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Computational techniques in Turan problems

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Computational Techniques in Turán Problems

By

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Computation Techniques in Turán Problems

ABSTRACT

A Turán set system, $T(n, l, k)$, is a k-uniform hypergraph on n points, such that any subset of l vertices contains at least one edge. The Turán number $T(n, l, k)$ is the minimal number of edges in any Turán set system $T(n, l, k)$. The known nontrivial values of Turán numbers are rare. Using the algorithm *turexp* for extending $T(n, l, k)$ systems to $T(n+1, l, k)$ systems and procedures *nauty* for determining the automorphism group of a graph, the new Turán numbers $T(13, 4, 3)$, $T(11, 5, 3)$, $T(12, 5, 3)$, $T(13, 5, 3)$ are determined, a new lower bound for $T(14, 5, 3)$ is given, the Turán numbers $T(10, 4, 3)$, $T(11, 4, 3)$, $T(12, 4, 3)$ are confirmed to be the same as the previous unpublished results of other authors, and all minimal Turán $T(n, 4, 3)$ $(n \leq 12)$, $T(n, 6, 5)$ $(n \leq 9)$, $T(n, 5, 3)$ $(n \leq 13)$ are obtained.

Parts of this thesis will appear in a joint paper 'On $(n, 5, 3)$-Turán Systems' with E. Boyer, D. Kreher, S. Radziszowski and A. Sidorenko.

CONTENTS

1. Introduction and Background

1.1 Problem Statement

A *Turán set system*, $T(n, l, k)$, is a k-uniform hypergraph on n points (vertices), such that any subset of l vertices contains at least one edge. The *Turán number*, $T(n, l, k)$, is the minimal number of edges in any Turán set system $T(n, l, k)$ (the context will clarify this abuse of notation). The known nontrivial values of Turán number $T(n, l, k)$ are rare. The following tables show some known small Turán numbers [1]. The numbers $T(10, 4, 3)$, $T(11, 4, 3)$ and $T(12, 4, 3)$ [1] were computed independently by three different authors, but the proofs are long and complex, and have not been published.

	l, k										
n	2,1	3,2	4,3	5,4	6,5	7,6	8,7	9,8	10,9	11,10	12,11
3	2										
4	3	2									
5	4	4	3								
6	5	6	6	3							
7	6	9	12	7	4						
8	7	12	20	14	11	4					
9	8	16	30	30	25	12	5				
10	9	20	45	50	?	30	17	5			
11	10	25	63	≤84	?	66	47	19	6		
12	11	30	84	≤126	?	132	113	57	24	6	
13	12	36	?	?	?	?	245	?	78	26	7

Table 1. Values of $T(n, l, l-1)$ for small n

	l, k						
n	3,1	4,2	5,3	6,4	7,5	8,6	9,8
4	2						
5	3	2					
6	4	3	2				
7	5	5	5	3			
8	6	7	8	6	3		
9	7	9	12	12	8	3	
10	8	12	20	20	17	9	4

Table 2. Values of $T(n, l, l-2)$ for small n

	l, k						
n	4,1	5,2	6,3	7,4	8,5	9,6	10,7
5	2						
6	3	2					
7	4	3	2				
8	5	4	4	2			
9	6	6	7	5	3		
10	7	8	10	10	6	3	
11	8	10	16	16~18	11	7	3

Table 3. Values of $T(n, l, l-3)$ for small n

	l, k						
n	5,1	6,2	7,3	8,4	9,5	10,6	11,7
6	2						
7	3	2					
8	4	3	2				
9	5	4	3	2			
10	6	5	6	4	2		
11	7	7	9	8	6	3	
12	8	9	12	12	11	6	3

Table 4. Values of $T(n, l, l-4)$ for small n

1.2 Previous Work

As mentioned before, some small Turán numbers have been found previously. The following two theorems resolve some general cases.

Theorem 1: $T(n, l, 1) = n - l + 1$.

Proof: If A is a $(n - l + 1)$-subset of n-set S, B is a l-subset of S, then $A \cap B \neq \phi$. Hence any $n - l + 1$ 1-subset of A form a $T(n, l, 1)$-system. For any $(n - l)$-subset C of S, $|S - C| = l$ and $C \cap S - C = \phi$. Hence no $(n - l)$ 1-subset can form a $T(n, l, 1)$-system; i.e., $T(n, l, 1) = n - l + 1$. □

The situation $T(n, l, 2)$ is completely solved by a well known Turán theorem [10].

Let

$$L(v, k, t) = \left\lceil \frac{v}{k} \left\lceil \frac{v-1}{k-1} \cdots \left\lceil \frac{v-t+1}{k-t+1} \right\rceil \cdots \right\rceil \right\rceil$$

Then we have the theorem:

Theorem 2: $T(n, l, 2) = L(n, n-2, n-l)$.

We have some theorems for general lower bound. The following theorem [5] is very useful in arranging the computations.

