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Combinatorial computing approach to selected extremal problems in geometry

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Combinatorial Computing Approach to Selected Extremal Problems in Geometry

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1 Acknowledgments

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Contents

2 Introduction

In the present thesis we investigate some open extremal problems in combinatorial geometry. It is difficult to give a satisfactory definition of combinatorial geometry. Classically, it is used to cover the theory of convex bodies and the related problems of covering, packing, and geometric inequalities. However, we will be mainly concerned with "Erdős-type" extremal problems involving finite sets of points in the plane of a highly combinatorial nature. These are problems that started out geometrical and soon *required* the language and techniques from combinatorics, because they appeared to be intractable by other means. According to Erdős's own words, such well-chosen innocently looking problems can isolate an essential difficulty in a particular area of mathematics.

The problems referred to in the title are:

1. (Proposed by Erdős)
   Denote by $g(n)$ the least number such that any set of $g(n)$ points in the plane in general position contains the vertices of an empty convex n-gon. Is it true that $g(n)$ always exists? And if it does, determine or estimate $g(n)$.
   Harborth [Har78] proved that $g(5)$ is finite. In fact, he showed that the exact value of $g(5)$ is 10. It was unexpectedly shown by Horton [Hor83] that $g(7) = \infty$. That is, there are arbitrarily large sets in the plane in general position containing no convex empty 7-gons. The existence of $g(6)$ is still open. We show that $g(6) \geq 27$ by giving a set of 26 points with no convex empty 6-gon.

2. (Proposed by Simmons)
   Given a set S of n points in general position in the plane, n is even, let a *halving line* be a line going through two of the points and cutting the remaining set of $n-2$ points in half. What is the maximum number $h(n)$ of halving lines that can be realized by a set of n points?
   There is a nontrivial gap remaining between the best currently known upper and lower bounds. We give the exact value for $h(12)$ and prove several theoretical results allowing to substantially restrict the computations needed to obtain the value of $h(n)$ for larger n.

Next chapter gives a historical account and the current state of the addressed problems. The first problem was destined to have a profound influence on the development of combinatorics (and Ramsey theory, in particular). It basically laid the groundwork for the latter. The guiding philosophy of all the problems in Ramsey theory is an inevitable occurrence of specific structures in some part of a large arbitrary structure.

In Chapter 4 we interpret the problems in the language of different combinatorial representations in the hope that this will suggest new approaches to their solution, as well as lead to new attractive theoretical problems which will themselves prove to be of independent interest. Classically, combinatorial geometry considers abstract incidence relations. However, many classes of geometric objects, in particular point sets, can be equipped with natural ordering

relations, and we will mainly manipulate combinatorial objects which encode the orientation properties of a geometric structure. We introduce the definitions and main properties of *counterclockwise systems* of Knuth defined as a set of triples obeying the counterclockwise relations on up to 5 points in the plane. We also introduce the *allowable n-sequences of permutations* of Goodman and Pollack that are very closely related to arrangements of pseudolines in the real projective plane, and show the connection between the two.

Chapter 6 is dedicated to the question of enumerating simple numbered configurations of n points in the plane. We naturally want to classify, in a reasonable and effective way, nondegenerate sets of n points in the plane into finitely many classes, such that sets within the same class are "essentially the same". Suppose, given n, we want to test each set of n points to determine whether it contains the vertices of a convex empty hexagon. How could we, even in principle, generate finitely many n-tuples of points such that any two are "essentially distinct"? What do we mean by "essentially distinct"? We consider several natural equivalence relations on point sets, and study the upper bounds on the number of equivalence classes formed by these relations. We also obtain D_{10}, which is an additional value for the enumeration table of Knuth representing the number of topologically distinct, simple arrangements of pseudolines with marked cell[1] or, equivalently, the number of weak equivalence/anti-equivalence classes, in the notation of Knuth.

Throughout the entire thesis we will try to show that combinatorial geometry is an elegant connection between algebraic and geometric intuition, and, as many connections between different areas, has proven to be fruitful.

The remainder of the thesis divides naturally into two parts, each one discussing theoretical results and computations concerning each of the problems. In those chapters we will make an excursion into the realm of massive computations used to obtain the results.

We have a recent genuine interest in the application of the probabilistic method to the problems addressed in the thesis. The guiding philosophy of the probabilistic method is to prove the existence of a configuration by creating a probability space whose points are configurations, and then showing a positive probability that the random configuration has the desired properties. We would like to use the probabilistic method to obtain the bounds on the asymptotic value of the extremal functions for the addressed problems.

3 The State of the Problems

3.1 Erdös-Szekeres problem

3.1.1 Definitions and Preliminary Remarks

We say that a set S of points in the plane is in *general position* if no three points of S lie on a line.

[1] as discussed by Goodman and Pollack.

A sequence of distinct vertices given by their Cartesian coordinates $\{(x_v, y_v) \mid v = 0, 1, \ldots, k, \;\; x_0 < x_1 < \ldots < x_k\}$ is said to be *convex* of length k (equivalently, the sequence is said to form a *k-cap*), if

$$\frac{y_v - y_{v-1}}{x_v - x_{v-1}} < \frac{y_{v+1} - y_v}{x_{v+1} - x_v} \qquad \text{for } v = 1, \ldots, k-1,$$

and *concave* of length k (equivalently, a *k-cup*), if the same is true with the inequality sign reversed. If we define the *counterclockwise* relation pqr, which states that the circle through points (p, q, r) is traversed counterclockwise when we encounter the points in cyclic order $p, q, r, p, \cdots$, we could give an alternative definition of a convex sequence: A sequence of distinct points $v_1, \cdots, v_k$ is *convex* of length k if

$$v_{i+1} v_i v_{i-1} \qquad \text{for } i = 1, \ldots, k, \;\; \text{where} \;\; i+1 \;\; \text{is taken mod } k,$$

and *concave* of length k if

$$v_{i-1} v_i v_{i+1} \qquad \text{for } i = 1, \ldots, k, \;\; \text{where} \;\; i+1 \;\; \text{is taken mod } k.$$

A *simple polygon* is a connected subset of the plane whose boundary is a closed chain of line segments with adjacent edges intersecting at their endpoints and no two non-adjacent intersecting edges. Concatenation of a convex chain and a concave chain with identical endpoints yields a *convex polygon.* Alternatively, a *convex polygon* can be defined as a simple polygon such that every pair of points interior to the polygon are mutually visible. Point y is said to be *visible* from z if the line segment yz lies entirely inside the polygon.

A simple polygon whose vertex set is a subset of a point set S is said to be *empty* if it contains no points of S in its interior.

3.1.2 Historical account

A talented young mathematician Esther Klein (to become Esther Szekeres the following year) observed that from any five points in the plane, no three collinear, it is always possible to select four that determine a convex quadrilateral, and suggested the following more general problem: Is it true that for all n, there is a least integer $f(n)$ so that any set of $f(n)$ points in the plane in general position must always contain the vertices of a convex n-gon?

In their seminal paper from 1935, Erdős and Szekeres [ES35] gave two proofs of the existence of $f(n)$ and established the following bounds:

$$2^{n-2} + 1 \leq f(n) \leq \binom{2n-4}{n-2} + 1.$$

They also conjectured that the lower bound is sharp for all n, i.e. $f(n) = 2^{n-2} + 1$. The conjecture still remains open[2], however, in their later paper,

[2]The conjecture is known to hold for $n \leq 5$: $f(3) = 3 = 2 + 1$, $f(4) = 5 = 2^2 + 1$, and $f(5) = 9 = 2^3 + 1$. The latter was proved by E. Makai and P. Turán.

Erdős and Szekeres [ES60], give an ingenious construction for a set of 2^{n-2} points which contains no convex n-gon.

In proving that $f(n)$ exists, Szekeres actually rediscovered Ramsey's theorem, which had only appeared (unknown to him) five years earlier. It was a genuinely combinatorial argument and it gave for $f(n)$ absurdly large value, not even close to the conjectured $2^{n-2}+1$. Erdős produced his "second proof" which was independent of Ramsey Theorem and gave a much more realistic value for $f(n)$.

The proof is based on several fundamental facts which illustrate the spirit of the Ramsey theory and have short elegant proofs. We couldn't resist presenting them here.

(i) For any sequence of n^2+1 distinct numbers, $x_1, x_2, \cdots, x_{n^2+1}$, there is always either an increasing subsequence of $n+1$ numbers (i.e., $x_{i_1} < x_{i_2} < \cdots < x_{i_{n+1}}$ where $i_1 < i_2 < \cdots < i_{n+1}$), or a decreasing subsequence (i.e. $x_{j_1} > x_{j_2} > \cdots > x_{j_{n+1}}$ where $j_1 < j_2 < \cdots < j_{n+1}$) of length $n+1$. (In other words, from n^2+1 points in the plane it is always possible to choose at least $n+1$ points with monotonically increasing abscissae and either monotonously increasing or monotonically decreasing ordinates.)

Proof. We put each number x_j in correspondence with a pair of integers (a_j, b_j) where a_j denotes the length of the longest increasing subsequence ending at x_j, and b_j denotes the length of the longest decreasing subsequence ending at x_j. It is easy to see that $(a_i, b_i) \neq (a_j, b_j)$ for $i \neq j$. Since there are n^2+1 numbers x_j, not all the (a_j, b_j) can satisfy $a_j, b_j \leq n$. Thus, there is a monotonic subsequence of length at least $n+1$. $\square$

(ii) For given positive integers m and n, any set of $\binom{n+m-2}{n-1}+1$ points in general position in the plane must contain either n points $x_1, \cdots, x_n$ with consecutive line segments $x_i x_{i+1}$ of increasing slopes, or m points with consecutive line segments of decreasing slopes.

Proof. Let $f(n, m)$ denote the largest number of points such that there is no n-cup (i.e. n points with consecutive line segments having increasing slopes), and there is no m-cap (i.e. m points with consecutive line segments having decreasing slopes). It suffices to show

$$f(n, m) \leq f(n, m-1) + f(n-1, m).$$

Suppose S is a set of $f(n, m)$ points containing no n-cup and no m-cap. We consider the set T of points x which are the right-hand endpoints of some $(n-1)$-cup. Clearly, x cannot be the left-hand endpoint of an $(m-1)$-cap. Therefore, we have

$$|T| \leq f(n, m-1).$$

Also,

$$|S \setminus T| \leq f(n-1, m),$$

which proves (ii). $\square$

Now the upper bound for $f(n)$ follows by using elementary properties of binomial coefficients from (ii) since an n-cup or n-cap forms a convex n-gon.

The lower bound $f(n) \geq 2^{n-2}+1$ was established by appropriately combining sets of sizes $f(\lfloor n/2 \rfloor - 2i, \lfloor n/2 \rfloor + 2i)$ for $i = -\lfloor n/2 \rfloor, \cdots, \lceil n/2 \rceil$.

Chung and Graham [CG98] have recently removed +1 from the upper bound $\binom{2n-4}{n-2} + 1 \approx c(4^n/\sqrt{n})$, for $n \geq 4$. This was the first, although microscopic, improvement since the original Erdős-Szekeres paper. Their proof immediately triggered a new bound $f(n) \leq \binom{2n-4}{n-2} - 2n + 7$ by Kleitman and Pachter [KP98], which was further improved by Tóth and Valtr [TV98] to $f(n) \leq \binom{2n-5}{n-3} + 2$. This is currently the best known upper bound improving over the previous bound roughly by a factor of 2.

It is believed that the lower bound $2^{n-2} + 1$ is the true value for $f(n)$, although there is little real evidence yet for this belief.

Erdös asked whether the following sharpening of the Erdös-Szekeres theorem is true: Is there a least integer $g(n)$ such that any set of $g(n)$ points in the plane in general position necessarily contains the vertices of an empty convex n-gon? He pointed out that $g(4) = 5$. Harborth [Har78] proved that $g(5) = 10$ [27]. It was unexpectedly shown by Horton [Hor83] that $g(7)$ is infinite by giving a construction of arbitrarily large configurations that do not contain convex empty 7-gons. Clearly, this result also holds for $n > 7$. The question about the existence of $g(6)$ is still open.

No wonder, the problem of finding the bounds on the number of empty polygons in planar point sets gained considerable attention. Given a set S of points in general position in the plane, denote by $f_k(S)$ the number of empty k-gons formed from the points of S and let

$$f_k(n) = \min_{|S|=n} f_k(S).$$

As it was just mentioned, Horton constructed configurations giving $f_k(n) = 0$ for $k \geq 7$. Bárány and Füredi proved

$$n^2 - O(n \log n) \leq f_3(n) \leq 2n^2,$$

$$\frac{1}{2}n^2 - O(n) \leq f_4(n) \leq 3n^2,$$

$$\lfloor n/10 \rfloor \leq f_5(n) \leq 2n^2,$$

$$f_6(n) \leq \frac{1}{2}n^2.$$

Valtr [Val95] showed constructions giving the following improved upper bounds:

$$f_3(n) < 1.8n^2, \qquad f_4(n) < 2.42n^2,$$

$$f_5(n) < 1.46n^2, \qquad f_6(n) < \frac{1}{3}n^2.$$

Using lengthy computer experiments based on the algorithm of Dobkin, Edelsbrunner and Overmars, Overmars detected several configurations on 26 points without convex empty 6-gons, therefore raising the lower bound on $g(6)$ to 27. No finite upper bound for $g(6)$ is known.

3.2 Halving Lines Graphs on Planar Point Sets

There are three kinds of mathematicians:
those who can count, and those who can't.
– Attributed to John Conway.

Simmons raised the following question: Given a set S of n points in general position in the plane, where n is even, let a *halving line* be a line going through two of the points and cutting the remaining set of $n-2$ points in half. What is the maximum number $h(n)$ of halving lines that can be realized by a set of n points?

Around 1970, Straus described a construction of a set of n points in the plane which determines $O(n \log n)$ halving lines. This was generalized by Erdös, Lovász, Simmons and Straus [ELSS73] (and later independently by Eddelsbrunner and Welzl [EW85]) to $\Omega(n \log k)$ lower bound on the maximum number of arbitrary k-sets. A subset S' of S is called a *k-set of S* if S' contains exactly k points and it can be cut off S by a straight line going through two points of $S \setminus S'$. Erdös et al. [ELSS73] define and show a number of structural properties of directed *k-graphs (dissection graphs)* induced by S, $G_k(S)$. The vertices of $G_k(S)$ are the points of S and the edges are the directed segments $\overline{pq}$ such that the directed line through p and q has exactly k points of S in the open half-space to its right. Obviously, $G_{n-k-2} = G_k$, hence it suffices to bound only the number of edges of G_k for $k \leq (n-2)/2$. Erdös et al. [ELSS73] show that each vertex of $G_k(S)$ has equal number of incoming and outgoing edges.

If n is even, $H(S) = G_{(n-2)/2}$ is the graph formed by halving segments of S. Clearly, each edge of $H(S)$ occurs in both directions, hence the graph can be considered undirected with each vertex having an odd degree.

Let $f_k(S)$ be the number of k-sets realized by S, i.e. the number of edges in $G_k(S)$, and let $f_k(n) = \max\{f_k(S) \mid S \text{ a set of } n \text{ points in } E^2\}$. The only values of k for which $f_k(n)$ is known exactly are $f_1(n) = n$ which is trivial to determine[3], and $f_2(n) = \lfloor 3n/2 \rfloor$ due to Edelsbrunner and Welzl [EW85]. Unaware of Erdös's lower bound, Edelsbrunner and Welzl [EW85] give a construction with $\lfloor \frac{1}{2} \log_2(2n/3) \rfloor$ halving lines, matching the lower bound of [ELSS73]. This lower bound is sharp for all even $n \leq 18$ and it is tempting to conjecture that it is sharp for all even n. Of course, a left cut of an infinite sequence does always reveal its true asymptotic behavior until the values become large.

[3] $f_1(n)$ is realized by a set of n points which are all extreme points of the set.

n	$\lfloor \frac{1}{2} n \log_2(2n/3) \rfloor$	$h(n)$	source
2	1	1	trivial
4	3	3	trivial
6	6	6	trivial
8	9	9	easily obtainable
10	13	13	Felsner [Fel97], Gerd Stöckl [Stö84]
12	18	18	[AAHP+98] and this paper
14	22	≥ 22	[Epp92] (probabilistic computer search)
16	27	≥ 27	[Epp92] (probabilistic computer search)
18	32	≥ 32	[Epp92] (probabilistic computer search)

Edelsbrunner and Welzl [EW85] also prove that any lower bound of the form $\Omega(nf(n))$ (for any $f(n)$) for the number of halving lines implies an $\Omega(nf(k))$ lower bound for k-sets.

The problem (in a dual setting of line arrangements) generalizes in a natural way to pseudoline arrangements (sacrifying the straightness, but preserving all combinatorial properties). In this generalized setting, an unpublished constructive lower bound of $n2^{\Omega(\sqrt{\log n})}$ is known [KPP82]. If the arrangement of pseudolines produced by this construction were shown to be realizable in plane, this would make the first dent in the lower bound on $h(n)$. We have just learned that very recently G. Tóth [Tót] presented a construction of a planar set of n points with $ne^{c\sqrt{\log n}}$ halving lines, thus matching the above lower bound of [KPP82] for pseudoconfigurations, now to become the best current bound for the plane.

Meanwhile, the best known upper bounds are much larger, and as was conjectured before, are most likely far from the true value. In 1971, Lovász [Lov71] proved that a set of n points in the plane has at most $O(n^{3/2})$ halving lines. The paper introduces Lovász crossing lemma, which became one of the most important tools for proving upper bounds on the number of k-sets. Erdös, Lovász, Simmons and Straus [ELSS73] generalized this result in 1973, showing that an n-point set in the plane has at most $O(nk^{1/2})$ k-sets. [ELSS73] conjectured that their lower bound is closer to the truth than their upper bound. In particular, they conjectured the the bound $h(n) > cn \log n$ cannot be substantially improved, and $h(n) = o(n^{1+\epsilon})$ for all $\epsilon > 0$. The upper bound given by Erdös et al. remained the best known upper bound (Edelsbrunner and Welzl [EW85] matched it using the allowable sequence of permutations, see Section 4.2.1.) until 1989, when Pack, Steiger, and Szemerédi slightly improved it to $O(nk^{1/2}/log^*n)$, using quite involved arguments, or as they put it in the paper, "a slightly stronger blend of geometry and combinatorics".

Very recently, Dey [Dey98] made the first significant improvement in the k-set problem since it was first posed, reducing the upper bound to $O(nk^{1/3})$. His proof was based on the notion of concave chains above an upper $(k-1)$-level in an arrangement of n lines. Another component of the proof is a probabilistic technique of Lázlo Székely. The proof is surprisingly simple, and we sketch it here.

Definition. The *crossing number* of a graph is the minimum possible number

of intersecting pairs of edges in any planar drawing.

