2002

Tournaments with a Transitive Subtournament as a Feedback Arc Set

Jennifer Baldwin
Rochester Institute of Technology

William Kronholm
Rochester Institute of Technology

Darren Narayan
Rochester Institute of Technology

Recommended Citation
J. Baldwin, W. Kronholm, and D. Narayan. Tournaments with a transitive subtournament as a feedback arc set. Congressus Numerantium 158 (2002), 51-58.

Tournaments with a transitive subtournament as a feedback arc set

Jennifer L. Baldwin* William C. Kronholm* Darren A. Narayan†

Department of Mathematics and Statistics
Rochester Institute of Technology, Rochester, NY 14623

May 4, 2002

Abstract

Given an acyclic digraph D, we seek a smallest sized tournament T that has D as a minimum feedback arc set. The reversing number of a digraph is defined to be $r(D) = |V(T)| - |V(D)|$. The case where D is a tournament T_n was studied by Isaak in 1995 using an integer linear programming formulation. In particular, this approach was used to produce lower bounds for $r(T_n)$, and it was conjectured that the given bounds were tight. We examine the class of tournaments where $n = 2^k + 2^{k-2}$ and show the known lower bounds for $r(T_n)$ are best possible.

AMS Classification: Primary: 05C20, Secondary: 90C47

1 Introduction

A *minimum feedback arc set* of a digraph is a smallest sized set of arcs whose reversal makes the resulting digraph acyclic. We consider the following problem posed by J. P. Barthélemy [2]: Given an acyclic digraph D determine the size of a smallest tournament T, having $A(D)$ as a minimum feedback arc set. The *reversing number of a digraph*, was defined to be $r(D) = |V(T)| - |V(D)|$. Reversing numbers have been well studied and

*Travel to the 33rd CGTC Conference supported by *JetBlue Airways* and a gift from Tony and Kay Carlisi.

†Partially supported by a 2001 RIT COS Dean's Summer Fellowship Grant.

calculated for many families of digraphs [2], [4], [5], [6], [7], and [8]. We focus on the case where D is an acyclic tournament T_n.

There is a close connection between minimum feedback arc sets and player rankings. Let the vertices of a tournament, T correspond to players in round-robin tournament and $(x, y) \in A(T)$ if and only if the player corresponding to vertex x beats the player corresponding to vertex y. A ranking of these players has an *inconsistency* whenever one player defeats another and the loser is ranked above the winner. The determination of a minimum feedback arc set is then equivalent to finding a ranking that minimizes the number of inconsistencies.

We will consider a ranking to be optimal if it contains a minimum number of inconsistencies. Then $r(T_n)$ equals the fewest number of extra players needed to create a tournament having an optimal ranking with n players that are all ranked wrong with respect to each other.

We continue by stating a well known elementary result involving minimum feedback arc sets.

Proposition 1 *Let $A(D)$ be a feedback arc set of T. If T contains a collection of $|A(D)|$ arc disjoint cycles then $A(D)$ is a minimum feedback arc set of T.*

It is easy to see that this proposition is true. If T contains a set of $|A(D)|$ arc disjoint cycles, then any feedback arc set of T must include at least one arc from each of these cycles. We note that in general, the converse is not true and has been well studied, starting in 1965 [3]. It was conjectured by Isaak that the converse is true if D is an acyclic tournament [1] and [4].

We restate a basic known result as our first lemma [2].

Lemma 2 *If D and D' are digraphs on the same number of vertices then $A(D') \subseteq A(D) \Rightarrow r(D') \leq r(D)$.*

An integer programming formulation was used to produce lower bounds for $r(T_n)$ [4]. We restate one of these results as our next lemma.

Lemma 3 *Let T_n denote the acyclic tournament on n vertices. Then $r(T_n) \geq 2n - 2 - \lfloor \log_2 n \rfloor$.*

In fact, a stronger result is known [4]. It was established that $r(T_n) \geq 2n - 2 - \lfloor \log_2 n \rfloor$ or $r(T_n) \geq 2n - 1 - \lfloor \log_2 n \rfloor$ depending on the binary expansion of n. However we will only need the weaker of these two bounds.

Upper bounds for $r(T_n)$ were obtained by Isaak [3] by explicitly describing a collection of $\binom{n}{2}$ arc disjoint cycles. It was shown that the lower bound was tight for $n = 2^k - 2^t$ and it was conjectured that the given lower bounds are tight in general. In this paper we present new results involving the reversing number of a tournament. We investigate $r(T_n)$ when $n = 2^k + 2^{k-2}$ and present upper bounds that match known lower bounds [4]. Furthermore, our methods used to obtain new upper bounds for $r(T_n)$ are not dependent upon the existence of a collection of $\binom{n}{2}$ arc disjoint cycles.

