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Minimal k-rankings and the a-rank number of a path

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Abstract

Given a graph G, a function $f : V(G) \rightarrow \{1, 2, ..., k\}$ is a k-ranking of G if $f(u) = f(v)$ implies every $u - v$ path contains a vertex w such that $f(w) > f(u)$. A k-ranking is *minimal* if the reduction of any label greater than 1 violates the described ranking property. The *a-rank number of* G, denoted $\psi_r(G)$ equals the largest k such that G has a minimal k-ranking. We establish new results involving minimal rankings of paths and in particular we determine $\psi_r(P_n)$, a problem suggested by Laskar and Pillone in 2000. We show $\psi_r(P_n) = \lfloor \log_2 (n+1) \rfloor + \left\lfloor \log_2 \left(n + 1 - \left(2^{\lfloor \log_2 n \rfloor - 1} \right) \right) \right\rfloor$.

1 Introduction

A labeling $f : V(G) \rightarrow \{1, 2, ..., k\}$ is a *k-ranking* of a graph G if and only if $f(u) = f(v)$ implies that every $u - v$ path contains a vertex w such that $f(w) > f(u)$. A k-ranking f is *minimal* if for all $v_i \in V(G)$, a function g satisfying $g(v) = f(v)$ when $v \neq v_i$ and $g(v_i) < f(v_i)$, is not a ranking. That is, if any label in a minimal ranking is replaced with a smaller label the new labeling is not a ranking. Note that for any ranking f there exists a minimal ranking h such that $h(v) \leq f(v)$ for every $v \in V(G)$. The rank number of a graph denoted $\chi_r(G)$, is defined to be the smallest k such that G has a minimal k-ranking, and the arank number of a graph denoted $\psi_r(G)$ is defined to be the largest k such that G has a minimal k-ranking. When the value of k is unimportant, we will refer to a k-ranking as simply a ranking.

The rank number of a graph has been well studied, partially due to its applications to VLSI (Very Large Scale Integration) Layouts and scheduling problems for manufacturing systems [1], [5], [8]. While the rank number has been determined for various families of graphs, the arank number is only known for a few classes of graphs, such as stars and split graphs. An important property of the arank number is that it implies a necessary condition for a given ranking to be minimal. That is, if a ranking contains a label greater than $\psi_r(G)$ it cannot be a minimal ranking.

The problem of determining the arank number of a path was suggested by Laskar and Pillone [7]. In Theorem 13 we provide a complete solution to this problem. In addition, we provide a general result involving necessary conditions for a ranking of a path to be minimal. In Theorem 7 we prove that more than half of the vertices in a minimal ranking of P_n must be labeled 1 or 2.

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2 Background

We will use P_n to denote the Hamiltonian path $v_1, v_2, ..., v_n$ and $\langle f(v_1), f(v_2), ..., f(v_n)\rangle$ to explicitly describe the labels in a ranking f. For a given ranking let S_i represent the independent set of all vertices labeled i. Given a graph G and a set $S \subseteq V(G)$ the *reduction* of G is a graph G^*such that $V(G^*) = V(G) - S$ and for vertices u and v, $(u, v) \in E(G^*)$ if and only if there exists a $u - v$ path in G. Note that if G is a path, G^* is also a path. An example of a reduction is given in Figure 1.

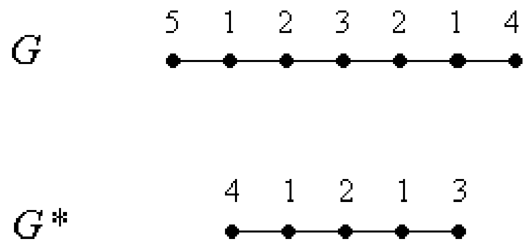

Figure 1: A reduction with $G = P_7$ and $S = S_1$.

For a ranking f of a graph G, $f^*_{|G^*}$ will represent the ranking of G^* where $f^*_{|G^*}(v) = f(v) - 1$ for all $v \in V(G)$ with $f(v) > 1$. For any other undefined notation, see the graph theory text by D. B. West [9].

We continue with a series of lemmas involving the frequency and locations of small labels that must appear in a minimal ranking. We restate the following two lemmas from [2].

