)}(t, r), \tag{7.9}$$

where  $\delta H_{0\pm 4\pm 4}^{(2)}$ ,  $\delta K_{4\pm 4}^{(2)}$ ,  $\delta h_{1\pm 4\pm 4}^{(2)(e)}$  and  $\delta G_{4\pm 4}^{(2)}$  are defined by the tensor harmonics expansion of the last two terms in the right hand side of Eq. (7.3), i.e.,  $(1/2)\mathcal{L}_{\xi^{(1)}}^2 g_{\mu\nu}^{(0)} - \mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu}^{(1)}$ . This includes only quadratic terms of the first order wave-function because  $h_{RW\mu\nu}^{(1)}$  and  $\xi^{(1)\mu}$  are the first order quantities written by the first order wave-function  $\psi_{2\pm 2}^{(1)}$  after we solve the gauge equations.

### A. About first order

First, we consider the asymptotic behavior of the metric perturbation at large  $r$  in the RW gauge. Using Eq. (4.8) and the asymptotic expansion in Eq. (6.1), the metric perturbation is given as follows,

$$\begin{aligned}
H_{0\pm 2\pm 2}^{(1)RW}(t, r) &= H_{2\pm 2\pm 2}^{(1)RW}(t, r) \\
&= \frac{1}{3} \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) r + \frac{2}{3} \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) M + \frac{2}{3} \frac{d^3}{dT_r^3} F_I(T_r) \\
&\quad + \left( \frac{d^2}{dT_r^2} F_I(T_r) + \frac{11}{6} \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) M + \frac{4}{3} \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) M^2 \right) \frac{1}{r} + O(r^{-2}), \\
H_{1\pm 2\pm 2}^{(1)RW}(t, r) &= -\frac{1}{3} \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) r - \frac{2}{3} \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) M - \frac{2}{3} \frac{d^3}{dT_r^3} F_I(T_r) \\
&\quad + \left( -\frac{4}{3} \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) M^2 - \frac{d^2}{dT_r^2} F_I(T_r) - \frac{11}{6} \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) M \right) \frac{1}{r} + O(r^{-2}), \\
K_{2\pm 2}^{(1)RW}(t, r) &= -\frac{1}{3} \frac{d^3}{dT_r^3} F_I(T_r) + \frac{1}{r^2} \left( \frac{1}{2} \left( \frac{d^2}{dT_r^2} F_I(T_r) \right) M + \frac{d}{dT_r} F_I(T_r) \right) + O(r^{-3}),
\end{aligned} \tag{7.10}$$

where we introduce  $T_r = t - r_*(r)$  for simplicity. Since we are interested in the asymptotic behavior now and the  $m = 2$  and  $m = -2$  modes have the same asymptotic behavior, we have used the same notation for the  $m = 2$  and  $m = -2$  modes.

On the other hand, the metric perturbation in an AF gauge should behave as

$$\begin{aligned}
H_{0\pm 2\pm 2}^{(1)AF}(t, r) &= H_{1\pm 2\pm 2}^{(1)AF}(t, r) = h_{0\pm 2\pm 2}^{(1)(e)AF}(t, r) = 0, \\
H_{2\pm 2\pm 2}^{(1)AF}(t, r) &= O(r^{-3}), \\
h_{1\pm 2\pm 2}^{(1)(e)AF}(t, r) &= O(r^{-1}),
\end{aligned}$$

$$\begin{aligned} K_{2\pm 2}^{(1)AF}(t, r) &= O(r^{-1}), \\ G_{2\pm 2}^{(1)AF}(t, r) &= O(r^{-1}). \end{aligned} \quad (7.11)$$

This asymptotic behavior will be also used for the second order calculation. Then, we can find that the gauge transformation has the following form.

$$\begin{aligned} V_0^{(1)}(t, r) &= -\frac{1}{6} \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) r - \frac{1}{3} \frac{d^2}{dT_r^2} F_I(T_r) - \frac{1}{3} \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) M \\ &\quad + \frac{1}{r} \left( -\frac{3}{4} \left( \frac{d^2}{dT_r^2} F_I(T_r) \right) M - \frac{2}{3} \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) M^2 - \frac{1}{2} \frac{d}{dT_r} F_I(T_r) \right) + O(r^{-2}), \\ V_1^{(1)}(t, r) &= -\frac{1}{6} \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) r - \frac{1}{2} \frac{d^2}{dT_r^2} F_I(T_r) + \frac{1}{r} \left( -\frac{1}{2} \frac{d}{dT_r} F_I(T_r) + \frac{1}{4} \left( \frac{d^2}{dT_r^2} F_I(T_r) \right) M \right) + O(r^{-2}), \\ V_2^{(1)}(t, r) &= -\frac{1}{6} \frac{1}{r} \frac{d^2}{dT_r^2} F_I(T_r) - \frac{1}{3} \frac{1}{r^2} \frac{d}{dT_r} F_I(T_r) + \frac{1}{r^3} \left( -\frac{1}{12} M \frac{d}{dT_r} F_I(T_r) - \frac{1}{2} F_I(T_r) \right) + O(r^{-4}). \end{aligned} \quad (7.12)$$

The above results are obtained iteratively for the large  $r$  expansion. Here, we note that the transverse-traceless tensor harmonics for the even parity part is  $\mathbf{f}_{\ell m}$  in Eqs. (3.1) and (3.11). Therefore the coefficient of the metric perturbation that is related to the gravitational wave is  $G_{\ell m}^{(1)AF}$ . With Eqs. (7.9) and (7.12), we obtain

$$G_{2\pm 2}^{(1)AF}(t, r) = \frac{1}{3} \frac{1}{r} \frac{d^2}{dT_r^2} F_I(T_r) + O(r^{-2}). \quad (7.13)$$

This can be shown in terms of  $\psi_{2\pm 2}^{(1)}$  with Eq. (6.1) as

$$G_{2\pm 2}^{(1)AF}(t, r) = \frac{1}{r} \psi_{2\pm 2}^{(1)}(t, r) + O(r^{-2}). \quad (7.14)$$

## B. About second order

In order to derive the gravitational wave amplitude for the second perturbative order, we also need to obtain the coefficient  $G_{4\pm 4}^{(2)AF}$  under the AF gauge as in the first order case. From Eq. (7.9), it is found that we must derive  $\delta G_{4\pm 4}^{(2)}$  and  $V_2^{(2)}$  which is calculated by using the solution of  $V_1^{(2)}$ . In the following, we first obtain  $V_1^{(2)}$  as Eq. (7.20) by solving the gauge equation in Eq. (7.7). Then,  $V_2^{(2)}$  is derived from Eq. (7.8) as in Eq. (7.23) and  $\delta G_{4\pm 4}^{(2)}$  is calculated as in Eq. (7.25).