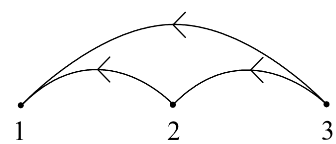
Our main result is presented in the following theorem.

Theorem 4 *If $n = 2^k + 2^{k-2}$ then $r(T_n) = 2n - 2 - \lfloor \log_2 n \rfloor$.*

2 Preliminaries

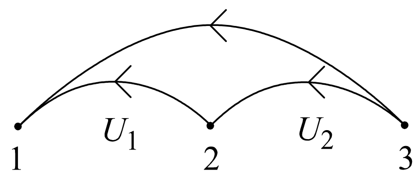
We begin with an example illustrating some of the key ideas that will be used throughout the paper. We will show $r(T_3) = 3$, that is, the smallest sized tournament having an optimal ranking with three players all of which are ranked wrong with respect to each other, has six players.

Example 5 *Let T_3 denote the acyclic tournament on three players, shown in Figure 1. By Lemma 2, extra players may be added so that no additional inconsistencies will be created.*



T_3

It follows that any extra player must fit in between the ranks of the original players. We use U_1 to represent extra players that lost to player 1, but defeated players 2 and 3, and U_2 represents extra players that lost to players 1 and 2, but defeated player 3. Let $x_i = |U_i|$. Then minimizing $x_1 + x_2$ over all optimal rankings yields $r(T_3)$. Any optimal ranking must have the form shown in Figure 2 (up to the sizes of x_1 and x_2) and hence any ranking has at least three inconsistencies.



All arcs not shown are directed from left to right.

Note that an optimal ranking must have at least one extra player ranked in between any of the two original players otherwise we could simply interchange the ranks of the two original players and reduce the number of inconsistencies. Hence, $x_1 \geq 1$ *and* $x_2 \geq 1$.

We also consider the ranking: $U_1 321 U_2$. *Here each player in* U_1 *is out of order with player* 1 *and each player in* U_2 *is out of order with player* 3. *This yields a total of* $x_1 + x_2$ *inconsistencies, and since each ranking has at least* 3 *inconsistencies, it follows that* $x_1 + x_2 \geq 3$.

To show the upper bound we give an example of a tournament T *on six vertices that has* $A(T_3)$ *as a feedback arc set and also contains a set of three arc disjoint cycles. Let* $u_{1,0}$ *and* $u_{1,1}$ *be extra vertices placed between 1 and 2, and let* $u_{2,0}$ *be the extra vertex placed between 2 and 3. Then* T *contains the collection of arc disjoint cycles,* $\{(1, u_{1,0}, 2), (1, u_{1,1}, 3), (2, u_{2,0}, 3)\}$.

Hence $r(T_3) = 3$.

2.1 The general case

Let $T(\sigma, \overrightarrow{x}, T_n)$ denote a tournament having the acyclic tournament T_n as a feedback arc set. Any tournament $T(\sigma, \overrightarrow{x}, T_n)$ having T_n as a feedback arc set must have this form for some set of extra vertices, $\overrightarrow{x}$. Let $V(T_n) = \{1, 2, ..., n\}$ and $A(T_n) = \{(j, i) \; | i < j\}$. Then $V(T(\sigma, \{x_i\}, T_n)) = V(T_n) \cup \{u_{i,j} \mid 1 \leq i \leq n-1, 1 \leq j \leq x_i\}$ and $A(T(\sigma, \overrightarrow{x}, T_n)) = A(T_n) \cup \{(u_{i,j}, u_{s,t}) : i < s \text{ or } i = s \text{ and } j < t\} \cup \{(i, u_{s,t}) \mid i \leq s\} \cup \{(u_{i,j}, s) \mid i < s\}$. That is, $V(T(\sigma, \overrightarrow{x}, T_n)) = V(T_n)$ along with a set of extra vertices dependent upon T_n, and the arc set consists of those arcs consistent with the ordering:

$$1, u_{1,1}, ..., u_{1,x_1}, 2, u_{2,1}, ..., u_{2,x_2}, 3, ..., n-1, u_{n-1,1}, ..., u_{n-1,x_{n-1}}, n$$

except for arcs between vertices i and j where $i < j$, which are inconsistent with the ordering. Since we are only concerned with the number of extra vertices between each i and $i+1$ we will simplify matters by grouping the x_i vertices into a single set, U_i.