Lemma 1 *Let G be a graph and f be a minimal ranking of G. If $x \in V(G)$ and $f(x) = 2$, then there exists a vertex u adjacent to x such that $f(u) = 1$.*

Lemma 2 *If x is a pendant vertex of a graph G and y is adjacent to x, then in any minimal ranking f of G, either $f(x) = 1$ or $f(y) = 1$.*

In the context of paths, this last lemma states that for any minimal ranking one of the first two vertices (or last two) must be labeled 1. If $n \geq 4$, we can use operation of reduction to show that one of the first four (or last four) vertices must be labelled 2. This is presented in our next lemma.

Lemma 3 *Let f be a minimum ranking of a path $P_n = v_1, v_2, ..., v_n$ with $n \geq 4$. Then $f(v_i) = 2$ for some $1 \leq i \leq 4$. Furthermore if $f(v_i) \neq 2$ for $1 \leq i \leq 3$, then $f(v_1) = f(v_3) = 1$.*

Proof. Assume the smallest i such that $f(v_i) = 2$ is greater than 4. Then at least two of the first four vertices in the path are labeled with integers greater than 2. It follows that in $f_{|P^*_n}$ an end vertex and its neighbor will both have labels greater than 1, contradicting Lemma 2. For the second part, assume $f(v_i) \neq 2$ for $1 \leq i \leq 3$ and $f(v_4) = 2$. Suppose that either $f(v_1) \neq 1$ or $f(v_3) \neq 1$. Then two of the vertices v_1, v_2 and v_3 will have labels greater than 2. Then again, the pedant vertex and its neighbor will be mapped to a value greater then 1 by $f_{|P^*_n}$, contradicting Lemma 2. ■

We next give a bound on the maximum size of a subpath with end vertices labeled w and all internal vertices labelled $z \neq w$.

Lemma 4 *If f is a minimal ranking of P_n then any subpath of order 2^{m+1} has a vertex v such that $f(v) = m$.*

Proof. The proof is by induction on m. The case where $m = 1$ was shown in [7]. The inductive step follows using reduction. ■

It is not difficult to show that if P' is an induced subpath of a path P, then $\psi_r(P') \leq \psi_r(P)$. We restate a lemma from [4] which shows that this monotonicity property holds in general.

Lemma 5 *Let H be an induced subgraph of graph G. Then $\psi_r(H) \leq \psi_r(G)$.*

Proof. An alternate proof is found in [4]. Let f be a minimal k-ranking of H. We construct a labeling of g where $g(v) = f(v)$ for all $v \in H$ and labeling all other vertices arbitrarily $k+1, k+2, ..., k+|V(G)|-|V(H)|$. To see that g is a ranking note that if two vertices in G have identical labels then both vertices must be in H, and use the fact that f is a ranking. Although g may not be a minimal ranking, no label of a vertex in H may can be replaced with a smaller label since f is a minimal ranking. Replacing labels in $V(G) - V(H)$ with smaller lables, if needed, will result in a minimal ranking of G that uses at least k labels. ■

We conclude this section by restating a lemma from [2] that will play a central role later in our proof of Theorem 7.

Lemma 6 *Let G be a graph and let f be a minimal ψ_r-ranking of G. If $S_1 = \{x : f(x) = 1\}$ then $\psi_r\left(G^*_{S_1}\right) = \psi_r\left(G\right) - 1$.*

3 Minimal k-rankings of paths

In our last section we noted many necessary conditions for a given ranking of a path to be minimal in lemmas , 2, 3, 4, and 6 All of these lemmas involve the proximity of vertices labeled 1 or 2 in a minimal ranking. This leads to our main result, which states that in any minimal ranking of a path, more than half of the vertices must be labeled 1 or 2.

Theorem 7 *If f is a minimal ranking of P_n then $|S_1 \cup S_2| > \frac{n}{2}$.*

Proof. Let $V(P_n) = v_1, v_2, ..., v_n$. The vertices in S_2 partition P_n into parts $F_1, F_2, ..., F_M$ where each $x \in S_2$ is the last vertex in some part F_i, $1 \leq i \leq M-1$ and F_M consists of the remaining vertices. We illustrate this in Figure 2.