First we consider to obtain  $V_1^{(2)}$  in terms of the asymptotic wave-functions. The equation

$$K_{4\pm 4}^{(2)AF}(t, r) = K_{4\pm 4}^{(2)RW}(t, r) - \frac{1}{r} V_1^{(2)}(t, r) + \delta K_{4\pm 4}^{(2)}(t, r), \quad (7.15)$$

is used as shown in Eq. (7.7). Here  $K_{4\pm 4}^{(2)RW}$  is derived from  $\chi_{4\pm 4}^{(2)}$ ,  $\mathcal{A}_{14\pm 4}$  and  $\mathcal{B}_{04\pm 4}$  in Eqs. (5.5). Since the wave-function  $\chi_{4\pm 4}^{(2)}$  is given by  $\chi_{4\pm 4}^{(2)reg}$  and  $\psi_{2\pm 2}^{(1)}$  in Eq. (6.3) and the coefficients  $\mathcal{A}_{14\pm 4}$  and  $\mathcal{B}_{04\pm 4}$  are also given by  $\psi_{2\pm 2}^{(1)}$  with Eqs. (3.14), (3.15) and (4.8), we may obtain  $K_{4\pm 4}^{(2)RW}$  in terms of wave-functions  $\chi_{4\pm 4}^{(2)reg}$  and  $\psi_{2\pm 2}^{(1)}$ . The asymptotic expansion of  $\psi_{2\pm 2}^{(1)}$  is given by Eq. (6.1) while the asymptotic expansion of  $\chi_{4\pm 4}^{(2)reg}$  is obtained from the Zerilli equation (6.4) as

$$\chi_{4\pm 4}^{(2)reg}(t, r) = F_I^{(2)}(T_r) + O(r^{-1}), \quad (7.16)$$

where  $F_I^{(2)}$  is some second order function. Here we consider only the leading behavior for large  $r$ , because this is sufficient in order to obtain the final gravitational waveform. If we consider the higher order calculation, it should note that there is a contribution from the second order source term of Eq. (6.5) in the above equation. Then, we can calculate the asymptotic expansion of  $\partial K_{4\pm 4}^{(2)RW}/\partial t$  in Eqs. (5.5) as

$$\frac{\partial}{\partial t} K_{4\pm 4}^{(2)RW}(t, r) = -\frac{d}{dT_r} F_I^{(2)}(T_r)$$

$$\begin{aligned}
& -\frac{\sqrt{70}}{3402\sqrt{\pi}} \left( \left( \left( \frac{d^5}{dT_r^5} F_I(T_r) \right) \frac{d^3}{dT_r^3} F_I(T_r) + \left( \frac{d^4}{dT_r^4} F_I(T_r) \right)^2 \right) M \right. \\
& \quad \left. + 9 \left( \frac{d^5}{dT_r^5} F_I(T_r) \right) \frac{d^2}{dT_r^2} F_I(T_r) - 15 \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) \frac{d^3}{dT_r^3} F_I(T_r) \right) + O(r^{-1}). \quad (7.17)
\end{aligned}$$

Integrating the above equation with respect to  $t$ , the second order metric component  $K_{4\pm 4}^{(2)RW}$  becomes

$$\begin{aligned}
K_{4\pm 4}^{(2)RW}(t, r) &= -F_I^{(2)}(T_r) \\
&\quad -\frac{\sqrt{70}}{3402\sqrt{\pi}} \left( 9 \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) \frac{d^2}{dT_r^2} F_I(T_r) - 12 \left( \frac{d^3}{dT_r^3} F_I(T_r) \right)^2 \right. \\
&\quad \left. + \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) M \right) + O(r^{-1}). \quad (7.18)
\end{aligned}$$

In Eq. (7.15), we also need the asymptotic expansion of  $\delta K_{4\pm 4}^{(2)}$ , which is defined by the tensor harmonics expansion of  $(1/2)\mathcal{L}_{\xi^{(1)}}^2 g_{\mu\nu}^{(0)} - \mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu}^{(1)}$  in Eq. (7.3). Using the first order metric perturbation under the RW gauge in Eq. (7.10) and the generator of the first order gauge transformation in Eq. (7.4) with Eq. (7.12),  $\delta K_{4\pm 4}^{(2)}$  is derived as

$$\delta K_{4\pm 4}^{(2)} = \frac{\sqrt{70}}{504\sqrt{\pi}} \left( \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) \frac{d^2}{dT_r^2} F_I(T_r) - 2 \left( \frac{d^3}{dT_r^3} F_I(T_r) \right)^2 \right) + O(r^{-1}), \quad (7.19)$$

From the above results and the AF gauge condition of  $K_{4\pm 4}^{(2)}$  in Eq. (7.11), in order to remove the  $O(r^0)$  terms from  $K_{4\pm 4}^{(2)RW}$  and  $\delta K_{4\pm 4}^{(2)}$  in Eq. (7.15), we obtain the pure second order gauge transformation  $V_1^{(2)}$  as

$$\begin{aligned}
V_1^{(2)}(t, r) &= -r F_I^{(2)}(T_r) \\
&\quad -\frac{\sqrt{70}}{13608\sqrt{\pi}} r \left( 9 \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) \frac{d^2}{dT_r^2} F_I(T_r) + 6 \left( \frac{d^3}{dT_r^3} F_I(T_r) \right)^2 \right. \\
&\quad \left. + 4 \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) M \right) + O(r^0). \quad (7.20)
\end{aligned}$$

