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Minimal rankings and the arank number of a path

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Abstract

Given a graph G, a function $f : V(G) \rightarrow \{1, 2, ..., k\}$ is a k-ranking of G if $f(u) = f(v)$ implies every $u - v$ path contains a vertex w such that $f(w) > f(u)$. A k-ranking is *minimal* if the reduction of any label greater than 1 violates the described ranking property. The *arank* number of a graph, denoted $\psi_r(G)$, is the largest k such that G has a minimal k-ranking. We present new results involving minimal k-rankings of paths. In particular we determine $\psi_r(P_n)$, a problem suggested by Laskar and Pillone in 2000.

1 Introduction

A labeling $f : V(G) \rightarrow \{1, 2, ..., k\}$ is a *k-ranking* of a graph G if and only if $f(u) = f(v)$ implies that every $u - v$ path contains a vertex w such that $f(w) > f(u)$. A k-ranking f is *minimal* if for all $v_i \in V(G)$, a function g satisfying $g(v) = f(v)$ when $v \neq v_i$ and $g(v_i) < f(v_i)$, is not a ranking. That is, if any label in a minimal ranking is replaced with a smaller label the new labeling is not a ranking. Note that for any ranking f there exists a minimal k-ranking h such that $h(v) \leq f(v)$ for every $v \in V(G)$. When the value of k is unimportant, we will refer to a k-ranking simply as a ranking.

Following along the lines of the chromatic number, the *rank number of a graph* $\chi_r(G)$ is defined to be the smallest k such that G has a minimal k-ranking. Similarly the concept of the achromatic number can be paralleled and the *arank number of a graph* $\psi_r(G)$ is defined to be the largest k such that G has a minimal k-ranking. We present examples involving $\chi_r(G)$ and $\psi_r(G)$ in Figures 1a and 1b.

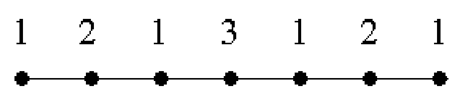

Figure 1a. Minimal χ_r-ranking of P_7

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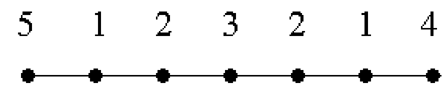

Figure 1b. Minimal ψ_r-ranking of P_7.

Early studies involving the rank number of a graph were sparked by its numerous applications including designs for very large scale integration (VLSI) layouts and Cholesky factorizations associated with parallel processing [2], [7], and [10]. Numerous papers have since followed [1], [3], [4], [5], [6], [8], and [9]. Ghoshal, Laskar, and Pillone can be credited with furthering much of the mathematical theory behind minimal rankings. They obtained precise rank numbers for many classes of graphs and also investigated the problem's complexity and extremal properties [3], [4], [8], and [9].

As our first theorem we will restate a well known result involving the rank number of a path [1].

Theorem 1 (Bodlaender et al.) *Let P_n be a path with vertices $v_1, v_2, ..., v_n$. Then $\chi_r(P_n) = \lfloor \log_2 n \rfloor + 1$.*

It is also noted that the explicit labeling can be constructed by letting $f(v_i) = 1 + \alpha(i)$ where $\alpha(i)$ is the highest power of 2 dividing i [1]. As a result a simple recursive process can be used for labeling paths with $2^n - 1$ vertices. Starting by labeling P_1 with a 1, and a desired labeling for P_{2^n-1}, the labeling for $P_{2^{n+1}-1}$ can be constructed in the following manner. Label the middle vertex with $n+1$ and then place one copy of the labeling for P_{2^n-1} on either side. As mentioned earlier minimal rankings have connections to parallel processing. One interesting relation involving $\chi_r(P_{2^n-1})$ is that the labels give the solution to the *Towers of Hanoi* problem. For a set of disks $d_1, d_2, ..., d_n$, listed in increasing size, instructions for which disk to move next can be found by reading the labels $f(v_1), f(v_2), ..., f(v_{2^n-1})$ in a χ_r-ranking of P_{2^n-1}. A label of i in the ranking would indicate to move disk d_i from one stack to another.