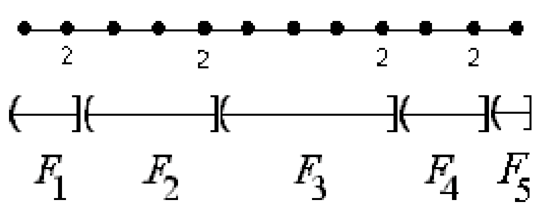

Figure 2. Partitioning of P_{12}.

We note that by Lemma 3, $|V(F_1)| \leq 4$ and by Lemma 4 $|V(F_i)| \leq 8$ for all $i = 2, 3, ..., M$. Our strategy will be as follows: we will prove that $|F_1 \cap (S_1 \cup S_2)| > \frac{|V(F_1)|}{2}$ and $|F_i \cap (S_1 \cup S_2)| \geq \frac{|V(F_i)|}{2}$ for all $i = 2, 3, ..., M$. Combining these inequalities will yield $|V(P_n) \cap (S_1 \cup S_2)| = |S_1 \cup S_2| > \frac{n}{2}$.

First we establish the inequality $|F_1 \cap (S_1 \cup S_2)| > \frac{|V(F_1)|}{2}$. By Lemma 3 the first 2 must appear somewhere among the first four vertices. We consider four cases and show the inequality holds in each one.

- $(f(v_1) = 2)$ Then $F_1 = v_1$ and $|V(F_1) \cap (S_1 \cup S_2)| > \frac{|V(F_1)|}{2}$.
- $(f(v_2) = 2)$ By Lemma 2 $f(v_1) = 1$ and $|V(F_1) \cap (S_1 \cup S_2)| > 1 = \frac{|V(F_1)|}{2}$.
- $(f(v_3) = 2)$ By Lemma 2, either $f(v_1) = 1$ or $f(v_2) = 1$. Hence $|V(F_1) \cap (S_1 \cup S_2)| > \frac{|V(F_1)|}{2}$.
- $(f(v_4) = 2)$ By Lemma 3, $f(v_1) = 1$ and $f(v_3) = 1$. Hence $|V(F_1) \cap (S_1 \cup S_2)| > \frac{|V(F_1)|}{2}$.

We use a similar argument for F_M to show $|V(F_M) \cap (S_1 \cup S_2)| \geq \frac{|V(F_M)|}{2}$. Next we show $|V(F_i) \cap (S_1 \cup S_2)| \geq \frac{|V(F_i)|}{2}$ for all $i = 2, 3, ..., M-1$. Consider F_i for some i, $2 \leq i \leq M$. Let $v_{i,1}, v_{i,2}, ..., v_{i,|V(F_i)|}$ be the vertices of F_i keeping the same ordering as in P_n. The inequality is clear when $|V(F_i)| = 2$. By Lemma 4, $|V(F_i)| \leq 8$. We consider cases for the various possible lengths of F_i. For completeness we include the details.

- $6 \leq |V(F_i)| \leq 8$. If $|F_i \cap S_1| < |V(F_i)| - 4$ then F_i contains at least four vertices with labels higher than 2. Then $f^*_{|P^*_n}$ contains labels for four consective vertices that are all greater than 1. By Lemma 4 $f^*_{|P^*_n}$ can not be a minimal ranking, a contradiction. Hence $|V(F_i) \cap S_1| \geq |V(F_i)| - 4$ and $|V(F_i) \cap (S_1 \cup S_2)| \geq |V(F_i)| - 3 \geq \frac{|V(F_i)|}{2}$.
- $|V(F_i)| = 5$. Bby Lemma 4 $|V(F_i) \cap S_1| \geq 1$ and the vertex labeled 1 can not be the first or fourth vertex of F_i. Assume, without loss of generality, the second vertex is labeled 1. We use a, b, and c to denote the first, third and fourth vertices of F_i respectively. If $f(c) > f(b)$, then $f(b)$ can be set to 2 and f still is a ranking; thus $f(c) < f(b)$, which implies $f(c)$ can only equal 1 if the ranking f is minimal. Hence$|V(F_i) \cap (S_1 \cup S_2)| \geq 3 \geq \frac{|V(F_i)|}{2}$.
- $|V(F_i)| = 3$ or 4. By Lemma 4, $|V(F_i) \cap S_1| \geq 1 \Rightarrow |V(F_i) \cap (S_1 \cup S_2)| \geq 2 \geq \frac{|V(F_i)|}{2}$.

