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Two new Ramsey numbers

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Two New Ramsey Numbers

by

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I would like to thank Professor Stanislaw Radziszowski for introducing me to the theory of Ramsey numbers and for sharing with me his experience and insight during my quest for new results in this field.

ABSTRACT

A graph with many vertices cannot be homogeneous,i.e., for any pair of integers (i,j) all large graphs must contain either a complete subgraph on i vertices or an independent set of size j. The Ramsey number for (i,j) is the smallest integer R such that all graphs with at least R vertices have this property. For example, the (3,3) Ramsey number is 6; if a graph has 6 or more vertices, then is must contain a triangle or an independent set of size 3. The (4,4) Ramsey number is 18, found in 1954 [GG]. The (5,5) Ramsey number is still unknown; it is between 43 and 52.

This thesis deals with subgraphs slightly different from the classical types. The subgraphs here are complete graphs with one edge missing and induced subgraphs with exactly one edge. The (4,6) and (4,7) Ramsey numbers for these types of subgraphs is computed. The method used is an exhaustive search, with many shortcuts employed to reduce computation time.

KEY WORDS

Ramsey, graph, Schläfli.

4. Table of Contents.

A. BACKGROUND

1. Notation.

In this thesis all graphs are undirected. A graph is complete if every pair of vertices forms an edge. A graph is n-colored if each edge is assigned one of n colors and a subgraph is monochromatic if all edges in the subgraph have the same color.

The Ramsey number $R(G,H)$ is the smallest integer n such that every 2-coloring of the complete graph on n vertices contains a monochromatic subgraph in the first color isomorphic to G or a monochromatic subgraph in the second color isomorphic to H. Equivalently, $R(G,H)$ can be defined as the smallest integer n such that for every graph F on n vertices either F contains a subgraph isomorphic to G or the complement of F contains a subgraph isomorphic to H. Both forms of this definition will be used henceforth, with the context making clear which form is appropriate for a particular argument. Note that $R(G,H)$ equals $R(H,G)$.

This thesis considers G and H of the form K_n, the complete graph on n vertices, or K_n-e, the complete graph on n vertices without one edge. The techniques used are similar to those in [RK1], [RK2], [Ra1], and [Ra2].

A graph F on n vertices is called a (G,H,n)-good graph if there is no G in F and no H in the complement of F. A graph F on n vertices is (G,H)-good if it is a (G,H,n)-good graph for some n. A graph is critical for $R(G,H)$ if its number of vertices is $R(G,H)$ -1.

The following notation is used throughout:

x = vertex in graph G
$\delta(x)$ = degree of x
G_x = subgraph of G induced by all vertices adjacent to x
H_x = subgraph of G induced by all vertices different from x and not adjacent to x.

The process of decomposing G into the triple (x,G_x,H_x) is called preferring the vertex x in G (Figure 1).

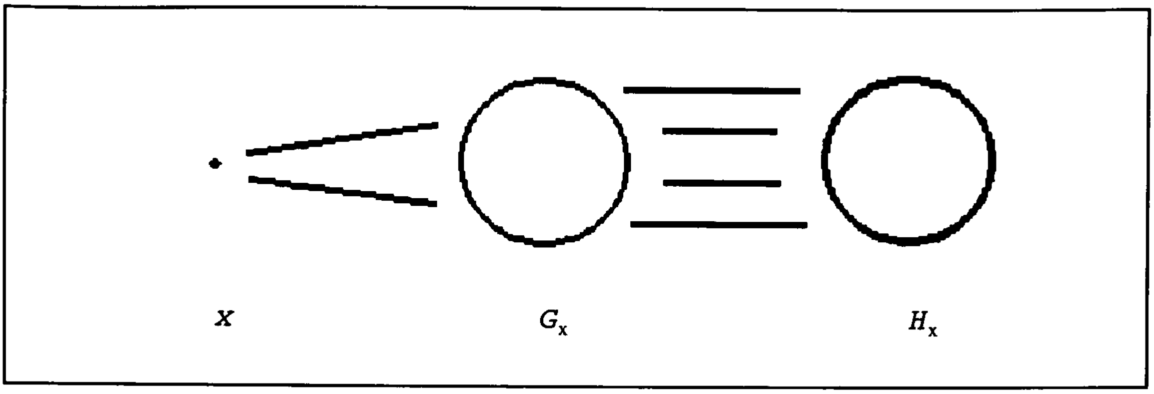

Figure 1. Preferring a vertex x.

2. History.

The theory of Ramsey numbers was begun in 1930 when Frank Plumpton Ramsey, an English mathematician, published a paper [Ram] entitled "On a Problem of Formal Logic" in which he proved:

Theorem 1. For every pair of integers i,j there exists an integer R such that for all integers n greater than or equal to R and all graphs F on n vertices, either F contains a subgraph isomorphic to K_i or the complement of F contains a subgraph isomorphic to K_j.

It follows that for every n less than R, there is some (K_i,K_j,n)-good graph. This result is the basis for finding Ramsey numbers by exhaustive search methods.

Ramsey's theorem has been generalized in three principal directions: using more than two colors, using subgraphs other than complete graphs, and using hypergraphs. There is only one known three-color Ramsey number: $R(3,3,3) = 17$ [GG], i.e., each 3-color graph with 17 or more vertices has at least one monochromatic triangle, and there is a 3-color graph on 16 vertices with no monochromatic triangle. A <u>hypergraph</u> on a set of vertices V is a collection of subsets, called <u>hyperedges</u>, of V. For example, a 3-graph is any collection of 3-sets from a set V. The only known Ramsey number for hypergraphs was found in 1991 by McKay and Radziszowski [MR]. Here is their result.

Theorem 2. Every 2-coloring of the 3-sets of a set of size at least 13 contains a monochromatic tetrahedron. In the notation of Ramsey theory:

$$R(K_4,K_4;3) \le 13$$

The existence of a 2-coloring of the 3-sets of a set of size 12 with no monochromatic tetrahedron was shown in 1969 by Isbell [Is], so Theorem 2 establishes 13 as the Ramsey number $R(K_4,K_4;3)$.

In 1935 P. Erdös and G. Szekeres published a paper [ES] with ideas similar to Ramsey's in which two famous theorems were established:

Theorem 3.

(a) For each integer n there exists an integer R such that every collection of R points in the plane contains a convex n-gon.

(b) Each sequence of length n^2+1 contains a monotone subsequence of length $n+1$.

The Erdös-Szekeres paper was widely read and created much interest in the theory and calculation of numbers related to Ramsey numbers. Recent interest in this field caused Graham, Rothschild, and Spencer to publish a second edition of their book *Ramsey Theory* [GRS].

B. CLASSICAL RAMSEY NUMBERS

1. Asymptotics.

The classical Ramsey numbers deal with G and H complete graphs. One of the most useful tools used in Ramsey theory comes from the analysis of the triple (x,G_x,H_x), where x is preferred in a (K_i,K_j,n)-good graph G. The following arguments were presented by Greenwood and Gleason in 1955 [GG].

Theorem 4. If i and j are at least 3, then

$$R(K_i,K_j) \le R(K_{i-1},K_j) + R(K_i,K_{j-1})$$

Furthermore, if $R(K_{i-1},K_j)$ and $R(K_i,K_{j-1})$ are both even, then strict inequality holds.

Proof. Let $n = R(K_{i-1},K_j) + R(K_i,K_{j-1})$, and assume G is a (K_i,K_j,n)-good graph. Let x be preferred. Then G_x is $(K_{i-1},K_j,\delta(x)-1)$ -good and H_x is $(K_i,K_{j-1},n-1-\delta(x))$ -good. Therefore $\delta(x) < R(K_{i-1},K_j)$ and $n-1-\delta(x) < R(K_i,K_{j-1})$. This implies that $n-1 < R(K_{i-1},K_j)-1 + R(K_i,K_{j-1})-1$, so that G cannot exist. Therefore the relation holds.

If $R(K_{i-1},K_j)$ and $R(K_i,K_{j-1})$ are both even, and $R(K_i,K_j) = R(K_{i-1},K_j) + R(K_i,K_{j-1})$, then let $n = R(K_i,K_j) -1$ and let G be a (K_i,K_j,n)-good graph. Since the number of vertices in G is odd, there is a vertex x with even degree. Preferring x decomposes the even number $n-1$ into the sum of $\delta(x)$ and $n-1-\delta(x)$. This forces $\delta(x)$ to be $R(K_{i-1},K_j)-1$, which is odd, contradicting the choice of x.

The early work on Ramsey numbers concentrated on estimating the asymptotic value of $R(K_k,K_k)$. If this function is denoted by R then theorem 4 can be used to show the following result.

Theorem 5.

$$R \le C\, 4^k\, k^{-1/2} \quad \text{for some constant } C$$

A lower bound for R is established using the method of finite probability spaces developed by Erdös [Er].

Theorem 6.

$$R \geq k\, 2^{k/2}\, [C + o(1)] \quad \text{for some constant } C$$

Proof. The following notation is needed for the proof:

a = the number of 2-sets in a n-set
b = the number of 2-sets in a k-set
c = the number of k-sets in a n-set
d = the number of k-sets in a R-set

Assume n is chosen so that $2c$ is less than 2^b. Note that the total number of ways to 2-color K_n is 2^a and the total number of ways to 2-color K_k is 2^b. If both colors are equally likely in a 2-coloring of K_n then the probability that a given k-set is monochromatic is $2(2^{-b})$.

Therefore the probability that some k- set is monochromatic is at most $2c(2^{-b})$. Since this probability is less than 1, there exists a 2-coloring of K_n with no monochromatic K_k. Therefore some (K_k, K_k, n)-good graph exists and implying R is greater then n.

