10-18-2001

A C~Diffeomorphism of R^2 that has a Cantor set that is a minimal set

William Basener

Carl Lutzer

A C^∞ DIFFEOMORPHISM OF $\mathbb{R}^2$ THAT HAS A CANTOR SET THAT IS A MINIMAL SET. - DRAFT OCTOBER 18, 2001

BILL BASENER AND CARL LUTZER

Abstract. We present a C^∞ diffeomorphism of $\mathbb{R}^2$ that has a Cantor Set that is a minimal Set. The Cantor Set is contained inside an annulus.

1. Introduction

For a homeomorphism $f : X \to X$ of a topological space X, a nonempty compact subset $Y \subset X$ is a minimal set if for every $y \in Y$ the orbit of y is dense in Y. Denjoy showed (see [4]) that any diffeomorphism of S^1 that has a Cantor set which is a minimal set cannot be C^2. Our example shows that this restriction does not hold for a diffeomorphism of the annulus.

This raises the question of whether diffeomorphisms of other manifolds can be smoother than C^2 and have a Cantor set as a minimal set. We answer this in the affirmative by constructing a C^∞ diffeomorphism of $\mathbb{R}^2$ that has a Cantor set which is a minimal set. We will refer to a Cantor Set that is a minimal set as a Cantor minimal set.

We need the following definition

DEFINITION 1. *For $F : \mathbb{R} \to \mathbb{R}$ any j-times differentiable map we define*

$$||F||_{C^j} = \sup_{x \in \mathbb{R}, 1 \le i \le j} \left| \frac{d^i F}{dx^i}(x) \right| + \sup_{x \in \mathbb{R}} |F(x)|.$$

and we need the following theorem.

THEOREM 1. *Let $f_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, 2, ...$ be a sequence of functions such that:*

1. *For every i, f_i is C^∞.*
2. *The sum $\sum_{i=1}^\infty ||f_i - f_{i+1}||_{C^i}$ converges.*

Then $f_i \to f : \mathbb{R}^2 \to \mathbb{R}^2$ such that f is C^∞.

The map can be loosely described as follows. Let $p_1, p_2, ...$ be an infinite sequence of positive integers which are pairwise relatively prime. Let D be the unit disk in $\mathbb{R}^2$. Let f_1 be a rotation of D by $2\pi/p_1$. For each $i = 0, 1, ..., p_1 - 1$ let $D_{(i)}$ be a closed disk contained in D such that f_1 takes $D_{(i)}$ to $D_{(i+1 \mod p_1)}$ and such that $D_{(i)} \cap D_{(i')} = \varnothing$ for

$i \neq i'$. (See Figure 1.) For each $i = 0, 1, ..., p_1 - 1$ define a closed disk $\overline{D_{(i)}}$ such that $D_{(i)} \subseteq \mathrm{intt}\overline{D_{(i)}}$ and $\overline{D_{(i)}} \cap \overline{D_{(i')}}$ for $i \neq i'$.

Let f_2 be a function that rotates each $D_{(i)}$ by $2\pi/p_2$ and is the identity outside of the $\overline{D_{(i)}}$. For each $i = 0, 1, ..., p_1 - 1, j = 0, 1, ..., p_2 - 1$ define a closed disk $D_{(i,j)}$ such that such that f_2 takes $D_{(i,j)}$ to $D_{(i,j+1 \mod p_2)}$. Hence $f_2 \circ f_1$ takes $D_{(i,j)}$ to $D_{(i+1 \mod (p_1), j+1 \mod (p_2))}$.

Continuing by induction, for every $i \in \mathbb{N}$, we define a homeomorphism $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

- f_i rotates every $D_{(x_1,...,x_{i-1})}$ by $2\pi/p_i$, where $(x_1, \ldots, x_{i-1}) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i - 1}$.
- f_i is the identity of off the $\overline{D_{(x_1,...,x_{i-1})}}$.