Crossing Lemma. *(Ajtai, Chvátal, Newborn and Szemerédi; Leighton)*[4] *Any graph with n vertices and $e > 4n$ edges has crossing number $\Omega(e^3/n^2)$.*

Proof. Consider a planar embedding of a graph G with n vertices, e edges, and c pairs of crossing edges. Euler's formula implies that $c \geq e - 3n + 6$.

> **Lemma 3.1.** $c \geq e - 3n + 6$.
>
> *Proof.* Case 1. Assume G is acyclic. Then $e = n - 1$, and the claim vacuously holds for all graphs with at least 3 vertices since $n-1-3n+6 = -2n+5 < 0$ for $n \geq 3$. Case 2. Assume G has cycles. Suppose G is drawn on the plane with c crossings. We can choose a minimal set of edges of G, call is $\bar{E}$, removing which will permit to embed G in the plane. Then clearly, $c \geq |\bar{E}|$, since for each crossing we could remove an edge involved in the crossing, producing a planar drawing of the remaining graph. Consider a subgraph of G, call it G', $V(G') = V(G), E(G') = E(G) \setminus \bar{E}$, where $V(G)$ is the set of nodes of graph G, $E(G)$ – the set of edges. G' is embeddable in plane, and it has a set of faces, each face is bounded by a cycle in G' of length at least L, $L \geq 3$. Let f be the number of faces of G'. Since each edge is shared between two faces, we have $Lf \leq 2(e-|\bar{E}|)$. At the same time Euler formula written for G' implies $n - (e - |\bar{E}|) + f = 2$. From the first inequality, $f \leq 2(e-|\bar{E}|)/L$. Substituting into the second yields $L(n - e + |\bar{E}|) + 2(e - |\bar{E}|) \geq 2L$, i.e. $|\bar{E}| \geq e - (n-2)L/(L-2)$, and $|\bar{E}| \geq e - 3(n-2)$ since $L \geq 3$. $c \geq |\bar{E}|$ shown above gives $c \geq e - 3n + 6$. □

This bound is good enough for values of e linear in n, but when e is larger compared to n, sharper bounds can be obtained from the following argument which is a particularly elegant application of the probabilistic method due to Erdös.

Take a random subset of the vertices, each vertex with probability p. The expected number of vertices, edges, and crossings in the induced subgraph are at least pn, p^2e, and p^4c, respectively. Thus, $p^4c > p^2e - 3pn$ by lemma above and by linearity of expectation. Thus, $c > e/p^2 - 3n/p^3$. Taking $p = 4n/e$ gives us $c > e^3/64n^2$ and this is it. □

Theorem 3.2. *(T. Dey) Any set of n points in the plane, n even, has at most $O(n^{4/3})$ halving lines.*

Proof. Let S be a set of n points in the plane in general position, n even. The

[4] According to János Pach, this probabilistic proof has been independently rediscovered several times, by Lovász, Matousek, Füredi, Alon, Seigel, and many others, but was not published since it is essentially the same as the original counting argument of Ajtai et. al. The proof was first published by Székely [Szé97]. Using a more complicated probabilistic argument, Pach and Tóth recently improved the constant from 1/64 to 4/135, which is currently the best known.

segments in $G_{(n-2)/2}$ can be decomposed into $n/2$ convex chains as follows. Start with a vertical line through one of the $n/2$ leftmost points p in S, and rotate this line clockwise around p until it contains a segment pq in $G_{(n-2)/2}$. Initially, there are less than $n/2$ points above the line; this number goes down whenever the line hits a point to the left of p, and goes up whenever it hits a point to the right of p. It follows that q must lie to the right of p. Continue rotating the line clockwise around q until it hits another segment in $G_{(n-2)/2}$ (which will lie to the right of q), and so on, until the line is vertical again. The sequence of segments hit by the rotating line forms a convex chain, and every segment in $G_{(n-2)/2}$ is in exactly one convex chain. The number of intersections between any two convex chains is no more than the number of upper common tangents between the same two chains. Any line between two points in S is an upper common tangent of at most one pair of chains. Thus, there are at most $O(n^2)$ intersections between the segments in $G_{(n-2)/2}$. An easy argument using Crossing Lemma proves that any graph with n vertices and crossing number $O(n^2)$ has at most $O(n^{4/3})$ edges, so S has at most $O(n^{4/3})$ halving lines. □

Another, slightly different proof of the bound $O(n^k 1/3)$ was given by Aronov and Welzl by exploiting identities relating the crossing number of G_k to degrees of the vertices of G_k. Throughout the thesis, let d_p denote the degree of vertex p, i.e. the number of halving segments incident to p, and let $cr(G)$ - denote the number of *pairs* of edges of G that cross in the minimal embedding of G. As always, let S be a set of n points in general position in the plane. Then we have:

$$cr(H(S)) + \sum_{p \in S} \binom{(d_p+1)/2}{2} = \binom{n/2}{2}$$

which immediately implies an upper bound of $O(n^{4/3})$ on the number of edges of $H(S)$. Since $cr(H(S)) = O(|E(H(S))|^3/n^2)$ while we have $cr(H(S)) = O(n^2)$, $|E(H(S))| = O(n^{4/3})$.

In 1992, Eppstein [Epp92] rediscovered constructions of [ELSS73] and [EW85] that give a lower bound of $1/2n\log_2(2n/3)$. He also performed massive probabilistic search which gave the following lower bounds: $h(12) \geq 18$, $h(14) \geq 22$, $h(16) \geq 27$, and $h(18) \geq 32$.

The configuration on 12 points with 18 halving lines given by Erdős, can be thought of as having 6 pairs of points in approximately similar positions to 6 points on Figure 2. This was generalized in a construction that produces configurations of $2n$ points with $2k+n$ halving lines from configurations of n points with k lines, satisfying some weak conditions.

Lemma 3.3. *(Eppstein) Let S be a configuration of n points, n even, and k halving lines for which the underlying halving lines graph is connected and contains a cycle. Then there is a configuration of $2n$ points with $2k+n$ halving lines for which the underlying graph is again connected and contains a cycle.*

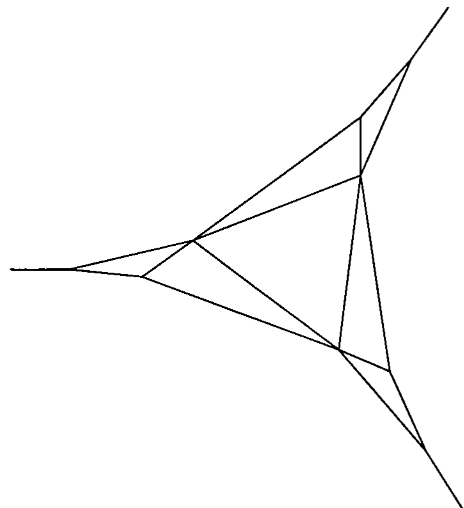


Figure 1: A set on 12 points with 18 halving lines

Proof. For each vertex in the underlying graph of S, pick an adjacent edge, such that no two vertices pick the same edge. This can be done as follows. Start by finding a cycle in the graph. For each vertex in the cycle, pick the next edge in some consistent orientation of the cycle. For each vertex not on the cycle, pick the edge on a shortest path from the vertex to the cycle.

Now replace each point p of the original configuration by two points p_1 and p_2, spaced at some small distance of ϵ from p, on either side of p along the line corresponding to the edge pq picked by p.

If ϵ is small, line $p_1p_2 = pq$ will be a halving line, since it cuts between q_1 and q_2, and since the number of other pairs of points on either side of the line in the new configuration is the same as the number of single points on either side of the line in the original configuration. Further, if p_2 is closer to q than p_1, lines p_2q_1 and p_2q_2 will halve the four-point configuration $p_1p_2q_1q_2$. If ϵ is small, these two lines will be halving lines of the configuration, since they will behave the same as pq with respect the the points outside this four-point configuration. Thus line pq has been replaced by three halving lines.

If some other line rs is not chosen by either r or s, then $r_1r_2s_1s_2$ will form a quadrilateral, which has two halving lines. As above, if ϵ is small enough these lines will be halving lines of the entire configuration.

Thus each of the n chosen halving lines is replaced by three such lines, and each line not chosen is replaced by two. The new edges p_2q_2 connect all vertices p_2, since the chosen edges of the original graph form a connected subgraph. Each vertex p_1 is then connected by the edge p_1p_2. The old configuration has at least n halving lines, so the new one must have more than $2n - 1$ and thus contains a cycle. $\square$

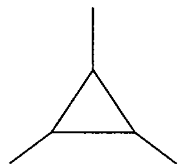


Figure 2: A set on 6 points maximizing the number of halving lines

Theorem 3.4. *(Eppstein) For any $i \geq 1$, there is a configuration of $3 \cdot 2^i$ points having $3(i+1)2^{i-1}$ halving lines.*

Proof. By induction, assume there is a configuration of $3 \cdot 2^{i-1}$ points with $3i \cdot 2^{i-2}$ lines, satisfying the conditions of Lemma 3.3. As a base case, the configuration maximizing the number of number of halving lines on 6 points, meets these conditions. Applying Lemma produces a new configuration with $2 \cdot 3 \cdot 2^{i-1} = 3 \cdot 2^i$ points, and $2 \cdot 3i \cdot 2^{i-2} + 3 \cdot 2^{i-1} = 3(i+1)2^{i-1}$ lines. □

So there is a nontrivial gap still remaining. The only exact values known are $h(2) = 1$, $h(4) = 3$, $h(6) = 6$, $h(8) = 9$, $h(10) = 13$, $h(12) = 18$. The latter is the result of [AAHP$^+$98] and this thesis.

Despite the pathetic state of the problem, we can obtain fairly sharp bounds for small values of n, see section 5.2.1.

3.2.1 Halving Lines on Dense Sets

Various combinatorial extremal problems that first were posed for arbitrary point sets have been recently reconsidered for dense sets. Dense sets are sets with bounded ratio of largest over smallest distance between any two points. More formally, a set S of n points in R^d is $\delta-dense$ if the ratio of the maximum over the minimum distance between any two points in S is at most $\delta n^{1/d}$.

The investigation of dense sets in motivated by the common discrepancy between the complexity of algorithms in the worst case and in practical cases. The complexity of a geometric algorithm typically depends on certain combinatorial parameters associated with the geometric data, and it is often the case that these parameters reach their extrema only for rare and bizarre sets of data. Extremal point sets constructed in the literature, and in the present thesis, in particular, exhibit large distance ratios, sometimes exponential in the number of points. This is related to the fact that there are combinatorial types of point configurations that require the distance ratio be at least double-exponential in the number of points. Such sets are unlikely to occur in reality. Thus, the notion of density can be seen as an attempt to make the theoretical analysis of algorithms more relevant to practice.

Edelsbrunner, Valtr and Welzl [EVW94] proved that the $\Omega(n \log n)$ lower bound on the number of halving lines is asymptotically unaffected by the density restriction, in particular, they showed by construction that for any even $n \geq 2$

there is a 2-dense set[5] of n points in the plane with at least $n/12\log_3 n - n$ halving lines.

Edelsbrunner, Valtr and Welzl [EVW94] showed that dense sets admit smaller bounds. They proved that the number of halving lines of a dense planar set of n points is at most $O(n^{5/4}/\log^* n)$. This improves over the current best upper bound of $O(n^{4/3})$ which holds generally without any density assumption.

Of course, similar questions have been posed in higher dimensions. Seidel showed that any lower bound of the form $\Omega(nf(n))$ for halving lines in the plane implies an $\Omega(n^{d-1}f(n))$ lower bound for halving hyperplanes for d-dimensional point sets. The best upper bound in three dimensions is $O(n^{8/3})$, due to Dey and Edelsbrunner [DE93] (improving earlier results of Bárány et. al., and Eppstein [Epp93]). For $d \geq 4$, the best known upper bounds are just slightly less than $O(n^d)$. Unfortunately, no generalization of Dey's proof technique to higher dimensions is known. The obvious generalization of Dey's "convex chains" to higher dimensions is not valid, since the latter are no longer necessarily convex!

Edelsbrunner, Valtr and Welzl [EVW94] proved that the number of halving planes of an δ-dense set of n points in R^3 is less than $20 \cdot \delta^{4/3} n^{7/3}$. The proof is based on a metric argument[6] that can be extended to $d \geq 4$ dimensions, where it leads to $O(n^{d-2/d})$ as an upper bound for the number of halving hyperplanes.

3.2.2 Motivation

Besides pure aesthetic beauty, the motivation for our interest in $f_k(n)$ stems from several applications in computational geometry, e.g. so-called *halfplanar range search problem*, which requires the number of points which lie in a specified query halfplane to be determined efficiently. Another problem related to $f_k(n)$ is *k-nearest neighbors problem* solved by Shamos and Hoey using order-k Voronoi diagrams. The order-k Voronoi diagram for S consists of a collection of nonoverlapping regions which covers the plane. Each subset of k points of S is assigned a potentially empty region R such that those k points are the k nearest to a query point q iff q resides in R. It is easily verified that the region assigned to k specific points is properly unbounded (i.e. there are two bounding rays which are not parallel) iff those points define a k-set of S. Thus $f_k(S)$ denotes the number of properly unbounded regions of the diagram and $f_k(n)$ denotes the maximal number of properly unbounded regions which can occur in any order-k Voronoi diagram for n points.

Finally, $f_k(n)$ applies to a problem which deals with a set S of n points on a horizontal line. Each one of those points moves with constant, but, in general, unique speed towards the left or the right. A point is said to be *at position* k at some point in time if it is the only kth point from the right. Let S be a set of n moving points on the line and let $L_k(S)$ denote the sequence of points at position k. Edelsbrunner and Welzl [EW85] proved that then the length of $L_k(S)$ is no greater than $f_{k-1}(n) + f_k(n) + 1$.

[5] In fact, they showed that the density factor can be reduced to $2\sqrt{\sqrt{3}/\pi} = 1.49$.

[6] an area argument for triangles spanned by points in S

4 Combinatorial Representations

In this section we introduce some combinatorial representations of geometric structures.

4.1 Counterclockwise systems

Everything should be made as simple
as possible, but not simpler.
– Albert Einstein

A *CC-system* [7] is defined as a relation on ordered triples of points that satisfy five axioms obeyed by the counterclockwise relation on up to 5 points in the plane. The *counterclockwise* relation pqr states that the circle through points (p, q, r) is traversed counterclockwise when we encounter the points in cyclic order $p, q, r, p, \ldots$ In standard geometrical terms, if points p, q, r have Cartesian coordinates $(x_p, y_p), (x_q, y_p)$, and (x_r, y_r), the counterclockwise predicate corresponds to the sign of a determinant:

$$pqr \iff \begin{vmatrix} x_p & y_p & 1 \\ x_q & y_q & 1 \\ x_r & y_r & 1 \end{vmatrix} > 0$$

Collinearity degeneracy implies $|pqr| = 0$ and is eliminated by Axiom 3.

Axiom 1 (Cyclic symmetry). $pqr \Longrightarrow qrp$.

Axiom 2 (Antisymmetry). $pqr \Longrightarrow \neg prq$.

Axiom 3 (Non-degeneracy). $pqr \vee prq$.

In all cases there is an implied quantification $(\forall p, q, r : p, q, r \in S \wedge p \neq q \wedge q \neq r \wedge r \neq p)$. Axioms 1 and 2 are simple consequences of the determinant identities $|pqr| = |qrp| = -|prq|$.

Axiom 4 (Interiority). $tqr \wedge ptr \wedge pqt \Longrightarrow pqr$.

Intuitively, the fact that t lies on the left of qr, rp and pq should imply that t lies inside the triangle pqr oriented counterclockwise.

Axiom 5 (Transitivity). $tsp \wedge tsq \wedge tsr \wedge tpq \wedge tqr \Longrightarrow tpr$.

The first three terms imply that points p, q, r lie to the left of ts. The last two state that q lies to the left of tp, while r lies to the left of tq, thus, t, s, p, q, r form a counterclockwise oriented sequence, and it is geometrically obvious that, by transitivity, r lies to the left of tp.

The same argument yields the dual axiom:

Axiom 5′ (Dual transitivity). $stp \wedge stq \wedge str \wedge tpq \wedge tqr \Longrightarrow tpr$.

[7] as introduced by D. Knuth [Knu92]

We can deduce Axiom $5'$ from Axioms 1-5. Assume Axiom $5'$ fails, that is,

$$stp \wedge stq \wedge str \wedge tpq \wedge tqr \wedge trp$$

We will show that $spq \Longrightarrow srp$.

Case 1. *pqr is true.* Then $spq \Longrightarrow srp$ certainly holds, since Axiom 5 says that $pqs \wedge pqt \wedge pqr \wedge pst \wedge ptr \Longrightarrow psr$, and we can always put t so that $pqt \wedge pst \wedge ptr$ holds.

Case 2. *pqr is false.* Suppose $spq \Longrightarrow srp$ doesn't hold, thus $spq \wedge spr \wedge prq$; then we must have rqs, otherwise Axiom 5 would say that $rsq \wedge rsp \wedge rst \wedge rqp \wedge rpt \Longrightarrow rqt$. But rqs causes another problem, since Axiom 5 also forces $qsr \wedge qst \wedge qsp \wedge qrt \wedge qtp \Longrightarrow qrp$.

Thus, this proves that $spq \Longrightarrow srp$. Symmetrically, $srp \Longrightarrow sqr$, and $sqr \Longrightarrow spq$, thus

$$spq \Longrightarrow srp \Longrightarrow sqr \Longrightarrow spq$$

and we have either $spq \wedge srp \wedge sqr$ or $sqp \wedge spr \wedge srq$. Both of these contradict Axiom 5, since

$$stp \wedge stq \wedge str \wedge spq \wedge sqr \Longrightarrow spr$$
$$stq \wedge stp \wedge str \wedge sqp \wedge spr \Longrightarrow sqr.$$

Thus, the original assumption fails, i.e. Axioms 5' can be deduced as a consequence of axioms 1, 2, 3 and 5. □

A similar argument shows that Axioms 1, 2, 3 and $5'$ imply Axiom 5 - we just have to complement the value of each triple containing s.

Quite naturally, we cannot expect axioms above to be strong enough to capture all the orientation properties of planar point sets, since they were obtained by considering configurations of at most 5 points.