Theorem 3 (Schonheim bound): $T(n, l, k) \geq \left\lceil \frac{n}{n-k} T(n-1, l, k) \right\rceil$.

Proof: Let $T(n, k, l)$ be a Turán set system on set S. For any subset A of S, let $F(A)$ be the number of blocks which do not intersect subset A. Then

$$(n - k)T(n, l, k) = \sum_{\alpha \in S} F(\alpha) \geq nT(n-1, l, k).$$

i.e.

$$T(n, l, k) \geq \left\lceil \frac{n}{n-k} T(n-1, l, k) \right\rceil$$

□

Corollary 1: $T(n, l, k) \geq L(n, l, k)$.

$T(n, l, k) = L(n, l, k)$ for many of the cases where $T(n, l, k)$ is known. When $T(n, l, k) > L(n, l, k)$, this is usually very difficult to prove. However there are certain situations where it is easy to increase the lower bound in Theorem 3.

Theorem 4: If $(n - k) \nmid nT(n-1, l, k)$ and if for some r, $2 \leq r \leq n - l$, we have

$$T(n-1, l, k) = \frac{\binom{n-1}{r-1}}{\binom{n-k-1}{r-1}} T(n-r, l, k),$$

then

$$T(n, l, k) \geq \left\lceil nT(n-1, l, k) + \frac{r}{n-k} \right\rceil.$$

Proof: For any r-element subsets A of our n-element subsets S we have

$$F(A) \geq T(n-r, l, k).$$

We distinguish two cases.

Case 1. $F(A) = T(n-r, l, k)$ for all n-element subsets A of S. Here we have

$$\binom{n-k}{r} T(n, l, k) = \sum F(A) = \binom{n}{r} T(n-r, l, k),$$

where the sum is the over all r-element subsets A of S. Therefore we have

$$(n-k)T(n, l, k) = n\frac{\binom{n-1}{r-1}}{\binom{n-k-1}{r-1}} T(n-r, l, k) = nT(n-1, l, k),$$

which is a contradiction.

Case 2. $F(B) > T(n-r, l, k)$ for some r-element subsets B. Here for any element β in B we have

$$\binom{n-k-1}{r-1} F(\beta) = \sum F(A) > \binom{n-1}{r-1} T(n-1, l, k),$$

where the summation is over all r-element subsets A of S that contain β, and therefore

$$F(\beta) > T(n-1, l, k)$$

by hypothesis. Thus we have

$$F(\beta) \geq T(n-1, l, k) + 1$$

for all β in B. We also have

$$F(\alpha) \geq T(n-1, l, k)$$

for all α in S. Hence we have

$$(n - k)T(n, l, k) = \sum_{\alpha \in S} F(\alpha) \geq nT(n-1, l, k) + r.$$

Since $T(n, l, k)$ must be an integer the theorem follows immediately. □

The next theorem describes the general upper bound.

Theorem 5: $T(n, l, k) \leq T(n-1, l, k) + T(n-1, l-1, k-1)$

Proof: Let $b_1, b_2, ..., b_{T(n-1,l,k)}$ be a $T(n-1, l, k)$ system and $b'_1, b'_2, \ldots, b'_{T(n-1,l-1,k-1)}$ be a $T(n-1, l-1, k-1)$ system. Then b_1, b_2, ..., $b_{T(n-1,l,k)}$, $b'_1 \cup \{n\}, b'_2 \cup \{n\}, ..., b_{T(n-1,l-1,k-1)} \cup \{n\}$ is a $T(n, l, k)$ system with $T(n-1, l, k) + T(n-1, l-1, k-1)$. The theorem follows. □

The numbers in the following table were known previously to our work:

n	5	6	7	8	9	10	11	12	13
$T(n, 4, 3)$	3	6	12	20	30	45	63	84	?

Table 5. $T(n, 4, 3)$

n	6	7	8	9	10
$T(n, 6, 5)$	1	4	11	25	?

Table 6. $T(n, 6, 5)$

n	6	7	8	9	10	11
$T(n, 5, 3)$	2	5	8	12	20	?

Table 7. $T(n, 5, 3)$

Reference [5] gives us some structure of Turán systems which can be used in the computations of Turán number.

The original purpose of *turexp* algorithm [2] was for searching the Ramsey number for hypergraphs, specifically in the study of $R(4, 4; 3)$. The Ramsey number $R(k, l; s)$ is defined to be the least n such that, in any coloring with two colors of the s-subsets of a set of n elements, there is a k-subsets all of whose s-subsets have the first color or there is an l-subset all of whose s-subsets have the second colors. There is a strong connection between the colorings with two colors of the triangles on n points and the special Turán set systems $T(n, 5, 4)$ called $TC(n)$ used in [2]. $TC(n)$ is formed by the systems $S \in T(n, 5, 4)$ such that every set of 5 vertices contains an odd number of blocks of S. We define $T(n, f) =$ $\{S \in T(n)$: *the number of blocks in* S *is* $f\}$. Given f and $G \in TC(n)$, *turexp* finds its extensions in $TC(n + 1, \leq f)$. Our algorithm will be based on *turexp* (and still named *turexp*.)