It follows that $2R$ is at least 2^b. Applying Stirling's formula to R yields the inequality stated in the theorem.

Theorems 5 and 6 combine to show

$$\sqrt{2} \leq \liminf R^{1/k} \leq \limsup R^{1/k} \leq 4$$

It is not yet known if $\lim R^{1/k}$ exists.

2. Exact values.

The exact values of Ramsey numbers are easily found for the following pairs of graphs:

Theorem 7.
(a) $R(K_2, K_j) = j$ for $j > 1$
(b) $R(K_3, K_3) = 6$

Proof. Since (K_2, K_j, n)-good graphs have no edges, n must be less than j, implying the Ramsey number is at least j. The graph with no edges and j vertices shows the Ramsey number is at most j.

To show that $R(K_3,K_3)$ is at most 6, consider a 2-coloring of K_6, and let v be any vertex. One may assume that at least 3 edges incident with v are red. If v belongs to a red triangle, the theorem is proved. Otherwise 3 vertices forming red edges with v comprise a green triangle. To show that $R(K_3,K_3)$ is greater than 5, color the edges of a pentagon red and the diagonals green. This coloring has no monochromatic triangle.

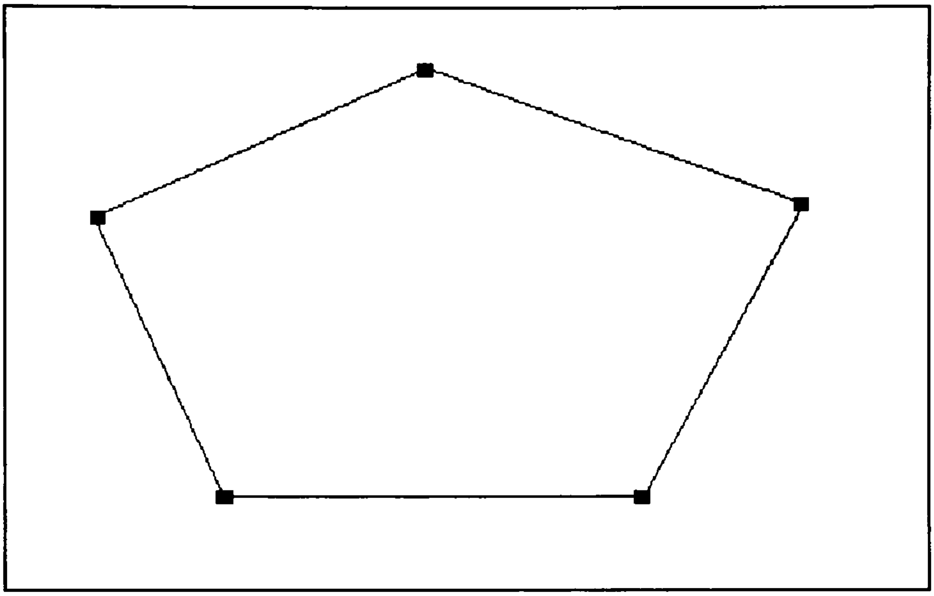

Figure 2. Unique critical graph for $R(K_3,K_3) = 6$.

Theorem 8.
(a) $R(K_3,K_4) = 9$
(b) $R(K_3,K_5) = 14$

Proof. Theorems 4 and 7 imply $R(K_3,K_4)$ is at most 9. The following graphs show $R(K_3,K_4)$ is at least 9 and $R(K_3,K_5)$ is at least 14. The latter graph has vertices Z_{13} with two vertices forming an edge if their difference is a cubic residue mod(13). Theorem 4 yields:

$$R(K_3,K_5) \le R(K_2,K_5) + R(K_3,K_4)$$

Since $R(K_2,K_5) = 5$, it follows that $R(K_3,K_5) = 14$ and $R(K_3,K_4) = 9$.

Figure 3 shows the three critical graphs for $R(K_3,K_4)$. These graphs have a relationship which occurs fairly often for Ramsey number critical graphs, viz., the smallest graph is a subgraph of the others.

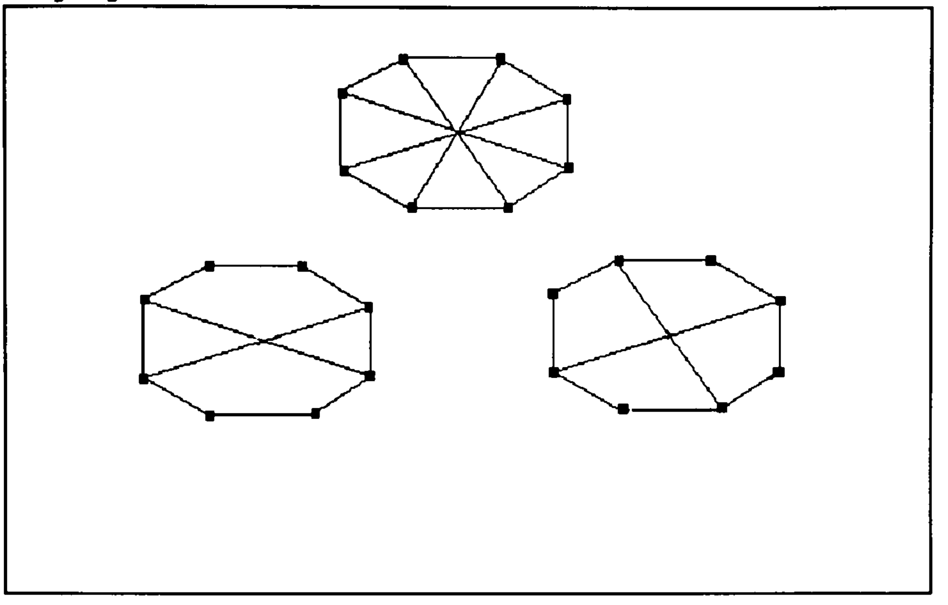

Figure 3. All critical graphs for $R(K_3,K_4) = 9$.

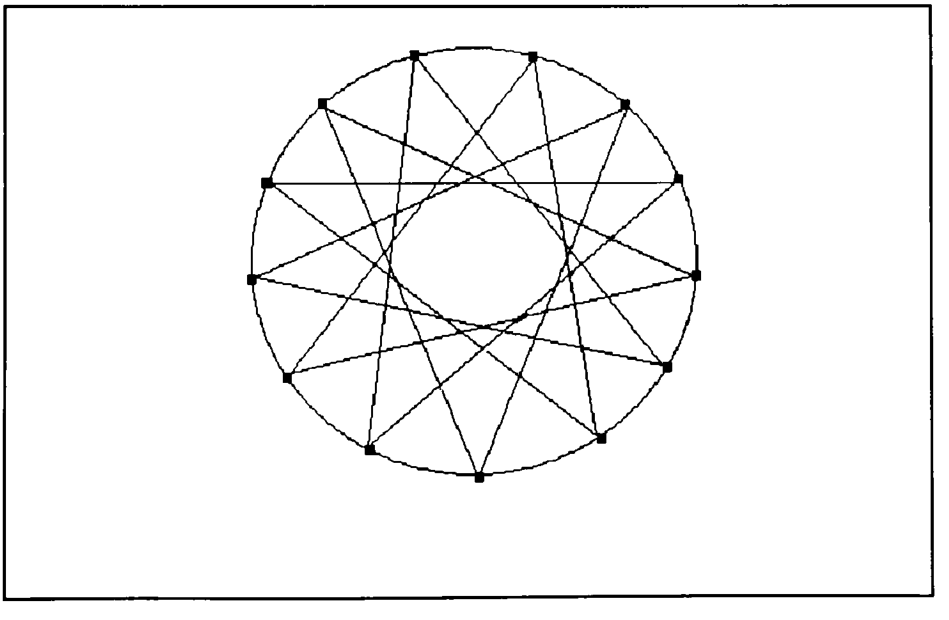

Figure 4. Unique critical graph for $R(K_3,K_5) = 14$.

Theorem 9. $R(K_4,K_4) = 18$

Proof. The relation $R(K_4,K_4) \leq R(K_3,K_4) + R(K_4,K_3)$ shows that $R(K_4,K_4)$ is at most 18 and the following graph shows $R(K_4,K_4)$ is at least 18. This graph has vertices Z_{17} with two vertices forming an edge if their difference is a quadratic residue mod(17).

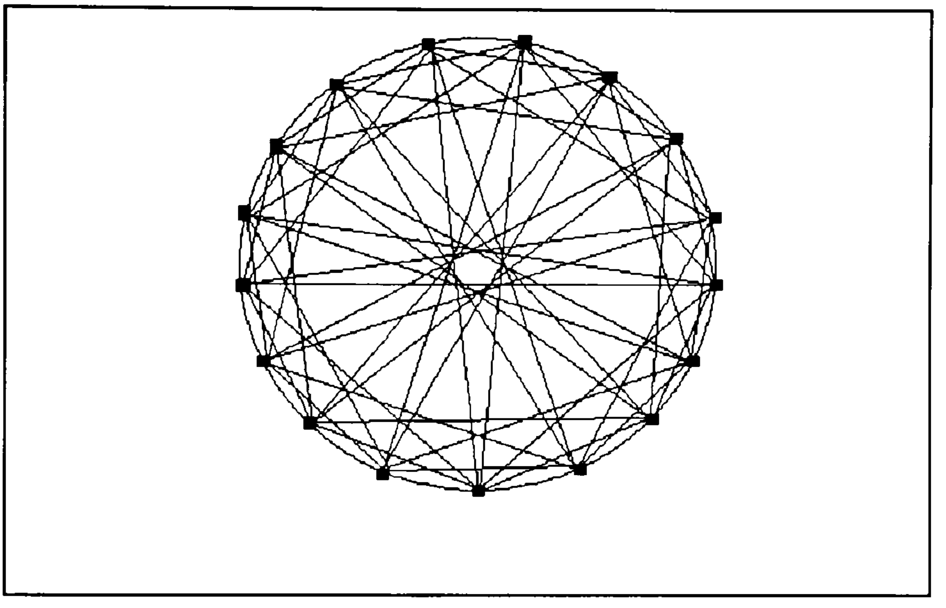

Figure 5. Unique critical graph for $R(K_4,K_4) = 18$.