However the arank number has only been determined precisely for only a few classes of graphs, such as stars and split graphs [4]. One important property of the arank number is that it implies a necessary condition to determine if given ranking is minimal. That is, if a ranking contains a label greater than $\psi_r(G)$, it can not be minimal. Furthermore the determination of $\psi_r(G)$ for various families of graphs may serve to refine algorithims for computing $\chi_r(G)$, since obviously $\chi_r(G)$ is bounded by $\psi_r(G)$.

The problem of determining the arank number of a path was suggested by Laskar and Pillone [9]. In Theorem 14 we provide a complete solution to this problem, showing that $\psi_r(P_n)$ is bounded by twice the size of $\chi_r(P_n)$. Furthermore, we present necessary conditions for a given ranking of a path to be minimal. In Theorem 8 we show that in any minimal ranking of P_n more than half of the vertices are labeled either 1 or 2.

2 Background

We use P_n to denote the Hamiltonian path $v_1, v_2, ..., v_n$ and $\langle f(v_1), f(v_2), ..., f(v_n) \rangle$ to explicitly describe the labels in a ranking f. For a given ranking S_i will represent the independent set of all vertices labeled i. Given a graph G and a set $S \subseteq V(G)$ the *reduction* of G is a graph G^*such that $V(G^*) = V(G) - S$ and for vertices u and v, $(u, v) \in E(G^*)$ if and only if there exists a $u - v$ path in G with all internal vertices belonging to S. Note that if G is a path, G^* is also a path. An example of a reduction is given in Figure 2.

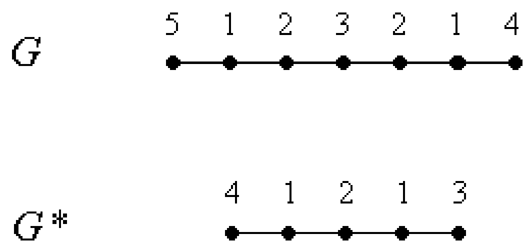

Figure 2. Reduction with $G = P_7$ and $S = S_1$.

In this paper we will have $S = S_1$. For a ranking f of a graph G, $f^*_{|G^*}$ will represent the ranking of G^* where $f^*_{|G^*}(v) = f(v) - 1$ for all $v \in V(G)$ with $f(v) > 1$. For any other undefined notation, see the graph theory text by D. B. West [11].

We continue with a series of lemmas involving the frequency and locations of small labels that must appear in a minimal ranking. We restate the following two lemmas from [3].

Lemma 2 *Let G be a graph and f be a minimal ranking of G. If $x \in V(G)$ and $f(x) = 2$ then there exists a vertex u adjacent to x such that $f(u) = 1$.*

Lemma 3 *If x is a pendant vertex of a graph G and y is adjacent to x, then in any minimal ranking f of G, either $f(x) = 1$ or $f(y) = 1$.*

In the context of paths, this last lemma states that for any minimal ranking one of the first two vertices (or last two) must be labeled 1. If $n \geq 4$, we can use the reduction operation to show that one of the first four (or last four) vertices must be labelled 2. This is presented in our next lemma.

Lemma 4 *Let f be a minimum ranking of a path $P_n = v_1, v_2, ..., v_n$ with $n \geq 4$. Then $f(v_i) = 2$ for some $1 \leq i \leq 4$. Furthermore if $f(v_i) \neq 2$ for $1 \leq i \leq 3$, then $f(v_1) = f(v_3) = 1$.*

Proof. Assume the smallest i such that $f(v_i) = 2$ is greater than 4. Then at least two of the first four vertices in the path are labeled with integers greater than 2. It follows that in $f_{|P_n^*}$ an end vertex and its neighbor will both have labels greater than 1, contradicting Lemma 3. For the second part, assume $f(v_i) \neq 2$ for $1 \leq i \leq 3$ and $f(v_4) = 2$. Suppose that either $f(v_1) \neq 1$ or $f(v_3) \neq 1$. Then two of the vertices v_1, v_2 and v_3 will have labels greater than 2. Then again, the pendant vertex and its neighbor will be mapped to a value greater then 1 by $f_{|P_n^*}$, contradicting Lemma 3. ■

We next give a bound on the maximum distance a vertex labelled m can be from the nearest vertex also labeled m.