Another important component of the software used is a general set-system automorphism group program *nauty* [3] written by B.D. McKay for determining the automorphism group of a graph, and optionally for canonically labeling it. Two graphs are isomorphic iff they have identical canonical labelings, thus *nauty* can be used as a powerful tool to detect isomorphs in large families of graphs, and indirectly via graphs in families of any reasonable finite objects, in particular Turán systems.

2. The Algorithm of Turexp

In order to search for Turán systems of our interest, we need to change the algorithm *turexp* from the case $T(n, 5, 4)$ to cases $T(n, 4, 3)$, $T(n, 6, 5)$, and $T(n, 5, 3)$. For the case of $T(n, 4, 3)$, we try to find all minimal $T(n, 4, 3)$ systems for $n \leq 12$. After obtaining the number $T(13, 4, 3)$, we find out that the number is same as the one in the Turán conjecture. For the case of $T(n, 6, 5)$, we will try to find all minimal $T(n, 6, 5)$ systems for $n \leq 9$. For the case of $T(n, 5, 3)$, the data structure of free variable (see 2.1) will be changed. We will use 6 bits to represent the 6 free variables in a free condition (see 2.1). After the program is obtained, we will try to find all minimal $T(n, 5, 3)$ systems for $n \leq 12$ and the numbers $T(11, 5, 3)$, $T(12, 5, 3)$. If $T(12, 5, 3) = 40$, as suggested by [6], then $T(13, 5, 3) = 52$.

2.1 The Description of Turexp

The following definition and theorem is used for cutting the search space.

Definition 1: If G is a $T(n, l, k)$ system and q_i $(i > 0)$ denotes the number of l- subsets of n points which contains i blocks of G, then the quantity

$$Q(G) = \sum_{i=2}^{l}(i-1)q_i$$

is called the Q-number of G.

Theorem 6: Let G be a $T(n, l, k)$ system with b blocks. Then

$$Q(G) = \binom{n-k}{l-k}b - \binom{n}{l} \tag{1}$$

Proof: It's easy to see that

$$\sum_{i=1}^{l} q_i = \binom{n}{l}$$
$$\sum_{i=1}^{l} iq_i = \binom{n-k}{l-k}b$$

Hence the equality (1) follows. □

Turexp is a recursive backtrack algorithm. The idea of the algorithm is that for each system G in $T(size, l, k)$ with b blocks and integer *total* the algorithm finds all systems in $T(size+1, l, k)$ which are extensions of G with $tot \le total$ blocks. The algorithm considers all possible sets of blocks containing the new vertex such that the union of the new blocks and G is a system in $T(size+1, l, k)$. The array variable $var[i]$ has k bits to describe the corresponding k-set passing through point $size + 1$. The global array $setvar[]$ provides an instant recovery of the index of variable given by a k-set, i.e.

$$setvar[var[i]] = i$$

Each l-set passing through point $size+1$ will be called a condition; $cond[i]$ stores similarly the l-set of condition i. At each level of recursion each variable and each condition is either closed or free. A closed variable has already a 0 or 1 assigned, a condition is closed if all of its $\binom{l}{k}$ k-subsets treated as variables are closed. Variables and conditions which are not closed are free. Initially we have b variables closed to

1's (and they stay closed during the computation - blocks of G), n free variables and m free conditions.

A. Recursive Operation of *Turexp*

The main recursive function is

$$rec(vf, cf, vfree, nis, degs, tot, qsum),$$

where:

1) vf, cf: lists of free variables and free conditions. All free variables are on the list $vf[0]$, such that the loop

for ($i = vf[0]$; i; $i = vf[i]$) {process free variable with index i}

processes all free variables. Similarly, free conditions are kept on the list $cf[0]$. For each free condition with index i, we keep the following information: (We sort all $(l-1)$-subsets of 1, 2,..., b in lexicographical order.)

2) $vfree[i]$: has bit j on iff $cond[i]$ with the jth $(l-1)$-subset dropped is a free variable, $vfree[i]$ has at most $\binom{l-1}{k-1}$ bits on.

3) $nis[i]$: integer in $[0, \binom{l}{k}]$ saying how many variables contained in $cond[i]$ have been closed with 1's.

4) $degs[i]$: for $1 \le i \le size$ is the current degree of point i.

5) tot: the number of variables already closed with 1's

6) $qsum$: Uses Theorem 2. $qsum$ is initialized as the Q-number of G. If $cond[i]$ contains q blocks, then it contributes $q-1$ to the $qsum$. Every time before the execution enters the recursion, the program compares $qsum$ with the Q-number

of the extended system H. If *qsum* reaches or exceeds the Q-number of H, then we exit this level of recursion.