Knuth showed that triples can satisfy many weird and wonderful identities such as

$$|pqw||rpv||qry||prx||quz| + |rpv||qru||rpz||qpy||qwx| +$$
$$|qru||pqw||rpz||prx||qvy| + |qru||pqw||rpz||qpy||rvx| +$$
$$|pqw||rpv||pqx||rqz||ruy| + |rpv||qru||pqx||qpy||rwz| +$$
$$|rpv||qru||pqx||rqz||pwy| + |qru||pqw||qry||prx||pvz| +$$
$$|pqw||rpv||qry||rqz||pux| = 0,$$

which corresponds to the theorem of Pappus. This identity implies that the counterclockwise triples of any nine points in the plane must satisfy the axiom

$$\neg((pqw \oplus rpv \oplus qry \oplus prx \oplus quz) \wedge (rpv \oplus qru \oplus rpz \oplus qpy \oplus qwx)$$
$$\wedge \ldots \wedge (pqw \oplus rpv \oplus qry \oplus rqz \oplus pux)),$$

since if each of the parenthesized clauses in the above axiom were true, each term on the left of the previous identity would be positive, and the sum could not be zero. It is in fact possible to construct a CC-system such that all of the triples $pqw, rpv, qry, \ldots, pux$ are true, violating the axiom above. CC-systems

are more general than the systems obtainable from point sets in Euclidean geometry. Again, it is not surprising since we could hardly expect such axioms to be strong enough to deduce the 9-point theorem of Pappus, which states that if eight of the triples of points $pux, pvx, pwy, qvy, qwy, quz, rwz, ruz, rvx$ are collinear, then the ninth triple is also collinear.

If a CC-system can arise from actual points in the plane, it is called *realizable.*

Since Axioms 1, 2, 3, and 5 imply Axiom 5′, it is natural to ask whether Axioms 1-5 are themselves independent, in the sense that no four axioms are strong enough to imply the fifth. It was shown by Knuth [Knu92] that they in fact are. Axiom 2 is clearly independent, since the other axioms hold if pqr in uniformly true for all distinct points p, q, r. Axiom 3 is also independent, since the other axioms happily hold if pqr is uniformly false. Axiom 1 is independent because Axioms 2 and 3[8] are valid for triples

$$abc, \neg acb, bac, \neg bca, cab, \neg cba,$$

for which Axiom 1, clearly, does not hold.

A predicate pqr satisfying Axioms 1-3 is unambiguously specified by the choice of orientation of each three-element subset. Since the triples are independent, there are $2^{\binom{n}{3}}$ ways to satisfy Axioms 1-3 over an n-element set. Clearly, not all of them are legal CC-systems.

To show that Axiom 4 is independent, consider triples

$$dbc, adc, abd, cba$$

and their cyclic symmetries that satisfy Axioms 1-3, but not Axiom 4. Axiom 5 holds vacuously establishing the independence of Axiom 4.

It remains to show the independence of Axiom 5. For each point p we can form a directed graph with arcs $q \to r$ iff pqr holds. In fact, each such digraph is a *tournament*[9], since precisely one of $q \to r$ or $r \to q$ appears for each pair of distinct vertices q and r.

Consider a set of triples on on $\{a, b, c, d, e\}$.

$$abc, dab, dbc, dca, eab, ebc, eca, ead, ebd, ecd,$$

and the corresponding tournaments:

$$a : \begin{tikzcd} b \arrow[r] \arrow[d] \arrow[dr] & c \\ d \arrow[r] \arrow[ur] & e \arrow[u] \end{tikzcd} , \quad b : \begin{tikzcd} c \arrow[r] \arrow[d] \arrow[dr] & a \\ d \arrow[r] \arrow[ur] & e \arrow[u] \end{tikzcd} , \quad c : \begin{tikzcd} a \arrow[r] \arrow[d] \arrow[dr] & b \\ d \arrow[r] \arrow[ur] & e \arrow[u] \end{tikzcd} , \quad d : \begin{tikzcd} a \arrow[r] & b \arrow[dl] \\ c \arrow[u] & e \arrow[l] \arrow[u] \arrow[ul] \end{tikzcd} , \quad e : \begin{tikzcd} a \arrow[r] \arrow[dr] & b \arrow[d] \arrow[dl] \\ c \arrow[u] \arrow[r] & d \end{tikzcd}$$

Tournaments guarantee that Axioms 2 and 3 are satisfied. In order to satisfy Axiom 1, the arc $q \to r$ should appear in the tournament for p iff the arcs $r \to p$ and $p \to q$ appear in the tournaments for q and r respectively.

[8]Axioms 4 and 5 vacuously hold on any tree-element set.

[9]A *tournament*, for purposes of this thesis, is a complete graph in which each edge is directed. That is, we have a number of players corresponding to the vertices of the tournament, every pair plays a game and no draws are allowed. We direct an edge from q to r if q beats r. The schedule of the tournament does not matter, only the results.

A tournament containing no 3-cycles is called *transitive.* Such tournaments contain no cycles at all, because any k-cycle $a_1 \to a_2 \to \cdots \to a_k \to a_1$[10] can always be shortened to $a_1 \to a_3 \to \cdots \to a_k \to a_1$. Axiom 4 says that any 3-cycle $p \to q \to r \to p$ in a tournament for t must correspond to a triple pqr, that is $q \to r$, $r \to p$, and $p \to q$ must appear in the tournaments for p, q, and r, respectively.

Axiom 5 says that the tournament for t must not contain a quadruple $\langle p, q, r, s \rangle$ forming a 3-cycle pqr and a source s:

$$t: \begin{array}{ccc} s & \to & p \\ \downarrow & \times & \uparrow \\ q & \to & r \end{array} .$$

Similarly, Axiom 5′ says that no tournament should contain a 3-cycle and a sink.

The tournaments for points a, b, c above define transitive linear orderings of the rest of the points ($b < d < e < c$, $c < d < e < a$, and $a < d < e < b$, respectively). However, the tournaments for d and e violate Axioms 5 and 5′, respectively. The only 3-cycle present in these tournaments is $a \to b \to c \to a$, and the triple abc does appear in the set of triples, so Axiom 4 holds. Thus Axioms 5 and 5′ cannot be derived from Axioms 1–4. Note that any construction satisfying Axioms 1–3 must satisfy both Axioms 5 and 5′ or neither of them.

4.1.1 Vortex-free tournaments

Let us temporarily drop Axiom 4 and consider ternary relations satisfying Axioms 1, 2, 3 and either 5 or 5′ (and therefore both). It turns out that Axioms 1, 2, 3, and 5 are enough to guarantee a rich geometric structure. Following Knuth we call such systems *pre-CC systems.* Axiom 5 says that the tournaments associated with individual points contain neither in-vortex nor out-vertex among their 4-point subtournaments.

Lemma 4.1. *(Knuth [Knu92])* A tournament is vortex-free if and only if it can be obtained from a transitive tournament by negating[11] a subset of its points.

Proof. Negating any point of an in-vortex produces an out-vortex, and conversely. Therefore negation preserves vortex-freeness; any tournament obtained from a transitive tournament by repeated negation must be vortex-free.

Corollary 4.2. A tournament on n points is vortex-free if and only if there is a string $\alpha_1 \alpha_2 \ldots \alpha_n$ containing each point or its complement, such that

$$\alpha_j \to \alpha_k \qquad \text{for} \quad 1 \le j < k \le n\,.$$

[10] W.l.o.g. we assume that all $a_1, a_2, \cdots, a_k$ are distinct, and $k > 3$.

[11] To facilitate the study of vortex-free tournaments, Knuth introduced the notion of *signed points* and considered the original points $a_1, \ldots, a_n$ as well as their complements $\bar{a}_1, \ldots, \bar{a}_n$. The relation $a_i \to a_j$ is now naturally extended to signed points by defining $\bar{a}_j \to a_i$, $a_j \to \bar{a}_i$, and $\bar{a}_i \to \bar{a}_j$ whenever $a_i \to a_j$ holds. Thus, negation of a signed point reverses the directions of all arcs that it touches. Double negation is defined as $\bar{\bar{a}} = a$.

Moreover, it is possible to construct such a string by examining the direction of only $O(n \log n)$ arcs.

Proof. Choose α_1 to be any signed point. Then if a partial string $\alpha_1 \ldots \alpha_k$ has been constructed representing a vortex-free subtournament on k points for some k, $1 \le k < n$, let p be any point distinct from $\alpha_1, \ldots, \alpha_k$ or their complements, and let $\alpha = p$ if $\alpha_1 \to p$, $\bar{p}$ otherwise. According to Lemma 4.1., there exists j, $1 \le j \le k$, such that $\alpha_i \to \alpha$ for $1 \le i \le j$ and $\alpha \to \alpha_i$ for $j < i \le k$. j can be determined using binary search by examining the direction of at most $\lceil \lg k \rceil$ arcs. This yields a string $\alpha'_1 \ldots \alpha'_{k+1} = \alpha_1 \ldots \alpha_j \, \alpha \, \alpha_{j+1} \ldots \alpha_k$ that represents a subtournament of $k + 1$ points. Iterate this process until $k = n$. □

This construction shows that there are precisely $2n$ strings $\alpha_1\alpha_2 \ldots \alpha_n$ that represent a given vortex-free tournament, since there is one string for each choice of α_1. The rest of the string is then uniquely determined. Observe that $\alpha_1\alpha_2 \ldots \alpha_n$ and $\alpha_2 \ldots \alpha_n \, \bar{\alpha}_1$ define the same vortex-free tournament, and hence, the set of all strings representing a given tournament is the set of all n-element substrings of the infinite periodic string $\alpha_1\alpha_2 \ldots \alpha_n \, \bar{\alpha}_1\bar{\alpha}_2 \ldots \bar{\alpha}_n \, \alpha_1\alpha_2 \ldots \alpha_n \, \bar{\alpha}_1 \ldots$.

This method of representation makes it trivial to observe that there are precisely $2^n n!/2n = 2^{n-1}(n-1)!$ ways to define a vortex-free tournament on n labeled points.

We can also count the number of nonisomorphic vortex-free tournaments, because there is one for every equivalence class of boolean[12] strings $\sigma_1\sigma_2 \ldots \sigma_n$ under the equivalence relation $\sigma_1\sigma_2 \ldots \sigma_n \equiv \sigma_2 \ldots \sigma_n\bar{\sigma}_1$.

For $n = 5$ the equivalence classes on the 32 boolean strings of length 5 are

$$00000 \equiv 00001 \equiv 00011 \equiv 00111 \equiv 01111 \equiv 11111 \equiv 11110 \equiv 11100 \equiv 11000 \equiv 10000;$$
$$00010 \equiv 00101 \equiv 01011 \equiv 10111 \equiv 01110 \equiv 11101 \equiv 11010 \equiv 10100 \equiv 01000 \equiv 10001;$$
$$00100 \equiv 01001 \equiv 10011 \equiv 00110 \equiv 01101 \equiv 11011 \equiv 10110 \equiv 01100 \equiv 11001 \equiv 10010;$$
$$01010 \equiv 10101.$$

It is easy to see that isomorphism of tournaments corresponds to the equivalence of boolean strings. Consider the vortex-free tournaments on $\{a_1, \ldots, a_n\}$ defined by the strings $\alpha_1\alpha_2 \cdots \alpha_n$ and $\beta_1\beta_2 \cdots \beta_n$. Let the corresponding boolean vectors be $b_1^\alpha b_2^\alpha \cdots b_n^\alpha$ and $b_1^\beta b_2^\beta \cdots b_n^\beta$, respectively. We will show that if $b_1^\alpha b_2^\alpha \cdots b_n^\alpha \equiv b_1^\beta b_2^\beta \cdots b_n^\beta$ [13], the two tournaments are indeed isomorphic by giving a permutation function π on $a_1, \ldots, a_n$ such that $\pi'(\alpha_1)\pi'(\alpha_2) \cdots \pi'(\alpha_n) = [\beta_1\beta_2 \cdots \beta_n]_{\gg j}$ for some $j < 2n$, where "$\gg j$" denotes applying the operation of "moving the first point of the string to the end and negating it" j times, π' is defined as $\pi'(\delta) = \mathrm{sign}(\delta)\pi(|\delta|)$, where $|\delta| = |\bar{\delta}| = \delta$ for any signed point δ.

Repeatedly move the first element of $b_1^\beta b_2^\beta \cdots b_n^\beta$ to the end, negating it until the resulting $b_1'^\beta b_2'^\beta \cdots b_n'^\beta$ equals $b_1^\alpha b_2^\alpha \cdots b_n^\alpha$. Set j to the number of "shift-negate"s we had to perform. Let $\beta'_1\beta'_2 \cdots \beta'_n = [\beta_1\beta_2 \cdots \beta_n]_{\gg j}$. Now define $\pi(|\alpha_i|) = |\beta'_i|$. It is easy to see that $\pi'(\alpha_1)\pi'(\alpha_2) \cdots \pi'(\alpha_n) = [\beta_1\beta_2 \cdots \beta_n]_{\gg j}$.

[12]The 0s correspond to positive variables and the 1s correspond to their negations in a string $\alpha_1\alpha_2 \ldots \alpha_n$ that represents a given tournament.

[13]under the equivalence relation $\sigma_1\sigma_2 \ldots \sigma_n \equiv \sigma_2 \ldots \sigma_n\bar{\sigma}_1$.

For example, consider the vortex-free tournaments on $\{a_1, \dots, a_5\}$ defined by the strings $a_1\bar{a}_2a_3\bar{a}_4a_5$ and $\bar{a}_3a_1\bar{a}_4a_2\bar{a}_5$. The corresponding boolean vectors, 01010 and 10101, are equivalent, so the two tournaments must to be isomorphic. Indeed, we obtain an isomorphism by mapping $(a_1, a_2, a_3, a_4, a_5)$ to $(a_1, a_4, a_2, a_5, a_3)$, i.e. by defining $\pi(a_1) = a_1$, $\pi(a_2) = a_4$, $\pi(a_3) = a_2$, $\pi(a_4) = a_5$, $\pi(a_5) = a_3$. We have $\pi'(|a_1|)\pi'(|\bar{a}_2|)\pi'(|a_3|)\pi'(|\bar{a}_4|)\pi'(|a_5|) = \pi(a_1)\overline{\pi(a_2)}\pi(a_3)\overline{\pi(a_4)}\pi(a_5) = a_1\bar{a}_4a_2\bar{a}_5a_3 = [\bar{a}_3a_1\bar{a}_4a_2\bar{a}_5]_{\gg 1}$.

Conversely, inequivalence of the boolean strings implies nonisomorphism of the tournaments. Prove the contrapositive by contradiction. Assume there exist isomorphic tournaments defined by strings $\alpha_1\alpha_2\cdots\alpha_n$ and $\beta_1\beta_2\cdots\beta_n$ such that the corresponding boolean vectors, $b_1^\alpha b_2^\alpha \cdots b_n^\alpha$ and $b_1^\beta b_2^\beta \cdots b_n^\beta$, are not equivalent. Since the tournaments are isomorphic, there must exist a permutation function π on $(a_1, a_2, \cdots, a_n)$ defining the isomorphism: $\pi'(\alpha_1)\pi'(\alpha_2)\cdots\pi'(\alpha_n) = [\beta_1\beta_2\cdots\beta_n]_{\gg j}$ for some $j < 2n$.[14] Let the corresponding boolean vectors be $B^{\pi'(\alpha)}$ and $B^{[\beta]_{\gg j}}$. Clearly, since the strings are identical, $B^{\pi'(\alpha)} = B^{[\beta]_{\gg j}}$ The boolean vectors corresponding to strings $\alpha_1\alpha_2\cdots\alpha_n$ and $\pi'(\alpha_1)\pi'(\alpha_2)\cdots\pi'(\alpha_n)$ are also identical by the virtue of the definition of π', denoted by $B^\alpha = B^{\pi'(\alpha)}$. Furthermore, since $[\beta_1\beta_2\cdots\beta_n]_{\gg j}$ is just a left "shift-negate" of $\beta_1\beta_2\cdots\beta_n$, the corresponding boolean vectors, $B^{[\beta]_{\gg j}}$ and B^β are equivalent under the equivalence relation $\sigma_1\sigma_2\dots\sigma_n \equiv \sigma_2\dots\sigma_n\bar{\sigma}_1$. By the transitivity of equality we have $B^\alpha \equiv B^\beta$, which contradicts the initial assumption. Hence, we proved that inequivalence of the corresponding boolean strings implies nonisomorphism of the tournaments.

For example, the tournament defined by $a_1\bar{a}_2a_3\bar{a}_4a_5$ is not isomorphic to, say, $a_2\bar{a}_1\bar{a}_4a_5a_3$, whose boolean string is $01100 \not\equiv 01010$. If the original a_k were mapped to $a_{\pi(k)}$, we could complement $a_{\pi(2)}$ and $a_{\pi(4)}$, getting a transitive tournament in which $a_{\pi(1)} \to \bar{a}_{\pi(2)} \to a_{\pi(3)} \to \bar{a}_{\pi(4)} \to a_{f(5)}$. The 10 strings $\alpha_1\dots\alpha_5$ representing that tournament cannot have the form $a_2\bar{a}_1\bar{a}_4a_5a_3$, because they correspond only to boolean strings of negation patterns that are equivalent to 01010.

Notice that the vortex-free tournament defined by $\alpha_1\alpha_2\cdots\alpha_n$ is transitive iff the corresponding boolean string is equivalent to $00\cdots0$, which is the case iff $\alpha_1\alpha_2\cdots\alpha_n$ has the form 0^k1^{n-k} or 1^k0^{n-k} for some k, $0 \geq k \geq n$. [15]

CC systems, which correspond to actual planar point sets satisfy all of the axioms considered above; hence every point p in a CC-system on n points has an associated vortex-free tournament defined by a string $\alpha_1\alpha_2\dots\alpha_{n-1}$ on the remaining points. This string has a natural geometric interpretation: it represents the order in which the remaining points are encountered when a straight line through p rotates counterclockwise through 180°. The positive elements of $\alpha_1\alpha_2\dots\alpha_{n-1}$ are the points to the left of the initial position of this sweep line; the negative elements are those to the right. If the sweep line is given a suitable direction, the positive elements are all encountered "ahead" of p and the negative ones are all encountered "behind" p. We have the counterclockwise

[14] π' is defined as $\pi'(\delta) = \text{sign}(\delta)\pi(|\delta|)$ as before.

[15] We assume $0^0 = 1^0 = \varepsilon$, where ε denotes an empty string.

triple $p\alpha_1\alpha_k$ iff α_1 and α_k have the same sign. When the sweep line passes α_1, point $|\alpha_1|$ passes to the other side, and the process continues in the same way on $\alpha_2 \ldots \alpha_{n-1}\bar{\alpha}_1$. The $2(n-1)$ different strings $\alpha_1\alpha_2 \ldots \alpha_{n-1}$ that define p's tournament correspond to the different initial positions and orientations of the sweep line.

Both Lemma 4.1. and Corollary 4.2. do not make use of Axiom 1, so vortex-free tournaments characterize all sets of triples that satisfy Axioms 2, 3, 5, and 5′. We call these *weak pre-CC systems* as discussed by Knuth. The number of weak pre-CC systems on n labelled points is exactly $\left(\frac{2^{n-1}(n-1)!}{2(n-1)}\right)^n = \left(2^{n-2}(n-2)!\right)^n =^{16} 2^{\Theta(n^2 \log n)}$, because this is the number of ways to define n independent vortex-free tournaments on $n-1$ points. This is substantially smaller than the total number $2^{\binom{n}{3}} \sim 2^{n^3}$ of triple systems that are required to satisfy only Axioms 2 and 3.