Two more classical Ramsey numbers were found during the 1960's.

Theorem 10.
(a) $R(K_3,K_6) = 18$ (1966) Kalbfleisch [Ka]
(b) $R(K_3,K_7) = 23$ (1968) Graver,Yackel [GY]

Proof. The lower bound for these Ramsey numbers are established by explicit graphs. The upper bounds require many technical lemmas. The paper by Graver and Yackel, for example, is 51 pages long.

3. Recent Results.

No more classical Ramsey numbers were found during the 1970's. Then computational methods were introduced in 1982 and more progress was made.

Theorem 11.

(a) $R(K_3,K_9) = 36$ (1982) Grinstead, Roberts [GR]
(b) $R(K_3,K_8) = 28$ (1991) McKay, Zhang [MZ]

Proof. The computer was used by Grinstead and Roberts to find the minimum number of edges in certain (K_3,K_7,n)-good graphs and (K_3,K_8,n)-good graphs. McKay and Zhang used the computer to show that no $(K_3,K_8,28)$-good graph exists by examining the H_x graph which would result from preferring a vertex x in such a graph.

C. RAMSEY NUMBERS OF THE FORM $R(K_i,K_j\text{-}e)$

1. Early results.

The elementary facts about $(K_i,K_j\text{-e})$ Ramsey numbers are summarized in the next theorem.

Theorem 12.

(a) $R(K_2,K_j\text{-e}) = j$ for $j > 1$
(b) $R(K_i,K_2\text{-e}) = 2$ for $i > 1$
(c) $R(K_i,K_3\text{-e}) = 2i-1$ for $i > 1$

Proof. Results (a) and (b) are immediate. To prove (c) consider a 2-coloring of K_n. If this 2-coloring is $(K_i,K_3\text{-e},n)$-good it must contain no red K_i and no adjacent green edges. The largest such graph consists of $2i-2$ vertices with $i-1$ vertex-disjoint green edges. Therefore the Ramsey number is $2i-1$.

The early research on this type of Ramsey number is summarized in Theorem 13.

Theorem 13.

(a)	$R(K_4,K_4\text{-e}) = 11$	(1972)	Chvatal, Harary [CH]
(b)	$R(K_3,K_5\text{-e}) = 11$	(1977)	Clancy [Cl]
(c)	$R(K_3,K_6\text{-e}) = 17$	(1980)	Faudree, Rousseau, Schelp [FRS1]
(d)	$R(K_5,K_4\text{-e}) = 16$	(1980)	Bolze, Harborth [BH]
(e)	$R(K_3,K_7\text{-e}) = 21$	(1982)	Grenda, Harborth [GH]

Proof. The method of proof for each of these results consists of the presentation of a critical graph to establish a lower bound and a series of technical lemmas to establish an upper bound. For each of these results, one of the critical graphs will be presented.

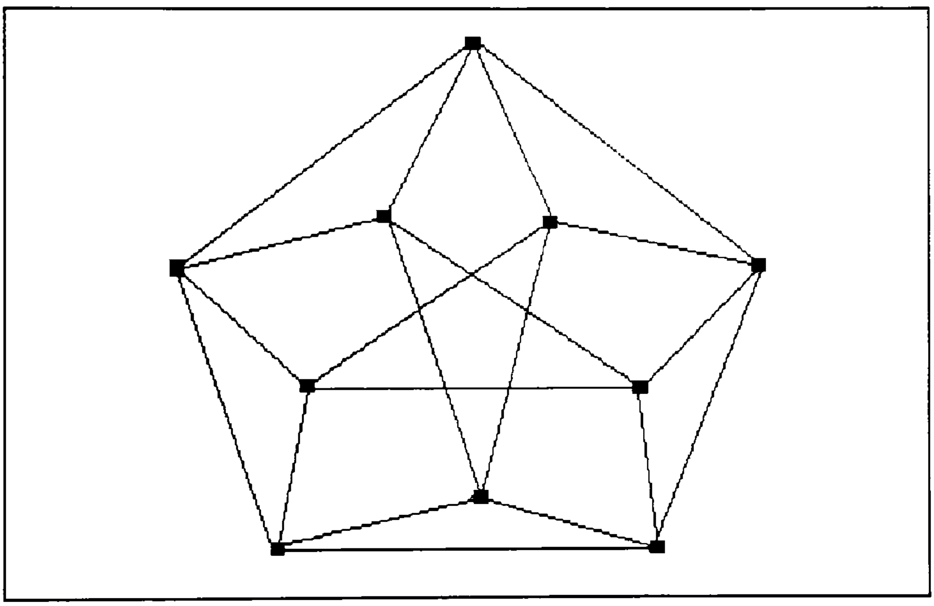

Figure 6. One critical graph for $R(K_4,K_4\text{-e}) = 11$

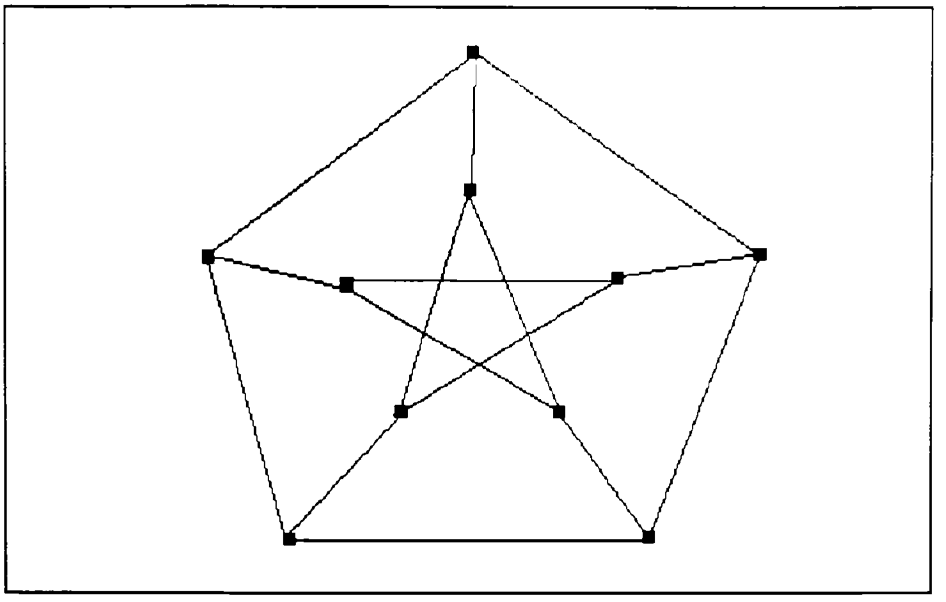

Figure 7. One critical graph for $R(K_3,K_5\text{-}e) = 11$
(This is the **Petersen** graph).

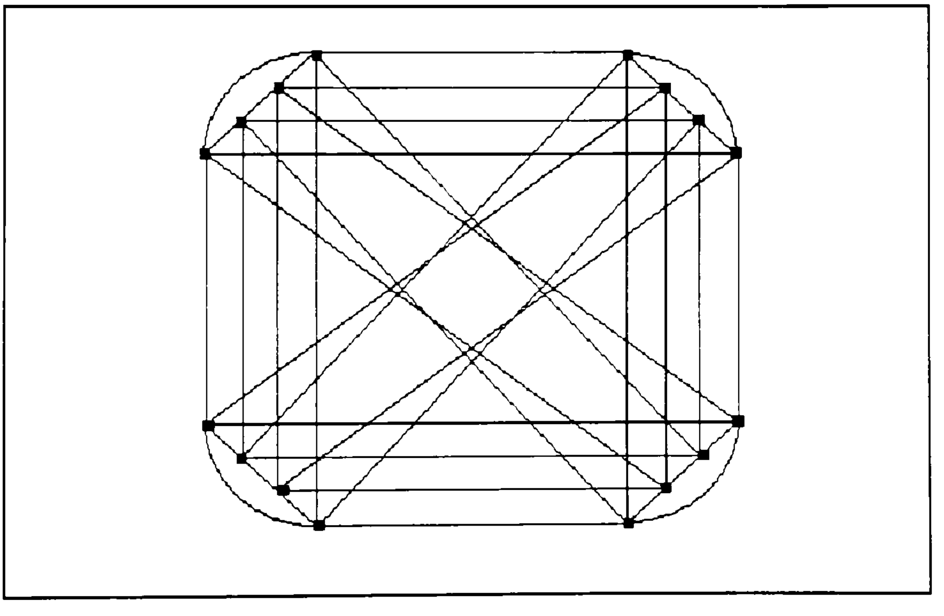

Figure 8. The unique critical graph for $R(K_3,K_6\text{-}e) = 17$

This graph is also critical for the Ramsey numbers $R(3,3,3)$ [GG] and $R(K_4\text{-}e,K_6\text{-}e)$ (see section F).