■

In our next section we use this result to completely determine the arank number of a path.

4 The a-rank number of a path

The a-rank number of a path denoted $\psi_r(P_n)$ has been determined for small values of n [2]. These values are given in Table 1.

n	1	2	3	4	5	6	7	8	9	10	11
$\psi_r(P_n)$	1	2	3	4	4	4	5	5	5	5	6

Table 1: a-rank numbers for small paths

A recursive construction was given in [7] for creating a minimal $(2m-1)$-ranking of path with $2^m - 1$ vertices and a minimal $(2m-2)$-ranking of path with $2^m - 2^{m-2} - 1$ vertices. The same construction was used for both familes of paths and it was conjectured that the rankings produced by this construction were ψ_r-rankings.

The case $m = 1$ is trivial and when $m = 3$, a minimal 3-ranking of a P_3 can be constructed simply by labeling the vertices $\langle 3, 1, 2 \rangle$. Starting with a k-ranking of a path on w vertices, first delete the two end vertices. We next join two copies of the resulting path with a P_3 with labels, $\langle k-1, k, k-1 \rangle$. Finally add one vertex to each end of the path and label one of these vertices $k+1$ and the other $k+2$. An example showing the construction of a minimal 6-ranking of P_{11} is shown in Figure 2.

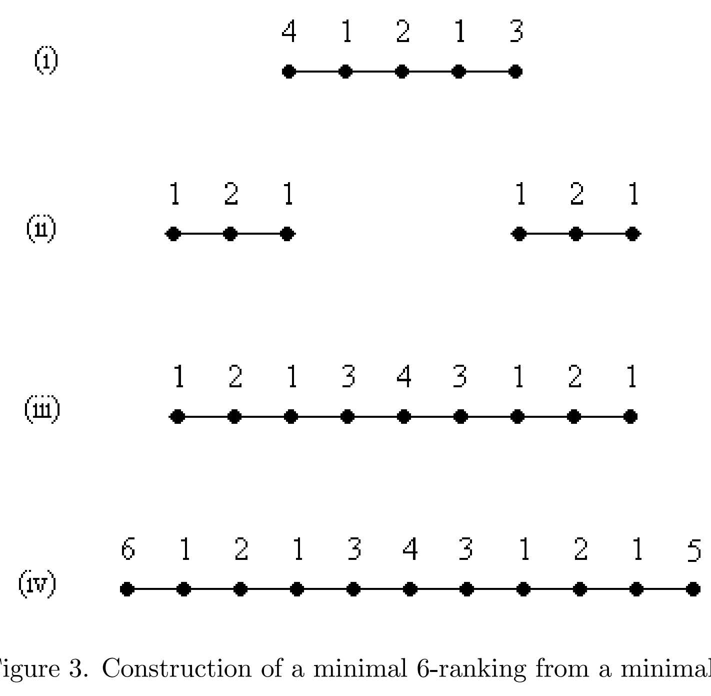

Figure 3. Construction of a minimal 6-ranking from a minimal 4-ranking.

A direct application of Lemma 6 can be used to show that the rankings produced by the construction are in fact ψ_r-rankings. We prove this in the following two lemmas.

Lemma 8 $\psi_r(P_{2^m-1}) = 2m-1$ *for all integers* $m \geq 2$.

Proof. We proceed by induction on m. As seen in Table 1, $\psi_r(P_{2^2-1}) = 2(2)-1 = 3$.
Assume the equality holds for m. Given a path on $2^{m+1}-1$ vertices, using the construction from Laskar and Pillone we can produce a $(2m+1)$-ranking. Hence $\psi_r(P_{2^{m+1}-1}) \geq 2m+1$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-1}) \geq 2m+2$. Then there exists a minimal $2m+2$-ranking for $P_{2^{m+1}-1}$, in which case reducing $P_{2^{m+1}-1}$ twice produces a path P with a $(2m)$-ranking. By Theorem 7, P must have less than 2^m-1 vertices. Then Lemma 5 implies $\psi_r(P_{2^m-1}) \geq 2m$ which contradicts our assumption. ■

Lemma 9 $\psi_r(P_{2^m-2^{m-2}-1}) = 2m-2$ *for all integers* $m \geq 2$.

