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2009

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Counting Triangles in Some Ramsey Graphs

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Abstract

We extend Goodman's result on the cardinality of monochromatic triangles in a 2-colored complete graph to the case of bounding the number of triangles in the first color. We apply it to derive the upper bounds on some non-diagonal Ramsey numbers. In particular we show that $R(K_4 - e, K_8) \leq 45$.

AMS Subject Classification: 05C55

1 Introduction

For any two simple graphs H_1 and H_2, the Ramsey number $R(H_1, H_2)$ is the least positive integer n such that for every n-vertex graph G, either H_1 is a subgraph of G or H_2 is a subgraph of $\overline{G}$. A graph G such that neither H_1 is a subgraph of G nor H_2 a subgraph of $\overline{G}$ is said to be (H_1, H_2)-good. An (H_1, H_2)-good graph is (H_1, H_2)-critical if there exists no (H_1, H_2)-good graph with more vertices. The standard Ramsey case involving two complete graphs is written as $R(m, n) = R(K_m, K_n)$. Ramsey numbers can also be generalized to cases involving more than two colors and hypergraphs [Rad06]. Such cases are not covered here.

To this day, only nine non-trivial exact values of classical Ramsey numbers are known. [Rad06]. In practice, often a lower bound for $R(m, n)$ is found by producing a graph which contains no cliques of order m and no independent sets of order n, or by a proof demonstrating that the probability of the existence of such graphs is non-zero. An upper bound for $R(m, n)$ is established by proving that every graph on a given number of vertices contains a clique of order m or an independent set of order n.

2 Related Work

Goodman [Goo59] proved that the number of monochromatic triangles in a simple graph is entirely determined by the degrees of each vertex. He showed that for $G = (V, E)$,

$$\sum_{v \in V} \binom{d(v)}{3} + \binom{v - 1 - d(v)}{3} = (v - 3)T \tag{1}$$

where T is the cardinality of K_3 and $I_3(= \overline{K_3})$ subgraphs in G. Several generalizations have been proposed afterwards, including that of Giraud, who gave a lower bound for the cardinality of monochromatic K_4 [Gir73]. Mackey [Mac94] later demonstrated how to apply Giraud's method to some cases of diagonal Ramsey numbers. Most notably, Mackey demonstrated the new upper bounds of $165, 540, 1870, 6625, 23854$ for the Ramsey numbers $R(6, 6)$

through $R(10, 10)$, respectively.

Our main result is a further generalization of this idea to allow one to bound the number of monochromatic K_3 or I_3 independently of each other. We demonstrate how to improve the upper bounds of some off-diagonal Ramsey numbers using this result.

3 Identities Involving Vertex Degrees

The edges of G will be denoted by E and the non-edges $\overline{E}$. We let $g_{m,n}$ denote the number of m-vertex subgraphs of G which are isomorphic to the graph labeled $G_{m,n}$ in figure 1.

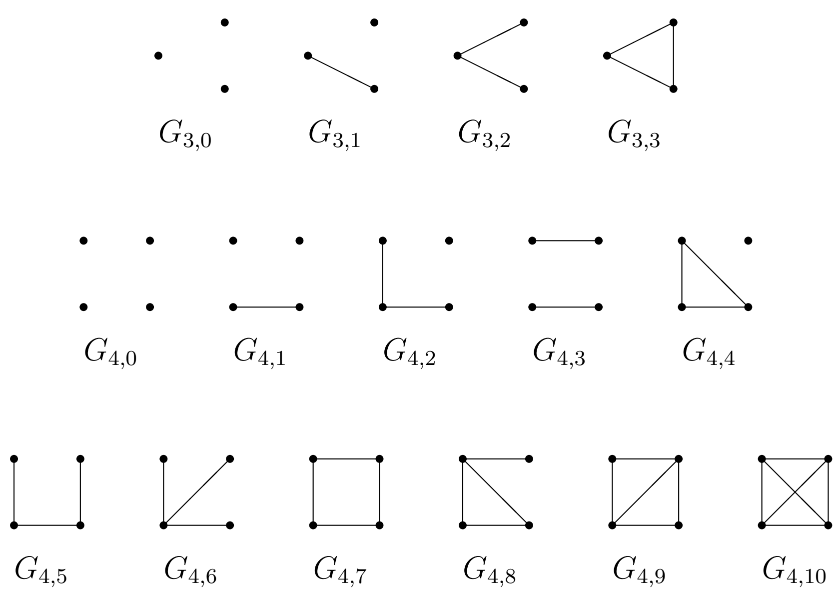

Figure 1. Labeling of graphs on 3 and 4 vertices.