Lemma 5 *If f is a minimal ranking of P_n then any subpath of order 2^{m+1} has a vertex v such that $f(v) = m$.*

Proof. The proof is by induction on m. The case where $m = 1$ was shown in [9]. The inductive step follows using reduction. ■

It is not difficult to show that if P' is an induced subpath of a path P, then $\psi_r(P') \leq \psi_r(P)$. We restate a lemma from [3] and [6] which show this monotonicity property holds in general.

Lemma 6 *Let H be an induced subgraph of graph G. Then $\psi_r(H) \leq \psi_r(G)$.*

Proof. An alternate proof is found in [6]. Let f be a minimal k-ranking of H. We construct a labeling of g where $g(v) = f(v)$ for all $v \in H$ and labeling all other vertices arbitrarily $k+1, k+2, ..., k+|V(G)|-|V(H)|$. To see that g is a ranking note that if two vertices in G have identical labels then both vertices must be in H, and use the fact that f is a ranking. Although g may not be a minimal ranking, no label of a vertex in H may can be replaced with a smaller label since f is a minimal ranking. Replacing labels in $V(G) - V(H)$ with smaller labels, if needed, will result in a minimal ranking of G that uses at least k labels. ■

We conclude this section by restating a result from [3] which will play a central role in establishing our main result.

Lemma 7 *Let G be a graph and let f be a minimal $\psi_r(G)$-ranking of G. Then $f^*_{|G^*}$ is a minimal $\psi_r(G^*)$-ranking of G^*.*

3 Minimal k-rankings of paths

Lemmas [?], [2], [3], and [4] provided necessary conditions for a given ranking of a path to be minimal in lemmas. All of these lemmas involved the frequency and location of vertices labeled 1 or 2 in a minimal ranking. This leads to our main result which states that in any minimal ranking of a path, more than half of the vertices must be labeled either 1 or 2.

Theorem 8 *If f is a minimal ranking of P_n then $|S_1 \cup S_2| > \frac{n}{2}$.*

Proof. Let $V(P_n) = v_1, v_2, ..., v_n$. We use the vertices in S_2 to partition P_n into parts $F_1, F_2, ..., F_M$ in the following manner. Each $x \in S_2$ is the last vertex in some part F_i, $1 \leq i \leq M-1$ and F_M consists of the remaining vertices. We illustrate this in Figure 3.

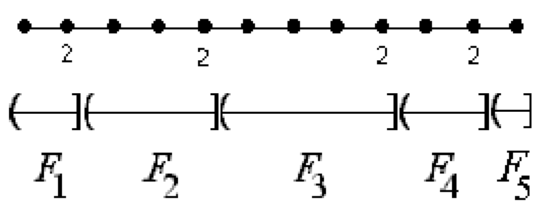

Figure 3. Partitioning of P_{12}.

By Lemma 4, $|V(F_1)| \leq 4$ and by Lemma 5 $|V(F_i)| \leq 8$ for all $i = 2, 3, ..., M$. Our strategy will be as follows. We will first prove that $|F_1 \cap (S_1 \cup S_2)| > \frac{|V(F_1)|}{2}$ and then show $|F_i \cap (S_1 \cup S_2)| \geq \frac{|V(F_i)|}{2}$ for all $i = 2, 3, ..., M$. Combining these inequalities will yield $|V(P_n) \cap (S_1 \cup S_2)| = |S_1 \cup S_2| > \frac{n}{2}$.

First we establish the inequality $|F_1 \cap (S_1 \cup S_2)| > \frac{|F_1|}{2}$. By Lemma 4 the first 2 must appear somewhere among the first four vertices. We consider four cases and show the inequality holds in each one. For completeness we provide the details.

