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Nathan Cahill

John D'Errico

Darren Narayan

Recommended Citation
The College Mathematics Journal 33N3 (2002) 221-225

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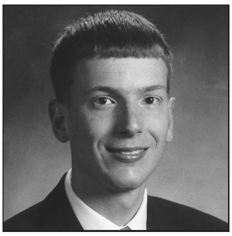
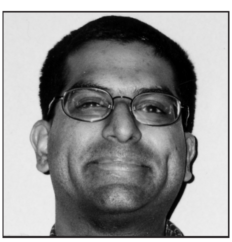
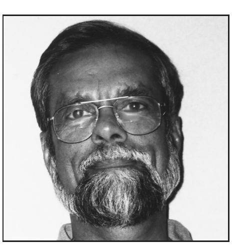


Nathan D. Cahill (cahill@kodak.com) is a Senior Research Scientist in the Imaging Science and Technology Laboratory of the Eastman Kodak Company research labs, specializing in 3-D imaging and computer vision. His mathematical interests include linear algebra and optimization theory.

John R. D'Errico is a Research Associate in the Color Science and Engineering Laboratory of the Eastman Kodak Company research labs, specializing in numerical analysis and mathematical modeling.



Darren A. Narayan (dansma@rit.edu) is an Assistant Professor of Mathematics at the Rochester Institute of Technology. His research interests include combinatorics and graph theory, with a particular interest in digraphs and tournaments. He is a 2000–01 Project NexT Fellow and has given talks at many universities and conferences nationwide.



Jack Y. Narayan is currently Dean of Graduate Studies and Research and a Distinguished Teaching Professor of Mathematics at the State University of New York, College at Oswego. He started teaching high school in Guyana in 1959 and has also held teaching positions at Lehigh, Syracuse, and the University of Arizona. Both he and his wife, Marion are graduates of Mount Allison University in New Brunswick, Canada and have both enjoyed careers in teaching. They have three sons, Dwayne, Drew and the co-author above.

We present yet another example where the Fibonacci sequence $\{f_n\} = \{1, 1, 2, 3, \ldots\}$ surprisingly appears. Let F_n be an $n \times n$ matrix where the terms on the main diagonal are 1, and the terms on the superdiagonal and subdiagonal are $i = \sqrt{-1}$, so,

$$F_n = \begin{pmatrix} 1 & i & 0 & 0 & \cdots & 0 \\ i & 1 & i & 0 & \cdots & 0 \\ 0 & i & 1 & i & \cdots & 0 \\ 0 & 0 & i & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & i \\ 0 & 0 & 0 & \cdots & i & 1 \end{pmatrix}.$$

This matrix is also presented in [**2**]. Calculation that $\det F_1 = 1$, $\det F_2 = 2$, $\det F_3 = 3$, $\det F_4 = 5$ and $\det F_5 = 8$. This progression looks familiar, and we will verify in the next section that in fact, $\det F_n = f_{n+1}$.

A *lower Hessenberg matrix*, A, is an $n \times n$ matrix where $a_{j,k} = 0$ whenever $k > j + 1$ and $a_{j,j+1} \neq 0$ for some j. That is, all entries above the superdiagonal are 0 but the matrix is not lower triangular. Throughout this paper we will refer to the following lower Hessenberg matrix

$$M_n = \begin{pmatrix} m_{1,1} & m_{1,2} & 0 & \cdots & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & \ddots & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & m_{n-1,n} \\ m_{n,1} & m_{n,2} & \cdots & m_{n,n-1} & m_{n,n} \end{pmatrix}.$$

We will consider the sequence $\{\det M_n, n \geq 1\}$. Our main result is stated in the following theorem.