For each edge or non-edge $e = \{a, b\}$ and $i = 0, 1$, or 2, we let $y_i(e)$ to be defined as the number of vertices not contained in e which are connected to exactly i members of e. We also let $y_{a,1}(e)$ be the number of vertices not contained in e which are connected to a and not connected to b, so $y_{a,1}(e) + y_{b,1}(e) = y_1(e)$. Although the latter definition depends on the labeling of e, there will be no ambiguity in expressions which is symmetric in $y_{a,1}(e)$ and $y_{b,1}(e)$.

In a simple graph, the sum of the degrees of each vertex is twice the number of edges. By simple subgraph counting, we can generate three identities involving quadratic sums of vertex degrees as follows.

$$\begin{aligned}
3g_{3,3} + g_{3,2} &= \sum_{v\in V} \binom{d(v)}{2} = \frac{1}{2}\sum_{v\in V} d(v)^2 - |E| \\
2g_{3,2} + 2g_{3,1} &= \sum_{v\in V} d(v)[|V| - 1 - d(v)] = -\sum_{v\in V} d(v)^2 + 2(|V| - 1)|E| \\
g_{3,1} + 3g_{3,0} &= \sum_{v\in V} \binom{|V| - 1 - d(v)}{2} \\
&= \frac{1}{2}\sum_{v\in V} d(v)^2 - (2|V| - 3)|E| + \frac{|V|(|V| - 1)(|V| - 2)}{2} \qquad (2)
\end{aligned}$$

The identities in in (2) allow one to write $g_{3,2}, g_{3,1}$, and $g_{3,0}$ in terms of four quantities: $|V|$, $|E|$, $\sum_{v\in V} d(v)^2$, and $g_{3,3}$, as follows.

$$\begin{aligned}
g_{3,2} &= -3g_{3,3} + \frac{1}{2}\sum_{v\in V} d(v)^2 - |E| \\
g_{3,1} &= 3g_{3,3} - \sum_{v\in V} d(v)^2 + |V||E| \\
g_{3,0} &= -g_{3,3} + \frac{1}{2}\sum_{v\in V} d(v)^2 - (|V| - 1)|E| + \frac{|V|(|V| - 1)(|V| - 2)}{6} \qquad (3)
\end{aligned}$$

Identity (4) can be verified by looking at subgraphs of four vertices.

$$
\begin{aligned}
& \sum_{e\in E}\left[\binom{y_{a,1}(e)}{2}+\binom{y_{b,1}(e)}{2}\right]+\sum_{\overline{e}\in\overline{E}}\left[\binom{y_{a,1}(\overline{e})}{2}+\binom{y_{b,1}(\overline{e})}{2}\right] \\
= \quad & g_{4,2}+3g_{4,4}+3g_{4,6}+g_{4,8} \\
= \quad & \sum_{e\in E} y_2(e)y_0(e)+\sum_{\overline{e}\in\overline{E}} y_2(\overline{e})y_0(\overline{e}) \qquad (4)
\end{aligned}
$$

By applying the Cauchy-Schwarz inequality to the LHS of (4), we have

$$
\begin{aligned}
\sum_{e\in E}\left[\binom{y_{a,1}(e)}{2}+\binom{y_{b,1}(e)}{2}\right] &= \frac{1}{2}\sum_{e\in E}\left[(y_{a,1}(e))^2+(y_{b,1}(e))^2\right]-\frac{1}{2}\sum_{e\in E} y_1(e) \\
&\geq \frac{1}{4}\sum_{e\in E}[y_1(e)]^2-\frac{1}{2}\sum_{e\in E} y_1(e) \\
&\geq \frac{1}{4|E|}(\sum_{e\in E} y_1(e))^2-\frac{1}{2}\sum_{e\in E} y_1(e) \\
&= \frac{g_{3,2}^2}{|E|}-g_{3,2} \qquad (5)
\end{aligned}
$$

Similarly,

$$
\sum_{e\in\overline{E}}\left[\binom{y_{a,1}(e)}{2}+\binom{y_{b,1}(e)}{2}\right]\geq\frac{g_{3,1}^2}{|\overline{E}|}-g_{3,1} \qquad (6)
$$

An upper bound for the RHS of (4) can be found by rearranging and applying Cauchy-Schwarz as follows.

Theorem 1

$$
\sum_{e\in E} y_2(e)y_0(e)\leq 3(v-2)g_{3,3}-\frac{9g_{3,3}^2}{|E|}-3g_{3,3}L_E
$$

and

$$
\sum_{\overline{e}\in\overline{E}} y_2(\overline{e})y_0(\overline{e})\leq 3(v-2)g_{3,0}-\frac{9g_{3,0}^2}{|\overline{E}|}-3g_{3,0}L_{\overline{E}}
$$

for any $L_E \leq min\{y_1(e)|e \in E\}$ *and* $L_{\overline{E}} \leq min\{y_1(e)|e \in \overline{E}\}$.