- Case (i) $(f(v_1) = 2)$ - Since $F_1 = v_1$, it is clear that $|F_1 \cap (S_1 \cup S_2)| > \frac{|F_1|}{2}$.

- Case (ii) $(f(v_2) = 2)$ - By Lemma 3 , $f(v_1) = 1$ and $|F_1 \cap (S_1 \cup S_2)| > 1 = \frac{|F_1|}{2}$.

- Case (iii) $(f(v_3) = 2)$ - By Lemma 3, either $f(v_1) = 1$ or $f(v_2) = 1$. Hence $|F_1 \cap (S_1 \cup S_2)| > \frac{|F_1|}{2}$.

- Case (iv)$(f(v_4) = 2)$ - By Lemma 4, $f(v_1) = 1$ and $f(v_3) = 1$. Hence $|F_1 \cap (S_1 \cup S_2)| > \frac{|F_1|}{2}$.

We use a similar argument for F_M to show $|F_M \cap (S_1 \cup S_2)| \geq \frac{|F_1|}{2}$. Next we show $|F_i \cap (S_1 \cup S_2)| \geq \frac{|F_i|}{2}$ for all $i = 2, 3, ..., M-1$. Consider F_i for some i, $2 \leq i \leq M$. Let $v_{i,1}, v_{i,2}, ..., v_{i,|F_i|}$ be the vertices of F_i keeping the same ordering as in P_n. The inequality is clear when $|F_i| = 2$. By Lemma 5, $|F_i| \leq 8$. We consider cases for the various possible lengths of F_i. For completeness we include the details.

- $6 \leq |F_i| \leq 8$. If $|F_i \cap S_1| < |F_i| - 4$ then F_i contains at least four vertices with labels higher than 2. Then $f^*_{|P^*_n}$ contains labels for four consecutive vertices that are all greater than 1. By Lemma 5 $f^*_{|P^*_n}$ can not be a minimal ranking, a contradiction. Hence $|F_i \cap S_1| \geq |F_i| - 4$ and $|F_i \cap (S_1 \cup S_2)| \geq |V(F_i)| - 3 \geq \frac{|F_i|}{2}$.

- $|F_i| = 5$. By Lemma 5 $|F_i \cap S_1| \geq 1$ and the vertex labeled 1 can not be the first or fourth vertex of F_i. Assume, without loss of generality, the second vertex is labeled 1. We use a, b, and c to denote the first, third and fourth vertices of F_i respectively. If $f(c) > f(b)$, then $f(b)$ can be set to 2 and f still is a ranking; thus $f(c) < f(b)$, which implies $f(c)$ can only equal 1 if the ranking f is minimal. Hence$|F_i \cap (S_1 \cup S_2)| \geq 3 \geq \frac{|F_i|}{2}$.

- $|F_i| = 3$ or 4. By Lemma 5, $|F_i \cap S_1| \geq 1 \Rightarrow |F_i \cap (S_1 \cup S_2)| \geq 2 \geq \frac{|F_i|}{2}$.

■

In our next section the above result will be used to completely determine the arank number of a path.

4 The a-rank number of a path

The a-rank number of a path denoted $\psi_r(P_n)$ has been determined for small values of n [3]. These values are given in Table 1.

n	1	2	3	4	5	6	7	8	9	10	11
$\psi_r(P_n)$	1	2	3	4	4	4	5	5	5	5	6

Table 1: a-rank numbers for small paths

A recursive construction was given in [9] for creating a minimal $(2m-1)$-ranking of path with $2^m - 1$ vertices and a minimal $(2m-2)$-ranking of path with $2^m - 2^{m-2} - 1$ vertices. The same construction was used for both families of paths and it was conjectured that the rankings produced by this construction were ψ_r-rankings.