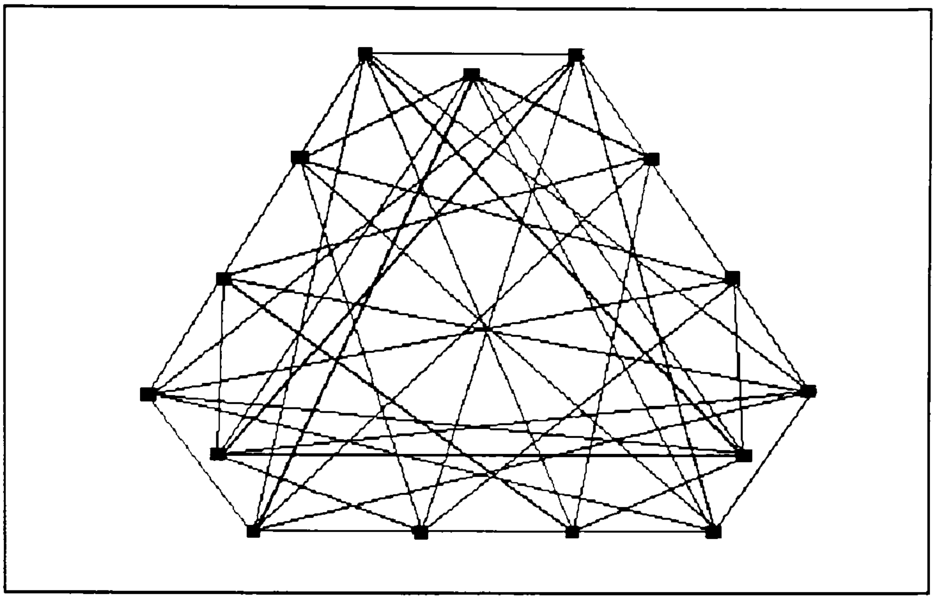

Figure 9. One critical graph for $R(K_5, K_4\text{-}e) = 16$

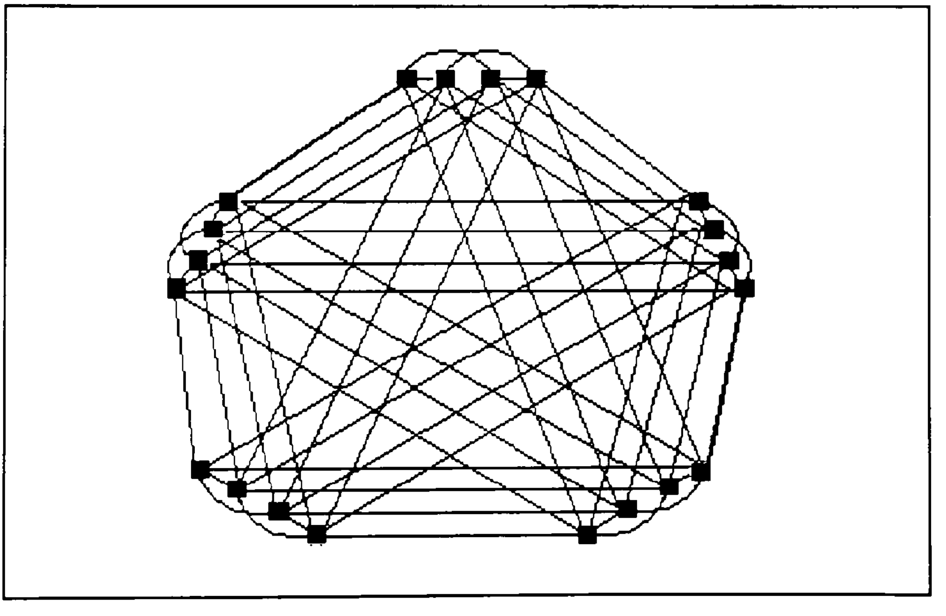

Figure 10. The unique critical graph for $R(K_3, K_7\text{-}e) = 21$

2. Recent results.

In 1988 Exoo, Harborth, and Mengersen [EHM] proved the following theorem.

Theorem 14. $R(K_4, K_5\text{-e}) = 19$

Proof. Their proof examines all critical colorings for $R(K_4, K_4\text{-e}) = 11$ and $R(K_3, K_5\text{-e}) = 11$ in order to prove that no 2-coloring of K_{19} can be missing a red K_4 and a green K_5-e. One critical graph for $R(K_4, K_5\text{-e})$ is shown in Figure 11.

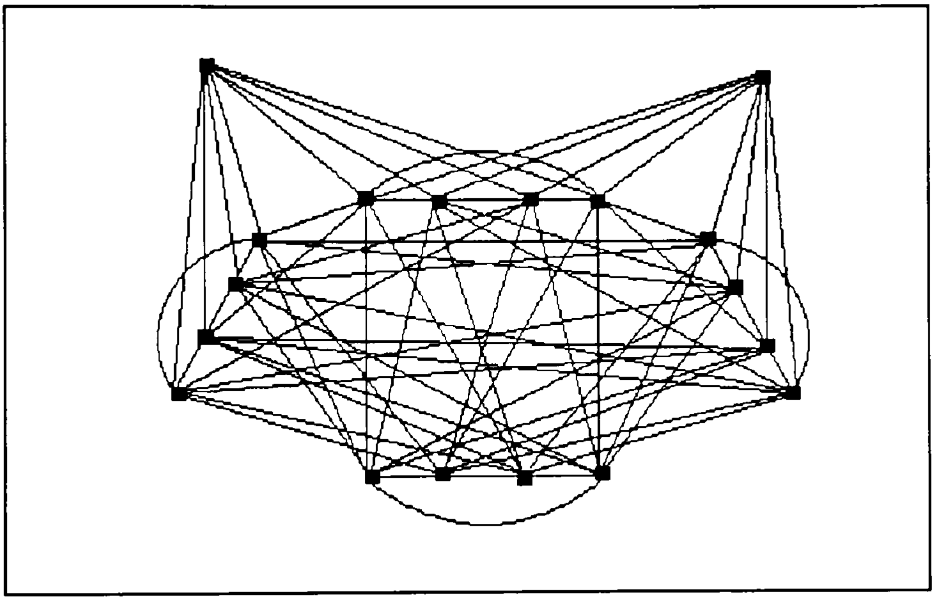

Figure 11. One critical graph for $R(K_4, K_5\text{-e}) = 19$

In 1990 Radziszowski [Ra1] proved the following theorem.

Theorem 15.
(a) $R(K_3, K_8\text{-e}) = 25$
(b) $R(K_3, K_9\text{-e}) = 31$

Proof. The proof presents a general construction for 2-colorings of K_n with no red triangles and no green K_j-e. This construction yields critical colorings for $R(K_3,K_6\text{-e})$, $R(K_3,K_7\text{-e})$, and $R(K_3,K_8\text{-e})$. A separate construction yields a critical coloring for $R(K_3,K_9\text{-e})$. Computer algorithms are used to establish the upper bounds. The numbers which must be calculated by computer are the lower bounds for the number of edges in a $(K_3,K_j\text{-e},n)$-good graph.

3. Enumerating small graphs.

This section contains tables showing the number of $(K_3,K_j\text{-e},n)$-good graphs, for j = 3,4,5, and 6, broken down by the number of vertices, n, and number of edges, e. Also minimal edge numbers are shown for j = 7 and 8. These results were recently obtained by Radziszowski [Ra1].

n=	1	2	3	4
e				
0	1	1		
1		1		
2			1	
3				
4				1

Table 1. Classification of all $(K_3,K_3\text{-}e)$-good graphs.

n=	1	2	3	4	5	6
e						
0	1	1	1			
1		1	1			
2			1	2		
3				2		
4				1	2	
5					2	
6					1	1
7						1
8						1
9						1

Table 2. Classification of all $(K_3,K_4\text{-e})$-good graphs.

$n=$ / e	1	2	3	4	5	6	7	8	9	10
0	1	1	1	1						
1		1	1	1						
2			1	2	2					
3				2	3	1				
4				1	4	4				
5					2	7				
6					1	7	5			
7						4	8			
8						2	12	2		
9						1	8	5		
10							1	14		
11							1	12		
12							1	10	1	
13								4	1	
14								2	3	
15								1	1	1
16								1	1	
17										
18										
19										
20										1

Table 3. Classification of all (K_3,K_5-e)-good graphs.

The following table shows one feature of the family of (K_3,K_7-e,n)-good graphs, viz., the minimal number of edges in this family, as a function of n.

For values of n less than 7 the minimal number of edges is 0.

$n=$	7	8	9	10	11	12	13	14	15	16	17	18	19	20
e=	2	3	4	5	8	11	15	19	24	30	37	43	54	60

Table 4. Minimal number of edges, e, over all (K_3,K_7-e,n)-good graphs, for n from 7 to 20

e \ n=	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	1	1	1	1											
1		1	1	1	1											
2			1	2	2	2										
3				2	3	4	1									
4				1	4	7	5	1								
5					2	9	11	1								
6					1	7	19	10								
7						4	20	25	1							
8						2	18	51	10							
9						1	11	64	33							
10							5	60	97	3						
11							1	38	167	11						
12							1	21	195	70						
13								9	150	204						
14								3	92	388	2					
15								2	42	445	23					
16								1	20	364	110					
17									8	217	261					
18									3	111	374	3				
19									1	50	330	9				
20									1	22	216	44				
21										10	101	71				
22										4	41	86				
23										2	12	56				
24										1	4	37	2			
25										1	1	22	5			
26												13	8			
27												6	5			
28												2	2			
29												1				
30														2		
31														2		
32																
33																
34																
35															1	
36																
37																
38																
39																
40																1

Table 5. Classification of all (K_3,K_6-e)-good graphs.

D. RAMSEY NUMBERS OF THE FORM $R(K_i\text{-e},K_j\text{-e})$, $i \leq j \leq 5$

1. First results.

Theorem 16. $R(K_3\text{-e},K_j\text{-e}) = 2j - 3$ for $j \geq 3$

Proof. Consider any two-coloring of K_n, for which the red subgraph has no K_3-e and the green subgraph has no K_j-e. Then the red subgraph has maximum degree 1, and the maximum number of red edges is j-2.