If we allow signed points to appear in triples, so that negating a point in a triple system complements the value of all triples that contain that point, we have the following theorem: A set of triples is a pre-CC system iff it can be obtained from a CC-system by negating a subset of its points[17]. Negating a point preserves Axioms 1, 2, and 3, and it interchanges Axioms 5 and 5′. Therefore, pre-CC systems are closed under negation, and any system obtained from a CC-system by repeated negation must be pre-CC. Conversely, let a and b be any points of a pre-CC system. Negate the remaining points p if necessary so that abp holds for all p. We will show that the resulting system is a CC-system. Axiom 5 implies that the tournament for a is transitive. It follows that Axiom 4 cannot be violated by four points that include the point a; we cannot have $(apq \wedge aqr \wedge arp \wedge rqp) \vee (aqp \wedge arq \wedge apr \wedge pqr)$ when the tournament for a is transitive. Moreover, any four points $\{p,q,r,t\}$ different from a can be ordered such that $p \to q \to r \to t$ in a's tournament. Suppose the tournament for t is defined by the string $\alpha_1 \ldots \alpha_{n-1}$, where $\alpha_1 = a$. Then p, q, r must occur in this string with a positive sign, because we have tap, taq, and tar. Hence the restriction of the tournament for t to $\{p,q,r\}$ is transitive, and $\{p,q,r,t\}$ cannot violate Axiom 4.

As a consequence, a pre-CC system for which at least one point is associated with a transitive tournament is a CC-system.[18] The converse is also true: Every CC-system with at least three points has at least three points associated with a transitive tournament. Such points may be called *extreme points*, since a point of a realizable CC system is associated with a transitive tournament if and only if it lies on the convex hull.

Two pre-CC systems are said to be *preisomorphic* if there is a signed bijection σ[19] that carries one into the other in such a way that pqr holds in the first

[16] $= \left(2^{n-2+\log(n-2)!}\right)^n \sim \left(2^{n+n\log n}\right)^n = 2^{\Theta(n^2 \log n)}$.

[17] due to D. Knuth.

[18] Thus, any system arising from planar point sets that satisfies axiom 5 automatically satisfies axiom 4.

[19] For the purposes of this thesis, a *signed bijection* is a one-to-one mapping from one set of signed points to another that maps $\alpha \mapsto \beta$ if and only if it maps $\bar{\alpha} \mapsto \bar{\beta}$. A *signed permutation*

iff $\sigma(p)\sigma(q)\sigma(r)$ holds in the second. We just showed that every pre-CC system is preisomorphic to a CC-system. Nonisomorphic CC-systems can sometimes be preisomorphic: there are exactly three isomorphism classes of CC-systems on 5 elements, all preisomorphic to each other. It follows that every pre-CC system on 5 elements can be obtained by negation and renaming of points from the CC-system that corresponds to the vertices of a pentagon.

It is easy to see which CC-systems are preisomorphic to n-gons, because we merely need to determine which points can be negated without violating Axiom 4. Axiom 4 applies to subsets of 4 points, and all 4-element subsystems of an n-gon are equivalent to the vertices of a square. If the four points are (a, b, c, d) in counterclockwise order, the valid triples are abc, bcd, cda, and dab; and it is easy to verify that the 16 possible negations all produce CC-systems except when we map (a, b, c, d) into $(\bar{a}, b, \bar{c}, d)$ or $(a, \bar{b}, c, \bar{d})$. Thus we obtain a CC-system from an n-gon by negation if and only if the negated vertices are consecutive. When we negate k consecutive vertices of an n-gon, the resulting CC-system is equivalent to the sets of points obtained by placing an $(n-k)$-cup sufficiently far above a k -cap. If $k > 1$ and $n - k > 1$, the $(n - k)$-cup and k -cap are both nondegenerate, and the convex hull of the system will be of size 4. Precisely $\lfloor (n+1)/2 \rfloor$ nonisomorphic CC-systems can be obtained in this way, because the negation of $n - k$ consecutive points is essentially the same as the negation of k.

An n-gon, $n > 4$, has exactly $2n$ preautomorphisms, generated by the cyclic shift $\sigma = (1, 2, \ldots, n) \mapsto (2, \ldots, n, 1)$ and by the negated reflection $\rho = (1, 2, \ldots, n) \mapsto (\bar{n}, \ldots, \bar{2}, \bar{1})$; the mapping $\sigma\rho\sigma\rho$ is the identity.

If p and q are signed points of a pre-CC system and if p' and q' are signed points of another, there is at most one preisomorphism σ with $\sigma(p) = p'$ and $\sigma(q) = q'$. For if the tournament for p is defined by the string $\alpha_1\alpha_2\ldots\alpha_{n-1}$ where $\alpha_1 = q$, and if the tournament for p' is defined by $\alpha'_1\alpha'_2\ldots\alpha'_{n-1}$ where $\alpha'_1 = q'$, then we must have $\sigma(\alpha_k) = \alpha'_k$ for all k. Notice that the tournament for p is defined by $\alpha_1\ldots\alpha_{n-1}$ iff the tournament for $\bar{p}$ is defined by $\alpha_{n-1}\ldots\alpha_1$.

Suppose two CC-systems are preisomorphic under the signed bijection σ, and suppose $\sigma(p)$ is positive for all extreme points. In other words, we are assuming that whenever p has a transitive tournament in the first CC-system, $\sigma(p)$ is a positive point of the second system. We can prove that $\sigma(p)$ must then be positive for all p. Let $\tau(p) = |\sigma(p)|$ be the ordinary (unsigned) bijection corresponding to σ; if the claim is false, we have $\tau(s) = \overline{\sigma(s)}$ for some s. Let p, q, r be extreme points; then $pqr \iff \tau(p)\tau(q)\tau(r)$, $pqs \iff \neg\,\tau(p)\tau(q)\tau(s)$, $qrs \iff \neg\,\tau(q)\tau(r)\tau(s)$, $rps \iff \neg\,\tau(r)\tau(p)\tau(s)$. Since s is not an extreme point, we can choose p, q, r so that $s \in \Delta pqr$ by letting q and r be the extreme points closest to s in the tournament for p. But then Axiom 4 is violated in the second CC-system.

If p is an extreme point of any CC-system, we obtain a preisomorphic CC-system by negating p (i.e., by mapping $p \mapsto \bar{p}$ and leaving all other points

is a signed bijection from a set of signed points to itself. Clearly, there are $2^n n!$ signed permutations on n elements, because there are $n!$ ways to choose the absolute values of the images and 2^n ways to choose the signs.

unchanged); this follows from the corollary above, because $\bar{p}$ has a transitive tournament.

Now suppose two CC-systems are preisomorphic under σ, and let k be the number of negated points. We call k the *distance* between the two systems under σ. If the original systems are not isomorphic, there must be an extreme point p in the first system for which $\sigma(p)$ is negative. Negating p gives us another CC-system whose distance from the second system is only $k-1$ under σ', where σ' is the mapping $\sigma'(x) = \sigma(x)$ if $|x| \neq p$, $\overline{\sigma(x)}$ if $|x| = p$. Therefore we can go from one CC-system to any other preisomorphic CC-system by repeatedly negating extreme points.

Naturally, now we would like to glue the tournaments for n individual points together in all possible compatible ways in order to generate all pre-CC systems on n points.

It appears that incrementally fitting tournaments together one by one is increasingly complex. The conditions that should be placed on strings so that they define compatible tournaments appear to be overrestrictive and unlikely to produce complete enumeration of all pre-CC systems. However, it turns out to be fairly easy to construct the tournaments in parallel. Instead of thinking of a single directed line that sweeps around one vertex at a time, let us imagine a family of parallel lines, one passing through each point, each directed consistently. If these lines revolve at the same rate, the moment when point p enters into the tournament for q will be the same as the moment when $\bar{q}$ enters the tournament for p; this occurs when the lines through p and q cross, with p visible in the positive direction from q and q visible in the negative direction from p. We can represent this situation by writing ${p \atop q}$.

As the parallel lines sweep through 180°, each pair of points $\{p,q\}$ will be encountered exactly once, either in the form ${p \atop q}$ or ${q \atop p}$. From these $\binom{n}{2}$ ordered pairs, we can write down strings defining the vortex-free tournaments associated with each point as before, appending p to string q and $\bar{q}$ to string p when the pair ${p \atop q}$ appears.

Of course, not every arrangement of ordered pairs will be legal, since we want to define a pre-CC system, not just a weak pre-CC system. Thus the tournament for p must contain the arc $q \to r$ iff the tournament for q contains $r \to p$. The three ordered pairs involving $\{p,q,r\}$ will have at least one variable (say p) occurring both on top and on the bottom, say as ${p \atop r}\ {q \atop p}$. Then p's tournament will contain $q \to r$; so the tournament for q will be consistent only if ${q \atop r}$ precedes ${q \atop p}$ or ${r \atop q}$ follows ${q \atop p}$, and the tournament for r will be consistent only if ${r \atop q}$ precedes ${p \atop r}$ or ${q \atop r}$ follows ${p \atop r}$. The only way to make both of them consistent is to have ${q \atop r}$ between ${p \atop r}$ and ${q \atop p}$. Similarly, if the pairs involving p are ${r \atop p}\ {p \atop q}$, we must have ${r \atop q}$ between them.

The above yields the necessity of the following *betweenness rule*, if $\binom{n}{2}$ ordered pairs are supposed to define a pre-CC system: if ${p \atop q}$ and ${r \atop p}$ occur (in either order), then ${r \atop q}$ occurs between them. Conversely, if an arrangement of $\binom{n}{2}$ ordered pairs obeys the betweenness rule, they define n strings for vortex-free tournaments in which all triples pqr, qrp, rpq have the same value. Therefore

they define a pre-CC system.

In fact, they define a CC-system. Given an arrangement of $\binom{n}{2}$ ordered pairs satisfying the betweenness rule, let's say that $p \succ q$ if ${p \atop q}$ appears. Then $r \succ p$ and $p \succ q$ implies $r \succ q$, so the relation is transitive. The points can therefore be listed in order $(p_1, p_2, \ldots, p_n)$ so that $p_j \succ p_k \iff j > k$. Point p_1 occurs only in the lower row, so its tournament is defined by a string with no negated entries. Thus p_1 has a transitive tournament, and we know that this guarantees a CC-system.

Suppose we are given n vortex-free tournaments associated with points labeled $\{1, \ldots, n\}$, where the string α_p defining the tournament for p contains the points $\{\bar{1}, \ldots, \overline{p-1}, p+1, \ldots, n\}$. The tournaments are assumed to be consistent; i.e., if $q \to r$ in α_p, then $r \to p$ in α_q and $p \to q$ in α_r. Each string α_p can be represented as a sequence of ordered pairs, using ${p \atop q}$ for $\bar{q}$ and ${q \atop p}$ for q. We will show how to construct an arrangement of all $\binom{n}{2}$ pairs, containing $\alpha_1, \alpha_2, \ldots, \alpha_n$ as subarrangements. The construction proceeds by induction on n: First we delete 'n' from $\alpha_1, \ldots, \alpha_{n-1}$ and arrange the $\binom{n-1}{2}$ pairs ${p \atop q}$ for $1 \le q < p < n$ in some manner consistent with the remainder of $\alpha_1, \ldots, \alpha_{n-1}$. Then we divide the pairs into two classes, assigning ${p \atop q}$ to class L if $\bar{p}$ follows $\bar{q}$ in α_n and to class R if $\bar{p}$ precedes $\bar{q}$ in α_n. Notice that ${p \atop q}$ is in R iff npq is true iff p follows n in α_q iff $\bar{q}$ follows n in α_p. Therefore we will be done if all pairs of L precede all pairs of R; the pairs of α_n can then all be inserted between L and R.

If the construction runs into trouble, there must be a pair ${p \atop q}$ of R immediately followed by a pair ${r \atop s}$ of L. We can interchange those pairs if p, q, r, s are distinct, obtaining an arrangement with one less problematic R before L, because the new arrangement will still be consistent with $\alpha_1, \ldots, \alpha_{n-1}$. If p, q, r, s are not distinct, suppose $p = r$. Then npq is true and nps is false, so $\bar{q}$ follows n and n follows $\bar{s}$ in α_p. Therefore $\bar{q}$ follows $\bar{s}$ in α_p, contradicting the fact that ${p \atop q}$ precedes ${p \atop s}$. Similarly, if $q = s$ we reach a contradiction after noting that p would have to follow n and n would have to follow r in α_q. The only other possibilities are $p = s$ or $q = r$; but these violate the betweenness condition, so such an arrangement cannot be consistent with $\alpha_1, \ldots, \alpha_{n-1}$. We have proved that the construction will eventually succeed, after possibly interchanging pairs that don't overlap.

For example, suppose

$$\alpha_1 = 3\,4\,2\,5\,, \quad \alpha_2 = 3\,4\,\bar{1}\,5\,, \quad \alpha_3 = \bar{2}\,\bar{1}\,5\,4\,, \quad \alpha_4 = \bar{2}\,\bar{1}\,5\,\bar{3}\,, \quad \alpha_5 = \bar{1}\,\bar{2}\,\bar{4}\,\bar{3}\,.$$

We begin by setting $n = 2$ and suppressing all entries that are at most 2, trivially obtaining ${2 \atop 1}$. Then n advances to 3, and we obtain ${3 \atop 2}\,{3 \atop 1}\,{2 \atop 1}$ since ${2 \atop 1} \in R$. Then n advances to 4; now we have ${3 \atop 2}$ and ${3 \atop 1} \in L$ and ${2 \atop 1} \in R$, so we obtain the sequence

$$\begin{matrix} 3 & 3 & 4 & 4 & 4 & 2 \\ 2 & 1 & 2 & 1 & 3 & 1 \end{matrix}\,.$$

Finally, when $n = 5$, all pairs except ${4 \atop 3}$ are in L, so we interchange ${4 \atop 3}$ with ${2 \atop 1}$ and obtain

$$\begin{matrix} 3 & 3 & 4 & 4 & 2 & 5 & 5 & 5 & 5 & 4 \\ 2 & 1 & 2 & 1 & 1 & 1 & 2 & 4 & 3 & 3 \end{matrix}$$

Theorem 4.3. *(Knuth)* Every arrangement of the $\binom{n}{2}$ distinct ordered pairs $\substack{p\\q}$ of n given points as an ordered list satisfying the betweenness condition defines a CC-system. Conversely, every CC-system can be defined by such an arrangement.

Proof. The first part has already been proven. To prove the converse, we use the fact that any CC-system has a point whose tournament is transitive. Thus we can number the points $\{1, \ldots, n\}$ so that the tournament associated with 1 is defined by the string $\alpha_1 = 2\,3\,\ldots\,n$, and the tournaments associated with $2, \ldots, n$ are defined by strings $\alpha_2, \ldots, \alpha_n$ beginning with $\bar{1}$. It follows that the elements of α_p are $\{\bar{1}, \ldots, \overline{p-1}, p+1, \ldots, n\}$, and the construction above can be used to produce the desired arrangement. $\square$

The arrangements in the preceding theorem are closely related to primitive sorting networks. A *sorting network* is a sequence of comparators (compare-exchange elements) that sort any given sequence $(x_1, \ldots, x_n)$. A sorting network is called *primitive* if only adjacent elements are allowed to be exchanged. It is a known result that if a primitive sorting network changes an array $(x_1, x_2, \ldots, x_n)$ into its reflection $(x_n, \ldots, x_2, x_1)$, it properly sorts an arbitrary sequence. Thus, primitive sorting network are equivalent to reflection networks[20] Any reflection network for n elements consists of exactly $\binom{n}{2}$ transpositions, because every adjacent transposition decreases the number of inversions by 1 if we begin with the array $(n, \ldots, 2, 1)$; this array has $\binom{n}{2}$ inversions, and the final array $(1, 2, \ldots, n)$ has none. We can construct reflection networks easily by starting with $(n, \ldots, 2, 1)$ and repeatedly exchanging any two adjacent elements that happen to be out of order; after $\binom{n}{2}$ steps we will surely arrive at $(1, 2, \ldots, n)$, and the sequence of operations performed will be a reflection network.

We have observed that every arrangement of $\binom{n}{2}$ pairs that satisfies the betweenness rule defines a linear order. In fact, reflection networks are in one-to-one correspondence with betweenness arrangements having a given linear order. If we number the points 1 to n according to the linear order, then the arrangement specifies a sequence of adjacent interchanges that converts $(n, \ldots, 2, 1)$ into $(1, 2, \ldots, n)$. For if the first pair is $\substack{p\\q}$, we must have $p = q + 1$; otherwise there would be an r such that $\substack{p\\r}$ and $\substack{r\\q}$ both appear, without $\substack{p\\q}$ between them. Therefore interchanging p with q is an adjacent interchange in $(n, \ldots, 2, 1)$. Removing $\substack{p\\q}$ from the left, placing $\substack{q\\p}$ at the right, and interchanging the labels of p and q allows us to repeat this argument $\binom{n}{2}$ times.

Pairs of transpositions are said to commute if they can be performed in either order, or simultaneously, without changing their effect. Similarly, two ordered pairs $\substack{p\\q}\,\substack{r\\s}$ can be interchanged in a betweenness arrangement without affecting the corresponding CC-system, if $\substack{p\\q}$ and $\substack{r\\s}$ have no points in common. We call two reflection networks *equivalent* if they can be obtained from each other by interchanging transpositions that commute.

[20] *Reflection network* is a sequence of adjacent transpositions $[i, i+1]$, which changes an array $(x_1, x_2, \ldots, x_n)$ into its reflection $(x_n, \ldots, x_2, x_1)$.

One way to eliminate the effect of commutation is to bring each transposition as far to the left as possible, producing a compressed canonical form. Equivalent networks have the same compressed form. The odd-even transposition sort has the shortest compressed form, among all reflection networks for n points.

The CC-system defined by an arrangement is unchanged if we remove the first pair $\substack{p\\q}$ and append $\substack{q\\p}$ as a new last pair. The corresponding operator on reflection networks removes the first transposition $[i, i+1]$ and appends $[n-i, n-i+1]$ at the end. Reflection networks are called *weakly equivalent* if they can be obtained from each other by commutativity and/or end-around moves.