The case $m = 1$ is trivial and when $m = 3$, a minimal 3-ranking of a P_3 can be constructed simply by labeling the vertices $\langle 3, 1, 2 \rangle$. Starting with a k-ranking of a path on w vertices, first delete the two end vertices. We next join two copies of the resulting path with a P_3 with labels, $\langle k-1, k, k-1 \rangle$. Finally add one vertex to each end of the path and label one of these vertices $k+1$ and the other $k+2$. An example showing the construction of a minimal 6-ranking of P_{11} is shown in Figure 4.

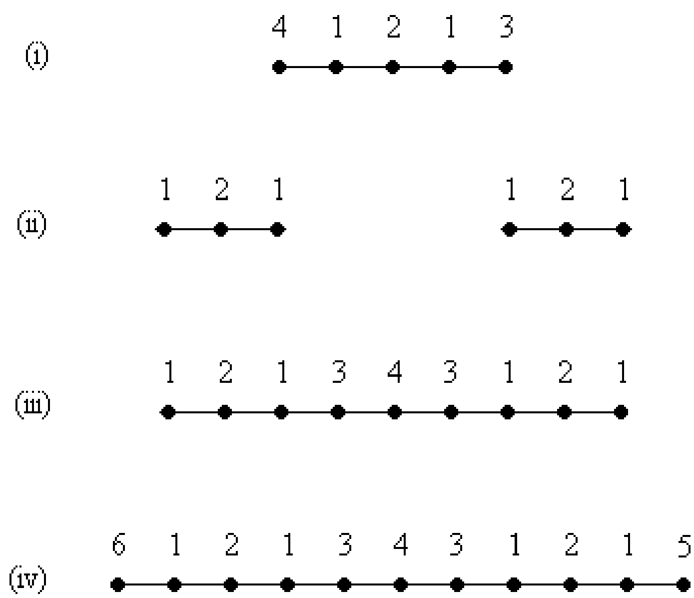

Figure 4. Construction of a minimal 6-ranking from a minimal 4-ranking.

A direct application of Lemma 7 can be used to show that the rankings produced by the construction are in fact ψ_r-rankings. We prove this in the following four lemmas.

Lemma 9 $\psi_r(P_{2^m-1}) = 2m - 1$ *for all integers* $m \geq 2$.

Proof. We proceed by induction on m. As mentioned earlier, a minimal 3-ranking of a P_3 can be constructed simply by labeling the vertices $\langle 3, 1, 2 \rangle$. Hence $\psi_r(P_{2^2-1}) = 2(2) - 1 = 3$.

Assume the equality holds for m. Given a path on $2^{m+1} - 1$ vertices we use the construction of Laskar and Pillone to produce a $(2m+1)$-ranking. Then $\psi_r(P_{2^{m+1}-1}) \geq 2m+1$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-1}) \geq 2m+2$. Then there exists a minimal k-ranking for $P_{2^{m+1}-1}$ where $k \geq 2m+2$. Reducing $P_{2^{m+1}-1}$ twice produces a path P with a $(k-2)$-ranking. By Theorem 8, P must have less than $2^m - 1$ vertices. Then Lemma 6 implies $\psi_r(P_j) \geq 2m$ for some $j \leq 2^m - 1$, which contradicts our assumption. ∎

Lemma 10 $\psi_r(P_{2^m-2^{m-2}-1}) = 2m-2$ *for all integers* $m \geq 2$.

Proof. We proceed by induction on m. For the base case, note that $\langle 1, 2\rangle$ is a minimal 2-ranking of a P_2. Given a path on $2^{m+1} - 2^{m-1} - 1$ vertices, we can construct a $2m$-ranking. Hence $\psi_r(P_{2^{m+1}-2^{m-1}-1}) \geq 2m$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-2^{m-1}-1}) \geq 2m+1$. Then there exists a minimal k-ranking for $P_{2^{m+1}-2^{m-1}-1}$ where $k \geq 2m+1$. Reducing $P_{2^{m+1}-2^{m-1}-1}$ twice produces a path P with a minimal $(k-2)$-ranking. By Theorem 8, P must have less than or equal to $2^m - 2^{m-2} - 1$ vertices. Application of Lemma 6, yields $\psi_r(P_j) \geq 2m-1$ for some $j \leq 2^m - 2^{m-2} - 1$ which contradicts our assumption. ■

Lemma 11 $\psi_r(P_{2^m-2^{m-2}-2}) = 2m-3$ *for all integers* $m \geq 2$.