B. Initialization:

Lists *vf* , *cf* are loaded. In *vfree* [*i*], there are $\binom{l-1}{k-1}$ bits. *nis* [*i*] = 1 if *cond[i]* − {*size* + *1*} is a block of G, otherwise *nis* [*i*]=0. *degs* [] is computed from G, *tot* = *b* and *qsum* = 0. The global answer vector *val* [] is loaded with blocks of G.

The call

$$rec(vf, cf, vfree, nis, degs, b, 0)$$

is done after proper initialization. Globals *total* = *blocks* (see 2.2) and *maxdeg* stay fixed, and they bound the total number of blocks and maximal degree in H, respectively. The global variable '*totqsum*' gets the value $Q(size{+}1, blocks) - Q(G)$ at the start of the program, is fixed afterwards, and must bound the number *qsum*.

C. Recursive Call

The recursive call

$$rec(vf, cf, vfree, nis, degs, tot, qsum)$$

does the following:

1) *lx* is the local name of parameter x, *lx*'s are used when entering the next level of recursion.

2)

$$cind = cf[0];$$
$$q = cond[cind];$$

cind is the index of the first free condition and q is the corresponding set.

3) Item 4) finds all feasible closures of condition *cind*. There are *cl* of them.

4) The loop

$$\text{for } (i=1;\ i \le cl;\ i\text{++})$$

processes each of *cl* 0-1 mappings of free variables in *cond*[*cind*] by:

4.1) Update *degs*. If for some *j* *degs*[*j*] > *maxdeg* then enter exit status (this refers to not entering further levels for recursion).

4.2) Update *tot*, load variables closed to ones in mapping *i* into *val*[].

4.3) Make local copies of data.

4.4) The loop with *dlist*(*lvf*, *vind*[*p*]) deletes free variables of mapping *i*, *dlist*(*lcf*, *cind*) deletes free condition *cind*.

4.5) $lqsum = qsum + ltot - tot - 1$ realizes the increment, under mapping *i*, of contribution of closed *cond*[*cind*] to *qsum*; see above for the description of *nis*.

4.6) For each closed variable in mapping *i*

$$upcond()$$

updates information of all free conditions, i.e. *nis*[] and *vfree*[]. It also detects possible Q-criterion exit status.

4.7) If a solution is still possible then

$$force()$$

enforces in the loop whatever is possible. *Force*() detects exit status and solutions.

4.8) If a solution is not possible or it has been detected by *force*() then process the next mapping, otherwise enter the next level of recursion.

D. The Function force()

Force () works in loop searching for and performing appropriate actions for the following situations, which enforce some update:

1) The number of blocks *tot* hits or exceeds the *total* - yields solution or exit status.

2) *qsum* reaches or exceeds *totqsum* - forces closure of all free conditions i.

3) Some condition has at most one free variable - close this condition.

4) $degs[i] \geq maxdeg$ - sets to zero all free variables containing point i, closes them and update conditions. $degs[i] > maxdeg$ causes exit status.

2.2 The Usage of Turexp

A call to *turexp* has the form

turexp desfile desno blocks maxdeg [*pr*]

where the parameters have meanings as defined below.

desfile: packed file of the Turán systems on n vertices.

desno: extend G, the system #desno in file *desfile*.

blocks: search for extensions H in Turán systems on $n + 1$ vertices for the number of the blocks $\leq$ *blocks*.

maxdeg: bounds maximal degree of points in H. Theorem 7 is used to determine *maxdeg*.

Theorem 7: Let G be a $T(n, l, k)$ system with b blocks. Then for any $T(n + 1, l, k)$ system H with b' blocks such that $G \subset H$, the maximum degree of H, m, satisfies

$$m \geq \left\lceil \frac{(k-1)b' + b}{n} \right\rceil \tag{2}$$

Proof: It's easy to see that

$$deg(n+1) = b' - b$$

and

$$nm \geq \sum_{x=1}^{n} deg(x) = kb + (k-1)deg(n+1)$$

Hence the equality (2) follows. □

2.3 Results

Turexp creates files SOLi.p (if nonempty) containing all extensions of G in Turán systems on $n+1$ vertices such that the number of blocks is at most *blocks*.

With *pr* missing, besides the header and tail, *turexp* prints three lines every 2000 calls to the recursive procedure; the first line gives statistics, the second line gives the sequence of single digits (without spaces) which are the degrees of vertices in the search tree on the path from root to the current node, the third line describes how much computation has advanced when compared with the second line. These messages are very useful when analyzing long computations.

3. The Results

3.1 Some Theoretical Results

3.1.1 About T(n, 4, 3)

Turán formulated an interesting conjecture regarding $T(n, 4, 3)$ numbers [4]. Let

us divide our set S into three disjoint subsets S_0, S_1, S_2 of sizes as nearly equal as possible. Take the set of all triples that are contained in one of the S_i. Then we add the set of triples which have two elements from one S_i and the third from S_{i+1}, where $i + 1$ is calculated modulo 3. Turán's conjecture says that this is a minimum Turán system $T(n,4,3)$, although not necessarily the only one. For example, suppose that S is the set of the first seven positive integers. We take $S_0 = \{1, 2, 3\}$, $S_1 = \{4, 5\}$, $S_2 = \{6, 7\}$. Then our set of triples is

123 124 125 134 135 234 235 456 457 167 267 367

We notice that our result (see 3.2.1) has confirmed Turán's conjecture for $n = 13$.