Proof:

We arrive at these bounds by making a simple substitution and applying the Cauchy-Schwarz inequality as follows.

$$\begin{aligned}
\sum_{e \in E} y_2(e) y_0(e) &= \sum_{e \in E} y_2(e)[v - 2 - y_2(e) - y_1(e)] \\
&= (v-2) \sum_{e \in E} y_2(e) - \sum_{e \in E} [y_2(e)]^2 - \sum_{e \in E} y_2(e) y_1(e) \\
&\leq (v-2) \sum_{e \in E} y_2(e) - \frac{[\sum_{e \in E} y_2(e)]^2}{|E|} - L_E \sum_{e \in E} y_2(e) \\
&= 3(v-2) g_{3,3} - \frac{9(g_{3,3})^2}{|E|} - 3 L_E g_{3,3} \qquad (7)
\end{aligned}$$

for any $L_E \leq min\{y_1(e)|e \in E\}$, and

$$\begin{aligned}
\sum_{e \in \overline{E}} y_2(e) y_0(e) &= \sum_{e \in \overline{E}} y_0(e)[v - 2 - y_0(e) - y_1(e)] \\
&= (v-2) \sum_{e \in \overline{E}} y_0(e) - \sum_{e \in \overline{E}} [y_0(e)]^2 - \sum_{e \in \overline{E}} y_0(e) y_1(e) \\
&\leq (v-2) \sum_{e \in \overline{E}} y_0(e) - \frac{[\sum_{e \in \overline{E}} y_0(e)]^2}{|\overline{E}|} - L_{\overline{E}} \sum_{e \in \overline{E}} y_0(e) \\
&= 3(v-2) g_{3,0} - \frac{9(g_{3,0})^2}{|\overline{E}|} - 3 L_{\overline{E}} g_{3,0} \qquad (8)
\end{aligned}$$

for any $L_{\overline{E}} \leq min\{y_1(e)|e \in \overline{E}\}$. $\diamond$

We thus obtain the following inequality among the two and three vertex subgraph cardinalities and L_E and $L_{\overline{E}}$.

$$
\begin{aligned}
& \frac{9(g_{3,3})^2+(g_{3,2})^2}{|E|}+\frac{(g_{3,1})^2+9(g_{3,0})^2}{|\overline{E}|}-\binom{v}{3}-(3v-7)(g_{3,3}+g_{3,0}) \\
+ \quad & 3L_E g_{3,3}+3L_{\overline{E}} g_{3,0} \leq 0
\end{aligned} \tag{9}
$$

Noting the relationships in (2), and then solving for $g_{3,3}$, we obtain the following bound:

$$
\begin{aligned}
g_{3,3} \geq \frac{1}{12v(v-1)}\{ \quad & (L_E-L_{\overline{E}})|E|(2|E|-v^2+v)+(8|E|+v^2-v)s \\
& -16(v-1)|E|^2+2v(v-1)(v-3)|E|-\Phi^{\frac{1}{2}}\}
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
\Phi \;=\; & (L_E-L_{\overline{E}})^2|E|^2(2|E|-v^2+v)^2 \\
& +2|E|L_E(2|E|-v^2+v)[8s|E|+v(v-1)s \\
& -16(v-1)|E|^2+2v(v-1)(v-3)|E|] \\
& -2|E|L_{\overline{E}}(2|E|-v^2+v)[8s|E|-5v(v-1)s \\
& -16(v-1)|E|^2+2v(v-1)(7v-9)|E|-2v^2(v-1)^2(v-2)] \\
& +64s^2|E|^2-32v(v-1)s^2|E|-v^2(v-1)^2s^2 \\
& -256(v-1)s|E|^3+8v(v-1)(19v-21)s|E|^2 \\
& -8v^2(v-1)^2(v-2)s|E|+256(v-1)^2|E|^4 \\
& -16v(v-1)^2(11v-13)|E|^3+4v^2(v-1)^3(5v-9)|E|^2, \\
s \;=\; & \sum_{v\in V} d(v)^2.
\end{aligned}
$$

It is important to note that this lower bound depends on only five graph parameters: v, E, $\sum_{v\in V} d(v)^2$, L_E, and $L_{\overline{E}}$.