Theorem 17.
(a) $R(K_4\text{-e},K_4\text{-e}) = 10$
(b) Furthermore, there is a unique $(K_4\text{-e},K_4\text{-e},9)$-good graph.

Proof. Theorem 4, which was originally proved for complete subgraphs, can be proved also for subgraphs which are complete with one edge missing. This version of theorem 4, together with theorem 16, gives an upper bound of 10 for $R(K_4\text{-e},K_4\text{-e})$. The unique $(K_4\text{-e},K_4\text{-e},9)$-good graph is the line graph of $K_{3,3}$ (the line graph of a graph G has vertices the edges of G and two vertices are adjacent if their corresponding edges are adjacent in G). To show this graph is unique, let G be any $(K_4\text{-e},K_4\text{-e},9)$- good graph. Since the Ramsey number $R(K_3,K_3)$ is 6, one may assume there is red triangle T in G. The remaining 6 vertices cannot contain a green cycle of length m, where m is 3, 4, or 5, as the following argument shows.

Assume there is a green cycle C on m vertices from the remaining 6 vertices. Since G has no red K_4-e, each vertex of C belongs to at least $2m$ green edges incident with T. For m equal to 3, 4, or 5, this implies one of the vertices in T must belong to at least m-1 green edges incident with C, forming a green K_4- e, which is not allowed.

Let $\{u_1,u_2,u_3\}$ be the red triangle R. Since the remaining 6 vertices do not have a green triangle, they must contain a red triangle, $\{u_4,u_5,u_6\}$. It is easy to see that the final 3 vertices must also form a red triangle, and that each of these 3 red triangles must have 3 red edges incident with each of the other 2, forming the line graph on $K_{3,3}$.

Table 6 shows all (K_4-e,K_4-e)-good graphs, classified by the number of vertices, n, and the number of edges, e. It was computed by Radziszowski in 1989 [Ra2].

e \ n=	1	2	3	4	5	6	7	8	9
0	1	1	1						
1		1	1						
2			1	2					
3			1	3					
4				2	3				
5					4				
6					3	2			
7						4			
8						4			
9						2			
10							2		
11							2		
12									
13									
14								1	
15									
16									
17									
18									1

Table 6. Classification of all (K_4-e,K_4-e)-good graphs.

2. Properties of (K_4-e,K_5-e,n)-good graphs.

Clancy [Cl] proved $R(K_4\text{-e},K_5\text{-e}) = 13$ in 1977. The 14 critical graphs for $R(K_4\text{-e},K_5\text{-e})$ were found by Faudree, Rousseau, and Schelp [FRS2] in 1985. Table 7 shows the number, *NE,* of edges and the size, *SA,* of the automorphism group for each of these critical graphs.

NE=	24	24	25	25	26	26	26	27	27	27	28	28	29	30
SA=	48	16	8	8	4	16	16	12	12	4	4	16	8	48

Table 7. Critical graphs for $R(K_4$-e,K_5-e).

The family of (K_4-e,K_5-e,12)-good graphs has the interesting property that each graph in this family contains one of the graphs with 24 edges as a subgraph.

Table 8 shows all (K_4-e,K_5-e)-good graphs, classified by the number of vertices, n, and the number of edges, e. It was computed by Radziszowski in 1989 [Ra2].

e \ n=	1	2	3	4	5	6	7	8	9	10	11	12
0	1	1	1	1								
1		1	1	1								
2			1	2	2							
3			1	3	4	1						
4				2	6	5						
5					5	11	1					
6					3	16	8					
7						12	21	1				
8						6	39	5				
9						2	39	18	1			
10							20	62	1			
11							6	102	3			
12							1	92	18			
13								37	70			
14								9	173			
15								1	176	3		
16								1	81	18		
17									16	74		
18									4	153		
19										116		
20										37	5	
21										6	19	
22											39	
23											32	
24											10	2
25											2	2
26												3
27												3
28												2
29												1
30												1

Table 8. Classification of all (K_4-e,K_5-e)-good graphs.

3. Recent Results.

Theorem 18. $R(K_5\text{-e},K_5\text{-e}) = 22$

This was proven in 1989 by Clapham, Exoo, Harborth, Mengersen, and Sheehan [CEH] using properties of certain graphs on 11 vertices. It was proven independently by Radziszowski [Ra2] using the enumeration of all $(K_5\text{-e},K_5\text{-e})$-good graphs on 20 and 21 vertices.

There is a unique graph which is critical for $R(K_5\text{-e},K_5\text{-e})$. It is regular of degree 10. Its 105 edges are partitioned into three sets of 35 edges in Figures 12 and 13.

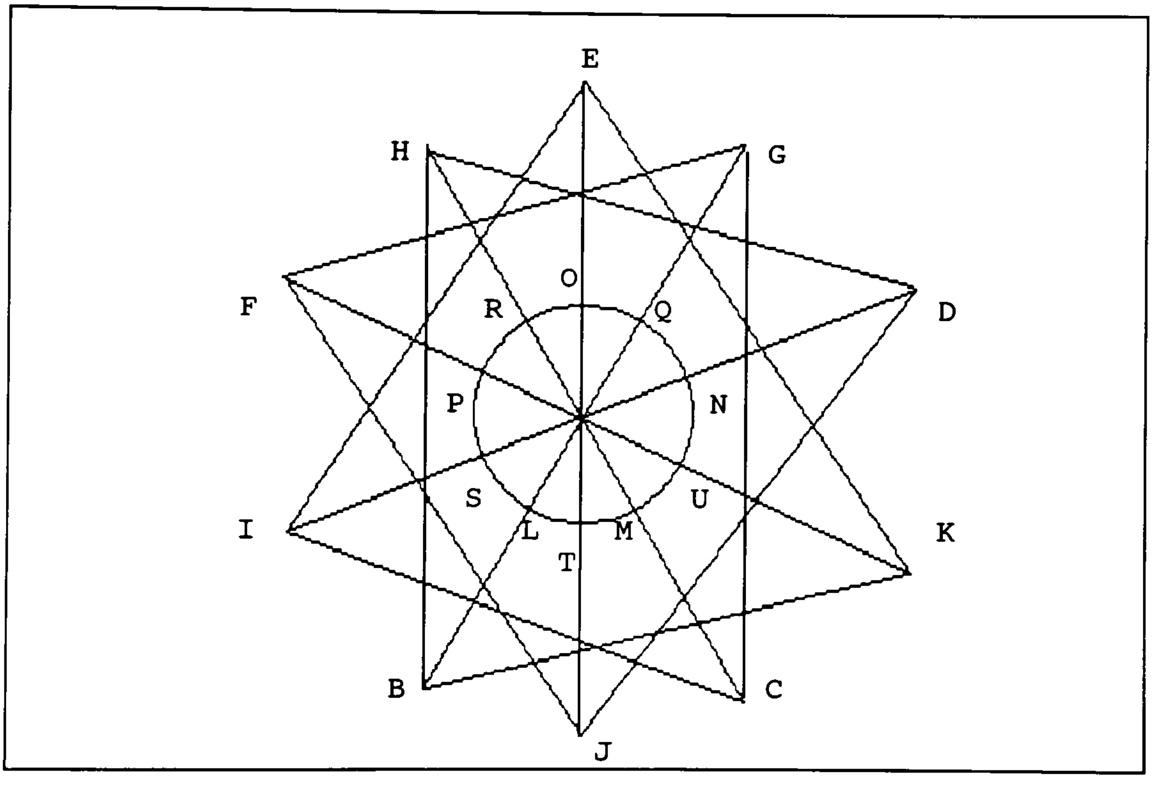

Figure 12. 35 edges from the unique critical graph for $R(K_5\text{-e},K_5\text{-e}) = 22$

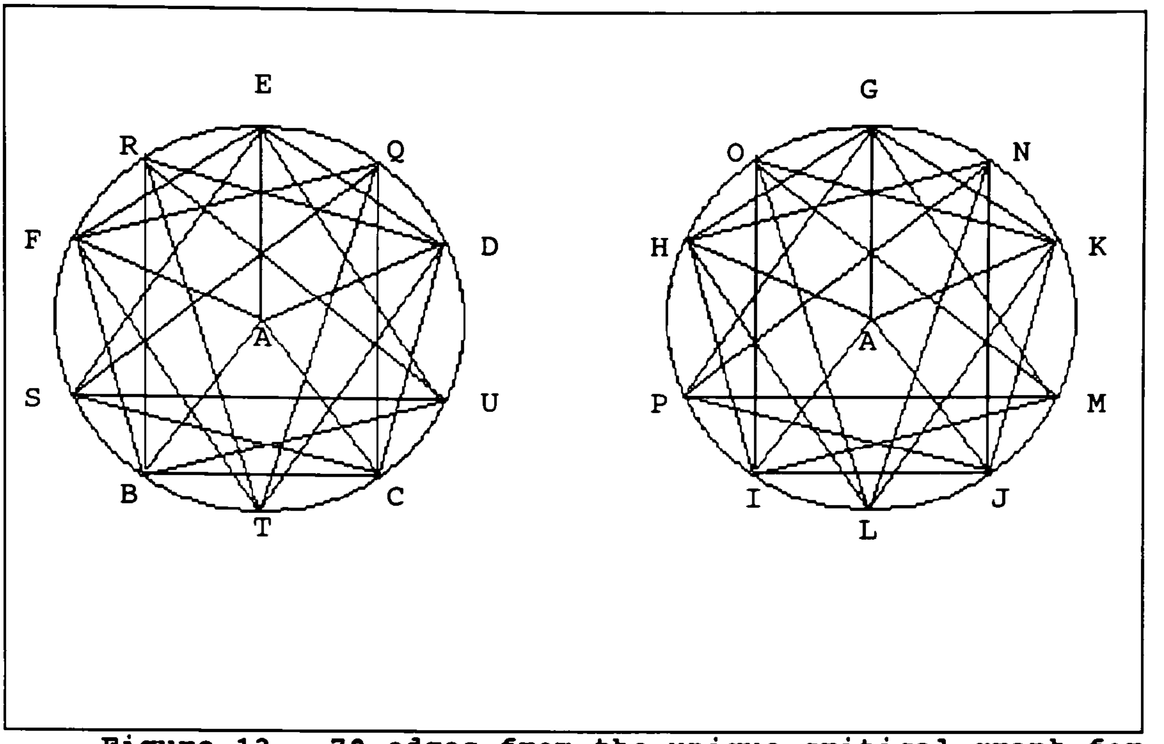

Figure 13. 70 edges from the unique critical graph for $R(K_5\text{-e},K_5\text{-e}) = 22$