Theorem 8 [8]: If there exists a $T(3k, 4, 3)$-system with $k(k - 1)(2k - 1)$ blocks, then there are at least 2^{k-2} of such systems.

By our computational result in 3.2.1, we notice that for $k \leq 4$ there are exactly 2^{k-2} $T(3k, 4, 3)$-systems with $k(k - 1)(2k - 2)$ blocks.

3.1.2 About T(n, 5, 3)

What follows is mostly an extraction from the joint paper [11] which has been submitted for publication.

For $T(n, 5, 3)$, Turán [8] proposed the following construction that yields $T(2r, 5, 3) \leq 2 \binom{r}{3}$ and $T(2r + 1, 5, 3) \leq \binom{r+1}{3} + \binom{r}{3}$. Split the point-set into two groups X and Y of nearly equal size, and take H to be all triplets contained in either X and Y. Turán suggested that this might be an optimal construction for $T(n, 5, 3)$. For n odd, there are three known exceptions: $T(11, 5, 3) = 29$ is given by the unique $T(11, 5, 3)$ Turán system with 29 blocks (see Table31), $T(9, 5, 3) \leq 12$ is given by

taking the affine plane of order three, and $T(13, 5, 3) \leq 52$ is obtained by taking the collinear triplets in the projective plane of order three as follows.

Definition 2: An *incidence structure* is a triple $D = (V, B, I)$ where V and B are any two disjoint sets and I is a binary relation between V and B, i.e. $I \subseteq V \times B$. The elements of V will be called *points*, those of B *blocks* and those of I *flags*. We use such geometric language as "the point p lies on the block B", "B passes through p", "p and B are incident".

Example 1: Take as point set $V=\{0, ..., 6\}$, as block set $B = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}$ and as incidence relation, the membership relation $\in$.

Definition 3: An incidence structure $D = (V, B, I)$ is called a *projective plane* if and only if it satisfies the following properties:

(a) Any two distinct points are joined by exactly one block.

(b) Any two distinct blocks intersect in a unique point.

(c) There exists a *quadrangle*, i.e. 4 points no three of which are on a common block.

The incidence structure in example 1 is a projective plane.

Theorem 9[14]: Let $D = (V, B, I)$ be a finite projective plane. Then there exists a natural number n, called the *order* of D, satisfying:

(a) $|(p)| = |(G)| = n + 1$ for all $p \in V$ and $G \in B$;

(b) $|V| = |B| = n^2 + n + 1$.

Proof: Consider any point p and any line G with $p \notin G$. By definition 3(a),(b), the mapping π: $(G) \rightarrow (p)$ with $q^{\pi} := pq$ for all $q \in G$ is a bijection (here pq denotes the unique block joining p and q). Hence $|(p)| = |(G)|$ whenever $p \notin G$. Thus (a)

follows if we can show that for any two distinct lines G, H there is a point p with $p \notin G$, $p \notin H$. But by definition 3(c) there exists a quadrangle q, r, s, t. If each of these points is on G or H, we may assume that $q, r \in G$ and $s, t \in H$. Then we can choose p to be the intersection of the blocks qs and rt. To verify (b) we again use counting in two ways. We choose a fixed point p and count all flags (q, G) with $p \in G$ and $p \neq q$. By definition 3(a), we obtain $|V| - 1$ such flags; and by (a) we have $(n+1)n$ such flags. Thus $|V| = n^2 + n + 1$. The assertion on $|B|$ follows similarly. □

Theorem 10: For each prime power q, there exists a projective plane of order q.

Proof: Let F be the Galois field on q elements and W the vector space of dimension 3 over F. Choose as points all 1-dimensional subspaces and as lines, all 2-dimensional subspaces of W. Using the dimension formula of linear algebra, one checks that the properties (a) and (b) in definition 3 are satisfied. For properties (c), one may choose the points e_1F, e_2F, e_3F and $(e_1 + e_2 + e_3)F$ where e_1, e_2, e_3, is any basis of W. This argument works in fact for any 3-dimensional vector space. The fact that F is the field on q elements is only needed to show that the resulting projective plane has order q: the number of 1-dimensional subspaces of W is then $(q^3 - 1)/(q - 1) = q^2 + q + 1$. □

Definition 4: An incidence structure $D = (V, B, I)$ is called an *affine plane* if and only if it satisfies the following properties:

(a) Any two distinct points are jointed by exactly one line.

(b) Given any point p and any block G with $p \notin G$, there is precisely one block H with $p \in H$ and not intersecting G.

(c) There is a *triangle*, i.e. 3 points not on a common block.

We say that two blocks G and H are *parallel* if $G = H$ or $|(G, H)| = 0$. Thus (b) is Euclid's parallel property.