Next, we derive  $V_2^{(2)}$  from the AF gauge condition of  $h_{14\pm 4}^{(2)(e)}$  in Eq. (7.11). The second order gauge transformation of this component is given by

$$h_{14\pm 4}^{(2)(e)AF}(t, r) = -\frac{r}{2(r-2M)} V_1^{(2)}(t, r) - \frac{r^2}{2} \frac{\partial}{\partial r} V_2^{(2)}(t, r) + \delta h_{14\pm 4}^{(2)(e)}(t, r), \quad (7.21)$$

as shown in Eq. (7.8). Here we have used the fact that  $h_{14\pm 4}^{(2)(e)RW} = 0$  under the RW gauge. We may also derive  $\delta h_{14\pm 4}^{(2)(e)}$ , which is defined by the tensor harmonics expansion of  $(1/2)\mathcal{L}_{\xi^{(1)}}^2 g_{\mu\nu}^{(0)} - \mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu}^{(1)}$  in Eq. (7.3), by Eqs. (7.10) and (7.4) with Eq. (7.12) as

$$\delta h_{14\pm 4}^{(2)(e)} = \frac{\sqrt{70}}{1008\sqrt{\pi}} r \left( \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) \frac{d^2}{dT_r^2} F_I(T_r) + \left( \frac{d^3}{dT_r^3} F_I(T_r) \right)^2 \right) + O(r^0). \quad (7.22)$$

Then,  $V_2^{(2)}$  is calculated from the above value and the result of  $V_1^{(2)}$  in Eq. (7.20) with the AF gauge condition in Eq. (7.11) as

$$\begin{aligned}
\frac{\partial}{\partial r} V_2^{(2)}(t, r) &= \frac{1}{r} F_I^{(2)}(T_r) \\
&\quad + \frac{\sqrt{70}}{13608\sqrt{\pi}} \frac{1}{r} \left( 36 \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) \frac{d^2}{dT_r^2} F_I(T_r) + 33 \left( \frac{d^3}{dT_r^3} F_I(T_r) \right)^2 \right. \\
&\quad \left. + 4 \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) M \right) + O(r^{-2}) \\
&= -\frac{\partial}{\partial t} V_2^{(2)}(t, r) + O(r^{-2}). \quad (7.23)
\end{aligned}$$

In the last line of the above equation, we have used the definition of  $T_r = t - r_*(r)$ .

At this stage, we can consider the metric perturbation related to the gravitational wave amplitude, i.e.,  $G_{4\pm 4}^{(2)AF}$ . The gauge transformation of the components  $G_{4\pm 4}^{(2)}$  is given by

$$G_{4\pm 4}^{(2)AF}(t, r) = -V_2^{(2)}(t, r) + \delta G_{4\pm 4}^{(2)}, \quad (7.24)$$

as shown in Eq. (7.9). Here  $\delta G_{4\pm 4}^{(2)}$ , which is defined by the tensor harmonics expansion of  $(1/2)\mathcal{L}_{\xi^{(1)}}^2 g_{\mu\nu}^{(0)} - \mathcal{L}_{\xi^{(1)}} h_{RW\mu\nu}^{(1)}$  in Eq. (7.3), is also obtained from Eqs. (7.10) and (7.4) with Eq. (7.12) as

$$\delta G_{4\pm 4}^{(2)} = -\frac{\sqrt{70}}{1512\sqrt{\pi}} \frac{1}{r} \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) \left( \frac{d^2}{dT_r^2} F_I(T_r) \right) + O(r^{-2}). \quad (7.25)$$

Finally, using Eq. (7.23) for  $V_2^{(2)}$  in Eq. (7.24), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} G_{4\pm 4}^{(2)AF}(t, r) &= \frac{1}{r} F_I^{(2)}(T_r) \\ &+ \frac{\sqrt{70}}{13608\sqrt{\pi}} \frac{1}{r} \left( 27 \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) \frac{d^2}{dT_r^2} F_I(T_r) + 24 \left( \frac{d^3}{dT_r^3} F_I(T_r) \right)^2 \right. \\ &\quad \left. + 4 \left( \frac{d^3}{dT_r^3} F_I(T_r) \right) \left( \frac{d^4}{dT_r^4} F_I(T_r) \right) M \right) + O(r^{-2}). \end{aligned} \quad (7.26)$$

By using  $\psi_{2\pm 2}^{(1)}$  and  $\chi_{4\pm 4}^{(2)reg}$ , the gravitational waveform is obtained as

$$\begin{aligned} \frac{\partial}{\partial t} G_{4\pm 4}^{(2)AF}(t, r) &= \frac{1}{r} \chi_{4\pm 4}^{(2)reg}(t, r) \\ &+ \frac{\sqrt{70}}{1512\sqrt{\pi}} \frac{1}{r} \left( 27 \psi_{2\pm 2}^{(1)}(t, r) \frac{\partial^2}{\partial t^2} \psi_{2\pm 2}^{(1)}(t, r) + 24 \left( \frac{\partial}{\partial t} \psi_{2\pm 2}^{(1)}(t, r) \right)^2 \right. \\ &\quad \left. + 4 M \left( \frac{\partial}{\partial t} \psi_{2\pm 2}^{(1)}(t, r) \right) \frac{\partial^2}{\partial t^2} \psi_{2\pm 2}^{(1)}(t, r) \right) + O(r^{-2}). \end{aligned} \quad (7.27)$$

It should be noted that we can show that all metric components satisfy the asymptotic flat gauge condition.