Proof. We proceed by induction on m. As seen in Table 1, $\psi_r(P_{2^4-2^2-1}) = \psi_r(P_{11}) = 6 = 2(4)-2$. Assume the equality holds for m. Given a path on $2^{m+1}-2^{m-1}-1$ vertices, we can construct a $2m$-ranking. Hence $\psi_r(P_{2^{m+1}-2^{m-1}-1}) \geq 2m$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-2^{m-1}-1}) \geq 2m+1$. Then there exists a minimal $2m+1$-ranking for $P_{2^{m+1}-2^{m-1}-1}$. Reducing $P_{2^{m+1}-2^{m-1}-1}$ twice produces a path P with a $(2m-1)$-ranking. By Theorem 7, P must have less than or equal to $2^m-2^{m-2}-1$ vertices. Application of Lemma 5, yields $\psi_r(P_{2^m-2^{m-2}-1}) \geq 2m-1$, which contradicts our assumption. ■

Lemma 10 $\psi_r(P_{2^m-2^{m-2}-2}) = 2m-3$ *for all integers* $m \geq 2$.

Proof. We proceed by induction on m. As seen in Table 1, $\psi_r(P_{2^4-2^2-2}) = \psi_r(P_{10}) = 5 = 2(4)-3$. Assume the equality holds for m. Given a path on $2^{m+1}-2^{m-1}-2$ vertices, we can construct a $(2(m+1)-3)$-ranking. Hence $\psi_r(P_{2^{m+1}-2^{m-1}-2}) \geq 2m-1$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-2^{m-1}-1}) \geq 2m$. Then there exists a minimal $2m$-ranking for $P_{2^{m+1}-2^{m-1}-2}$. Reducing $P_{2^{m+1}-2^{m-1}-2}$ twice produces a path P with a $(2m-2)$-ranking. By Theorem 7, P must have less than or equal to $2^m-2^{m-2}-2$ vertices. Then by Lemma 5 we have $\psi_r(P_{2^m-2^{m-2}-2}) \geq 2m-2$, a contradiction. ■

Lemma 11 $\psi_r(P_{2^m-2}) = 2m-2$ *for all integers* $m \geq 2$.

Proof. We proceed by induction on m. As seen in Table 1, $\psi_r(P_{2^2-2}) = 2(2)-2 = 2$.
Assume the equality holds for m. Given a path on $2^{m+1}-2$ vertices, using the construction from Laskar and Pillone we can produce a $2m$-ranking. Hence $\psi_r(P_{2^{m+1}-2}) \geq 2m$. To show the reverse inequality, we assume that $\psi_r(P_{2^{m+1}-2}) \geq 2m+1$. Then there exists a minimal $(2m+1)$-ranking for $P_{2^{m+1}-2}$, in which case reducing $P_{2^{m+1}-2}$ twice produces a path P with a minimal $(2m)$-ranking. By Theorem 7, P must have less than or equal to 2^m-2 vertices. Application of Lemma 5 $\psi_r(P_{2^m-2}) \geq 2m$, a contradiction. ■

As mentioned Laskar and Pillone established an upperbound for the arank number of a path. In our next theorem we combine the above four lemmas with Lemma 5 to show that their upper bounds from [7] are in fact tight.

Theorem 12 (*arank* **number of** P_n)

(i) $\psi_r(P_s) = 2m - 2$ for all integers s, $2^m - 2^{m-2} - 1 \leq s \leq 2^m - 2$.

(ii) $\psi_r(P_t) = 2m - 1$ for all integers t, $2^m - 1 \leq t \leq 2^{m+1} - 2^{m-1} - 2$.

Following algebraic manipulation, the above theorem can be restated as follows to give an explicit formula for the arank number of a path.

Theorem 13 *Let P_n denote on a path on n vertices. Then* $\psi_r(P_n) = \lfloor \log_2 (n+1) \rfloor + \left\lfloor \log_2 \left(n + 1 - \left(2^{\lfloor \log_2 n \rfloor - 1}\right)\right) \right\rfloor$.

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