Theorem. Let M_n be as above for all $n \geq 1$ and define $\det M_0 = 1$. Then $\{\det M_n, n \geq 0\}$ satisfies:

$$\det M_0 = 1, \quad \det M_1 = m_{1,1} \quad \text{and}$$
$$\det M_n = m_{n,n} \cdot \det M_{n-1} + \sum_{r=1}^{n-1} \left((-1)^{n-r} m_{n,r} \prod_{j=r}^{n-1} m_{j,j+1} \cdot \det M_{r-1} \right), \; n \geq 2.$$

We can obtain the family of matrices $\{F_n\}$ by letting $m_{j,j} = 1$ and $m_{j,j+1} = m_{j+1,j} = i = \sqrt{-1}$, $1 \leq j \leq n - 1$ in the above theorem. Then $\det M_1 = 1$, $\det M_2 = 2$ and $\det M_n = \det M_{n-1} + \det M_{n-2}$, which is exactly the Fibonacci recurrence.

Our main interest will not be in the general theorem, but rather in some very interesting special cases such as $\{F_n\}$. We will provide additional examples where the sequence of matrix determinants contains a pattern of recurrence present in subsequences of the Fibonacci sequence which serve as a source of entertaining exercises for students in a matrix algebra or discrete mathematics class. We will prove the general theorem later, but will leave the individual verification of the special cases to the reader.

Familiar recurrences

In this section we will consider Hessenberg and tridiagonal matrices obtained by restricting the entries of M_n. By choosing values for the entries of M_n we can obtain matrices where the sequence $\{\det M_n, n \geq 1\}$ follows a pattern of recurrence found in subsequences of the Fibonacci sequence.

Example 1. *Let $n \geq 1$ and let B_n be an $n \times n$ matrix where $b_{1,1} = 2$, all other terms on the main diagonal are* 2*, the terms on the superdiagonal are* -1 *and all terms below the main diagonal are* 1*, so*

$$B_n = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 2 & -1 & \ddots & 0 \\ 1 & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}.$$

Let C_n be an $n \times n$ matrix where all of the terms on the main diagonal are 2, the terms on the superdiagonal are -1 and all terms below the main diagonal are 1, so

$$C_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ 1 & 2 & -1 & \ddots & 0 \\ 1 & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}.$$

We inspect the first few determinants.

$$\det B_1 = \det(1) = 1 \qquad \det C_1 = \det(2) = 2$$

$$\det B_2 = \det\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = 3 \qquad \det C_2 = \det\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = 5$$

$$\det B_3 = \det\begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix} = 8 \qquad \det C_3 = \det\begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{pmatrix} = 13$$

The sequences of determinants appear to be every other term of the Fibonacci sequence. Indeed, $\det B_n = f_{2n}$ and $\det C_n = f_{2n+1}$, for all $n \geq 1$. A short proof verifies this fact for both families of matrices. The key is to observe the following relationship between the two families: for all $n \geq 1$, C_n is the $(1, 1)$-cofactor of B_{n+1}, and B_n is the $(1, 2)$-cofactor of C_{n+1}.

Next we modify C_n by replacing the -1's on the superdiagonal with 1's to create a new matrix D_n.

Example 2. *Let $n \geq 1$ and D_n be an $n \times n$ matrix where all of the terms on the main diagonal are 2, the terms on the superdiagonal are 1 and all terms below the main diagonal are 2, so*

$$D_n = \begin{pmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & 0 \\ 1 & 1 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}.$$

The determinants of the first few matrices are $\det D_1 = 2$, $\det D_2 = 3$ and $\det D_3 = 5$. It turns out that this sequence is precisely $\{f_n\}$ starting at $n = 3$.

Tridiagonal matrices. Recall a matrix is called *tridiagonal* if all of its non-zero entries appear only on the main diagonal, superdiagonal or subdiagonal, so every tridiagonal matrix is a Hessenberg matrix. We will investigate connections between the determinants of tridiagonal matrices and familiar sequences.