### C. Power of Gravitational waves

In this subsection, we summarize the power of gravitational waves which is derived from the gravitational waveform. The power  $P$  per solid angle is given by [42] as

$$\frac{dP}{d\Omega} = \frac{1}{16\pi r^2} \left\langle \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial t} h_{\theta\phi} \right)^2 + \frac{1}{4} \left( \frac{\partial}{\partial t} h_{\theta\theta} - \frac{1}{\sin^2 \theta} \frac{\partial}{\partial t} h_{\phi\phi} \right)^2 \right\rangle, \quad (7.28)$$

where  $\langle \dots \rangle$  means an averaging over a spacetime region which is sufficiently larger than the characteristic wavelength of gravitational waves. Therefore, for the  $\ell = 2$ ,  $m = \pm 2$  modes of the first order metric perturbation, we have the following equation.

$$\begin{aligned} \frac{dP^{(1)}}{d\Omega} &= \frac{r^2}{64\pi} \left\langle \frac{1}{\sin^2 \theta} \left( \frac{\partial}{\partial t} G_{22}^{(1)AF}(t, r) X_{22}(\theta, \phi) + \frac{\partial}{\partial t} G_{2-2}^{(1)AF}(t, r) X_{2-2}(\theta, \phi) \right)^2 \right. \\ &\quad \left. + \left( \frac{\partial}{\partial t} G_{22}^{(1)AF}(t, r) W_{22}(\theta, \phi) + \frac{\partial}{\partial t} G_{2-2}^{(1)AF}(t, r) W_{2-2}(\theta, \phi) \right)^2 \right\rangle \\ &= \frac{r^2}{64\pi} \left[ 2 \left\langle \left( \frac{\partial}{\partial t} G_{22}^{(1)AF}(t, r) \right) \left( \frac{\partial}{\partial t} G_{2-2}^{(1)AF}(t, r) \right) \right\rangle \left( \frac{1}{\sin^2 \theta} X_{22}(\theta, \phi) X_{2-2}(\theta, \phi) + W_{22}(\theta, \phi) W_{2-2}(\theta, \phi) \right) \right] \end{aligned}$$



$$\begin{aligned}
& + \left\langle \left( \frac{\partial}{\partial t} G_{22}^{(1)AF}(t, r) \right)^2 \right\rangle \left( \frac{1}{\sin^2 \theta} (X_{22}(\theta, \phi))^2 + (W_{22}(\theta, \phi))^2 \right) \\
& + \left\langle \left( \frac{\partial}{\partial t} G_{2-2}^{(1)AF}(t, r) \right)^2 \right\rangle \left( \frac{1}{\sin^2 \theta} (X_{2-2}(\theta, \phi))^2 + (W_{2-2}(\theta, \phi))^2 \right) \Big] \\
& = \frac{r^2}{64\pi} \sum_{m=\pm 2} \left[ \left| \frac{\partial}{\partial t} G_{2m}^{(1)AF}(t, r) \right|^2 \left( \frac{1}{\sin^2 \theta} |X_{2m}(\theta, \phi)|^2 + |W_{2m}(\theta, \phi)|^2 \right) \right. \\
& \quad \left. + \left\langle \left( \frac{\partial}{\partial t} G_{2m}^{(1)AF}(t, r) \right)^2 \right\rangle \left( \frac{1}{\sin^2 \theta} (X_{2m}(\theta, \phi))^2 + (W_{2m}(\theta, \phi))^2 \right) \right], \tag{7.29}
\end{aligned}$$

in terms of the coefficient of the tensor harmonics in Eq. (3.1). In the above equation, we use  $G_{22}^{(1)AF} = G_{2-2}^{(1)AF*}$ , and the angular functions  $X_{\ell m}$  and  $W_{\ell m}$  are given by Eqs. (3.12), (3.13) and (4.6). We note that it is meaningless to distinguish the power of  $m = \pm 2$  modes since they appear as the cross term in the second equality of the above equation. It is also noted that the averaging  $\langle (\partial G_{2m}^{(1)AF} / \partial t)^2 \rangle$  does not vanish in this calculation. Integrating the above equation with respect to the angular directions, we obtain

$$\begin{aligned}
P^{(1)} &= \frac{3}{8\pi} r^2 \sum_{m=\pm 2} \left| \frac{\partial}{\partial t} G_{2m}^{(1)AF}(t, r) \right|^2 \\
&= \frac{3}{8\pi} \sum_{m=\pm 2} \left| \frac{\partial}{\partial t} \psi_{2m}^{(1)}(t, r) \right|^2 \\
&= \frac{3}{4\pi} \left| \frac{\partial}{\partial t} \psi_{22}^{(1)}(t, r) \right|^2, \tag{7.30}
\end{aligned}$$

where we have used the result in Eq. (7.14) in the second line and the fact that  $\psi_{2-2}^{(1)} = \psi_{22}^{(1)*}$  in Eq. (4.7) in the last line. The relation between  $m = 2$  and  $m = -2$  modes are summarized in the appendix.

When we set  $\psi_{22}^{(1)} = A \exp(-i\omega_{\ell=2, m=2}^{(1)}(t - r_*))$ , where we choose the amplitude  $A$  of  $\psi_{22}^{(1)}$  at the origin of  $t - r_*$ , the total radiated energy is

$$\begin{aligned}
E^{(1)} &= \int dt P^{(1)} \\
&= \frac{3}{4\pi} \int dt \left| \frac{\partial}{\partial t} \psi_{22}^{(1)}(t, r) \right|^2 \\
&= \frac{3}{4\pi} |A|^2 |\omega_{22}^{(1)}|^2 \int dt e^{2\Im(\omega_{22}^{(1)})(t - r_*)} \\
&= \frac{3}{8\pi} |A|^2 |\omega_{22}^{(1)}|^2 \frac{1}{|\Im(\omega_{22}^{(1)})|}. \tag{7.31}
\end{aligned}$$