Then define disjoint disks $D_{(x_1,...,x_{i-1},x_i)}$, where $(x_1, \ldots, x_{i-1}, x_i) \in \mathbb{Z}_{p_1} \times \cdots \times \mathbb{Z}_{p_i - 1} \times \mathbb{Z}_{p_i}$, and disks $\overline{D_{(x_1,...,x_{i-1},x_i)}}$ with $D_{(x_1,...,x_{i-1},x_i)} \subset \mathrm{intt}\overline{D_{(x_1,...,x_{i-1},x_i)}}$,such that

- $\overline{D_{(x_1,...,x_{i-1},x_i)}} \cap \overline{D_{(y_1,...,y_{i-1},y_i)}} = \varnothing$ for $(x_1, \ldots, x_{i-1}, x_i) \neq (y_1, \ldots, y_{i-1}, y_i)$
- $f_i \circ f_{i-1} \circ \cdots \circ f_2 \circ f_1$ takes $D_{(x_1,...,x_{i-1},x_i)}$ to $D_{(x_1+1 \mod (p_1),...,x_{i-1}+1 \mod (p_{i-1}), x_i+1 \mod (p_i))}$
- f_i is the identity of off the $\overline{D_{(x_1,...,x_{i-1},x_i)}}$.

We show that the map $f = \cdots \circ f_i \circ f_{i-1} \circ \cdots \circ f_2 \circ f_1$ is continuous in Section 2.

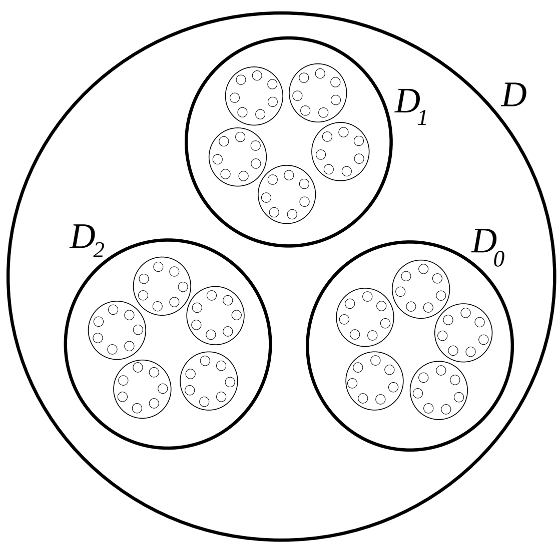

FIGURE 1. The first three steps in creating the Cantor set C for $p_1 = 3$, $p_2 = 5$, and $p_3 = 7$.

The points in the Cantor set $C = \cap_{i=1}^{\infty} (\cup_{(x_1,x_2,...,x_i) \in \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_i}} D_{(x_1,x_2,...,x_i)})$ can be indexed by the group

$$G = \times_{i=1}^{\infty} \mathbb{Z}_{p_i}$$

using the map

$$(x_1, x_2, x_3, \dots) \mapsto D_{(x_1)} \cap D_{(x_1,x_2)} \cap D_{(x_1,x_2,x_3)} \cap \cdots .$$

We can now prove that C is a minimal set for f.

PROPOSITION 1. *The set C is a minimal set for the map f.*

Proof. It follows from our definition that for $(x_1, x_2, x_3, \dots) \in C$,

$$(1) \quad f(x_1, x_2, \dots) = (x_1 + 1 \mod (p_1), x_2 + 1 \mod (p_2), \dots) \in C$$

For any $\epsilon > 0$ there exists an N such that for two points $(x_1, x_2, x_3, \dots)$, $(y_1, y_2, y_3, \dots) \in C$, $|(x_1, x_2, x_3, \dots) - (y_1, y_2, y_3, \dots)| < \epsilon$ if $x_i = y_i$ for all $i < N$. This follows because $\mathrm{diam} D_{(x_1,\dots,x_i)} \to 0$ as $i \to \infty$. So to show that C is a minimal set for f it suffices to show that for any $(x_1, x_2, x_3, \dots), (y_1, y_2, y_3, \dots) \in C$ and positive integer N there exists a positive integer k such that the first N entries of $f^k(x_1, x_2, x_3, \dots)$ agree with the first N entries of $(y_1, y_2, y_3, \dots)$. This follows easily from Formula 1 because the p_i are pairwise relatively prime. □

2. The Map f can be C^∞

For convienence we will use $\mathbb{C}$ instead of $\mathbb{R}^2$. We begin with a technical but useful lemma.