E. THE RAMSEY NUMBER $R(K_4\text{-e},K_6\text{-e})$ is 17.

This section contains the main results of this thesis. The lower bound of 17 for the Ramsey number $R(K_4,K_6\text{-e})$ was already known, since $R(K_4,K_6\text{-e}) \geq R(K_3,K_6\text{-e}) = 17$. That 17 is also an upper bound is proved by two different methods. The first method is an exhaustive search by computer to determine if there are any $(K_4\text{-e},K_6\text{-e},17)$-good graphs. The second method is a computer count of how many $(K_4\text{-e},K_5\text{-e})$-good graphs have certain properties related to the existence of a $(K_4\text{-e},K_6\text{-e},17)$-good graph. These two methods are detailed below.

1. A lower bound for $R(K_4\text{-e}, K_6\text{-e})$ is 17.

There are four $(K_4\text{-e}, K_6\text{-e}, 16)$-good graphs. The smallest, with 40 edges (Fig. 8), has been shown above to be critical for $R(K_3, K_6\text{-e})$. The other $(K_4\text{-e}, K_6\text{-e}, 16)$-good graphs were previously known to Geoffrey Exoo [Ex]. There are 3 of them, with number of edges equal to 48, 49, and 50, and automorphism group sizes of 48, 24 and 48, respectively. The graph with 48 edges (Fig. 12), is regular of degree 6 and is a subgraph of the two larger graphs.

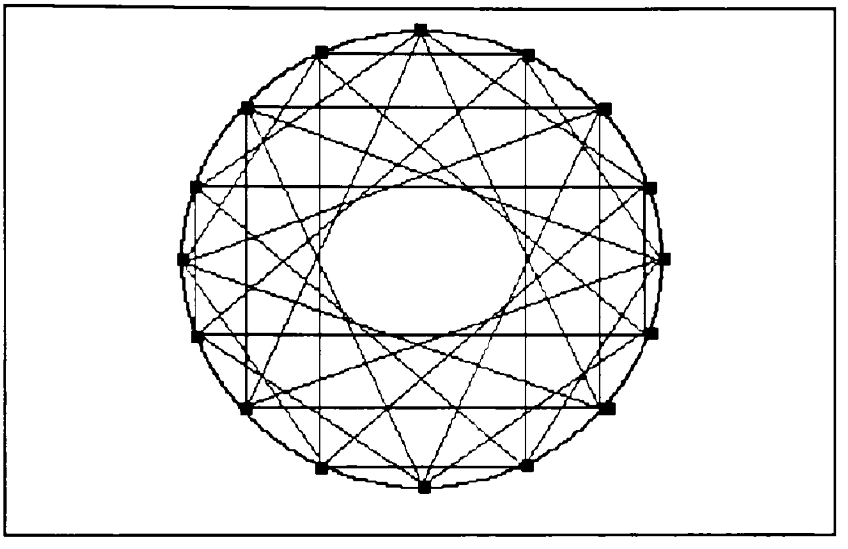

Figure 14. Another critical graph for $R(K_4\text{-}e, K_6\text{-}e)$.

2. Search for all $(K_4\text{-e},K_6\text{-e},17)$-good graphs.

The following notation is used in all of part F:

G = $(K_4\text{-e},K_6\text{-e},17)$-good graph

x = any vertex in G

S = support set
= any subset of vertices of H_x satisfying:
(S1) no triangle in H_x has 2 vertices in S
(S2) S induces in H_x a subgraph with maximum degree at most 1
(S3) no independent 4-set in H_x is disjoint from S

OKN = binary relation on the family of support sets consisting of those pairs (S,T) such that:
(OKN1) no subgraph of H_x which is induced by 4 vertices and has only 1 edge is disjoint from the union of S and T
(OKN2) no independent 4-set in H_x has 3 vertices outside the union of S and T, and 1 vertex in $S - T$ or $T - S$.

Note that the vertices in H_x adjacent to a vertex y in G_x form a support set, called the support set rooted at y. Note further that every G_x is a $(K_3\text{-e},K_6\text{-e},\delta(x))$-good graph and every H_x is a $(K_4\text{-e},K_5\text{-e},16-\delta(x))$-good graph.

The exhaustive-search proof that no $(K_4\text{-e},K_6\text{-e},17)$-good graph exists proceeds as follows. If such a G exists, and some vertex x is preferred, then the H_x part of the resulting triple (x,G_x,H_x) is a $(K_4\text{-e},K_5\text{-e},n)$-good graph. The computer program "fillJ4J6" examines all $(K_4\text{-e},K_5\text{-e},n)$-good graphs and shows that no such triple exists.

The counting proof also examines the decomposition of G into (x,G_x,H_x). Necessary properties of H_x are detailed and the number of $(K_4\text{-e},K_5\text{-e},n)$-good graphs with these properties are counted. The next two sections deal with the counting proof.

3. Properties of $(K_4\text{-}e, K_6\text{-}e)$-good graphs.

Many of the results below rely on the properties of support sets. Since H_x is a $(K_4\text{-e}, K_5\text{-e}, \delta(x))$-good graph, no support set can have more than 6 vertices. Since G_x is a $(K_3\text{-e}, K_6\text{-e}, 16-\delta(x))$-good graph, and has maximum degree at most 1, G_x has at most 8 vertices. Moreover, if G_x has more than 5 vertices, then at most 1 vertex does not belong to an edge. It is clear that support sets rooted at adjacent vertices of G_x are disjoint and support sets rooted at non-adjacent vertices of G_x are OKN-related.

An edge in H_x is called a <u>support edge</u> if its vertices form a support set. The first proposition characterizes support edges and shows that H_x has relatively few edges which can occur as subsets of support sets.

Proposition 1. If an edge in H_x has both vertices in the same support set, then it is a support edge.

Proof. Let $\{x,y\}$ be an edge in H_x with x and y in a support set S. It suffices to show that $\{x,y\}$ is incident with every independent 4-set I in H_x. Assume neither x nor y belongs to I. Since the complement of H_x has no K_5-e, both x and y must be adjacent to at least 2 vertices in I. Since $\{x,y\}$ is not in a triangle, by (S1), there must be exactly 2 vertices in I adjacent to x and the remaining 2 vertices in I must be adjacent to y. One of the vertices of I must belong to S, by (S3). This causes 2 edges in S to be incident, contradicting (S2).

The second proposition relates to vertices in G of degree 4, 5, or 6.

Proposition 2.

(a) If H_x has 12 vertices, then H_x has no support sets.

(b) If H_x has 11 vertices, then
- (1) H_x has at most 4 support edges
- (2) H_x has at most 3 support sets which are pairwise *OKN*.

(c) If H_x has 10 vertices, then
- (1) H_x has at most 9 support edges
- (2) H_x has no pairwise *OKN* collection of 4 support sets S,T,U,V satisfying:
  - (i) S and T have size at least 5
  - (ii) U and V have size at least 4
  - (iii) U and V contain at least 1 edge each.

Proof. Four computer programs, described in Appendix A, have been written to do the counting required to establish this result. All 4 programs examine all graphs in an input file consisting of all $(K_4\text{-e},K_5\text{-e},n)$-good graphs for a given n. The programs are: "countS", which counts all support sets; "countE", which counts all support edges; "hxOKN", which counts all pairwise *OKN* sequences of support sets; and "OKN4E5", which counts those pairwise *OKN* sequences of support sets satisfying the conditions in (c)(2).

4. Main Results.

Theorem 19.
If G exists, then G has minimum degree at least 5.

Proof. The Ramsey number $R(K_4\text{-e},K_5\text{-e})$ is 13, so each H_x has size at most 12. Proposition 2(a) shows that no H_x has size 12. Thus the maximum size of H_x is at most 11 and the minimum degree in G is at least 5.

Theorem 20.
If G exists, then G has minimum degree at least 6.

Proof. Assume that some vertex x has degree 5. Then H_x has 11 vertices. If x belongs to fewer than 2 triangles, then G_x has an independent set of size 4 and H_x has 4 pairwise *OKN* support sets, which is not allowed by Proposition 2(b)(2). Therefore each vertex of degree 5 belongs to exactly 2 triangles. The properties of G_x mentioned above then imply that all vertices of G belong to at least 2 triangles.

Now let y be a vertex of degree 5. Consider the 5 support sets in H_y rooted at the vertices adjacent to y. These vertices must each belong to a triangle not containing y, so the 5 support sets they generate must each contain one or more edges. This causes H_y to have at least 5 support edges, contradicting Proposition 2(b)(1). Thus no vertex in G has degree 5.

Theorem 21.
If G exists, then G has minimum degree equal to 6.

Proof. If the minimum degree is greater than 6 then the only degrees are 7 and 8, since no G_x has size greater than 8. If every degree is 8, then every vertex belongs to 4 triangles, and in every H_x the 8 support sets break up into 4 pairs of support sets, with each pair consisting of

disjoint support sets containing 3 support edges each. This requires 12 vertices in an H_x with 8 vertices and cannot happen.