For the second perturbative order, we also have the following formula for the power of gravitational waves from the result of the second order gravitational waveform in Eq. (7.27),

$$\begin{aligned}
P^{(2)} &= \frac{45}{8\pi} r^2 \sum_{m=\pm 4} \left| \frac{\partial}{\partial t} G_{4m}^{(2)AF}(t, r) \right|^2 \\
&= \frac{45}{4\pi} \left| \chi_{44}^{(2)reg}(t, r) + \frac{\sqrt{70}}{1512\sqrt{\pi}} \left( 27 \psi_{22}^{(1)}(t, r) \frac{\partial^2}{\partial t^2} \psi_{22}^{(1)}(t, r) + 24 \left( \frac{\partial}{\partial t} \psi_{22}^{(1)}(t, r) \right)^2 \right. \right. \\
& \quad \left. \left. + 4 M \left( \frac{\partial}{\partial t} \psi_{22}^{(1)}(t, r) \right) \frac{\partial^2}{\partial t^2} \psi_{22}^{(1)}(t, r) \right) \right|^2. \tag{7.32}
\end{aligned}$$

Here, we have calculated the angular integration to derive the first line, and used the relation that the second order metric components of the  $m = 4$  and  $-4$  modes are complex conjugate, i.e.,  $\chi_{44}^{(2)reg} = \chi_{4-4}^{(2)reg*}$ . This complex conjugate relation can be shown with Eqs. (4.7) and (6.4). We should note that when the first and second order perturbations

TABLE I: The first order QNM frequencies for the  $\ell = 2$  mode. We have calculated them with higher-precision because it is necessary to calculate the second perturbative order. The QNM frequencies do not depend on  $m$  for a Schwarzschild black hole.

$n$	$\Re\omega_{2m}^{(1)}$	$\Im\omega_{2m}^{(1)}$
0	0.37367168441804183579349200298	-0.08896231568893569828046092718
1	0.34671099687916343971767535973	-0.27391487529123481734956022214
2	0.30105345461236639380200360888	-0.47827698322307180998418283072

belong to the same harmonics  $(\ell, m)$  mode, there are the cross-term contributions even after the angular integration, such as  $G_{\ell m}^{(1)AF} \times G_{\ell m}^{(2)AF*}$  and  $G_{\ell m}^{(1)AF*} \times G_{\ell m}^{(2)AF}$  from Eq. (7.28). However, we do not have these terms in our case that  $G_{\ell m}^{(1)AF}$  and  $G_{\ell m}^{(2)AF}$  have different harmonics  $(\ell, m)$ .

### VIII. NUMERICAL CALCULATION: LEAVER'S METHOD FOR SECOND-ORDER QNMS

In this section, we consider a numerical method to obtain the particular solution  $\chi_{44}^{(2)reg}(t, r)$  for the regularized second order Zerilli equation in Eq. (6.4). Here we only consider the  $\ell = 4, m = 4$  mode for the second order calculation because  $\chi_{4-4}^{(2)reg}$  may be obtained by  $\chi_{4-4}^{(2)reg} = \chi_{44}^{(2)reg*}$ . In order to solve the second-order Zerilli equation with several digits, we need to know many digits of the first order QNM frequencies. These are given in Table I.

The following method is basically a modified version of the Leaver's continued fraction method [22]. We will finally calculate the second order amplitude at infinity and the horizon as the following forms,

$$\chi_{44}^{(2)reg}(t, r = \infty) = C_I \left[ \omega \psi_{22}^{(1)}(t, r = \infty) \right]^2, \quad (8.1)$$

$$\chi_{44}^{(2)reg}(t, r = 2M) = C_H \left[ \omega \psi_{22}^{(1)}(t, r = 2M) \right]^2, \quad (8.2)$$

where  $\omega$  denotes the first order QNM frequency  $\omega_{22}^{(1)}$  and we consider not only the fundamental  $n = 0$  mode, but also the overtone  $n = 1$  or 2. And then, we may calculate the energy of the second order QNM as a function of the first order one,

$$\frac{E^{(2)}}{M} = C_E M |\Im\omega| \left( \frac{E^{(1)}}{M} \right)^2, \quad (8.3)$$

where Eq. (7.32) gives

$$C_E = 20\pi \left| C_I - \frac{\sqrt{70}}{1512\sqrt{\pi}} (51 - 4iM\omega) \right|^2. \quad (8.4)$$

The coefficients,  $C_I$ ,  $C_H$  and  $C_E$ , are summarized in Table II.

By Fourier transforming,

$$\begin{aligned} \psi_{22}^{(1)}(t, r) &= \int d\omega \psi_{22\omega}^{(1)}(r) e^{-i\omega t}, \\ \chi_{44}^{(2)reg}(t, r) &= \int d\omega \chi_{44\omega}^{(2)reg}(r) e^{-i\omega t}, \end{aligned} \quad (8.5)$$

we may solve the second order Zerilli equation with a source term,

$$\left[ \frac{d^2}{dr_*^2} + \left( \omega^{(2)} \right)^2 - V_Z(r) \right] \chi_{44\omega^{(2)}}^{(2)reg}(r) = S_{44\omega^{(2)}}^{reg}(r), \quad (8.6)$$

where  $\omega^{(2)} = 2\omega$  ( $\omega_{44}^{(2)} = 2\omega_{22}^{(1)}$ ;  $n = 0, 1$  or 2) and the source term is given by

$$S_{44\omega^{(2)}}^{reg}(r) = \frac{i\omega}{126} \sqrt{\frac{70}{\pi}} \left\{ \left[ -\frac{r(7r + 4M)\omega^4}{r - 2M} \right. \right.$$