Theorem 4.4. *(Knuth)* Two arrangements of $\binom{n}{2}$ pairs satisfying the betweenness condition yield the same CC-system if and only if one can be obtained from the other by interchanging disjoint pairs and/or removing the first pair $\substack{p\\q}$ and appending $\substack{q\\p}$ at the end. The number of CC-systems on $\{1, 2, \ldots, n\}$ such that npq holds iff $n > p > q$ is the number of equivalence classes of reflection networks on $n-1$ elements. The number of nonisomorphic CC-systems on n points is the number of weak equivalence classes of reflection networks on n elements.
Proof. If two arrangements define the same CC-system, we can use the stated operations to transform the associated networks into almost-canonical form for some extreme point x. These forms must be identical, or different tournaments will be defined. A CC-system on $\{1, 2, \ldots, n\}$ with the stated property defines and is defined by a unique arrangement beginning with $\substack{n\\n-1} \cdots \substack{n\\2}\,\substack{n\\1}$, up to interchange of disjoint pairs. $\square$

The left-right mirror image of a reflection network is a reflection network that corresponds to an *anti-isomorphic/* CC-system: If pqr is true in the CC-system defined by the original network, then pqr is false in the new system, and conversely. All reflection networks on 5 or fewer elements are weakly equivalent to their mirror images.

Reflection networks are said to be *preweakly equivalent* if they can be transformed into each other by using this flip operation together with the operations associated with weak equivalence. From the analysis earlier in the chapter it follows that any two preisomorphic CC-systems correspond to reflection networks that are preweakly equivalent.

Reflection networks have an important relationship to *simple arrangements of pseudolines*. Such an arrangement consists of n simple closed curves in the projective plane, with the property that every pair of curves intersects exactly once; the $\binom{n}{2}$ intersection points must also be distinct.

An arrangement of pseudolines is called *stretchable* if the lines can all be made straight without changing the topological configuration of cells. Stretchable arrangements correspond to realizable CC-systems; hence a CC-system that is preisomorphic to a realizable CC-system is realizable. Grünbaum conjectured that every arrangement of eight pseudolines is stretchable. Goodman and Pollack [GP80b] proved Grünbaum's conjecture using the idea of allowable sequences of permutations (to be introduced in the next section).

Corollary 4.5. The number of nonpreisomorphic CC-systems on n points is the number of preweak equivalence classes of reflection networks on n elements, and it is also the number of topologically different simple arrangements of n pseudolines in a projective plane.

CC-systems also turn out to be equivalent to uniform acyclic oriented matroids of rank 3 [Knu92]. There is a 2-to-1 correspondence between CC-systems on a set of labeled points and all such oriented matroids defined on the same set. The two CC-systems with the same image under this correspondence are obtained from each other by negating the value of every triple pqr.

4.1.2 Remark about the complexity of CC-systems

Certainly, we would like to know the complexity of decision problems involving CC-systems. Unfortunately, the axioms, although simple, can lead to situations that are difficult to decide in general. Knuth showed that the problem of deciding whether a set of fewer than $\binom{n}{3}$ triples can be completed to a full set that satisfies Axioms 1-5 is NP-complete. In fact, seemingly much simpler problem of deciding whether additional arcs can be added to make a directed graph into a vortex-free tournament, is NP-complete.

4.2 Allowable Sequences of Permutations

In a number of papers ([GP84], [GP92], [GP80a]), Goodman and Pollack have introduced the notion of an *allowable sequence of permutations*, which encodes in combinatorial terms the order properties of a point configuration or arrangement of lines. This notion has since proven fruitful in solving a number of problems on configurations and arrangements[21], and, as we shell see, it provides a natural setting for investigating the problems addressed in the title.

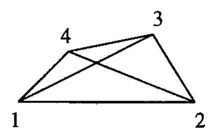
Let $C = \{p_1, \ldots, p_n\}$ be a numbered configuration of n points in the Euclidean plane, and consider a directed line L not orthogonal to any direction determined by any two points of C. If we project C onto L orthogonally, we get a permutation of $[1, n]$. As L rotates counterclockwise, the permutation changes whenever L passes through a direction orthogonal to that determined by points of C. Allowing L to continue rotating about a fixed point we obtain a periodic sequence of permutations, called the *circular sequence of permutations induced by* C.

For example, the following sequence is associated with the quadrilateral on the figure below:

$$\cdots 1432 \overset{32}{\to} 1423 \overset{42}{\to} 1243 \overset{12}{\to} 2143 \overset{43}{\to} 2134 \overset{13}{\to} 2314 \overset{14}{\to}$$

$$2341 \overset{23}{\to} 3241 \overset{24}{\to} 3421 \overset{21}{\to} 3412 \overset{34}{\to} 4312 \overset{31}{\to} 4132 \overset{41}{\to} 1432 \cdots$$

[21] For example, Ungar completely solved the problem on the minimum number of directions determined by n points using allowable sequences. Goodman and Pollack proved Helly's theorem, Randon's theorem, and some others entirely in the context of allowable sequences.



Goodman and Pollack noticed that the infinite periodic sequence of permutations of $[1, n]$ arising this way has the following properties:

(i) The move from one term to the next consists of reversing one or more nonoverlapping substrings of the term.

(ii) If a move results in the reversal of a pair ij then every other pair of indices that is not reversing together with ij is reversed in some move before the next move in which the pair ij is reversed again. This guarantees that each period of the sequence breaks up into two half-periods, with each move of the first half reversed in the second.

Clearly, if C is nondegenerate (no three points collinear), the associated sequence has period of precisely $2\binom{n}{2} = n(n-1)$, and successive permutations differ only in having the order of two adjacent numbers switched.

Definition. Any sequence of permutations of $[1, n]$ satisfying two properties above is called an *allowable sequence of permutations.*

The allowable sequence Π associated with a configuration C encodes many of its geometric properties. For example,

1. The points p_i,p_j,p_k are collinear iff the indices i, j, k occur in the same string reversal.

2. p_i is an extreme point iff i is the first index in some permutation of Π (and therefore the last in some other). Furthermore, if the extreme points of C occupy first indices in Π in the circular order $\cdots, i_1, \cdots, i_k, i_1, \cdots$, this circular order is the counterclockwise order of the extreme points.

3. The lines p_ip_j and p_kp_m are parallel iff the indices i, j and k, m are in disjoint substrings which reverse in the same move.

4. p_i is in the convex closure of $p_{i1}, \ldots, p_{ik}$ iff there is no term of Π in which i precedes (hence follows) all of $i_1, \ldots, i_k$.

5. $p_ip_jp_k$ has counterclockwise orientation iff $(ij \prec ik)$, where the notation $(i_1j_1 \prec i_2j_2)$ means that within the *same* half-period of Π following the move in which the indices i_1 and j_1 (in that order) switch, the indices i_2 and j_2 (in that order) switch in the same or in the subsequence move. Note that $(\forall i, j, k)[(ij \prec jk) \Rightarrow (ij \prec ik \prec jk)]$.

6. Line p_ip_j separates p_k from p_m iff when i and j switch, k and m are on opposite sides of the substring containing i and j which reverses.

With each point we associate its *local sequence*, which is the sequence of switches involving this point. Each term of the local sequence specifies the order in which the points lie on a directed line which rotates counterclockwise through i. We will use this ordering for constructing fan graphs in the Erdös-Szekeres problem.

Even though there are allowable sequences that cannot be realized by planar point sets (thus, by duality, by planar line arrangements), every allowable sequence is realizable by an *arrangement of pseudolines* in real projective plane, i.e. a finite collection of topological lines, none of which separates the plane, and any two of which meet exactly once (and necessarily cross there).

A nice visualizations of arrangements of pseudolines are given by their *wiring diagrams.* Let W be a wiring diagram of a simple arrangement of size n. An arrangement is *simple* if no three pseudolines have a common point of intersection. For each vertical cut of W where no crossing takes place, the vertical order of the pseudolines at the point of the cut is a permutation of $[1, n]$. Assuming that no two crossings of W have the same abscissa position, we obtain $\binom{n}{2} + 1$ different permutations. The sequence of these permutations is, in fact, a simple allowable sequence.[22] Conversely, a simple allowable sequence is easily transformed into a wiring diagram and, hence, an arrangement of pseudolines. Note, however, that many allowable sequences may correspond to the same arrangement. Consecutive pairs of crossings that have no pseudoline in common can be interchanged without changing the topology of the arrangement. Moving a crossing from the left to the right (and turning it upside down) preserves the arrangement as well. The "mirror flip" transformation also preserves the arrangement.

Simple allowable sequences have 1-to-1 correspondence with the reflection networks discussed in the previous section. Alternatively, they correspond to the maximal chains in the weak Bruhat order of the symmetric group. In the last context their number has been determined by Stanley [Sta84] (to be discussed when studying different equivalence relations in the next chapter).

The main purpose of our formulation of problems in combinatorial terms is that it is, at least in principle, finite and testable on a computer. Clearly, whatever we can prove about allowable sequences or CC-systems will immediately follow about configurations of points; the converse as we have already seen, is false, since there are allowable sequences/CC-systems that are not associated with any planar configurations. Nevertheless, it has proven to be fruitful to examine planar problems in the context of allowable sequences/CC-systems: if the problem is already solved, such an examination may reveal what is essential in the solution; if not, it may provide new approaches to the problem.

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[22] Note that (i) the first element of Π is the identity permutation $(1, 2, \cdots, n)$ and the last element of Π is the reverse permutation $n, n-1, \cdots, 1$; (ii) two consecutive permutations in Π differ by the reversal of an adjacent pair. Also observe that the pseudolines in a simple arrangement divide the projective plane into $\binom{n}{2} + 1$ cells, one to the right of each crossing and one at the exterior.

4.2.1 Proving an upper bound on the number of halving lines in the context of allowable sequences

Although the combinatorial problems connected to CC-systems and allowable sequences of permutations are of interest in their own rights, we want to go back to our original problems and see how our combinatorial reformulation may suggest fresh approaches to the problems.

Let S be a nondegenerate configuration of n numbered points in the plane. Let C be a circular sequence of permutations induced by S. Recall that the circular sequence of C has period $2\binom{n}{2} = n(n-1)$. Two properties of circular sequences are important for the subsequent discussion: (i) Two successive permutations of a circular sequence differ only by having the order of two adjacent numbers switched. Again, this follows from the assumption that the points are is general position. Thus, a permutation is obtained from its predecessor as the sweeping line rotates through the direction perpendicular to the line through two points i and j. Clearly, i and j are adjacent in both permutations while their order differs. The positions of the other points cannot change simultaneously. (ii) Each of the $n(n-1)/2$ switches occurs exactly once in any subsequence of $n(n-1)/2+1$ permutations of a circular sequence for n points. This property corresponds to the fact that as the sweeping line rotates through 180° it defines $n(n-1)/2+1$ permutations and rotates perpendicular through each line defined by two points of the set.

Definition. An infinite sequence of permutations of the numbers $1, 2, \cdots, n$ satisfying two properties above is called a *simple allowable circular sequence.* Note that the two properties above imply a period of length $n(n-1)$. As an immediate consequence of the second property, the ordered switch ij occurs exactly $n(n-1)/2$ steps after the ordered switch ji. The number of halving lines in S corresponds to the number of flips occurring between the middle two elements of the permutations.

Definition. A subset A of $\{1, 2, \cdots, n\}$, n is even, is called an *allowable half-set* of an allowable circular sequence C of permutations of $1, 2, \cdots, n$ if A contains $n/2-1$ numbers and there exists a permutation in C such that the $n/2-1$ numbers of A occur at the leftmost $n/2-1$ positions.

Denote by $g(C)$ the number of allowable half-sets realized by the circular sequence C. Let $g(n) = \max_{|C|=n} g(C)$. Evidently, $h(n) \leq g(n)$. Thus, an upper bound on $g(n)$ is also an upper bound on $h(n)$.

Observe that $g(C)$ equals the number of switches at adjacent middle positions $n/2-1$ and $n/2$ among $n(n-1)+1$ successive permutations of C.

Clearly, each switch at positions $n/2-1$ and $n/2$ defines a resulting allowable half-set. It is also easy to see that no allowable half-set is defined more than once. Thus, we eventually consider the maximum number of switches that can place at positions $n/2-1$ and $n/2$ during $n(n-1)+1$ successive permutations.

Let $C_{\mathcal{M}}$ denote an allowable circular sequence of $1, 2, \cdots, n$ maximizing[23] the

[23]that is realizing $g(n)$ switches

number of switches at positions $n/2-1$ and $n/2$. Let C denote a subsequence of $C_{\mathcal{M}}$ which consists of $n(n-1)/2+1$ successive permutations such that at least $g(n)/2$ switches occur at the middle positions. Recall that each possible not ordered switch occurs exactly once in C. Let $P = 1, 2, \cdots, n$ be the first permutation of C and let $Y = i, i+1, \cdots, j$ for $1 \leq i \leq j \leq n$ denote a subsequence of P. The subsequences to the left and to the right of Y are denoted by X and Z, respectively. Hence, $P = XYZ$. Let y denote the length of Y, that is, the number of numbers in Y. We will see that y contributes to switches at the middle positions.

Lemma 4.6. *(Edelsbrunner and Welzl [EW85])* Let C be an allowable circular sequence of $1, 2, \cdots, n$. Then at most $\binom{y}{2} + \min\{n-y, n-2\}$ of the switches in C at positions $n/2-1$ and $n/2$ involve a number of Y.

Proof. It is trivial to observe that at most $\binom{y}{2}$ of the relevant switches involve two numbers of Y. After those switches, Y is totally reversed. It remains to derive a bound on those relevant switches which involve exactly one number of Y. To this end we perform certain transformations on C such that the contribution of the numbers in Y does not change and such that the switches involving two numbers of X or two numbers of Z are performed as late as possible, i.e. C can be transformed such that (i) the contribution of the numbers in Y does not change, and (ii) no switch involving two numbers of either of X and Z is performed before the last switch (at positions $n/2-1$ and $n/2$) involving a number of Y, is completed. Let us establish this fact. To facilitate the proof, consider the smallest positive integer such that the switch leading from the fth permutation P_f in C to the $(f+1)$st permutation P_{f+1} involves two numbers i and $i+1$ of X for some i. We delete P_{f+1} from C and append a permutation P^*_{f+1} to the end of C. In all permutations between P_f and P^*_{f+1}, the numbers i and $i+1$ are replaced by each other. P^*_{f+1} is chosen to differ from its predecessor only by having the order of i and $i+1$ changed. Applying this local transformtation repeatedly proves the assertion. Thus, in the transformed sequence of permutations, no number of X is able to move to the left before Y has completed its last contribution to the switches at positions $n/2-1$ and $n/2$. Analogously, no number of Z is able to move to the right before this happens. It follows immediately that each one of the number in X and Z can only once be the partner of a number in Y when it is involved in a switch at the middle positions. In addition, at most $n/2-1$ numbers of X can move from position $n/2-1$ to $n/2$, before the last contribution of Y to the number of switches at these positions took place. This completes the proof. □

Theorem 4.7. *(Edelsbrunner and Welzl [EW85])* $h(n) = O(n^{3/2})$.

Proof. Let $P = 1, 2, \cdots, n$ be the first permutation of C. Partition P into subsequences with $\lfloor (n/2-1)^{1/2} \rfloor$ or $\lfloor (n/2-1)^{1/2} \rfloor - 1$ numbers each. Due to Lemma above, the contributions to the number of switches at the middle positions of any one of those subsequences is at most $\binom{(n/2-1)^{1/2}}{2} + (n-2) < 3(n/2-1)$. There are at most $\lfloor n/(\lfloor (n/2-1)^{1/2} \rfloor - 1) \rfloor$ subsequences for $(n/2-$

$1) \geq 4$, i.e. for $n \geq 10$, and at most n subsequences for $(n/2 - 1) < 4$, i.e. for $n < 10$. $g(n) = O(n^{3/2})$ follows since $g(n)$ is at most twice the number of switches in C at the relevant positions. Since $h(n) \leq g(n)$, $h(n) = O(n^{3/2})$. □

4.3 Equivalence classes and Enumeration

I don't believe in mathematics.
– Albert Einstein.

An outstanding problem problem in combinatorial geometry has long been to classify, in a reasonable way, non-degenerate configurations of n points in the plane (or in Euclidean space of any dimension) into finitely many "essentially distinct" classes. Any classification scheme can be described by mapping the set of nondegenerate configurations of n points into some finite set and identifying configurations with the same image. Clearly, the main advantage of such mapping is that the number of images (equivalence classes formed by the mapping) is finite and, thus, at least in principle, testable on a computer!

Enumeration of different equivalence classes suggested itself as deserving to be studied in its own right.

Let A_n be the total number of reflection networks on n elements, alternatively, the total number of simple allowable sequences on n numbered points, and let B_n, C_n, D_n, E_n be the corresponding number of equivalence classes, weak equivalence classes, weak equivalence/anti-equivalence classes, and preweak equivalence classes, as discussed by Knuth. B_n is also the number of non-isomorphic simple arrangements of size n. We have seen that the number C_n is the number of nonisomorphic CC-systems on n points. D_n is also the number of nonisomorphic uniform acyclic oriented matroids of rank 3 on n elements. Alternatively, D_n is the number of topologically distinct, simple arrangements of pseudolines with marked cell, as discussed in the previous section.

Stanley [Sta84] proved that the number of simple allowable sequences on n indices is precisely

$$A_n = \frac{\binom{n}{2}!}{\prod_{k=1}^{n-1}(2n - 2k - 1)^k},$$

as the number of maximal chains in the weak Bruhat order of the symmetric group. Other combinatorial and algebraic explanations of this formula have been found by Edelman and Greene [EG87] [24].

Knuth gives an enumeration of the above equivalence classes for $n \leq 9$.

n	1	2	3	4	5	6	7	8	9
A_n	1	1	2	16	768	292864	1100742656	48608795688960	29258366996258488320
B_n	1	1	2	8	62	908	24698	1232944	112018190
C_n	1	1	1	2	3	20	242	6405	316835
D_n	1	1	1	2	3	16	135	3315	158830
E_n	1	1	1	1	1	4	11	135	4382

[24] who prove this formula via a combinatorial bijection between different types of tableaux.

The numbers B_n and C_n are related by

$$B_{n-1}/n \le C_n \le B_{n-1}\,,$$

at least because a weak equivalence class on n elements has at most n almost-canonical forms, and there are B_{n-1} almost-canonical forms. Obviously

$$C_n/2 \le D_n \le C_n\,.$$

We also have

$$D_n \bigg/ \binom{n}{2+1} \le E_n \le D_n\,,$$

because we get at most $\binom{n}{2}+1$ preweakly inequivalent networks from a given weak equivalence/anti-equivalence class by moving the "exterior" into each cell of the corresponding pseudoline arrangement. (This is much better than the obvious bound $C_n/2^n \le E_n$ that we get by simply counting the number of ways to negate points. Most point-negations give a pre-CC system that violates Axiom 4.)