Proof. We proceed by induction on m. The base case is trivial, $\langle 1\rangle$ is a minimal ranking of P_1. Assume the equality holds for m. Given a path on $2^{m+1}-2^{m-1}-2$ vertices, we can construct a $(2(m+1)-3)$-ranking. Hence $\psi_r(P_{2^{m+1}-2^{m-1}-2}) \geq 2m-1$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-2^{m-1}-1}) \geq 2m$. Then there exists a minimal k-ranking for $P_{2^{m+1}-2^{m-1}-2}$ where $k \geq 2m$. Reducing $P_{2^{m+1}-2^{m-1}-2}$ twice produces a path P with a $(k-2)$-ranking. By Theorem 8, P must have less than or equal to $2^m - 2^{m-2} - 2$ vertices. Then by Lemma 6 we have $\psi_r(P_j) \geq 2m-2$ for some $j \leq 2^m - 2^{m-2} - 2$, a contradiction. ■

Lemma 12 $\psi_r(P_{2^m-2}) = 2m-2$ *for all integers* $m \geq 2$.

Proof. We proceed by induction on m. For the base case, note that $\langle 1, 2\rangle$ is a minimal 2-ranking of a P_2. Next, assume the equality holds for m. Given a path on $2^{m+1} - 2$ vertices, using the construction from Laskar and Pillone we can produce a $2m$-ranking. Hence $\psi_r(P_{2^{m+1}-2}) \geq 2m$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-2}) \geq 2m+1$. Then there exists a minimal k-ranking for $P_{2^{m+1}-2}$ where $k \geq 2m+1$, in which case reducing $P_{2^{m+1}-2}$ twice produces a path P with a minimal $(k-2)$-ranking. By Theorem 8, P must have less than or equal to $2^m - 2$ vertices. Application of Lemma 6 $\psi_r(P_j) \geq 2m$ for some $j \leq 2m-2$, a contradiction. ■

As mentioned Laskar and Pillone established an upper bound for the arank number of a path [9]. In our next theorem we combine the above four lemmas with Lemma 6 to show that the known upper bounds are in fact tight.

Theorem 13 (The *arank* **number of** P_n**)**

(i) $\psi_r(P_s) = 2m-2$ for all integers $s \geq 2$, $2^m - 2^{m-2} - 1 \leq s \leq 2^m - 2$.

(ii) $\psi_r(P_t) = 2m-1$ for all integers $t \geq 2$, $2^m - 1 \leq t \leq 2^{m+1} - 2^{m-1} - 2$.

Finally we use a change of variable to give an explicit formula for the arank number of a path.

Theorem 14 *Let* P_n *denote on a path on* n *vertices. Then* $\psi_r(P_n) = \lfloor \log_2 (n+1) \rfloor + \left\lfloor \log_2 \left(n+1-\left(2^{\lfloor \log_2 n \rfloor - 1}\right)\right)\right\rfloor$.

Noting that $\left\lfloor \log_2 \left(n+1-\left(2^{\lfloor \log_2 n \rfloor - 1}\right)\right)\right\rfloor \leq \lfloor \log_2 (n+1) \rfloor \leq \lfloor \log_2 n \rfloor + 1 = \chi_r(P_n)$, we see that $\psi_r(P_n) \leq 2\chi_r(P_n)$.

5 Conclusion

The arank number is only known for a few families of graphs including paths, split graphs, and stars. We propose the following problems.

Problem 15 *Determine ψ_r for a tree.*

In this paper we have stated several necessary conditions for determining if a given ranking of a path is in fact minimal. It would be an interesting problem to determine a set of simple necessary conditions that are also sufficient.

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