Therefore some vertex y has degree 7. Its H_y has 3 pairs of support sets with each pair consisting of disjoint support sets having at least 5 vertices each. This requires 10 vertices in an H_y with 9 vertices, again impossible. Thus the minimum degree is neither 8 nor 7.

Theorem 22. If G exists, then every vertex of G belongs to at least three triangles.

Proof. The only vertices which can belong to fewer than 3 triangles are the vertices of degree 6. Assume x is such a vertex and y,z are the 2 vertices in G_x which do not lie in any triangle with x. The support sets in H_x rooted at y and z have size at least 5. Choose 2 non-adjacent vertices u,v in G_x distinct from y and z. The 2 support sets rooted at u and v each have size at least 4 and at least 1 edge. The 4 support sets rooted at y,z,u,v satisfy the conditions of Proposition 2(c)(2) and hence cannot exist. Therefore all degree 6 vertices belong to 3 triangles, implying the theorem.

Theorem 23. The Ramsey number $R(K_4\text{-e},K_6\text{-e})$ is 17.

Proof. It suffices to show G does not exist. Assume that G does exist and that x is a vertex in G of degree 6. Theorem 15 implies that the 6 support sets in H_x have at least 2 edges each, requiring 12 support edges. Proposition 2(c)(1) shows this is impossible, since H_x has size 10.

Table 9 shows the classification of all $(K_4\text{-e},K_6\text{-e},n)$-good graphs by the number of edges, e. The notation u is used to denote an unknown number (not yet computed).

e \ n=	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	1	1	1	1											
1		1	1	1	1											
2			1	2	2	2										
3			1	3	4	5	1									
4				2	6	9	6	1								
5					5	14	16	2								
6					3	17	34	15	1							
7						12	49	49	4							
8						6	55	122	25	1						
9						2	45	210	101	5						
10							22	260	355	23	1					
11							6	223	853	104	3					
12							1	136	1399	529	12	1				
13								49	1537	2066	49	1				
14								12	1163	5567	*u*	4				
15								2	582	9713	*u*	14				
16								1	187	11072	*u*	*u*				
17									38	8261	*u*	*u*				
18									9	4020	*u*	*u*				
19									1	1238	*u*	*u*				
20									1	252	*u*	*u*	2			
21										41	*u*	*u*	26			
22										7	*u*	*u*	447			
23										2	*u*	*u*	*u*			
24										1	*u*	*u*	*u*			
25										1	47	*u*	*u*			
26											4	*u*	*u*	40		
27											1	*u*	*u*	605		
28												*u*	*u*	*u*		
29												78	*u*	*u*		
30												7	*u*	*u*	7	
31													*u*	*u*	24	
32													*u*	*u*	151	
33													250	*u*	589	
34													16	*u*	1645	
35														*u*	3063	
36														*u*	4105	
37														831	4030	
38														71	3100	
39														7	1812	
40															747	1
41															207	0
42															36	0
43															5	0
44																0
45																0
46																0
47																0
48																1
49																1
50																1

Table 4. Classification of (K_4-e,K_6-e)-good graphs.

F. THE RAMSEY NUMBER $R(K_4-e,K_7-e)$ is 28.

1. Strongly regular graphs.

The following exposition of strongly regular graphs is taken from Seidel [Se]. A vertex in a graph is a <u>between-vertex</u> if it is adjacent to at least two vertices. A graph is <u>strongly regular</u> with parameters (k,l,m) if

(a) each vertex is adjacent to k vertices;
(b) each adjacent pair of vertices has l between-vertices; and
(c) each non-adjacent pair of vertices has m between-vertices.

There is a relationship between these three numbers and the number of vertices, n:

(d) $(n-k-1)m = k(k-1-l)$

Examples of strongly regular graphs are the pentagon (Fig. 2) with parameters (2,0,1) and the Petersen graph (Fig. 7) with parameters (3,0,1).

Strongly regular graphs can be characterized by properties of their adjacency matrices. If A is the adjacency matrix of a strongly regular graph G, then the square of A has k for its diagonal entries and either l or m for each off-diagonal entry. If B is the adjacency matrix of the complement of G, I is the identity matrix, and J is the square matrix of all ones, then

(e) $A + B + I = J$
(f) $A^2 = kI + lA + mB$
(g) $A^2 + (m-l)A - (k-m)I = mJ$

It can be shown that the quadratic form in equation (f) has real roots r,s. The four parameters (n,k,r,s) are used to classify strongly regular graphs, because they determine m and l as follows:

(h) $m = k + rs$
(i) $l = r + s + m$

Various combinations of the four parameters have been completely analyzed. For example, there are only four possibilities for strongly regular graphs if $l = 0$ and $m = 1$:

$(n,k,r,s) = (5,2,1/2+\sqrt{5}/2,1/2-\sqrt{5}/2)$ <--> pentagon
$(n,k,r,s) = (10,3,1,-2)$ <--> Petersen graph
$(n,k,r,s) = (50,7,2,-3)$ <--> Hoffman-Singleton graph
$(n,k,r,s) = (3250,56,7,-8)$ <--> unknown

There are only two strongly regular graphs with $r = 1$ and *Krein parameter*(2,2,2)= 0(see Seidel [Se]). They are:

(n,k,r,s) = (27,10,1,-5) <--> complement of Schläfli graph
(n,k,r,s) = (275,112,2,-28) <--> McLaughlin graph

2. The Schläfli graph.

The Schläfli graph has 27 vertices, 216 edges, and is regular of degree 16. Its complement, *CS*, has 135 edges and is regular of degree 10. Equations (h) and (i) show that each edge in *CS* belongs to exactly one triangle and each non-adjacent pair of vertices in *CS* have exactly 5 between-vertices.

The graph *CS* can be constructed from the graph 5*T*, consisting of five triangles with one vertex in common, and the graph, *U*16, the unique (K_3,K_6-e,16)-good graph (Fig. 8). These graphs are shown in Figure 15.

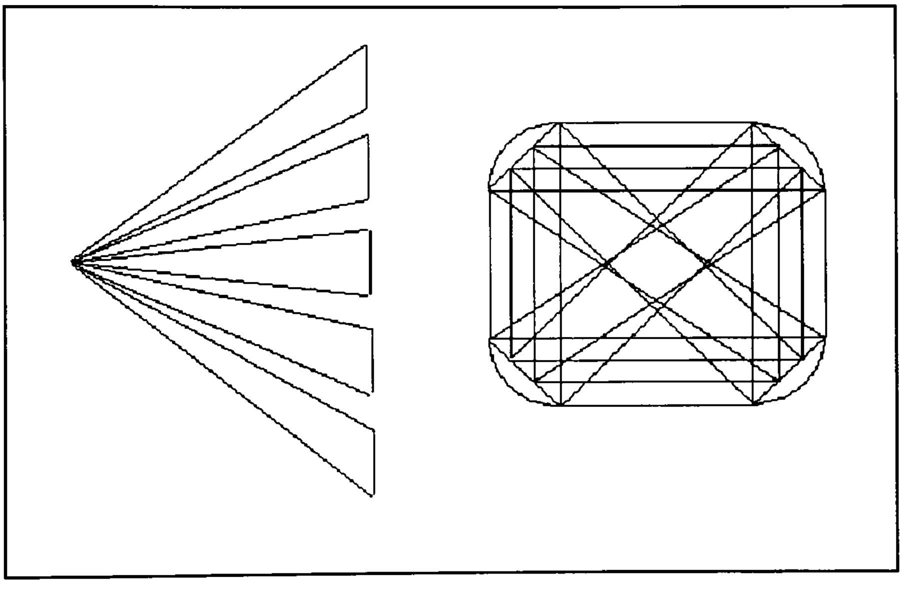

Figure 15. The graphs 5*T* and *U*16.

*U*16 contains 40 edges and each edge belongs to four 4-cycles. Call two edges 4-opposite if they form opposite sides of a 4-cycle. Then the 40 edges of *U*16 can be partitioned into five sets of eight edges each, { e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 }, such that the edges 4-opposite with e_1 are { e_2, e_4, e_6, e_8 } and the edges 4-opposite with e_2 are { e_1, e_3, e_5, e_7 }. Figure 16 shows one such set of eight edges.

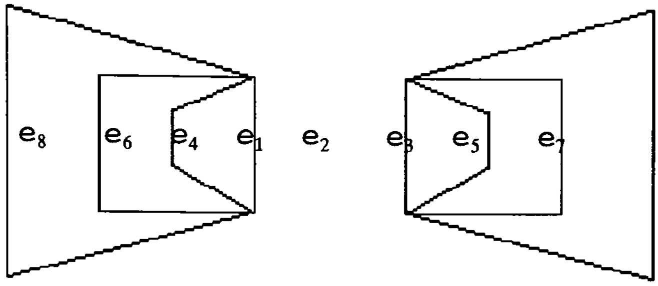

Figure 16. Eight edges from *U16*.

The five triangles in $5T$ are connected to $U16$ by connecting each of the five edges in $5T$ not containing the vertex of degree 10 as follows: one end forms four triangles with the edges 4-opposite from e_1 and the other end forms four triangles with the edges 4-opposite from e_2. This process is repeated for all five triangles in $5T$, resulting in the graph CS.

Theorem 24. CS, the complement of the Schläfli graph, is the unique $(K_4\text{-e},K_7\text{-e},27)$-good graph.