LEMMA 1. *For any positive integers p, k, real numbers $0 < a < b < 1$, and any real number $\epsilon > 0$ there exists a c^∞ diffeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ such that:*

(1) *$\phi(z) = ze^{2\pi i/p'}$ for all $z \in \mathbb{C}$ such that $|z| \le a$ and for some prime number $p' > p$.*
(2) *$\phi(z) = z$ for all $z \in \mathbb{C}$ such that $|z| \ge b$.*
(3) *$||\phi(z) - z||_{C^k} < \epsilon$.*

Proof. Let $\rho : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\rho(r) = 1$ for $r < a$, ρ is monotonically decreasing on (a, b), and $\rho(r) = 0$ for $r > b$. For any prime number $p' > p$ the function

$$(2) \qquad \phi(z) = ze^{2\pi i\rho(|z|)/p'}.$$

satisfies (1) and (2) from the theorem. We will show that if p' is chosen large enough then $\phi(z)$ from Equation 3 also satisfies (3).

Using Definition 1,

$$(3) \quad ||\phi(z) - z||_{C^k} = \sup_{z \in \mathbb{C}, 1 \le i \le j} \left| \frac{d^i[\phi(z) - z]}{dx^i} \right| + \sup_{x \in \mathbb{C}} |ze^{2\pi i\rho(|z|)/p'} - z|$$

We will show that each of the terms on the right side of this equation can be made less than $\epsilon/2$ if p' is chosen small enough.

We first show this for the term $\sup_{x\in\mathbb{C}} |ze^{2\pi i\rho(|z|)/p'} - z|$. Since $\rho(r) = 0$ for $r > 1$, $|ze^{2\pi i\rho(|z|)/p'} - z| = 0$ if $|z| > 1$. So

$$\sup_{z \in \mathbb{C}} |ze^{2\pi i\rho(|z|)/p'} - z| = \sup_{|z| \le 1} |ze^{2\pi i\rho(|z|)/p'} - z| \le \sup_{|z| \le 1} |e^{2\pi i\rho(|z|)/p'} - 1|.$$

Since $|e^{2\pi i\rho(|z|)/p'} - 1| \to 0$ as $p' \to \infty$, we can choose p' so that

$$\text{(4)} \qquad \sup_{z\in\mathbb{C}} |ze^{2\pi i\rho(|z|)/p'} - z| \leq \sup_{|z|\leq 1} |e^{2\pi i\rho(|z|)/p'} - 1| < \frac{\epsilon}{4}.$$

Note that we bound this term by $\epsilon/4$.

Now we show that the term $\sup_{z\in\mathbb{C},1\leq i\leq j}\left|\frac{d^i[\phi(z)-z]}{dx^i}\right| < \epsilon/2$ if p' is chosen small enough. As before, $\phi(z) - z = 0$ if $|z| > 1$ so it suffices to prove that $\sup_{|z|\leq 1,1\leq i\leq j}\left|\frac{d^i[\phi(z)-z]}{dx^i}\right| < \epsilon/2$ if p' is chosen large enough. We demonstrate this by showing that if p' is large enough then $\sup_{|z|\leq 1}\left|\frac{d^i[\phi(z)-z]}{dx^i}\right|$ for every $1 \leq i \leq j$. For the case $i = 1$, (using the triangle inequality and Equation 4.)