$$\begin{aligned}
& +3 \frac{(276 r^7 + 476 r^6 M - 1470 r^5 M^2 - 1389 r^4 M^3 - 816 r^3 M^4 - 800 r^2 M^5 - 555 r M^6 - 96 M^7) \omega^2}{r^3 (3r - M)^2 (2r + 3M)^2 (r - 2M)} \\
& + \frac{9(r - 2M)}{r^7 (3r + M)^2 (2r + 3M)^4} (2160 r^9 + 11760 r^8 M + 30560 r^7 M^2 + 41124 r^6 M^3 + 31596 r^5 M^4 \\
& + 11630 r^4 M^5 - 1296 r^3 M^6 - 4182 r^2 M^7 - 1341 r M^8 - 144 M^9) \left( \psi_{22\omega}^{(1)}(r) \right)^2 \\
& + \left[ -4 \frac{(r - 2M)^2 M^2 \omega^2}{r^2 (3r + M)^2} + \frac{6(r - 2M)}{r^6 (3r + M)^2 (2r + 3M)^3} (144 r^8 + 4116 r^7 M + 2154 r^6 M^2 \right. \\
& \left. - 2759 r^5 M^3 - 8230 r^4 M^4 - 9512 r^3 M^5 - 3540 r^2 M^6 - 1119 r M^7 - 144 M^8) \right] \left( \frac{d}{dr} \psi_{22\omega}^{(1)}(r) \right) \psi_{22\omega}^{(1)}(r) \\
& + \left[ -\frac{(r - 2M)(7r + 4M) \omega^2}{r} - \frac{3(r - 2M)}{r^5 (3r + M)^2 (2r + 3M)^2} (228 r^7 + 8 r^6 M - 370 r^5 M^2 \right. \\
& \left. + 142 r^4 M^3 - 384 r^3 M^4 - 514 r^2 M^5 - 273 r M^6 - 48 M^7) \right] \left( \frac{d}{dr} \psi_{22\omega}^{(1)}(r) \right)^2 \Bigg\}, \tag{8.7}
\end{aligned}$$

where  $\omega = \omega_{22}^{(1)}$ .

We solve the above equation under the boundary conditions with purely ingoing at the horizon and purely outgoing at infinity. This gives the two-point boundary value problem. However a simple numerical integration like a Runge-Kutta method does not work because the QNM frequency has a negative imaginary part and hence the QNMs diverge exponentially towards infinity and the horizon. We need unrealistic precision to resolve the unwanted solution, which is decreasing exponentially as we approach the boundaries. Furthermore, since the infinity is an irregular singularity [22], the expansion of the basic equation around boundaries does not behave well, although this method was applied to the first-order [43]. In the first-order case the most famous method to solve this problem is the Leaver's continued fraction method [22]. In the following we modify the Leaver's method and apply it to the second-order.

In order to use the Leaver's method, we need to transform the Zerilli equation to the Regge-Wheeler one, and vice versa. This is possible by the Chandrasekhar transformations [44] for general  $(\ell, m)$  modes,

$$\psi_{\ell m \omega}^{(1)}(r) = \frac{1}{\frac{4}{9} \lambda^2 (\lambda + 1)^2 + 4M^2 \omega^2} \left( \left( \frac{2}{3} \lambda (\lambda + 1) + \frac{6M^2(r - 2M)}{r^2 (\lambda r + 3M)} \right) \psi_{\ell m \omega}^{(1)RW}(r) + 2M \frac{d}{dr_*} \psi_{\ell m \omega}^{(1)RW}(r) \right), \tag{8.8}$$

$$\psi_{\ell m \omega}^{(1)RW}(r) = \left( \frac{2}{3} \lambda (\lambda + 1) + \frac{6M^2(r - 2M)}{r^2 (\lambda r + 3M)} \right) \psi_{\ell m \omega}^{(1)}(r) - 2M \frac{d}{dr_*} \psi_{\ell m \omega}^{(1)}(r). \tag{8.9}$$

We also have similar equations for  $\chi_{44\omega^{(2)}}^{(2)reg}(r)$  with replacing  $\omega$  with  $\omega^{(2)} = 2\omega$  ( $\omega_{44}^{(2)} = 2\omega_{22}^{(1)}$ ;  $n = 0, 1$  or  $2$ ). With the Chandrasekhar transformations, the Zerilli equation (8.6) becomes the Regge-Wheeler one,

$$\begin{aligned}
\left[ \frac{d^2}{dr_*^2} + \left( \omega^{(2)} \right)^2 - V_{RW}(r) \right] \chi_{44\omega^{(2)}}^{(2)reg,RW}(r) &= S_{44\omega^{(2)}}^{reg,RW}(r); \\
V_{RW}(r) &= 2 \left( 1 - \frac{2M}{r} \right) \frac{(\lambda + 1)r - 3M}{r^3}, \tag{8.10}
\end{aligned}$$

where the source term  $S_{44\omega^{(2)}}^{reg,RW}(r)$  is also transformed by the Chandrasekhar transformations, but has a similar form as  $S_{44\omega^{(2)}}^{reg}(r)$  in Eq. (8.7),

$$S_{44\omega^{(2)}}^{reg,RW}(r) = (\dots) \left( \psi_{22\omega}^{(1)}(r) \right)^2 + (\dots) \psi_{22\omega}^{(1)}(r) \left( \frac{d\psi_{22\omega}^{(1)}(r)}{dr} \right) + (\dots) \left( \frac{d\psi_{22\omega}^{(1)}(r)}{dr} \right)^2.$$

Here we can obtain the first order wave-function  $\psi_{22\omega}^{(1)}(r)$  from  $\psi_{22\omega}^{(1)RW}(r)$  with Eq. (8.8).

The Leaver's method provides the first order Regge-Wheeler function  $\psi_{\ell m \omega}^{(1)RW}(r)$  in the form,

$$\psi_{22\omega}^{(1)RW}(r) = 2MA_\psi(r) \sum_{n=0}^{\infty} a_n \left( \frac{r - 2M}{r} \right)^n, \tag{8.11}$$

TABLE II: The coefficients for the amplitude and energy of the second order QNMs in Eqs. (8.1), (8.2) and (8.3). The first order QNM has  $\ell = 2, m = 2$  with the overtone  $n$ .