Example 3. *Let $n \geq 1$ and E_n be an $n \times n$ matrix where all of the terms on the main diagonal are* 3*, the terms on the superdiagonal and subdiagonal are* -1*, so*

$$E_n = \begin{pmatrix} 3 & -1 & 0 & \cdots & 0 \\ -1 & 3 & -1 & \ddots & 0 \\ 0 & -1 & 3 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ 0 & 0 & \cdots & -1 & 3 \end{pmatrix}.$$

This family of matrices and its correspondence to the Fibonacci sequence is noted in [**1**]. The first three determinants are $\det E_1 = 3$, $\det E_2 = 8$ and $\det E_3 = 21$. A short induction argument verifies that $\det(E_n) = f_{2n+2}$ for all $n \geq 1$.

A sequence intimately related to the Fibonacci sequence is the Lucas sequence $\{l_n\} = \{1, 3, 4, 7, \ldots\}$. With a slight change we obtain a family of matrices whose determinants are numbers that appear in the Lucas sequence. We present this in our final example.

Example 4. *Let $n \geq 1$ and let L_n be an $n \times n$ matrix where $l_{2,2} = 2$, all other the terms on the main diagonal are* 1 *and the terms on the superdiagonal and subdiagonal are $i = \sqrt{-1}$, so*

$$L_n = \begin{pmatrix} 1 & i & 0 & \cdots & 0 \\ i & 2 & i & \ddots & 0 \\ 0 & i & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & i \\ 0 & 0 & \cdots & i & 1 \end{pmatrix}.$$

Calculation of this sequence of determinants yields $\det L_1 = 1$, $\det L_2 = 3$, $\det L_n = \det L_{n-1} + \det L_{n-2}$ for all $n \geq 3$. This, of course, is the Fibonacci recurrence starting with 1 and 3, which generates the Lucas sequence.

Proof of the general theorem. We conclude with the proof of our theorem which will follow by induction on n.

Proof. We first prove the base cases, $n = 2$. $\det M_2 = m_{2,2}m_{1,1} - m_{2,1}m_{1,2}$. Next, we assume the theorem is true for $n = k$ and show it is true for $n = k + 1$. Then

$$\begin{aligned} &\det M_{k+1} \\ &\quad = m_{k+1,k+1} \cdot \det M_k \\ &\qquad - \left(m_{k,k+1}m_{k+1,k} \cdot \det M_{k-1} + \sum_{r=1}^{k-1} \left((-1)^{k-r} m_{k+1,r} \prod_{j=r}^{k-1} m_{j,j+1} \cdot \det M_{r-1} \right) \right) \end{aligned}$$

$$
\begin{aligned}
&= m_{k+1,k+1} \cdot \det M_k - m_{k,k+1} m_{k+1,k} \cdot \det M_{k-1} \\
&\quad + m_{k,k+1} \sum_{r=1}^{k-1} \left((-1)^{k-r+1} m_{k+1,r} \prod_{j=r}^{k-1} m_{j,j+1} \cdot \det M_{r-1} \right) \\
&= m_{k+1,k+1} \cdot \det M_k - m_{k,k+1} m_{k+1,k} \cdot \det M_{k-1} \\
&\quad + \sum_{r=1}^{k-1} \left((-1)^{k-r+1} m_{k+1,r} \prod_{j=r}^{k} m_{j,j+1} \cdot \det M_{r-1} \right) \\
&= m_{k+1,k+1} \cdot \det M_k + \sum_{r=1}^{k} \left((-1)^{k-r+1} m_{k+1,r} \prod_{j=r}^{k} m_{j,j+1} \cdot \det M_{r-1} \right)
\end{aligned}
$$

as desired.

Acknowledgement. Research was partially supported by a grant from the Reidler Foundation.

References

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2. N. D. Cahill, J. R. D'Errico, J. Spence, Complex factorizations of the Fibonacci and Lucas numbers, to appear in the *Fibonacci Quarterly*.

What is This?

Marvin Johnson (eng511@exglk.clc.cc.il.us), of the College of Lake County, sends an item in the tradition of $\int \frac{d(\text{cabin})}{\text{cabin}} = \text{houseboat}$. He asks what, in other words, is $f(g(\text{hung}))$? The answer appears on page 229.