Theorem 11: Let $D = (V, B, I)$ be an affine plane. Then parallelism is an equivalence relation on B. If D is finite, there exists a natural number n (called the *order* of D) satisfying:

(a) $|(p)| = n + 1$ for all points p;

(b) $|(G)| = n$ for all blocks G;

(c) $|V| = n^2$, $|B| = n^2 + n$.

Proof: We only need to check the transitivity of parallelism. Without loss of generality we may suppose that G, H and K are mutually distinct. If G and K are not parallel there would be a point $p \in G, K$; but then G and K would be two parallels through p to H, contradicting definition 4(b). The remaining assertion follows as in the proof of Theorem 9. □

Definition 5: Let $D = (V, B, I)$ be an incidence structure and $Q \subseteq V$, and $C \subseteq B$. Then the incidence structure *induced* by D on Q and C is $D' = (Q, C, I \mid Q \times C)$, and D' is called an *induced substructure* of D. Instead of $I \mid Q \times C$ we will usually again write I.

Theorem 12[14]: Let $D = (V, B, I)$ be a projective plane and G a line of D. Then the substructure $D_G = (V \setminus (G), B \setminus \{G\}, I)$ is an affine plane. Conversely, every affine plane may be obtained in this way from a projective plane. In the finite case, the orders of D and D_G are the same.

Proof: Let D be a projective plane, and remove a block U together with all its points. Then D_U satisfies Definition 4(a), because D satisfies Definition 3(a). Let p be a point of D_U and G a block of D_U with $p \notin G$. If q is the point of intersection of G and U in D, then the block containing p, q is the desired parallel to G through p. Certainly, it is unique. Now, Definition 4(c) follows from Definition 3(c) and the existence of a point which is on neither of two given blocks.

Conversely, given an affine plane, we can obtain a projective plane. Extend the point set by choosing an additional point corresponding to each parallel class of blocks. Then choose one additional block ("the block at infinity"). The new incidence extends the old incidence as follows: The line at infinity contains each new point and a new point is contained in each of the lines i, the corresponding parallel class. (This can be thought of as decreeing that each parallel class meets at some "point at infinite".) Clearly the given affine plane arises from the new structure by deleting the line at infinity. It is easy to prove that the enlarged structure is indeed a projective plane. □

As an immediate consequence of Theorem 10 and 12 we have:

Corollary: For any prime power q, there exists an affine plane of order q.

Theorem 13: The affine plane of order 3 is a $T(9, 5, 3)$-system with 12 blocks.

Proof: By the Corollary of Theorem 12, there is an affine plane of order 3. By Theorem 11, it has 9 points and 12 blocks. Define b_i to be the number of blocks intersection with S in i points. If for a 5-point set S, there are no 3-subsets of S in the same blocks of the affine plane, then we have $b_0 + b_1 + b_2 = 12$, $b_1 + 2b_2 = 4 \cdot 5$ and $b_2 = \binom{5}{2}$. Hence $b_0 = 2$ and $b_1 = 0$, i.e. besides S, there 4 points in 2 blocks of

the affine plane. Assume that the 4 points are x_1, x_2, x_3, x_4, and $\{x_1, x_2, x_3\}$ is a block of the by Definition 4. The second block will intersect at least two points with the first block, which contradicts Definition 4. □

Theorem 14: The set of collinear triplets in the projective plane of order 3 is a $T(13, 5, 3)$-system with 52 blocks.

Proof: By Theorem 10, there is a projective plane of order 3. By Theorem 9, it has 13 points and 13 blocks. We call the set of collinear triplets in the projective plane of order 3 as T. Then $|T| = 13$ and there are $13 \cdot \binom{4}{3} = 52$ blocks in this system. Define b_i = the number of blocks intersecting with S in i points. If for a 5-points set S, there are no 3-subsets of S in same blocks of the projective plane, then we have $b_0 + b_1 + b_2 = 13$, $b_1 + 2b_2 = 4 \cdot 5$ and $b_2 = \binom{5}{2}$. Hence $b_0 = 3$ and $b_1 = 0$, i.e. besides S, there 8 points in 3 blocks of the projective plane. Assume that the 8 points are x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, and $\{x_1, x_2, x_3, x_4\}$, $\{x_1, x_5, x_6, x_7\}$ are blocks of the projective plane by Definition 3. The third block will intersect with one of the other two blocks at least two points, which contradicts Definition 3. □

Lemma 1: Any $T(13, 5, 3)$-system with 52 blocks is regular of degree 12.