Proof. If y is a vertex in a $(K_4\text{-e},K_7\text{-e})$-good graph, F, and F is decomposed into (y,G_y,H_y) by preferring y, then G_y is $(K_3\text{-e},K_7\text{-e})$-good and H_y is $(K_4\text{-e},K_6\text{-e})$-good. Therefore G_y has at most 10 vertices and H_y has at most 16 vertices, implying F has at most 27 vertices. Thus 28 is an upper bound for the Ramsey number $R(K_4\text{-e},K_7\text{-e})$.

Note that CS has no K_4-e subgraphs. In the Schläfli graph each vertex belongs to 16 K_6's. The largest intersection between any two of these K_6's is a K_3, so there are no K_7-e subgraphs in the Schläfli graph and CS is $(K_4\text{-e},K_7\text{-e})$-good, establishing 28 as the Ramsey number $R(K_4\text{-e},K_7\text{-e})$.

The computer program "fillJ4J6", modified to construct $(K_4\text{-e},K_7\text{-e})$- good graphs, was used to extend all four of the $(K_4\text{-e},K_6\text{-e},16)$-good graphs to all possible $(K_4\text{-e},K_7\text{-e},27)$-good graphs. Only the graph CS was produced, proving its uniqueness as a critical graph for the Ramsey number $R(K_4\text{-e},K_7\text{-e})$.

3. Other (K_4-e,K_7-e)-good graphs.

Some (K_4-e,K_7-e,n)-good graphs have been computed for n less than 27. The following table shows the smallest number of edges observed in a (K_4-e,K_7-e,n)-good graph, for $8 < n < 27$. For n less than 15, the minimal graph has maximum degree < 3. For n between 10 and 15, the minimal graph has five connected components, n-10 triangles, and 15-n edges. For n less than 10, the minimal graph has maximum degree < 2 (except for n = 7, where one vertex has degree 2). For n less than 7, the minimal graph is independent.

n = number of vertices
e = upper bound for minimal number of edges

n	8	9	10	11	12	13	14	15	16	17
e	3	4	5	7	9	11	13	15	26	32

n	18	19	20	21	22	23	24	25	26	27
e	38	49	65	75	87	97	106	115	125	135

Figure 17. Smallest number of edges, e, found for a (K_4-e,K_7-e,n)-good graph.

G. COMPUTATIONAL TECHNIQUES

1. Need for computer.

There are 2^{136} graphs with 17 vertices. If each one could be tested for $(K_4\text{-e},K_6\text{-e})$-goodness in one second, it would take 10^{36} days to test them all.

The total number of $(K_3\text{-e},K_6\text{-e})$-good graphs is 15, since these graphs have maximum degree 1 and at most 4 edges. The total number of $(K_4\text{-e},K_5\text{-e})$-good graphs is 1,623, as seen in Fig. 19. To build a $(K_4\text{-e},K_6\text{-e},17)$-good graph from, say, a $(K_3\text{-e},K_6\text{-e},6)$-good graph and a $(K_4\text{-e},K_5\text{-e},10)$-good graph entails choosing 6 subsets from each of 407 graphs on 10 vertices. This is 2,500,608 choices; at one second per graph, it would take 29 days to find all $(K_4\text{-e},K_6\text{-e},17)$-good graphs having a vertex of degree 6.

To build a $(K_4\text{-e},K_7\text{-e},25)$-good graph from, say, a $(K_3\text{-e},K_7\text{-e},9)$-good graph and a $(K_4\text{-e},K_6\text{-e},15)$-good graph entails choosing 9 subsets from each of 19,521 graphs on 15 vertices. This is 5.7 million choices and would take 183 years. The algorithms outlined below permit these constructions to be done in a matter of days.

2. Description of main algorithm.

The idea of the algorithm is to construct a $(K_4\text{-e},K_6\text{-e},m+n+1)$-good graph F, from a vertex x, a $(K_3\text{-e},K_6\text{-e},m)$-good graph G, and a $(K_4\text{-e},K_5\text{-e},n)$-good graph H, so that the triple (x,G,H) becomes the triple (x,G_x,H_x) which results from preferring x in F.

There are five principal methods used to shorten computation time:

(a) store each graph as a one-dimensional array formed by converting its adjacency matrix into a string of hexadecimal digits,
(b) represent subgraphs by 32-bit integers,
(c) store all support sets in one array,
(d) store the *OKN* relation between support sets,
(e) have the minimum degree of the constructed graph be the degree of the preferred vertex.

The algorithm used in the program "fillJ4J6" is:

0. read a $(K_4\text{-e},K_5\text{-e})$-good graph from a file.
1. compute all support sets with size greater than $\delta(x)-2$ and store in array *SU*[].
2. compute the *OKN* relation and store it in array *OK*[][].
3. for each edge in G choose a disjoint pair of support sets.

4. for each vertex in G not in an edge choose one support set.
5. check all pairs of independent vertices in G to make sure their support sets are *OKN*-related.
6. construct the edges from G to x and to the support sets.
7. test if the resulting graph is $(K_4\text{-e},K_6\text{-e})$-good.
8. store the graph in a file if it is good.

After all graphs from the input file have been processed, the output file is shortened by removing graphs isomorphic to other graphs in the file.

2. Storage of *OKN* relation.

The *OKN* relation is a large sparse matrix, 320 by 320, and is stored as a 320 by 10 matrix, using 32-bit words consisting only of powers of 2. Each index in the array of support sets is written in the form:

$$i = 32(d-1) + r$$

where d and r is positive, and r is < 32. Suppose $j = 32(x-1) + y$. Then support_set[i] is *OKN*-related to support_set[j] if and only if

$$\text{OKN}[i][[x] = 2^{y-1} \text{ and } \text{OKN}[j][[d] = 2^{r-1}$$

3. Graph isomorphism algorithm.

The graph isomorphism algorithm is described in [RK1]. The first test compares the degree sequences of two graphs. The second test compares the "3-independent set" sequences of the two graphs (this sequence consists of the number of 3-independent sets containing each vertex). If these two tests do not show the two graphs to be non-isomorphic, then a full isomorphism algorithm is run.

4. Run times.

The longest run, using a 3B1 UNIX PC, took 117 hours. This run processed all of the 1,623 $(K_4\text{-e},K_5\text{-e})$-good graphs and computed all of the 19,521 $(K_4\text{-e},K_6\text{-e},15)$-good graphs.

H. COMPUTER PROGRAMS

The primary computer programs used in this thesis are presented here in schematic form.

```
program:     countS(H)
arguments:   H = R(K4-e,K5-e)-good graph
purpose:     compute the number of support sets in H
code:        1. call support(U,H)
             2. return number of nonzero entries in U

program:     countE(H)
arguments:   H = R(K4-e,K5-e)-good graph
purpose:     compute the number of support edges in H
code:        1. call support(U,H)
             2. for each edge E in H:
                  2a. if E belongs to some set in U
                        increment NUM
             3. return NUM

program:     OKN4E5(H)
arguments:   H = R(K4-e,K5-e)-good graph
purpose:     compute the maximum size of a family of support
                  sets in H satisfying:
                  (a) 2 sets have size 5
                  (b) all sets have size 4 or 5 and at least
                        1 edge
                  (c) all sets are pairwise OKN
code:        1. call support(U,H)
             2. remove from U support sets of size less than
                  4 or greater than 5
             3. remove from U support sets without an edge
             4. for each pair (S1,S2) from U with S1 OKN S2
                  and size(S1) = size(S2) = 5:
                        4a. form the array C of all support
                        sets from U of size 4 which are OKN
                        with S1 and S2
                        4b. define length(S1,S2) = maxOKN(C)
             5. return the maximum value of length(S1,S2)

program:     support(U,H)
arguments:   U = array to hold all support sets in H
             H = R(K4-e,K5-e)-good graph
purpose:     compute the family of support sets in H
code:        1. build array A of all adjoining edges in H
             2. build array T of all triangles in H
             3. build array I of all independent 4-sets in H
             4. for each set S of vertices of H:
                  if S contains no A[i] and S meets each
                  T[i] in fewer than 2 vertices and S meets
                  each I[i] in at least 1 vertex then adjoin
                  S to the array U
```

program: **hxOKN(C)**
arguments: *C* = array of support sets
purpose: compute *MAXOKN* = the maximum number of support sets in *C* which are pairwise *OKN*
code:
1. define *MAXOKN* = current value of the maximum number of support sets in *C* which are pairwise *OKN*
2. build array *flag* of 0's of same length as *C*
3. call cluster(&*MAXOKN*,*flag*,*C*,1)
4. return *MAXOKN*

program: **cluster(*ptr*,*flag*,*C*,*index*)**
arguments:
ptr = pointer to integer variable *MAX*
flag = array of 0's and 1's showing families of support sets which are pairwise *OKN*
C = array of support sets
index = index in array *C*
purpose: recursively construct all families of support sets which are pairwise *OKN* and record the maximum size of such families
code:
1. if *index* > length(*C*)
   { update *MAX*; return; }
2. if *C*[*index*] is *OKN* with all preceding flagged support sets
   { *C*[*index*]= 1;
     call cluster(*ptr*,*flag*,*C*,*index*+1); }
3. *C*[*index*]= 0
4. call cluster(*ptr*,*flag*,*C*,*index*+1)

program: **fillJ4J6(*min*,*H*)**
arguments:
min = integer
H = (K_4-e,K_5-e)-good graph
purpose: construct all (K_4-e,K_6-e)-good graphs with preferred triple (y,G_y,H_y) using G_y with size *min*, *H* as H_y, and minimum degree *min*
code:
1. call support(*U*,*H*)
2. for each number of edges in G_y and each assignment of support sets from *U* to the vertices of G_y:
   2a. test if the support sets for adjacent vertices are disjoint and the support sets for independent vertices are *OKN*
   2b. test if the resulting graph is (K_4-e,K_6-e)-good with minimum degree equal to *min*

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