$$\begin{aligned}
&\sup_{|z|\leq 1}\left|\frac{d^i[\phi(z)-z]}{dx^i}\right| \\
=&\sup_{|z|\leq 1}\left|e^{2\pi i\rho(|z|)/p'} + z\left(\frac{2\pi i}{p'}\frac{\mathrm{d}\rho(|z|)}{\mathrm{d}z}\right)e^{2\pi i\rho(|z|)/p'} - 1\right| \\
<&\sup_{|z|\leq 1}\left|e^{2\pi i\rho(|z|)/p'} - 1\right| + \sup_{|z|\leq 1}\left|z\left(\frac{2\pi i}{p'}\frac{\mathrm{d}\rho(|z|)}{\mathrm{d}z}\right)e^{2\pi i\rho(|z|)/p'}\right| \\
<&\frac{\epsilon}{4} + \frac{1}{p'}\sup_{|z|\leq 1}\left|z\left(2\pi i\frac{\mathrm{d}\rho(|z|)}{\mathrm{d}z}\right)e^{2\pi i\rho(|z|)/p'}\right|
\end{aligned}$$

The function $\left|z\left(2\pi i\frac{\mathrm{d}\rho(|z|)}{\mathrm{d}z}\right)e^{2\pi i\rho(|z|)/p'}\right|$ is continuous on $|z| \leq 1$ so it achieves its max $M = \sup_{|z|\leq 1}\left|z\left(2\pi i\frac{\mathrm{d}\rho(|z|)}{\mathrm{d}z}\right)e^{2\pi i\rho(|z|)/p'}\right|$ on this set. Hence choosing $p' > \frac{4M}{\epsilon}$ gives

$$\begin{aligned}
&\sup_{|z|\leq 1}\left|\frac{d^i[\phi(z)-z]}{dx^i}\right| \\
<&\frac{\epsilon}{4} + \frac{1}{p'}\sup_{|z|\leq 1}\left|z\left(2\pi i\frac{\mathrm{d}\rho(|z|)}{\mathrm{d}z}\right)e^{2\pi i\rho(|z|)/p'}\right| \\
<&\frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.
\end{aligned}$$

For $i > 1$, we can write

$$\sup_{|z|\leq 1}\left|\frac{d^i[\phi(z)-z]}{dx^i}\right| = \frac{1}{p'}\sup_{|z|\leq 1}|F(z)|$$

where $F : \mathbb{Z} \to \mathbb{Z}$ is a continuous function. Hence $|F(z)|$ achieves its max on $|z| \leq 1$ and if p' is large enough,

$$\sup_{|z|\leq 1}\left|\frac{d^i[\phi(z)-z]}{dx^i}\right| = \frac{\epsilon}{2}.$$

This proves that if p' is large enough then both of the terms on the right hand side of Equation 4 are less than $\epsilon/2$, which finishes the proof of (3). $\square$

Denote the function $\phi : \mathbb{C} \to \mathbb{C}$ associate with positive integers p, k, real numbers $0 < a < b < 1$, and $\epsilon > 0$ by

$$\phi_{p,k,(a,b),\epsilon} : \mathbb{C} \to \mathbb{C}.$$

Denote the n^{th} roots of unity by

$$\{u_k^n = e^{k2\pi i/n}\}_{k=1}^n$$

Define

$$f_1(z) = \phi_{1,1,(1,1.1),1/2}.$$

So f_1 rotates the unit disk by $2\pi/p'_1$ for some prime number $p'_1 > 1$, f_1 is the identity outside of the disk of radius 1.1 centered at the origin, and $||f_1(z) - z||_{C^1} < 1/2$. Choose $0 < a_1 < b_1 < 1$ such that $|\frac{1}{2}u_i^{p'_1} - \frac{1}{2}u_j^{p'_1}| > 2b_1$ for all $i \neq j$. Define the points

$$c_i = \frac{1}{2}u_i^{p'_1}, \text{ for } i = 0, ..., p'_1 - 1,$$

and the disks

$$\begin{aligned} D_i &= B_{a_1}(c_i), \text{ for } i = 0, ..., p'_1 - 1, \\ \overline{D_i} &= B_{b_1}(c_i), \text{ for } i = 0, ..., p'_1 - 1, \end{aligned} \tag{5}$$