$n$	$C_I$	$C_H$	$C_E$
0	$0.221 - 0.489i$	$-0.221 + 1.19i$	15.0
1	$0.397 - 0.384i$	$0.262 - 1.16i$	12.7
2	$0.459 - 0.226i$	$0.604 - 0.811i$	8.97

where

$$\begin{aligned} A_\psi(r) &= \left(\frac{r}{2M} - 1\right)^\rho \left(\frac{r}{2M}\right)^{-2\rho} e^{-\rho(r-2M)/(2M)}; \\ \rho &= -2iM\omega. \end{aligned} \quad (8.12)$$

The coefficients  $a_n$  are determined by three-term recurrence relations,

$$\begin{aligned} \alpha_0 a_1 + \beta_0 a_0 &= 0, \\ \alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} &= 0, \quad n = 1, 2, \dots, \end{aligned} \quad (8.13)$$

where  $a_0$  is arbitrary and  $\alpha_n, \beta_n$  and  $\gamma_n$  are given by

$$\begin{aligned} \alpha_n &= n^2 + (2\rho + 2)n + 2\rho + 1, \\ \beta_n &= -(2n^2 + (8\rho + 2)n + 8\rho^2 + 4\rho + 2\lambda - 1), \\ \gamma_n &= n^2 + 4\rho n + 4\rho^2 - 4. \end{aligned} \quad (8.14)$$

Similarly the second order solution can be written in the form,

$$\chi_{44\omega^{(2)}}^{reg,RW}(r) = [A_\psi(r)]^2 \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{r-2M}{r}\right)^n. \quad (8.15)$$

The coefficients  $a_n^{(2)}$  are determined by the following three-term recurrence relations,

$$\begin{aligned} \alpha_0 a_1^{(2)} + \beta_0 a_0^{(2)} &= b_0, \\ \alpha_n a_{n+1}^{(2)} + \beta_n a_n^{(2)} + \gamma_n a_{n-1}^{(2)} &= b_n, \quad n = 1, 2, \dots, \end{aligned} \quad (8.16)$$

where we replace  $\rho$  with  $2\rho$  in  $\alpha_n, \beta_n$  and  $\gamma_n$  in Eqs. (8.14) and the source terms  $b_n$  are determined by expanding  $S_{44\omega^{(2)}}^{reg,RW}$  in Eq. (8.10) as

$$\frac{r^3}{r-2M} S_{44\omega^{(2)}}^{reg,RW}(r) = [A_\psi(r)]^2 \sum_{n=0}^{\infty} b_n \left(\frac{r-2M}{r}\right)^n. \quad (8.17)$$

We can first obtain the source terms  $b_n$  algebraically, and then evaluate them numerically with the *Maple* calculator. Once we have numerical values of  $b_n$ , we can solve the recurrence relations. The solution that satisfies the boundary condition at infinity is determined by adjusting  $a_0^{(2)}$  for which  $\sum a_n^{(2)}$  exists and is finite.

## IX. SUMMARY AND DISCUSSIONS

In this paper, we have investigated the second order QNMs of a Schwarzschild black hole, as summarized in the following.

1. We consider the  $\ell = 2, m = \pm 2$  even parity modes with the QNM frequencies  $\omega_{2\pm 2}^{(1)}$  for the first order perturbations because these modes dominate for binary black hole mergers.

2. We have considered the  $\ell = 4, m = \pm 4$  even parity modes for the second perturbative order that are driven by the first order perturbations. We have employed the second order Regge-Wheeler-Zerilli formalism with the source term written by quadratic terms of the first order wave-function.
3. We have regularized the source term because this does not behave well at the boundaries.
4. Based on the regularized source, the second order wave-function have been obtained by using the modified Leaver's continued fraction method.
5. We have explicitly derived the gauge transformation of the metric perturbation into an asymptotic flat gauge to extract physical information from the first and second order wave-functions. We have also formulated the power of gravitational waves for the first and second perturbative order.
6. As the result, the second order QNM frequencies are found to be  $\omega_{4\pm 4}^{(2)} = 2\omega_{2\pm 2}^{(1)}$  and the gravitational wave amplitude could go up to  $\sim 10\%$ . This means that we can in principle detect and identify the higher order QNMs since their frequencies differ from any first order ones. The detectability of a two-mode ringdown wave has been discussed in [45].

The detection of the second order QNMs would have the following advantages.

1. The second order QNMs would provide a new test of general relativity, in particular of the no-hair theorem. According to the no-hair theorem, the astrophysical BHs are completely characterized by their mass and angular momentum. Then the mass and angular momentum derived from the second order QNMs should coincide with that from the first order ones.
2. The first and second order QNM amplitudes include information about the total radiated GW energy  $E^{(1)}$  and  $E^{(2)}$ , respectively. Using these ratio, this could provide distance indicators from a data analysis of only the QNM gravitational waves, since the observed GW amplitude is  $h \sim (E/M)^{1/2}(M/r)$ . This is important for the case that the inspiralling phase is out of the detector frequency range. To be precise, since the observed GW amplitude also depends on the zenith angle  $\theta$ , the parameters are degenerated and we can determine only the order-of-magnitude distances. A possible solution to this problem is presented below.
3. We may also use the first-to-second amplitude ratio to reject fake events in the QNM search in which there are many fake events [10]. In [10], a method to remove fake events has been proposed by using the overtone QNM of the same  $(\ell, m)$  mode. The amplitude of the overtone QNM is determined by two factors, the 'excitation coefficient' of the QNM and the 'initial condition' of a perturbation. We can derive the excitation coefficient theoretically [46, 47, 48, 49, 50], but we must give the initial condition which depends on how QNMs are excited. Thus, it has been difficult to remove fake events by using the overtone QNM with an undetermined amplitude. On the other hand, the second order QNMs have the predictable amplitude except for the zenith angle dependence. This may ease the fake event rejection.