Proof: Because the unique $T(13, 5, 3)$-system with 52 blocks is extended from $T(12, 5, 3)$-systems with 40 blocks, the degree of any element is at most $52 - 40 = 12$. Since

$$\sum_x deg(x)/13 = 3 \cdot 52/13 = 12.$$

Hence the lemma is true. □

The unique (13, 5, 3)-Turán system on 52 blocks has 5616 automorphisms, and it can be obtained by taking all triplets of collinear points in the well known projective plane of order 3. It seems that further progress in the study of $T(n, 5, 3)$ Turán systems could be achieved by a careful analysis of the 16 $T(12, 5, 3)$ systems on 40 blocks. 15 of them are regular of degree 10, and one has 6 points of degree 9 and 6 points of degree 11. These systems have automorphism groups of surprisingly large size, namely: 48, 64 (2 systems), 96, 128, 144, 256 (3 systems), 288, 384, 432, 768, 4608, 5184, and 1036800.

Now, without computation, we can obtain the number $T(12, 5, 3)$ from the uniqueness of $T(11, 5, 3)$-system with 29 blocks and obtain the number $T(13, 5, 3)$ form the number $T(12, 5, 3)$ and the result in [4].

Checking the unique $T(11, 5, 3)$-system with 29 blocks (Table 31), we have:

Lemma 2: $T(11, 5, 3)$-system with 29 blocks has the following properties:

(1) There is a subset of 4 elements of degree 9 which does not contain any block of the system.

(2) For any element x of degree 9 and any other element y there is a block containing x and y.

Theorem 15: $T(12, 5, 3) = 40$.

Proof: Suppose G is $T(12, 5, 3)$-system with 39 blocks. Since $39 \cdot 3/12 > 9$, there is an element x of degree 10. According to Lemma 2 (1), $G \backslash x$ contains elements y_1, y_2, y_3, y_4 of degree 9 and they form no block. Since G is a Turán (5, 3)-system, one must contain at least one block of type $\{x, y_i, y_j\}$ $1 \le i, j \le 4$. Suppose, for definiteness, block $\{x, y_1, y_2\}$ presents in G. We know that y_1 is contained in 9

blocks without x and in block $\{x, y_1, y_2\}$. Since the degree of y_1 is at most 10, it means that $\{x, y_1, y_2\}$ is the only block containing pair $\{x, y_1\}$. The same is true for $\{x, y_2\}$. System $G \backslash y_1$ contains 29 blocks, and x has degree 9 there. Note that $G \backslash y_1$ does not have a block containing $\{x, y_2\}$. This contradicts to Lemma 1 (2). □

Theorem 16: $T(13, 5, 3) = 52$.

Proof: By Theorem 1, $T(13, 5, 3) \geq 13T(12, 5, 3)/10 = 13 \cdot 40/10 = 52$. By Theorem 17 $T(13, 5, 3) \leq 52$, therefore we have $T(13, 5, 3) = 52$. □

3.2 Computational Results

3.2.1 Calculations of T(n, 4, 3)

Reference [1] tells us that for $n = 10, 11, 12$, $T(n, 4, 3)$ are solved independently by three different authors, but the proofs are long and complex and have not been published. We confirmed the values of these three numbers using *turexp*. Because the goal is searching for the $T(13, 4, 3)$ systems, by theorem 3, the parameters of maximum degree of extending systems are listed in the following table:

n	4	5	6	7	8	9	10	11	12
maxdeg	1	3	6	12	20,21	30,31	45	63	84,85

Table 8.

By the algorithms, we have generated all of the $T(n, 4, 3)$ systems with the following parameters.

n	blocks	number of systems
8	20	6
9	30	2
9	31	23
10	43	0
10	44	0
10	45	20
11	62	0
11	63	40
12	84	4
12	85	80
13	110	0
13	111	0

Table 9.

By the Schonheim bound and the table, we have

Theorem 17: $T(10, 4, 3) = 45$.

Theorem 18: $T(11, 4, 3) = 63$.

Theorem 19: $T(12, 4, 3) = 84$.

Theorem 20: $T(13, 4, 3) = 112$.

All minimal $T(n,4,3)$ ($8 \leq n \leq 12$) systems are given in the following tables (in the tables we use x for 10, y for 11, z for 12, t for 13).

No.	Turán system
1	456 356 256 156 146 145 245 246 345 346 123 128 127 137 138 237 238 578 678 478
2	245 145 345 135 134 234 235 457 126 127 136 236 128 467 567 468 568 178 278 378
3	234 236 235 345 245 246 346 128 127 137 138 145 146 278 378 478 178 156 567 568
4	234 134 128 127 126 145 245 135 235 136 236 345 178 278 478 467 468 378 567 568
5	234 134 347 346 145 135 235 245 126 127 128 367 467 358 458 168 178 268 278 567
6	123 234 237 236 136 126 127 137 245 345 145 148 258 358 158 467 468 478 567 678

Table 10. Minimal $T(8, 4, 3)$ Systems

No.	Turán system
1	789 589 489 689 679 678 457 347 247 147 157 257 357 468 469 568 569 128 138 238 129 139 239 145 245 345 123 126 136 236
2	789 678 679 689 589 279 347 349 148 138 139 149 357 457 167 157 238 248 128 267 269 258 259 346 345 356 456 156 123 124