4 Proof that $R(K_4-e,K_8)\leq 45$

It is currently known that $R(K_4-e,K_n)=7,11,15,21$ for $n=3,4,5,6$, respectively [CH72] [BH81] while the best known bounds for $R(K_4-e,K_7)$

and $R(K_4-e,K_8)$ are only ones obtained by general inequalities, such as 2.3(a) of [Rad06], and the fact that Ramsey numbers are monotonic increasing. We see that $R(K_4-e,K_7) \geq R(K_4-e,K_7-e) = 28$ [MR91] and $R(K_4-e,K_7) \leq R(P_3,K_7)+R(K_4-e,K_6) = 34$. Similarly, $R(K_4-e,K_8) \leq R(K_4-e,K_7)+R(P_3,K_8) \leq 21+13 = 49$ and $R(K_4-e,K_8) \geq R(K_4-e,K_8-e) \geq R(K_4-e,K_7-e)+3 = 31$. The latter bound is given by observing that any (K_4-e,K_7-e)-good graph can be extended to (K_4-e,K_8-e)-good by adding an isolated K_3.

Next, we demonstrate how to apply inequality (10) to improve this bound, and show that $R(K_4-e,K_8) \leq 45$. Our proof is by contradiction.

Theorem 2 $R(K_4-e,K_8) \leq 45$.

Proof: Suppose that $G=(V,E)$ is a 45-vertex (K_4-e,K_8)-good graph.

For each vertex $v \in V$, by the obvious Ramsey constraints of the subgraphs induced by the vertices either connected to or not connected to v, we see that $11 \leq d(v) \leq 14$, since $R(K_4-e,K_7) \leq 34$ and $R(K_3-e,K_8) = 15$.

For each $e \in E$, $y_2(e) \leq 1$ and $y_{a,1}(e)+y_0(e) \leq R(K_4-e,K_7)-1 \leq 33$. Thus, $y_{b,1}(e) \geq 9$. In a similar way, $y_{a,1}(e) \geq 9$, and thus $y_1(e) \geq 18$. Similarly, for $e \in \overline{E}$, $y_0(e) \leq R(K_4-e,K_6)-1 \leq 20$, $y_2(e)+y_{a,1}(e) \leq R(K_3-e,K_8)-1 = 14$, and thus $y_1(e) \geq 18$. Thus, $L_E = L_{\overline{E}} = 18$ are suitable values for the above bound, which then becomes:

$$\begin{aligned} g_{3,3} \geq\ & \frac{1}{2970}(\sum_{v\in V} d(v)^2)|E| + \frac{1}{12}(\sum_{v\in V} d(v)^2) - \frac{4}{135}|E|^2 + 7|E| \\ & -\frac{1}{23760}\{ \\ & (64|E|^2 - 63360|E| - 3920400)(\sum_{v\in V} d(v)^2)^2 \\ & -(11264|E|^2 - 14065920|E| + 2195424000)(\sum_{v\in V} d(v)^2)|E| \\ & +495616|E|^4 - 747141120|E|^3 + 247832006400|E|^2 \\ & -24032365632000|E|\}^{\frac{1}{2}} \end{aligned} \tag{11}$$

Since $11 \leq d(v) \leq 14$ for each vertex $v \in V$, we can obtain the bounds $248 \leq |E| \leq 315$. Using the upper bound of $d(v)$ as well as the Cauchy-Schwarz inequality, we obtain $\frac{4}{45}|E|^2 = \frac{4|E|^2}{v} \leq \sum_{v \in V} d(v)^2 \leq 28|E|$.

In a $(K_4 - e)$-free graph, it is clear that for all $e \in E$, $y^2(e) \leq 1$ and thus $3g_{3,3} \leq |E|$. However, looping through the above constraints of $|E|$ and $\sum_{v \in V} d(v)^2$ and looking at the lower bound of $g_{3,3}$, as given in (11), we see that this is not satisfiable. Thus G does not exist and $R(K_4 - e, K_8) \leq 45$. $\diamond$

5 Conclusions

The best known bound for $R(K_4 - e, K_7)$ is the bound of $R(K_4 - e, K_7) \leq R(K_3 - e, K_7) + R(K_4 - e, K_6) = 13 + 21 = 34$, using identity 2.3(a) of [Rad06]. Our approach of theorem 2 does not appear to improve it. It is also not likely to improve any other bounds of the form $R(K_4 - e, K_n)$ for $n \geq 9$. In a similar fashion, this approach also confirms that $R(3, 10) \leq 43$, but does not seem to improve the bound for any cases of $R(3, n)$ for $n \geq 11$.

This method demonstrates a generalization of Goodman's bound on the cardinality of 3-vertex cliques to that of off-diagonal cases, and provides a means to bound some cases of off-diagonal Ramsey numbers. In particular, cases in which the cardinality of 3-vertex cliques is bounded relative to the cardinality of edges, such as those involving the book graphs B_n, are likely to give way to such approach.

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