Future problems include

1. It is necessary to formulate the odd parity mode case in the Schwarzschild background. The odd parity mode appears when BHs have spin before mergers.
2. We also need to discuss coupling contributions between different harmonics  $(\ell, m)$ .
3. As discussed in Sec. V, the product of  $\ell = 2, m = \pm 2$  even modes gives not only the  $m = \pm 4$  modes, which is studied in this paper, but also the  $m = 0$  mode. Although the  $m = 0$  mode does not oscillate as a function of time, the amplitude would be comparable to that of the  $m = \pm 4$  modes. Since the metric perturbation of the  $m = 0$  mode does not return to the initial value, this mode represents the gravitational memory effect [51, 52]. If the  $m = 0$  mode can be also detected, the amplitude ratio of the  $m = 0$  mode to the  $m = \pm 4$  modes would determine the zenith angle to the observer since the zenith angle dependence is different. This could resolve the parameter degeneracy in the distance measurements with the second order QNMs discussed above.
4. We have to extend the analysis to the Kerr BH case. When BHs have no spin before mergers, the final spin of the remnant BH is  $a \sim 0.7$  [14] and hence the Kerr effects may not be so large as inferred from the fact that the QNM frequencies shift by only a small factor. However, the final spin becomes large in the case of highly-spinning BH binaries [53]. Since the master equation for Kerr BHs also has a source term that is quadratic in the first order function [54], we may expect similar results.

5. It is also important to discuss a mathematically rigorous definition of second order QNMs like the first order ones that use the Laplace transformation rather than the Fourier transformation [9, 46].
6. The third order formulation is interesting. As suggested by [20], the QNM frequencies will also blueshift up to  $(\psi^{(1)}/M)^2 \sim 1\%$  at this third order.
7. In order to proof that the second-order QNMs actually exist and stand out of the GW tail, we have to find the second-order QNMs directly in the numerical simulations. Such simulations are challenging because the mesh size should be less than  $\sim 1\% \times M$  to resolve  $\sim 1\%$  metric perturbations. There are detail analyses of nonlinear mode-coupling effects on a waveform by using the full nonlinear code [55, 56], but the second-order QNMs have not been identified.

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### APPENDIX A: QNM FREQUENCIES

In this appendix, we consider quasi-normal frequencies of a black hole in the first perturbative order in order to clarify their complex nature and relation between positive and negative  $m$  modes. First, we discuss the quasi-normal frequencies of a Kerr black hole, and then go back to the Schwarzschild case.

With the spacetime symmetry, we may use the spheroidal (spherical in the Schwarzschild case) harmonics expansion which has the labels,  $\ell$  and  $m$ . There are infinite overtone frequencies of the QNM for each  $(\ell, m)$  mode. This is labeled by  $n$ . Even if we fix  $\ell, m$  and  $n$ , there exist two quasi-normal frequencies as we can see from Fig. 1 and 3 in [22]. One has a positive real part  $\omega_{\ell mn(+)}$  while the other has a negative one  $\omega_{\ell mn(-)}$ .

For  $m \geq 0$ , the QNM frequency with a positive real part,  $\omega_{\ell mn(+)}$ , represents more slowly damped modes than that with a negative real part,  $\omega_{\ell mn(-)}$ , while for  $m \leq 0$ ,  $\omega_{\ell mn(-)}$  represents more slowly damped modes than  $\omega_{\ell mn(+)}$ . Since we concentrate on the most dominant modes, we consider  $\omega_{\ell mn(+)}$  and  $\omega_{\ell -mn(-)}$  ( $m \geq 0$ ) in this paper. For example, although the integration in Eq. (4.3) picks up the two set of frequencies,  $\omega_{\ell mn(+)}$  and  $\omega_{\ell mn(-)}$ , we adopt the most slowly damped mode as

$$\psi_{\ell m}(t, r) = e^{-i\omega_{\ell m0(+)}t} \psi_{\ell m \omega_{\ell m0(+)}}(r) + e^{-i\omega_{\ell m0(-)}t} \psi_{\ell m \omega_{\ell m0(-)}}(r) \quad (\text{A1})$$

$$\sim \begin{cases} e^{-i\omega_{\ell m0(+)}t} \psi_{\ell m \omega_{\ell m0(+)}}(r), & \text{for } m \geq 0, \\ e^{-i\omega_{\ell m0(-)}t} \psi_{\ell m \omega_{\ell m0(-)}}(r), & \text{for } m < 0, \end{cases} \quad (\text{A2})$$

where we consider the fundamental  $n = 0$  mode. We note that the QNM frequencies have the complex conjugate symmetry,

$$\omega_{\ell mn(+)} = -\omega_{\ell -mn(-)}^*, \quad (\text{A3})$$

as discussed in Sec. 4 of [22].

Next, we discuss the Schwarzschild case. In this case, the  $m$  modes degenerate due to the spherical symmetry. Therefore, the label  $m$  have no meaning and we may write

$$\omega_{\ell mn(+)} = -\omega_{\ell mn(-)}^*.$$

This equations means that the frequencies  $\omega_{\ell mn(+)}$  and  $\omega_{\ell mn(-)}$  have the same damping time. The wave-function  $\psi_{\ell m}(t, r)$  of the fundamental QNM is formally written by the same equation in Eq. (A1). The two terms in the right hand side of Eq. (A1) have the same damping time in the Schwarzschild case, but we do not consider the term with  $\omega_{\ell mn(-)}$  for a positive  $m$  mode and  $\omega_{\ell mn(+)}$  for a negative  $m$  mode, because these modes are not the more slowly damped mode in the Kerr case. Therefore, we have adopted the following wave-function for the first perturbative order,

$$\begin{aligned} \psi_{\ell m}(t, r) &= e^{-i\omega_{\ell m}^{(1)}t} \psi_{\ell m \omega_{\ell m}^{(1)}}(r), \quad \text{for } m \geq 0, \\ \psi_{\ell m}(t, r) &= e^{i\omega_{\ell m}^{(1)*}t} \psi_{\ell m -\omega_{\ell m}^{(1)*}}(r), \quad \text{for } m < 0, \end{aligned} \quad (\text{A4})$$

where  $\omega_{\ell m}^{(1)} = \omega_{\ell mn(+)}$  and we consider  $n = 0, 1$  or  $2$  for comparison in this paper. These wave-functions satisfy Eq. (4.7) ensuring real metric components.

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