Given T_n we investigate inequalities involving the number of extra vertices specified by $\overrightarrow{x}$. We note $r(T_n)$ equals the minimum $\sum_{i=1}^{n-1} x_i$ such that a tournament on $n + \sum_{i=1}^{n-1} x_i$ vertices has $A(T_n)$ as a minimum feedback arc set. These properties were formulated as an integer linear program [3] and [5] as shown in our next theorem.

Theorem 6 *Let T_n be the acyclic tournament on n vertices.*

Then $r(T_n)$ equals the objective value for an optimal solution to the integer linear program, $ILP(n)$:

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{n-1} x_i \\ \text{subject to} & \sum_{i=1}^{(h-j)/2} i(x_{j+i-1} + x_{h-i}) \geq \binom{n}{2} \text{ for } h-j \text{ even} \\ & \sum_{i=1}^{(h-j-1)/2} i(x_{j+i-1} + x_{h-i}) \geq \binom{n}{2} \text{ for } h-j \text{ odd} \\ & \text{where the } \sum \text{ term is 0 if } h-j=1. \\ & x_i \geq 0 \text{ and integer} \end{array}$$

The constraints of $ILP(n)$ place conditions on the extra vertices. These conditions were shown to be necessary in and used to generate lower bounds for $r(T_n)$ [4]. Upper bounds were obtained by explicitly describing a collection of $\binom{n}{2}$ arc disjoint cycles contained in the host tournament T. Later it was shown that the necessary conditions involving the extra vertices were in fact sufficient [6]. Using this stronger result $r(T_n)$ can be obtained by either by solving $ILP(n)$, or by finding a feasible $\overrightarrow{x}$ where $\sum_{i=1}^{n-1} x_i$ matches a known lower bound.

As part of a larger result, $r(T_n)$ was determined for all $n = 2^k$ [4]. Furthermore an explicit pattern for $\overrightarrow{x}$ was given. These results are restated in the following definition and theorem.

Definition 7 *Let* $\widehat{x}_i = 1 + \max\left\{j : 2^j \mid i\right\}$

Theorem 8 $(\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_{n-1})$ *is an optimal solution to $ILP(n)$ for $n = 2^k$.*

Example 9 $(\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_{15}) = (1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1)$ *is an optimal solution to $ILP(16)$, and $r(T_{16}) = \sum_{i=1}^{15} \widehat{x}_i = 26$.*

We note that $(\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_{2^k-1})$ is symmetric. In addition the first $2^{k-1} - 1$ entries are an optimal solution to $ILP(2^{k-1})$, and $\widehat{x}_{2^{k-1}} = k$.

3 The reversing number of a tournament

In this section we provide the proof for Theorem 4.

Theorem 10 *If* $n = 2^k + 2^{k-2}$ *then* $r(T_n) = 2n - 2 - \lfloor \log_2 n \rfloor$.

Proof. We will show $r\left(T_{2^k+2^{k-2}}\right) = 2^{k+1} + 2^{k-1} - k - 2$. By Theorem 3, $r\left(T_{2^k+2^{k-2}}\right) \geq 2^{k+1} + 2^{k-1} - k - 2$. To show the reverse inequality we construct a feasible solution $\overrightarrow{x} = (x_1, x_2, ..., x_{2^k+2^{k-2}-1})$ with minimum value $2^{k+1} + 2^{k-1} - k - 2$. To simplify matters, for a tournament T_n the set of constraints in $ILP(n)$ will be simply referred to as T_n*-inequalities.*

We start with an optimal solution to $ILP(2^k)$, $(\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_{2^k-1})$, and let $x_i = x_{2^k+2^{k-2}-i} = \widehat{x}_i$ for $i \leq 2^{k-1} + 2^{k-3}$. Each of the inequalities corresponding to the subtournaments on vertices $1, 2, ..., 2^k$ and $2^{k-2}, 2^{k-2} + 1, ..., 2^k + 2^{k-2}$ are immediately satisfied since $x_i \geq \widehat{x}_i$ and $x_{i+2^{k-2}} \geq \widehat{x}_i$ for $i \leq 2^k - 1$. The remaining cases involve inequalities containing both x_s and x_t where $s \leq 2^{k-2}$ and $t \geq 2^k$.