A_n is asymptotically $2^{\Theta(n^2\log n)}$ since $A_n = 2^{\log\binom{n}{2}! - \log\prod_{k=1}^{n-1}(2n-2k-1)^k}$, very roughly $\log\binom{n}{2}! \sim \binom{n}{2}\log\binom{n}{2} \sim n^2\log n$, and we also have

$$\begin{aligned}\log\prod_{k=1}^{n-1}(2n-2k-1)^k &= \sum_{k=1}^{n-1} k\log(2n-2k-1)\\ &< \sum_{k=1}^{n/2-1} k\log 2n + \sum_{k=1}^{n/2}(n-k)\log n\\ &< \sum_{k=1}^{n/2} n\log 2n = \tfrac{1}{2}\,n^2\log 2n\,.\end{aligned}$$

The table shows that B_n is substantially smaller than A_n, and indeed it can easily be seen that

$$B_n < 2^{n^2+n}\,,$$

based on the canonical forms of reflection networks. Every canonical form is a sequence of transpositions of adjacent elements $[i_1, i_1+1]\ldots[i_{\binom{n}{2}}, i_{\binom{n}{2}}+1]$ where $i_{k+1} \le i_k + 1$ for $1 \le k < \binom{n}{2}$. There are fewer than $4^{\binom{n}{2}+n} \sim 2^{n^2+n}$ such sequences with $i_1 < n$, no matter if they actually correspond to reflection networks or not. This follows from the fact that if we write down i_1 left parentheses, then for $1 \le k < l$ append $i_k - i_{k+1} + 1$ right parentheses and another left parenthesis, and finish with $i_{\binom{n}{2}}$ right parentheses, we obtain a balanced string of $\binom{n}{2} + i_1 - 1$ matched parenthesis pairs from which $i_1 \ldots i_{\binom{n}{2}}$ can be reconstructed. The number of such strings with m matched pairs is the Catalan number $\frac{1}{m+1}\binom{2m}{m}$, which is less than 4^m [Grü71]. Thus $\lim_{n\to\infty}\log B_n/\log A_n = 0$. We have seen that weak pre-CC systems have the same asymptotic behavior

as A_n, namely $2^{\Theta(n^2 \log n)}$, hence Axiom 1 has a strong effect on the total number of systems.

Knuth proves lower and upper bounds for B_n:

$$2^{\frac{n^2}{6} - \frac{5n}{2}} \leq B_n \leq 3^{\binom{n+1}{2}}$$

giving $b_n = \log B_n \leq 0.7924(n^2 + n)$, and also reports on computations for small n supporting his conjecture of $b_n \leq \binom{n}{2}$. Bern gives an improved lower bound $b_n \leq 0.7194n^2$ obtained from the sharpest version of the zone theorem. Felsner [Fel97] proposes a new encoding of arrangements from which he obtains

$$B_n \leq \prod_{k=0}^{n-1} \binom{n}{k} \prod_{1 \leq i \leq \frac{n}{2} < j \leq n} \left(1 - \frac{i(n-j)}{n^2}\right)$$

by the probabilistic argument. This gives $b_n \leq 0.6974n^2$.

Hence, the numbers B_n, C_n, D_n, and E_n all grow asymptotically as $2^{\Theta(n^2)}$.

Clearly the above discussion classifies *generalized* configurations of n points (in which the points are connected by an arrangement of pseudlines); hence we are clearly overcounting the number of *genuine* configurations that actually correspond to point sets in the Eucledian plane. The question is: By how much? The surprising answer is: by a lot!

Goodman and Pollack [GP86] have proved that the number of different *realizable* CC-systems is only $2^{\Theta(n \log n)}$. Their upper bound depends on Milnor's theorem of algebraic geometry, which implies that the zeroes of a polynomial of degree d in k real variables always partition R^k into at most $(2 + d)(1 + d)^{k-1}$ connected components. Consider the polynomial in $(x_1, y_1, \ldots, x_n, y_n)$ that we obtain by multiplying $\binom{n}{3}$ distinct determinants $|pqr|$ together; this polynomial will vanish at the "boundaries" between nonisomorphic realizable CC-systems. Hence the number of such systems is at most $\left(2 + 2\binom{n}{3}\right)\left(1 + 2\binom{n}{3}\right)^{2n-1}$ A comparison of this result with the value of $C(n) = 2^{cn^2 \log n}$ immediately shows that most (in a *very strong sense*) allowable sequences are not geometrically realizable.

Quite surprisingly, Goodman and Pollack's lower bound [GP86], which was obtained by a very elementary counting argument, is close to their upper bound, in fact agreeing with it in the highest order term in the exponent. Given a simple configuration of n points in R^2, draw $\binom{n}{2}$ lines connecting them, thereby obtaining

$$\binom{\binom{n}{2}}{2} + \binom{n}{2} + 1 - n\binom{n-2}{2}$$

cells. (A simple configuration of n points in R^d determines $\binom{n}{d}$ hyperplanes, and these in turn determine

$$\binom{\binom{n}{d}}{d} + \binom{\binom{n}{d}}{d-1} + \cdots + \binom{\binom{n}{d}}{0} - n\binom{\binom{n-1}{d-1} - 1}{d}$$

cells [Zas75]). Any n-point configuration can be extended to an $(n+1)$-point configuration by placing a new points in any one of these cell (clearly we are undercounting, since different realizations of the same set of semispaces may have noncorresponding cells). Thus gives a lower bound of roughly

$$\frac{(n!)^{2^2}}{2^{3n}} \sim n^{4n-3n\log_n 2} \sim n^{4n+O(n\log n)}$$

on the number of realizable nonisomorphic configurations. This shows that the upper bound of n^{6n} is quite close to truth, at least asymptotically, which is rather surprising.

A similar lower bound follows, in fact, from a simple direct construction. Replace each point of an n-gon by a pair of points extremely close together. Rotating each pair of points independently gives at least $(n-1)^n/n$ nonisomorphic, realizable CC-systems on $2n$ points.

4.4 New values

We obtain a complete enumeration of D_n for $n = 10$ which corresponds to the number of isomorphism classes of marked arrangements of 10 pseudolines as discussed by Goodman and Pollack[GP84] (pseudoline arrangements with a marked 2-cell), which is also the number of isomorphism classes of generalized configurations of 10 points (modulo semispace equivalence), also the number of uniform acyclic oriented matroids of rank 3 on 10 elements, and the number of weak equivalence/anti-equivalence classes of reflection networks on $[1, 10]$.

$D_{10} = 14,320,182$. This gives an additional value for the table of Knuth. It was obtained by a program that, given a system S on n points, generated all possible extensions of S by extreme points. Extensions correspond to all possible ways to throw a $(n+1)$st point into the exterior of the system. For $n \leq 9$ this resulted in the number D_n given by Knuth.

As a byproduct of the counting we also found complete statistics on the number of nonisomorphic systems on 10 points realizing different number of halving lines.

Number of halving lines	Total number of extensions	Number of nonisomorphic extensions	Ratio
5	2,247,826	517,423	0.23
6	10,596,609	2,584,235	0.24
7	19,204,602	4,865,400	0.25
8	16,482,171	4,290,426	0.26
9	6,578,464	1,757,011	0.27
10	1,021,892	283,580	0.28
11	73,972	21,389	0.29
12	2,326	713	0.31
13	14	5	0.36

The ratio tells us the average number of extreme points that systems with certain number of halving lines tend to have. For example, systems on 10 points with 7 halving lines tend to have 4 extreme points on average, while those with 12-13 halving lines – 3 extreme points.

Massive computations themselves became a very nontrivial and intriguing task (both in terms of time and space). [25] We had to produce extensions in pieces in the way that would guarantee that all isomorphic systems would be generated in the same piece.

Isomorph rejection was considerably facilitated by an accustomed program **nauty** [26] which is capable of producing the canonical labeling of a CC-system.

Having completed an enumeration of D_{10}, we get an almost free ride with C_{10}. $C_{10} = 2D_{10} - R_{10}$. The residue subtracted is the number of non-isomorphic self-complementary (isomorphic to their mirror image) CC-systems on 10 points, i.e. $R_{10} = |\{\alpha \in D_{10} \mid \bar{\alpha} \equiv \alpha\}|$. We obtain $R_{10} = 13103$, hence $C_{10} = 2 * 14320182 - 13103 = 28627261$.

Felsner [Fel97] enumerated all topologically different *simple* arrangements of 10 pseudolines. $B_{10} = 18,410,581,880.$, and he obtained a general upper bound of $B_n \leq 2^{0.6974n^2}$, which is currently the best known.

Let us now return to our original problems.

5 Theoretical and Computation Results

5.1 Erdös-Szekeres Problem

Naturally, in order to improve the lower bound on $g(6)$ we used incremental approach: starting with a configuration of n points that doesn't contain a convex empty 6-gon, we tried to add an $(n+1)$st point that doesn't introduce the latter. We repeat this procedure until we fail at constructing a configuration. To find a place for a new point p, we used probabilistic methods.

Our main goal is to construct an efficient algorithm for the following problem: Given a set of points V without an empty convex 6-gon, and a point $p \notin V$ test whether $V' = V \cup \{p\}$ contains an empty convex 6-gon. Obviously, when V' contains an empty convex 6-gon, p must be one of the vertices. Perform a geometric counterclockwise sorting of all the points of V in respect to p. We can imagine a sweeping ray that starts rotating counterclockwise[27] around p inducing a circular sorted sequence of points $\ldots, p_1, \ldots p_n, p_1, \ldots$. Construct a polygon $P = pp_1 \ldots p_n$. Let r be a line going through p and bisecting p_1p_n. Unless p belongs to the convex hull of V', r will also bisect some other pair of subsequent points in the circular sorted sequence, p_i, p_{i+1}. Construct $P' = pp_{i+1} \ldots p_n p_1 \ldots p_i$.

[25] The author is unlimitedly grateful to James Craig for providing facilitating equipment.

[26] written by Brendan McKay for determining the automorphism group of a vertex-colored graph. **nauty** stands for no **automorphisms, yes?**

[27] with ranodom initial position.

It is easy to see that if p is not an extreme point of V', and V' contains an empty convex 6-gon H, H lies either inside P or inside P'. Since p has to be one of the points of H, the later will have the form $pp_{i_1} \dots p_{i_5}$, where $p_{i_1} \dots p_{i_5}$ appear in this order in the circular sorted sequence above. Since p lies in the interior of V', edges of H cannot intersect r on both sides of p. Assume this is not true, then there exist edges of H, ab and cd, that intersect r on opposite sides of p, where points a, b, c, d appear in this order in the circular counterclockwise traversal of H. This yields $abp \wedge cdp \wedge dap \wedge bcp$, which implies that p lies inside the convex 4-gon $abcd$ oriented counterclockwise, which contradicts the fact that H is convex. Hence edges of H can never intersect edges of both P and P'.

This leads to the following algorithm for the following problem: Given a polygon V of n vertices with one extra vertex p in the interior, determine whether $V \cup \{p\}$ contains an empty convex 6-gon that has p as one of its vertices.

1. Construct a counterclockwise ordering of points of V in respect to p.
2. Form P and P' as described above.
3. Let the *visibility graph* of a polygon $P = p_1 \dots p_n$ be the graph with $p_1, \dots, p_n$ as nodes and an edge between p_i and p_j if the segment $p_i p_j$ lies entirely inside or on the boundary of P.

   Compute the visibility graph of P.
4. Check whether the visibility graph contains a convex chain of length 4, such that $p_{i_1} \dots p_{i_5}$ such that $p_{i_5} p p_{i_1}$ is true.
5. If no convex empty hexagon was found, repeat the procedure for P'.

We start with a configuration represented by Cartesian coordinates and convert it to CC-system. Lower bound of a conversion of a point configuration to corresponding CC-system is $c\binom{n}{3} = \Theta(n^3)$, which is just a combinatorial complexity of the output. Insertion of a new point into CC-system takes $c\binom{n}{2} = \Theta(n^2)$, since we need to determine the direction of all triples involving p. Computing visibility graph and determining convex chains takes $O(n^2)$. However, in practice, this point of the algorithm behaves almost linear, since the visibility graph tends to have a size much smaller than n^2.

Experiments were scheduled on the network of almost 100 Sun Ultra 1 machines, and several hours of computation resulted in several configurations of 26 points. No configuration on 27 points was found after several weeks of computations.

Theorem. $g(6) \geq 27$.

We believe that $g(6)$ is finite and the lower bound obtained is not far from the true value. There should be some theoretical explanation for such strong threshold behavior observed experimentally.

Many theoreticians strongly react against massive computations, but there are many arguments for making use of it. When no improvements can be made

we study the record-breaking configurations. Many construction methods have been found by this approach, although the role of the computer is seldomly mentioned in the published results. Massive computations can isolate the true difficulty of the problem, providing with an appropriate feeling for its behavior.

Below we give a configuration on 26 points without convex empty hexagons in order to give some insight into the structure of "good" sets. It is noticeable that all "good" sets, that is sets with no convex empty hexagons, are very sparse, in the sense that the ratio of the largest over the smallest distance between any two points of the set very large.

```
1767 895
2015 1225
1694 1378
2939 1268
2036 1987
2080 479
2174 861
3202 357
1070 1472
1613 436
1002 2369
1551 3544
3642 -538
3333 2448
734 612
2367 -2584
3862 1008
11911 -8
2651 -1365
5227 5521
-3 2810
5745 -3855
-5639 8338
-2529 -7175
2763 -8137
112 12753
```

5.2 The Halving Lines

5.2.1 Basic Observations

This section is intended to provide an appropriate feeling for the behavior of bisection graphs.

First, we extend the definition of halving lines to odd number of points. Let S be a configuration of n points in the plane in general position. A *halving line* of S is a directed line going through two points of S and having $\lfloor n/2 \rfloor - 1$ points of S on its right side. Let $G_{\lfloor n/2 \rfloor - 1}(S)$ be a halving lines graph induced by S. Clearly, $G_{\lfloor n/2 \rfloor - 1}(S) = -G_{\lceil n/2 \rceil - 1}(S)$, that is, $G_{\lceil n/2 \rceil - 1}(S)$ with all edges reversed[28], thus w.l.o.g. we consider only the first case. Throughout the thesis, by $E(G)$ and $V(G)$ we denote the edge-set and the vertex-set of G, respectively.

We can construct the halving lines graph $G_{\lfloor n/2 \rfloor - 1}(S)$ as follows. Let l be any oriented line containing no points of S and having $\lfloor n/2 \rfloor$ points of S on its right side. Translate l to its left until it meets a point of S. Call this point p_1 and this line $l(0)$. Now rotate l counterclockwise about p_1 by Θ into $l(\Theta)$ until it meets a second point of S, p_2. Call the resulting line $l(\Theta_1) = l_1$. Now rotate $l(\Theta)$ counterclockwise about p_2 until it meets p_3 of S at $l(\Theta_2) = l_2$, etc. This way we get a circular sequence of not necessarily distinct points $\cdots, p_1, p_2, \cdots, p_N, p_1, \cdots$ of S and a circular sequence of directed lines $\cdots, l_1, l_2, \cdots, l_N, l_{N+1}, l_1, \cdots$. The halving lines graph $G_{\lfloor n/2 \rfloor - 1}(S)$ consists of those vertices p_i and those directed edges $p_{i+1}p_i$ for which the orientation p_ip_{i+1} is opposite to that of the line l_i. Clearly, the number $N(\Theta)$ of points on the right side of $l(\Theta)$ remains the same in any interval which does not contain one of the angles Θ_i. If p_ip_{i+1} has the same direction as l_i then for small $\epsilon > 0$ we have $N(\Theta_i - \epsilon) = N(\Theta_i) = N(\Theta_i + \epsilon)$ since the points p_i, p_{i+1} are either on or to the left of $l(\Theta)$ for $\Theta_i - \epsilon \leq \Theta \leq \Theta_i + \epsilon$. If p_ip_{i+1} has the direction opposite to l_i then $N(\Theta_i) = N(\Theta_i - \epsilon) - 1 = N(\Theta_i + \epsilon) - 1$ for small $\epsilon > 0$, since one of p_i, p_{i+1} is to the right of $l(\Theta)$ in $\Theta_i - \epsilon \leq \Theta \leq \Theta_i + \epsilon$ except for $\Theta = \Theta_i$ when both are on l_i. Thus we have $N(\Theta) = \text{constant} = \lfloor n/2 \rfloor$ for all $\Theta \neq \Theta_i$ and $N(\Theta_i) = \lfloor n/2 \rfloor$ or $\lfloor n/2 \rfloor - 1$ according to whether p_ip_{i+1} has the same direction as l_i or not. Finally, note that $E(G_{\lfloor n/2 \rfloor - 1}) \subseteq \{l(\Theta)\}$, since any line l' not included in $l(\Theta)$ has the same direction as $l(\Theta')$ so that $N(l') - N(\Theta') \neq 0$ since $l(\Theta')$ contains a point of S. If $\Theta' = \Theta_i$ and p_ip_{i+1} has the direction opposite to l_i, this proves that $N(l') \neq \lfloor n/2 \rfloor - 1$. Otherwise $N(\Theta') = \lfloor n/2 \rfloor$ and l' passes through two points on the right of $l(\Theta')$ so that $N(l') \leq \lfloor n/2 \rfloor - 2$.

Note that this argument perfectly generalized for k-sets.

Lemma 5.1. Let l be a line containing no points of S and dividing S into two nonempty sets S_1 and S_2 with cardinalities m and $n - m$, respectively. W.l.o.g. assume that $m \leq n - m$. Then l intersects exactly m edges of $G_{\lfloor n/2 \rfloor - 1}$

[28] As before we denote by $G_k(S)$ *k-graphs of S* – graphs on S whose edges are directed segments $\vec{pq}$ such that the directed line $\vec{pq}$ has k points of S on its right side. Halving lines (bisection) graphs are just special cases of k-graphs when $k = \lfloor n/2 \rfloor - 1$. Remark: If n is even, each edge of $G_{(n-2)/2}$ occurs in both orientations and $G_{(n-2)/2}$ can therefore be considered undirected. Let $H(S)$ denote the set of edges of $G_{\lfloor n/2 \rfloor - 1}(S)$.

going from S_1 to S_2 and the same number of edges going from S_2 to S_1. As a consequence, any halving line l of S intersects exactly $\lfloor n/2 \rfloor - 2$ edges of $G_{\lfloor n/2 \rfloor - 1}$ going in each direction.

Proof. Since a small perturbation of l does not affect the hypotheses we may assume that l is not parallel to any line pq connecting two points $p, q \in S$. Pick the point $p_1 \in S_2$ and the directed line $l(0)$ of the family defined above through p_1; S_1 lies on the left side of $l(0)$. As Θ increases from 0 to π, the number of points of S_1 on the right side of $l(\Theta)$, denoted by $N(\Theta, S_1)$, monotonically increases from 0 to m.