where $B_r(c)$ is the ball of radius r centered at the point c. Notice that

$$f_1(D_i) = D_{i+1 \mod (p'_1)}.$$

Define

$$\psi_i(z) = \phi_{p'_1,2,(a_1,b_1),1/4}(z - c_i) + c_i.$$

for some prime number $p'_2 > p'_1$, and let

$$f_2(z) = \psi_{p'_1} \circ \cdots \circ \psi_0(z).$$

So f_2 rotates each disk D_i by $2\pi/p'_2$ for some prime number $p'_2 > p'_1$, f_2 is the identity outside of the disks $\overline{D_i}$, and $||f_2(z) - z||_{C^2} < 1/4$. For each $i = 0, 1, ..., p'_1$ and $j = 0, 1, ..., p'_2$, define

$$c_{(i,j)} = c_i + \frac{a_1}{2}u_j^{p'_2}$$

Notice that $f_1(c_{(i,j)}) = c_{(i+1 \mod (p'_1),j)}$ and $f_2(c_{(i,j)}) = c_{(i,j+1 \mod (p'_2))}$. Hence, $f_2 \circ f_1(c_{(i,j)}) = c_{(i+1 \mod (p'_1),j+1 \mod (p'_2))}$. Choose $0 < a_2 < b_2 < 1$ such that $|c_{(i_1,j_1)} - c_{(i_2,j_2)}| > 2b_2$ for all $(i_1, j_1) \neq (i_2, j_2)$. Define the disks

$$\begin{aligned} D(i,j) &= B_{a_2}(c_{(i,j)}), \text{ for } i = 0, ..., p'_2 - 1, \\ \overline{D(i,j)} &= B_{b_2}(c_{(i,j)}), \text{ for } i = 0, ..., p'_2 - 1. \end{aligned} \tag{6}$$

Notice that $f_2(D_{(i,j)}) = D_{(i,j+1 \mod (p'_2)}$ and hence,

$$f_2 \circ f_1(D_{(i,j)}) = D_{(i+1 \mod (p'_1),j+1 \mod (p'_2)}.$$

We continue by induction as described in Section 1. Suppose maps $f_1, f_2, \ldots, f_i - 1$ are given with disks $D_{(x_1,x_2,\ldots,x_{i-1})}$ and $\overline{D_{(x_1,x_2,\ldots,x_{i-1})}}$, with $D_{(x_1,x_2,\ldots,x_{i-1})}$ centered at $C_{(x_1,x_2,\ldots,x_{i-1})}$, such that

- Each f_j rotates each $D_{(x_1,x_2,\ldots,x_j)}$ by $2\pi/p'_j$ for some prime number $p'_j > p'_{j-1}$.
- For every f_j, $||f_j(z) - z||_{C^j} < 1/2^j$.
- Each f_j is the identity outside of the disks $\overline{D_{(x_1,x_2,\ldots,x_j)}}$.
- For $(x_1, x_2, \ldots, x_j) \neq (y_1, y_2, \ldots, y_j)$, $\overline{D_{(x_1,x_2,\ldots,x_j)}} \cap \overline{D_{(y_1,y_2,\ldots,y_j)}} = \varnothing$.
- For every $(x_1, \ldots, x_j) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_j}$,

  $$f_j \circ \cdots \circ f_1(D_{(x_1,\ldots,x_j)}) = D_{(x_1+1 \mod (p'_1),\ldots,x_j+1 \mod (p'_j))}.$$

- There exist $0 < a_{i-1} < b_{i-1} < 1$ such that $|c_{(x_1,x_2,\ldots,x_{i-1})} - c_{(y_1,y_2,\ldots,y_{i-1})}| > 2b_{i-1}$ for all $(x_1, x_2, \ldots, x_{i-1}) \neq (y_1, y_2, \ldots, y_{i-1})$.