To help illustrate the general structure of $\overrightarrow{x}$ and $\overrightarrow{\widehat{x}}$, the case where $k = 5$ is shown in Table I. ■

$\overrightarrow{x}$	121312141213121512131215**5**12131214**4**1213121
$\overrightarrow{\widehat{x}}$	1213121412131215121312141213121**6**1213121

Table 1: Comparison of $\overrightarrow{x} = (x_1,\, x_2, ..., x_{39})$ and $\overrightarrow{\widehat{x}} = (\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_{39})$

Proof. Since $\overrightarrow{\widehat{x}}$ is part of a feasible solution for $ILP(2^{k+1})$, it must also a feasible solution for $ILP(2^k + 2^{k-2})$. An important fact is that $\overrightarrow{x}$ and $\overrightarrow{\widehat{x}}$ differ in only two places, $i = 2^k - 2^{k-2}$ and $i = 2^k$. To see this note that $\overrightarrow{x}$ was constructed using two copies of $(\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_{2^{k-1}-1})$ which are aligned so their starting positions are off set by a multiple of 8. Hence $\overrightarrow{x}$ and $\overrightarrow{\widehat{x}}$ will agree whenever they have a value of 3 or less, and also will agree for all $i \leq 2^{k-1} + 2^{k-3}$. Then $x_{2^k - 2^{k-2}} = k$ in $\overrightarrow{x}$ and $k - 1$ in $\overrightarrow{\widehat{x}}$, and $x_{2^k} = k - 1$ in $\overrightarrow{x}$ and $k + 1$ in $\overrightarrow{\widehat{x}}$.

Next, consider the constraints of $ILP(2^k + 2^{k-2})$ containing both x_s and x_t, where $s \leq 2^{k-2}$ and $t \geq 2^k$. Let a equal the coefficient of $x_{2^k - 2^{k-2}}$ and b equal the coefficient of x_{2^k}. In each inequality, $a \leq 2^{k-1}$, $b \leq 2^{k-2}$, and $a - b = 2^{k-2}$. To transform $\overrightarrow{\widehat{x}}$ to $\overrightarrow{x}$ we need to increase $x_{2^k - 2^{k-2}}$ from $k - 1$ to k, and decrease x_{2^k} from $k + 1$ to $k - 1$. Then the total change

in $\sum_{i=1}^{2^k+2^{k-2}-1} x_i$ is $a - 2b$. To show the inequalities still hold after the transformation we must verify $a - 2b \geq 0$. This follows since $a - 2b = a - 2(a - 2^{k-2}) = 2^{k-1} - a \geq 0$.

Hence $\overrightarrow{x} = (x_1, x_2, ..., x_{2^k+2^{k-2}-1})$ is a feasible solution for $ILP(2^k + 2^{k-2})$ and by Theorem 6, $r\left(T_{2^k+2^{k-2}}\right) \leq \sum_{i=1}^{2^k+2^{k-2}-1} x_i = 2^{k+1} + 2^{k-1} - k - 2$. This completes the proof. ■

4 Conclusion

As mentioned earlier, all of the upper bounds for $r(T_n)$ were established by explicitly describing an appropriately sized collection of arc disjoint cycles. Our construction is unique in that it is independent of whether or not such a collection exists. We restate the following conjecture given by Isaak [4].

Conjecture 11 *If T is a tournament with a minimum feedback set of arcs which form a (smaller) acyclic tournament then the maximum number of arc disjoint cycles in T equals the minimum size of a feedback arc set.*

We remark that although we have presented a new result involving the reversing number of a tournament the above conjecture still remains open.

Acknowledgements

The authors thank Garth Isaak and Stanislaw Radziszowski for valuable discussions and input.

E-mail addresses:

jellybean26@hotmail.com (J. L. Baldwin)
some1hitapossum@hotmail.com (W. C. Kronholm)
dansma@rit.edu (D. A. Narayan)

References

[1] J. Bang-Jensen and G. Gutin, **Digraphs: Theory, Algorithims and Applications**, (Springer-Verlag, London, 2001), 562.

[2] J. -P. Barthélemy, O. Hudry, G. Isaak, F. S. Roberts and B. Tesman, *The reversing number of a digraph*, Discrete Appl. Math, **60** (1995) 39-76.

[3] P. Erdös and J. W. Moon, *On sets of consistent arcs in a tournament*, Canad. Math Bull. **8** (1965), 269-271.

[4] G. Isaak, *Tournaments as Feedback Arc Sets*, Elec. J. Combin. **R20** (1995) 19 pp.

[5] G. Isaak and D. A. Narayan, *A complete classification of all tournaments having a disjoint union of directed paths as a minimum feedback arc set*, submitted.

[6] G. Isaak and D. A. Narayan, *An integer linear program formulation for minimum feedback arc sets of tournaments,* to be submitted.

[7] D. A. Narayan, *Powers of directed Hamiltonian paths as feedback arc sets*, submitted.

[8] D. A. Narayan, *The reversing number of a digraph; A disjoint union of directed stars*, Congr. Numer. (2000) **145**, 153-164.