Now the number $N(\Theta, S_1)$ is clearly constant in any interval which does not contain Θ_i. If both points p_i, p_{i+1} of $l(\Theta_i)$ are in S_2 then $N(\Theta_i - \epsilon, S_1) = N(\Theta_i, S_1) = N(\Theta_i + \epsilon, S_1)$ for small $\epsilon > 0$. Similarly, if the directed line $p_i p_{i+1}$ is the direction of $l(\Theta_i)$ then $N(\Theta_i - \epsilon, S_1) = N(\Theta_i, S_1) = N(\Theta_i + \epsilon, S_1)$ for small $\epsilon > 0$. If p_i, p_{i+1} both belong to S_1 and the directed line $p_i p_{i+1}$ has direction opposite to the one of $l(\Theta_i)$ then the point p_{i+1} is to the right of $l(\Theta_i - \epsilon)$ and p_i is to the left of $l(\Theta_i - \epsilon)$ for small $\epsilon > 0$, while p_{i+1} is to the left of $l(\Theta_i + \epsilon)$ and p_i is to the right of $l(\Theta_i + \epsilon)$ for small $\epsilon > 0$. Thus, in this case, we have $N(\Theta_i - \epsilon, S_1) = N(\Theta_i + \epsilon, S_1)$. Finally, if p_i, p_{i+1} are in opposite sides of l and the directed line $p_i p_{i+1}$ has direction opposite to the one of $l(\Theta_i)$ for $0 < \Theta_i < \pi$ then $p_i \in S_1, p_{i+1} \in S_2$ and $N(\Theta_i + \epsilon, S_1) = N(\Theta_i, S_1) + 1 = N(\Theta_i - \epsilon, S_1) + 1$ for small $\epsilon > 0$, since the point p_i is to the right of $l(\Theta_i + \epsilon)$ but on $l(\Theta_i - \epsilon)$. Thus $N(\Theta, S_1)$ increases by one in the interval $0 < \Theta_i < \pi$ whenever l is intersected by a directed segment $p_{i+1} p_i$ of $G_{\lfloor n/2 \rfloor - 1}$ going from S_2 to S_1. Since $N(\Theta, S_1)$ increases to m there must be m such segments of $G_{\lfloor n/2 \rfloor - 1}$.

As Θ increases from π to 2π, the number $N(\Theta, S_1)$ decreases from m to 0 and in a manner entirely analogous to that used above we see that $N(\Theta, S_1)$ decreases by one whenever l is intersected by a directed segment $p_{i+1} p_i$ of $G_{\lfloor n/2 \rfloor - 1}$ going from S_1 to S_2. Thus there must be m such segments.

Corollary 5.1. A halving line l of S intersects exactly $\lfloor n/2 \rfloor - 2$ other halving lines of S going in each direction.

Lemma 5.2. If we construct a counterclockwise ordering of the oriented lines of the edges of $G_{\lfloor n/2 \rfloor - 1}$ at a vertex, v, then between any two lines containing outgoing edges there is a line containing an incoming edge, and between any two lines containing an incoming edge there is a line containing an outgoing edge.

Proof. Let l_1 and l_2 be successive oriented lines through v containing outgoing edges of $G_{\lfloor n/2 \rfloor - 1}$. Then as l rotates from l_1 to l_2 we have $\lfloor n/2 \rfloor$ points of S on the right side of l if l is near to l_1 and $\lfloor n/2 \rfloor - 1$ points of S on the right side of l if l is near to l_2. Since the number of points of S on the right side of l increases by one each time l passes through a point p of S in the oriented angle $\widehat{(l_1, l_2)}$ and decreases by one each time l passes through a point p of S in the opposite vertical angle $\widehat{(-l_2, -l_1)}$, it follows that at some stage of the rotation the number of points on the right side of l decreases from $\lfloor n/2 \rfloor$ to $\lfloor n/2 \rfloor - 1$ so

that l contains an incoming edge $\vec{pv}$ of $G_{\lfloor n/2\rfloor-1}$. The argument for successive lines containing incoming edges is entirely analogous.

Corollary 5.2.1. At each vertex of $G_{\lfloor n/2\rfloor-1}$, the number of incoming edges is equal to the number of outgoing edges. Thus each component of $G_{\lfloor n/2\rfloor-1}$ has an oriented Euler circuit. Hence, if n is odd, the degree of each vertex of $G_{\lfloor n/2\rfloor-1}$ is even; if n is even, the degree of each vertex of undirected $G_{\lfloor n/2\rfloor-1}$ is odd.

Since each vertex of $G_{\lfloor n/2\rfloor-1}$ is incident to at least two directed edges, the number of directed edges can be no less than the number of its vertices.

Corollary 5.2.2. If n is odd, the graph $G_{\lfloor n/2\rfloor-1}$ has at least n directed edges, if n is even - $n/2$ undirected edges.

It is trivial to observe, that this lower bound is indeed realized by a set of n points which all are extreme points of the set.

Let $\bar{h}(n)$ be the minimum number of halving lines realized by any set of n points in the plane. Then $\bar{h}(n) = n$ is n is odd, and $n/2$ otherwise.

Lemma 5.3. Let G be a halving lines graph of S with $2n$ vertices and $|H(S)|$ undirected edges, $n \geq 3$, then each component of G has at least 6 vertices.

Proof. Since $|H(S)| \geq 2n$, each union of components of G is itself a bigraph. Thus if G has a component G' with no more than 4 vertices we can write $G = G' \cup G''$ where G' has $2i \leq 4$ vertices and therefore $\leq 2i-1$ edges and G'' has $2n-2i$ vertices and therefore $\leq h(2(n-i))$ edges. This implies $h(2n) \leq h(2(n-i)) + 2i - 1$, contradicting the obvious fact that $h(2n+2) \geq h(2n) + 2$.[29]

Lemma 5.4. Let n be odd, then $h(2n) \geq 2h(n) + n$.

Proof. Let S be a set of n vertices, n is odd. Associate an outgoing edge e_p to each vertex p of $G_{\frac{n-3}{2}}(S)$ and construct a set S' with $2n$ points by splitting each point p into two points p' and p'' at a small distance ϵ from p with $p'p$ and pp'' in the direction of e_p. Consider $G_{2\frac{n-3}{2}+1}(S') = G_{n-2}(S')$. For each point p of S, corresponding $p'p''$ is an edge of $G_{n-2}(S')$ since each point on the positive side of e_p has been split in two. If $e_p = pq$, then exactly one of q', q'' is on the right side of e_p. Furthermore, both $p''q'$ and $p''q''$ are edges of $G_{n-2}(S')$ whose right sides contain the points arising from those on the right side of e_p and, respectively, the point p' or that point q', q'' which lies on the positive side of e_p. Finally, if pq is an edge of $G_{\frac{n-3}{2}}(S)$ other than e_p, then the edge which connects p' or p'' on the left side of pq to the point q'

[29] *Proof.* Let G be a bisection graph of S. By an affine transformation assume that all points of S are close to the x axis and that all edges of G make small angles with the x axis. Now we add two points p, q to S, where p has sufficiently large positive y-coordinate and q has sufficiently large negative y coordinate and both have, say, x-coordinates smaller than the x coordinates of the points of S. Then all the edges of G are also edges of G', where G' is a bisection graph of $S \cup \{p, q\}$, and since pq is not an edge of G, there are two new edges incident to p and q, respectively. Thus, $h(2n+2) \geq H(S \cup \{p, q\}) = h(2n)$.

or q'' on the right side of pq as well as the edge which connects p' or p'' on the right side of pq to the point q' or q'' on the left side of pq are edges of $G_{n-2}(S')$. Thus in this splitting process each edge of $G_{\frac{n-3}{2}}(S)$ yields two edges of $G_{n-2}(S')$ and the n edges e_p of $G_{\frac{n-3}{2}}(S)$ yield an additional edge $p'p''$. Thus $h(2n) \geq |H(G_{n-2}(S'))| = 2h(n) + n = 2\max_{|S|=n} |H(G_{\frac{n-3}{2}}(S))| + n$.

Lemma 5.5. Let S be a set of $2n$ vertices realizing $h(2n)$ halving lines, $n \geq 3$, and let $G(S)$ be a bisection graph of S. Then it is possible to associate to each vertex p an edge e_p of $G(S)$ so that $e_p \neq e_q$ for all $p \neq q$.

Proof. We already know that each component of $G(S)$ contains an Euler circuit C. If we arrange the vertices of C in cyclic order $\cdots, p_1, p_2, \cdots, p_s, p_1, \cdots$, then we can associate with each p_i the edge p_ip_{i+1}. If there are vertices not included in C then p_i have immediate neighbors, q, in C. With each such neighbor we associate the edge qp_i. If this still does not exhaust the vertices of the component we get additional vertices joined to the immediate neighbors of C, etc.

Theorem 5.6. For $n \geq 2$, we have $h(4n) \geq 2h(2n) + 2n$

Proof. For $n = 2$, we do this by inspection since $h(4) = 3$ is obtained whenever the vertices form a non-convex quadruple and $h(8) = 9$ by a modified splitting procedure. For $n \geq 3$, we use Lemma 5.5 to associate to each vertex a different incident edge of $G(S)$ and then apply the splitting process used above to complete the proof.

By iterated application of the above theorem we get the configuration on Figure 1.

In spite of the pathetic state of the halving lines problem, we can obtain fairly sharp bounds for small values of n.

Throughout the thesis, let (n,k)*-set* denote a set of n points with k halving lines, and let $(d_1, d_2, \cdots, d_n)$ denote a non-decreasing sequence of degrees of vertices of $H(S)$. Let n_i also denote the number of vertices of degree i in $H(S)$, i is odd. We will use a short form notation for degree sequences: $(n_1; n_3; \cdots; n_{d_n}) = (\underbrace{1, \cdots, 1}_{n_1}, \underbrace{3, \cdots, 3}_{n_3}, \cdots, \underbrace{d_n, \cdots, d_n}_{n_{d_n}})$.

The following inequality is immediately implied by the main identity in [AAHP+98]:

$$\sum_i n_i \binom{(i+1)/2}{2} \leq \binom{n/2}{2},$$

We also certainly have

$$\sum_i i n_i = 2h(n),$$

since every edges contributes 2 to the sum of all degrees, and just by counting vertices according to their degrees, we trivially have

$$\sum_i n_i = n.$$

We also know that at least three points have degree one (since there are at least 3 extreme points), thus $n_1 \geq 3$.

This facts, in conjunction with the following observations, readily derive improved bounds for small values of n:

Observation 1. If we want to show the upper bound $h(n) < c$ for some n, we must prove the nonexistence of an (n, c)-set.

Observation 2. If there does not exist an $(n, \geq c)$-set having n_1 extreme points, there does not exist an $(n, \geq c)$-set having $n_1' > n_1$ extreme points.

Observation 3. The sequence of degrees produced by the following greedy algorithm minimizes the sum $\sum_{\substack{i=2j+1,\ j\in\mathbb{N} \\ i\leq d_n}} n_i\binom{(i+1)/2}{2}$ for a given n and c:

> Let $d = (2c-n)/2$. Assign $n_1 = 3$, $n_p = n_{2\lceil d/(n-n_1)\rceil+1} =$
> $d \bmod (n-n_1)$, $n_q = n_{2\lfloor d/(n-n_1)\rfloor+1} = n - n_1 - n_p$.
> The sequence produced is
>
> $$(1,1,1,\underbrace{q,\cdots,q}_{n_q},\underbrace{p,\cdots,p}_{n_p})$$

This immediately implies the following upper bounds for small values of n:

$h(18) \leq 18$, in conjunction with the known lower bound this implies the equality $h(12) = 18$. The only degree sequence that a $(12,18)$-set may have is $(3;6;3)$.

$h(14) \leq 23$, thus $22 \leq h(14) \leq 23$. Any $(14,22)$-set must have at most 5 extreme points. $h(14) = 23$ if and only if there exist a stretchable set of 14 points with the degree sequence $(3;6;5)$, otherwise $h(14) = 22$, realized by sets with degree sequences from the following set $\{(3;7;4),(3;8;2;1),(3;9;0;2),(4;5;5),(4;6;3;1),(5;3;6)\}$.

$h(16) \leq 28$, thus $27 \leq h(16) \leq 28$. The existence of a realizable set of 16 points with one of the three degree sequences $(3;6;7),(3;7;5;1), or (4;4;8)$ would imply $h(16) = 28$. Any $(16,28)$-set must have at most 6 extreme points.

$h(18) \leq 34$, hence $32 \leq h(18) \leq 34$. Analogous analysis can clearly be done.

In the following sections we give another proof of $h(12) = 18$, and show that the set of 12 points maximizing the number of halving lines is unique. Moreover, we claim that all sets of ≤ 12 points maximizing the number of halving pseudolines are realizable in plane[30], and we give a complete enumeration of all topologically distinct halving lines graphs with $h(n)$ edges for $n \leq 12$.

[30]For $n \leq 8$, this claim vacuously holds since Grünbaum's conjecture on the strechability of all arrangements of fewer than nine pseudolines was shown to be true [GP80b].

5.2.2 Properties discovered. The Main Symmetry.

Let S be a set of n points constituting a legal CC-system, and consider all one-point extensions of S by extreme points (before filtering out isomorphs). Extensions correspond to all possible ways to throw a $(n+1)$st point into the exterior of the system. Denote the set of all extensions by $E(S)$. Let $\{E_i(S)\}$ be a partition of E into equivalence classes by the number of halving lines i. Let $h(E) = \max_{e \in E} |H(e)|$, $\bar{h}(E) = \min_{e \in E} |H(e)|$, and $e_i = |E_i|$.

Theorem 5.7. For odd n we have:

$$e_{\lfloor x \rfloor - i} = e_{\lceil x \rceil + i}, \qquad \text{for integer} \quad i,\ 0 \le i \le h(E) - x,$$

where x is a fixed center of symmetry

$$x = \frac{|H(S)|}{2} + 1 = \frac{h(E) + \bar{h}(E)}{2},$$

Hence, knowing the minimum number of halving lines realized by any extension of S, $|S|$ odd, implies knowing the maximum number of halving lines realized by any extension of S, and vice versa. Moreover, the problem of finding the number of extensions of S maximizing the number of halving lines is equivalent the the problem of finding the number of extensions minimizing the number of halving lines.

Proof of Theorem 5.7. We need the following lemmas.

> **Lemma 5.8.** Any two halving lines of S necessarily cross, and the crossing point is necessarily in the interior (convex closure) of S.
>
> *Proof.* Assume the contrary. Suppose there are lines $pq, rs \in H(S)$ that are either parallel or cross in the exterior of S. Then points $\{p, q, r, s\}$ form a convex quadrilateral. We have to consider the following cases:
> Case 1. $rsqp$ is oriented counterclockwise[31]: Denote the subset of points of S to the left of pq by S_1, the subset of points of S to the right of rs by S_2. Let $m_1 = |S_1|$, $m_2 = |S_2|$, $m_3 = |S \setminus \{S_1 \cup S_2 \cup \{p, q, r, s\}\}|$. Since pq is a halving line, (i) $m_1 = m_3 + m_2 + 3$, if n is odd, and $m_1 = m_3 + m_2 + 2$, if n is even. On the other hand, since rs is a halving line, (ii) $m_2 = m_3 + m_1 + 1$, if n is odd, and $m_2 = m_3 + m_1 + 2$, if n is even. Substituting (ii) into (i) yields $2m_3 + 4 = 0$ which cannot be true, since $m_3 \ge 0$.
> Case 2. $srqp$ is oriented counterclockwise. Denote the subset of points of S to the left of pq by S_1, and the subset of points of S to the left of rs by S_2. Let m_1, m_2, m_3 be as before. Since pq is a halving line, we have (i) $m_1 = m_3 + m_2 + 3$, if n is odd, and $m_1 = m_3 + m_2 + 2$, if n is even. On the other hand, since rs is a

[31] $rsqp \Leftrightarrow rsq \wedge sqp \wedge qpr \wedge prs$.

halving line, (ii) $m_2 = m_3+m_1+1$, if n is odd, and $m_2 = m_3+m_1+2$, if n is even. Substituting (ii) into (i) yields $2m_3+4 = 0$ which cannot be true, since $m_3 \geq 0$.
Case 3. $rspq$ is oriented counterclockwise. Denote the subset of points of S to the right of pq by S_1, and the subset of points of S to the right of rs by S_2. Let m_1, m_2, m_3 be as before. Since pq is a halving line, we have (i) $m_1 = m_3 + m_2 + 1$, if n is odd, and $m_1 = m_3 + m_2 + 2$, if n is even. On the other hand, since rs is a halving line, we have (ii) $m_2 = m_3 + m_1 + 1$, if n is odd, and $m_2 = m_3 + m_1 + 2$, if n is even. Substituting (ii) into (i) yields $2m_3 + 2 = 0$, if n is odd, and $2m_3 + 4$, if n is even. Again, both cannot be true since $m_3 \geq 0$. □

Corollary 5.9. $H(S)$ partitions the exterior of S into $2|H(S)|$ cells.

Proof of Theorem 5.7 (cont.) According to Corollary 5.9, halving lines divide the exterior of S into $2|H(S)|$ cells. Extensions obtained by throwing an $(n+1)$st point, v, in the same cell have the same number of halving lines. Let the *mark* of the cell be the number of halving lines of S that would become halving lines of $S' = \{S \cup v\}$ if v were placed in this cell. Notice that marks have the following property: when we move between two adjacent cells, the mark can either stay the same or change by at most 1. Corresponding to each cell is an antipodal cell. We say that a pair of antipodal cells is *atomic* if the cells are not cut by any lines determined by points of S.

Lemma 5.10. Let $\langle \alpha, \beta \rangle$ be a pair of atomic antipodal cells with marks m_α and m_β. Then $m_\alpha + m_\beta = |H(S)|$.

Proof. Let the halving lines determining α and β be l_α and l_β. Let v_α be an arbitrary point thrown into the cell α. Analogously define v_β. Note that v_α and v_β lie on opposite sides of any line $l \in H(S)$. Assume this is not true, and there is a line $l \in H(S)$ that crosses l_α and l_β[32] in such a way that v_α and v_β lie on one side of l. Then α and β would split into two pairs of antipodal cells, one formed by l_1 and l, and the other – by l_2 and l, contradicting the assumption that $\langle \alpha, \beta \rangle$ form pair of atomic cells. It follows, that $H(S)$ can be partitioned into two disjoint subsets with cardinalities m_α and m_β: one is realized when we put v in cell α, the other – when we put v in cell β. Hence, $m_\alpha + m_\beta = |H(S)|$. □

Proof of Theorem 5.7 (cont.) Since v is an extreme point, it defines a linear ordering of the points of S, therefore introducing exactly one new halving line, v_e. Clearly, $(\forall e : e \in E)[H(e) \setminus \{v_e\} \subset H(S)]$, and $|H(S' = S \cup v)| = m_v + 1$, where m_v denotes the mark of v's cell.

[32] According to Lemma 5.8., any pair of halving lines necessarily crosses.

Symmetry occurs because for every extension obtained by putting an $(n+1)$st point in some cell α with mark m_α there is an extension obtained by putting an $(n+1)$st point in the cell antipodal to α with mark $|H(S)| - m_\alpha$. Clearly the center of symmetry, x, occurs when $x = \frac{|H(S)|}{2} + 1$. $+1$ appears since v brings one extra halving line into every extension, thus raising x by 1. □

The symmetry observed gives a *considerable* restriction on the extension process that needed to be done in order to obtain the exact value of $h(12)$.