For each $(x_1, \ldots, x_{i-1}) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_{i-1}}$ define

$$\psi_{(x_1,\ldots,x_{i-1})}(z) = \phi_{p'_{i-1},i-1,(a_{i-1},b_{i-1}),1/2^i}(z - c_{(x_1,\ldots,x_{i-1})}) + c_{(x_1,x_2,\ldots,x_{i-1})}.$$

for some prime number $p'_i > p'_{i-1}$, and let

$$f_i(z) = \circ_{(x_1,\ldots,x_{i-1}) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_{i-1}}} \psi_{(x_1,\ldots,x_{i-1}) \in \mathbb{Z}_{p'_1}}(z).$$

That is, f_i is the composition of all of the $\psi_{(x_1,\ldots,x_{i-1})}$, where $(x_1, \ldots, x_{i-1}) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_{i-1}}$, and the order of composition does not matter because for any $(x_1, \ldots, x_{i-1}) \neq (y_1, \ldots, y_{i-1})$, the set of points for which $\psi_{(x_1,\ldots,x_{i-1})}$ is not the identity is disjoint form the set of points for which $\psi_{(y_1,\ldots,y_{i-1})}$ is not the identity. So f_i rotates each disk $D_{(x_1,\ldots,x_{i-1})}$ by $2\pi/p'_i$ for some prime number $p'_i > p'_{i-1}$, f_i is the identity outside of the disks $\overline{(x_1, \ldots, x_{i-1})}$, and $||f_i(z) - z||_{C^i} < 1/2^{i+1}$. For each $(x_1, \ldots, x_i) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_i}$ and define

$$c_{(x_1,\ldots,x_{i-1},x_i)} = c_{(x_1,\ldots,x_{i-1})} + \frac{a_{i-1}}{2} u^{p'_i}_{x_i}$$

Notice that $f_i(c_{(x_1,\ldots,x_{i-1},x_i)}) = c_{(x_1,\ldots,x_{i-1},x_i+1 \mod (p'_i))}$. Hence,

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(c_{(x_1,\ldots,x_{i-1},x_i)}) = c_{(x_1+1 \mod (p'_i),\ldots,x_{i-1}+1 \mod (p'_{i-1}),x_i+1 \mod (p'_i))}.$$

Choose $0 < a_i < b_i < 1$ such that $|c_{(x_1,\ldots,x_{i-1})} - c_{(y_1,\ldots,y_{i-1})}| > 2b_i$ for all $(x_1, \ldots, x_{i-1}) \neq (y_1, y_2, \ldots, y_{i-1})$. Define the disks

(7)

$$D_{(x_1,\ldots,x_{i-1},x_i)} = B_{a_i}(c_{(x_1,\ldots,x_{i-1},x_i)}), \text{ for each } (x_1, \ldots, x_i) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_i},$$

$$\overline{D_{(x_1,x_2,\ldots,x_{i-1},x_i)}} = B_{b_i}(c_{(x_1,x_2,\ldots,x_{i-1},x_i)}), \text{ for each } (x_1, \ldots, x_i) \in \mathbb{Z}_{p'_1} \times \cdots \times \mathbb{Z}_{p'_i},$$

Notice that $f_i(D_{(x_1,\ldots,x_{i-1},x_i)}) = D_{(x_1,\ldots,x_{i-1},x_i+1 \mod (p'_i))}$. Hence,

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(D_{(x_1,\ldots,x_{i-1},x_i)}) = D_{(x_1+1 \mod (p'_i),\ldots,x_{i-1}+1 \mod (p'_{i-1}),x_i+1 \mod (p'_i))}.$$

References

[1] C. Camaho and A. L. Neto, Geometric Theorey of Foliations, Birkhäuser Boston, Inc., 1985

[2] W. H. Gottschalk, Minimal Sets: An Introduction to Topological Dynamics, Bull. Amer. Math. Soc. **64** (1958) 336-351

[3] G. R. Hall, A C^∞ Denjoy Counterexample, Ergod. Th. & Dynam. Sys. **1** (1981) 261-272

[4] A. Katok and B. Hasselblatt, Introduction to the Modern Theorey of Dynamical Systems (Cambridge University Press, 1995)

[5] Z. F. Zang, A Topological Dynamical System in $\mathbb{R}^3$ with Antoine's Necklace as a Minimal Set, Sci. Sinica Ser. A **25** (1982), no. 9, 932-941