Let $f(n,k)$ be the largest integer such that any system of n points with at least k halving lines is necessarily an extension of a system of $n-1$ points with at least $f(n,k)$ halving lines. In particular, if n is even, $x = \frac{f(n,k)}{2} + 1$. On the other hand we certainly have $x \geq \frac{n/2+k}{2}$, hence $f(n,k) \geq \frac{n}{2} + k - 2$. Thus, to obtain all set on 12 points with at least 18 halving lines, we only need to extend sets on 11 points with ≥ 22 halving lines. It gives a considerable reduction on the number of systems that need to be extended, since $\bar{h}(11) = 11,\ h(11) = 24$.

A similar symmetry is preserved when we consider the set of all one-point extensions of a configuration of even number of points.

Theorem 5.11. For even n we have:

$$e_{\lfloor x \rfloor - i} = e_{\lceil x \rceil + i}, \quad \text{for integer } i,\ 0 \leq i \leq h(E) - x,$$

where x is a variable center of symmetry

$$x = \frac{h(E) + \bar{h}(E)}{2} \leq 3|H(S)| + 2$$

Proof of Theorem 5.11. We would like to get an analog of Lemma 5.8.

> **Lemma 5.12.** Any two distinct lines of $\{H(G_{\frac{n-2}{2}-1}(S)) \cup H(G_{\frac{n-2}{2}}(S))\}$ necessarily cross, and the crossing point is necessarily in the interior (convex closure) of S.[33]
>
> *Proof.* Assume the contrary. Suppose there are lines $pq, rs \in \{H(G_{\frac{n-2}{2}-1}(S)) \cup H(G_{\frac{n-2}{2}}(S))\}$ that are either parallel, or cross in the exterior of S.
> Case 1. $rsqp$ is oriented counterclockwise[34]: Denote the subset of points of S to the left of pq by S_1, the subset of points of S to the right of rs by S_2. Let $m_1 = |S_1|$, $m_2 = |S_2|$, $m_3 = |S \setminus \{S_1 \cup S_2 \cup \{p, q, r, s\}\}|$.
>
> We have four cases:

[33] As before, by $H(G)$ we denote a set of edges of G.

[34] $rsqp \Leftrightarrow rsq \wedge sqp \wedge qpr \wedge prs$.

1. $pq \in H(G_{\frac{n-2}{2}})$, $rs \in H(G_{\frac{n-2}{2}})$. This case is covered by Lemma 1.

2. $pq \in H(G_{\frac{n-2}{2}})$, $rs \in H(G_{\frac{n-2}{2}-1})$. The first says that $m_1 = m_3 + m_2 + 2$, while the second yields $m_2 = m_3 + m_1 + 2 - 2$. Substituting second into the first gives $2m_3 + 2 = 0$, which can not be true since $m_3 \geq 0$.

3. $pq \in H(G_{\frac{n-2}{2}-1})$, $rs \in H(G_{\frac{n-2}{2}})$. The first says that $m_1 = m_3 + m_2 + 2 + 2$, while the second yields $m_2 = m_3 + m_1 + 2$. Substituting second into the first gives $2m_3 + 6 = 0$, which can not be true since $m_3 \geq 0$.

4. $pq \in H(G_{\frac{n-2}{2}-1})$, $rs \in H(G_{\frac{n-2}{2}-1})$. The first says that $m_1 = m_3 + m_2 + 2 + 2$, while the second yields $m_2 = m_3 + m_1 + 2 - 2$. Substituting second into the first gives $2m_3 + 4 = 0$, which can not be true since $m_3 \geq 0$.

Case 2. *srqp* is oriented counterclockwise. Denote the subset of points of S to the left of pq by S_1, the subset of points of S to the left of rs by S_2. Let be m_1, m_2, m_3 as before.
Case 3. *rspq* is oriented counterclockwise. Denote the subset of points of S to the right of pq by S_1, the subset of points of S to the right of rs by S_2. Let be m_1, m_2, m_3 as before.
The argument for cases 2 and 3 is entirely analogous. □

Edges of $G_{\frac{n-2}{2}-1}$ can potentially become halving lines of $S' = S \cup \{v\}$, hence we have to consider the partitioning of the exterior of S into cells formed by $L = \{H(G_{\frac{n-2}{2}-1}(S)) \cup H(G_{\frac{n-2}{2}}(S))\}$. According to Lemma 5.12 above, every pair of L necessarily crosses in the interior of S, and the rest of the proof proceeds by exact analogy with the proof for odd n.

Note, however, that x is not fixed in this case, since it is a function of $|L|$, which is not fixed for all extensions.

We can obtain a trivial upper bound on x by an easy counting argument: Clearly, any one-point extension of S preserves all halving lines of S. Since $(n+1)$st point, v, is extreme, it introduces exactly two new directed halving lines into an extension. Each halving line of S can potentially clone 4 new halving lines in the extension (obtained by rotating around each end point in each direction), hence $x \leq 4|H(S)|/2 + |H(S)| + 2$, $x \leq 3|H(S)| + 2$. Clearly, we are overcounting, but the question is: by how much? Although very trivial, this upper bound is not absurdly large: Computer experiments for small n show that no substantial restriction seems possible. For example, one-point extensions of some $(10, \geq 6)$-sets do produce $(11, 22)$-sets while $\bar{h}(10) = 5$, $h(10) = 13$. However, this doesn't immediately imply that $f(11, 22) = 6$. We would certainly like to prove that any (n, k)-set obtained by extending an $(n-1, < f(n, k))$-set is equivalent (modulo isomorphism) to some configuration obtained by extending configurations with at least $f(n, k)$ halving lines.

Note that all the proofs in this section depend only on the combinatorial properties of configurations, and thus are perfectly valid for generalized pseudoconfigurations.

5.3 $h(8)$

There are 4 nonisomorphic CC-systems on 8 points that maximize the number of halving lines. They are certainly realizable, since Grünbaum's conjecture that every arrangement of eight pseudolines in stretchable was shown to be true. Corresponding configurations[35] and halving lines arrangements are shown below.
8 points with 9 halving lines *(System 1)*

```
021 031 032 132 041 042 124 034 134 234 051 052 125 053 135
235 054 145 245 354 061 062 126 063 136 236 064 146 246 364
065 156 256 365 456 071 072 172 073 173 273 074 174 274 374
075 175 275 375 475 076 176 276 376 476 576
```

8 points with 9 halving lines *(System 2)*

```
021 031 032 132 041 042 124 034 134 234 051 025 125 035 135
235 045 145 245 354 016 026 126 036 136 236 046 146 246 364
056 156 256 365 465 071 072 172 073 173 273 074 174 274 374
075 175 275 375 475 076 176 276 376 476 576
```

8 points with 9 halving lines *(System 3)*

```
021 031 032 132 041 042 124 034 134 234 015 025 125 035 135
235 045 145 245 354 016 026 126 036 136 236 046 146 246 364
056 156 256 365 465 071 072 172 073 173 273 074 174 274 374
075 175 275 375 475 076 176 276 376 476 576
```

8 points with 9 halving lines *(System 4)*

```
021 031 032 132 041 042 124 034 134 234 015 025 125 035 135
235 045 145 245 354 016 026 126 036 136 236 046 146 246 364
056 156 256 365 465 071 072 172 073 173 273 074 174 274 374
075 175 275 375 475 076 176 276 376 476 576
```

Systems *1,2 and 4* induce isomorphic bisection graphs, although their CC-systems are not isomorphic.

[35] represented as sets of triples in canonic labeling.

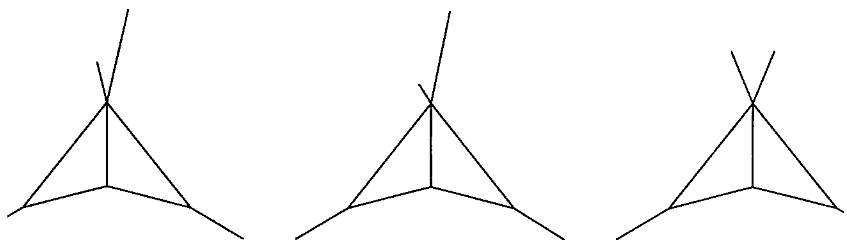

Figure 3: Systems on 8 points with 9 halving lines (*Systems 1,2 and 4*)

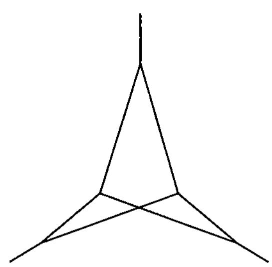

Figure 4: System on 8 points with 9 halving lines (*System 3*)

Number of halving lines	Number of nonisomorphic extensions	Number of all extensions	Ratio
4	284	1473	0.19
5	1018	4683	0.22
6	1299	5423	0.24
7	633	2500	0.25
8	77	291	0.26
9	4	10	0.4

Figure 5: Statistics on $7 \leadsto 8$ extensions

Number of halving lines	Number of nonisomorphic extensions
9	673
10	4814
11	17215
12	36308
13	47038
14	35539
15	14423
16	2590
17	220
18	10

Figure 6: Statistics on $9 \rightsquigarrow 10$ extensions

5.4 $h(10)$

There are 5 nonisomorphic systems on 10 points with 13 halving lines. All of them turned out to be realizable.
10 points with 13 halving lines *(System 1)*

021 031 032 132 041 042 142 043 143 243 051 052 125 053 135
235 045 145 245 345 061 062 126 063 136 236 064 146 246 346
065 156 256 365 465 071 072 127 073 137 237 074 147 247 347
075 157 257 357 475 076 167 267 367 476 567 081 082 128 083
138 238 084 148 248 348 085 158 258 358 485 086 168 268 368
486 568 087 178 278 378 487 578 687 091 092 192 093 193 293
094 194 294 394 095 195 295 395 495 096 196 296 396 496 596
097 197 297 397 497 597 697 098 198 298 398 498 598 698 798

10 points with 13 halving lines *(System 2)*

021 031 032 132 041 042 142 043 143 243 051 052 125 035 135
235 045 145 245 345 061 026 126 036 136 236 046 146 246 346
056 156 256 356 465 071 027 127 037 137 237 047 147 247 347
057 157 257 375 475 067 167 267 376 476 576 018 028 128 038
138 238 048 148 248 348 058 158 258 385 485 068 168 268 386
486 586 078 178 278 387 487 587 678 091 092 192 093 193 293
094 194 294 394 095 195 295 395 495 096 196 296 396 496 596
097 197 297 397 497 597 697 098 198 298 398 498 598 698 798

10 points with 13 halving lines *(System 3)*

021 031 032 132 041 042 142 043 143 243 051 052 125 035 135
235 045 145 245 345 061 026 126 036 136 236 046 146 246 346
056 156 256 356 465 017 027 127 037 137 237 047 147 247 347
057 157 257 375 475 067 167 267 376 476 576 018 028 128 038

138	238	048	148	248	348	058	158	258	385	485	068	168	268	386
486	586	078	178	278	387	487	587	678	091	092	192	093	193	293
094	194	294	394	095	195	295	395	495	096	196	296	396	496	596
097	197	297	397	497	597	697	098	198	298	398	498	598	698	798

10 points with 13 halving lines *(System 4)*

021	031	032	132	041	042	142	043	143	243	051	052	125	035	135
235	045	145	245	345	016	026	126	036	136	236	046	146	246	346
056	156	256	356	465	017	027	127	037	137	237	047	147	247	347
057	157	257	375	475	067	167	267	376	476	576	018	028	128	038
138	238	048	148	248	348	058	158	258	385	485	068	168	268	386
486	586	078	178	278	387	487	587	678	091	092	192	093	193	293
094	194	294	394	095	195	295	395	495	096	196	296	396	496	596
097	197	297	397	497	597	697	098	198	298	398	498	598	698	798

10 points with 13 halving lines *(System 5)*

021	031	032	132	041	042	142	043	143	243	015	025	125	035	135
235	045	145	245	345	016	026	126	036	136	236	046	146	246	346
056	156	265	365	465	017	027	127	037	137	237	047	147	247	347
057	157	257	375	475	067	167	267	376	476	567	018	028	128	038
138	238	048	148	248	348	058	158	258	385	485	068	168	268	386
486	568	078	178	278	387	487	578	687	091	092	192	093	193	293
094	194	294	394	095	195	295	395	495	096	196	296	396	496	596
097	197	297	397	497	597	697	098	198	298	398	498	598	698	798

5.5 $h(12)$

We completed the extension process needed to obtain the exact value of $h(12)$. $h(12) = 18$. We claim that the CC-system on 12 points maximizing the number of halving lines is unique (modulo isomorphism). The system is realizable, and its halving lines graph is shown in Figure 1. As expected, the degree sequence is $(1, 1, 1, 3, 3, 3, 3, 3, 3, 5, 5, 5)$.

5.6 Attacking $h(14)$

Lemma 5.6.1. If there exists a planar embedding of a halving edges graph H, $|V(H)| = n$, $|E(H)| = \dot{h}(n)$, there exists a planar drawing of H with straight edges if and only if

$$\dot{h}(n) \leq 2n - n_1 - 3.$$

If the above holds with equality, all bounded faces of H are 3-faces, and the unbounded face is an $(n + n_1)$-face.

Proof. Since there exist a planar embedding of H, we can use Euler's formula to find the number of faces of H, f:

$$f = \dot{h}(n) - n + 2.$$

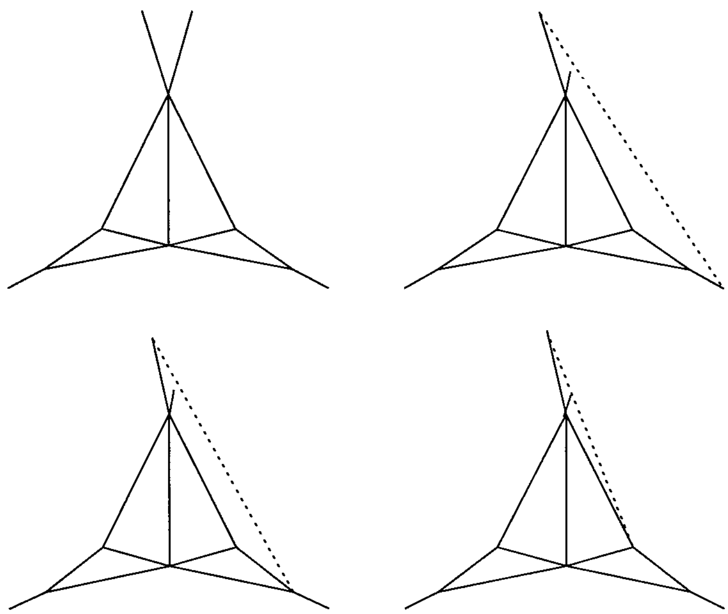

Figure 7: Systems on 10 points with 13 halving lines *(Systems 1, 2, 3, and 4)*

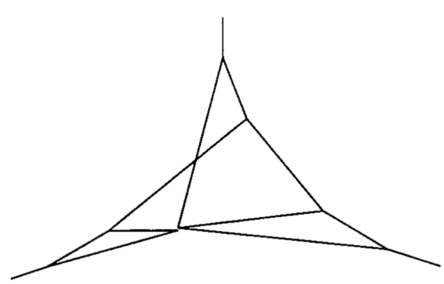

Figure 8: System on 10 points with 13 halving lines *(System 5)*

Next we count the faces of H according to their number of sides. Let a k-face be a face bounded by k edges, where edges that border the same face on both sides are counted twice. Let f_k be the number of k-faces. We certainly have the following identities:

$$\sum_{i\geq 3} f_i = f = \dot{h}(n) - n + 2,$$

$$\sum_{i\geq 3} if_i = 2\dot{h}(n)$$

If there exists a planar embedding of H with straight edges, the unbounded face of the embedding if clearly an $(n + n_1)$-face. Thus,

$$\sum_{i\geq 3}^{i<n+n_1} if_i = 2\dot{h}(n) - n - n_1.$$

Each of the remaining $f - 1$ bounded faces should have at least 3 sides each, i.e.

$$3(\dot{h}(n) - n + 1) \leq 2\dot{h}(n) - n - n_1,$$

which yields

$$\dot{h}(n) \leq 2n - n_1 - 3.$$

Prove in other direction is omitted. □

We have shown that $22 \leq h(14) \leq 23$. The upper bound is tight if and only if there exist a stretchable set of 14 points with degree sequence coefficients $(n_1; n_3; n_5) = (3; 6; 3)$. Using lemma above, it is easy to see that if such set exists, it cannot be drawn with straight lines. Let Q be a hypothetical $(14, 23)$-set. Since $23 \leq 3 \cdot n - 6 = 3 \cdot 14 - 6$, $H(Q)$ would certainly be planar, and by Euler's formula is should have $f = 23 - 14 + 2 = 11$ faces. Suppose the contrary, that is, there is a planar drawing of $H(Q)$ with straight edges. Then the unbounded face should be a 17-face. From the identities in Lemma above,

$$3 \cdot f_3 + 4 \cdot f_4 + \cdots + 17 \cdot 1 = 46,$$

$$f_3 + f_4 + \cdots + f_{16} = 10.$$

From the first one,

$$3 \cdot f_3 + 4 \cdot f_4 + \cdots + 16 \cdot f_{16} = 29,$$

which can not be true, since no 10 planar faces can be bordered by 29 edges (even if they all were 3-faces, we would need at least 30 edges!). Thus, the original assumption that $H(Q)$ can be drawn on the plane with straight edges, is wrong.

It is easy to see, however, that there exists a planar embedding of $H(Q)$ with all but one edges straight.

Lemma 5.6.2. If a halving edges graph H can be drawn with straight lines on the plane, and it's degree sequence coefficients are known to be $(n_1; n_3; \cdots)$,

where n_1 is odd, then H is necessary an extension of a halving edges graph on $n-2$ points with at least

$$\lfloor\frac{(n_1-1)(n-n_1)}{n_1}\rfloor+(n_1-2)+\lfloor n/n_1\rfloor-2$$

halving lines, in particular when $n_1=3$, H can be obtained as an extension of a set of $n-2$ points with at least

$$\lfloor\frac{2n-3}{3}\rfloor+\lfloor n/3\rfloor-2$$

halving lines.

This gives a considerable restriction of the extension process that needs to be done to obtain exact values of $h(n)$ for small n. For example, all sets of 12 points maximizing the number of halving lines can be obtained just by extending sets of $(10,\geq 9)$ sets, which is just 15% of all nonisomorphic sets of 10 points!!!!!!!

Similar bound is easily obtainable if we allow at most one line in a planar embedding to be curved. This bound can be implied to reduce the number of systems that need to be extended in order to obtain all $(14,23)$-sets (if the latter exist).

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