Table 11. Minimal $T(9, 4, 3)$ Systems

No.	Turán system
1	156 157 167 12x 13x 14x 148 149 139 138 124 123 59x 58x 89x 589 79x 78x 69x 68x 789 689 67x 567 25x 259 258 289 279 278 269 268 457 357 456 356 467 367 24x 23x 345 349 348 347 346
2	156 157 138 139 13x 145 147 146 128 12x 129 123 58x 589 59x 89x 78x 789 79x 68x 689 69x 678 67x 679 358 35x 359 367 48x 489 49x 457 456 257 256 267 347 346 238 23x 239 245 247 246
3	156 157 13x 138 139 14x 145 147 146 129 128 123 59x 58x 89x 589 79x 78x 69x 68x 789 689 67x 679 678 357 356 367 489 457 456 25x 259 258 27x 26x 267 34x 345 347 346 239 238 249 248
4	156 178 125 126 139 14x 149 13x 147 148 138 137 69x 59x 68x 67x 689 679 58x 57x 589 579 89x 79x 678 578 256 29x 28x 27x 289 279 456 356 478 378 246 236 245 235 346 345 34x 349 234
5	678 67x 68x 78x 789 689 679 389 379 369 478 468 467 49x 578 568 567 58x 57x 56x 19x 29x 34x 35x 238 138 237 137 236 136 459 24x 14x 249 149 259 159 128 127 126 12x 345 123 245 145
6	179 17x 16x 169 158 168 167 145 124 134 135 125 89x 49x 59x 39x 29x 48x 489 47x 479 57x 579 67x 679 38x 28x 389 289 478 378 278 56x 569 23x 239 568 567 238 237 345 245 346 246 236
7	179 18x 189 17x 178 125 126 136 146 145 135 156 78x 789 29x 49x 39x 69x 59x 68x 58x 689 589 67x 57x 679 579 678 578 24x 23x 34x 249 239 349 248 238 348 247 237 347 234 256 456 356
8	689 789 679 78x 68x 67x 678 39x 367 489 479 469 48x 58x 57x 56x 578 568 567 19x 29x 347 346 359 239 139 23x 13x 238 138 45x 458 247 147 246 146 259 159 12x 128 127 126 235 135 124
9	689 68x 67x 679 89x 79x 36x 369 378 49x 467 47x 479 56x 569 568 58x 589 19x 29x 278 178 348 356 35x 359 238 138 237 137 458 246 146 24x 14x 249 149 257 157 126 128 127 345 123 125
10	789 78x 79x 89x 57x 67x 679 579 678 578 49x 48x 489 69x 59x 68x 58x 689 589 367 357 34x 349 348 456 356 147 247 23x 13x 239 139 238 138 246 146 245 145 256 156 127 12x 129 128 123
11	789 78x 79x 89x 49x 59x 69x 48x 68x 58x 47x 67x 57x 489 689 589 479 679 579 478 678 578 456 146 246 346 145 345 245 156 356 256 13x 12x 23x 139 129 239 138 128 238 137 127 237 123
12	57x 67x 68x 69x 59x 58x 79x 78x 89x 679 678 689 579 578 589 789 46x 45x 456 356 379 378 389 34x 346 345 156 256 23x 13x 247 147 249 149 248 148 236 136 235 135 12x 127 129 128 124
13	67x 678 679 89x 689 47x 479 478 46x 567 579 578 569 568 17x 16x 29x 39x 38x 28x 189 389 289 457 459 458 14x 146 346 246 35x 25x 159 158 137 127 237 23x 136 126 239 238 134 124 235

Table 12. Minimal $T(10, 4, 3)$ Systems (I)

No.	Turán system
14	78x 789 69x 67x 68x 19x 29x 39x 48x 58x 57x 47x 689 679 678 589 489 579 479 578 478 13x 12x 23x 45x 139 129 239 138 128 238 137 127 237 123 156 146 356 346 256 246 456 145 345 245
15	78x 39x 68x 69x 67x 19x 29x 48x 58x 57x 47x 389 689 678 679 289 189 578 478 579 479 23x 13x 12x 45x 238 138 239 139 237 137 127 458 459 457 356 346 345 126 256 246 156 146 125 124
16	78x 39x 69x 67x 68x 19x 29x 48x 58x 57x 47x 378 689 679 678 278 178 589 489 579 479 23x 13x 12x 45x 239 139 238 138 237 137 129 459 458 457 356 346 345 126 256 246 156 146 125 124
17	78x 79x 89x 789 57x 67x 69x 59x 68x 58x 489 679 579 678 578 689 589 34x 13x 23x 24x 14x 12x 349 348 239 139 238 138 367 357 247 147 127 129 128 123 356 246 146 245 145 456 256 156
18	78x 79x 89x 789 59x 69x 68x 58x 67x 57x 489 479 478 689 589 679 579 678 578 14x 24x 34x 13x 12x 23x 139 129 239 138 128 238 137 127 237 146 346 246 145 345 245 456 123 156 356 256
19	49x 479 48x 489 47x 478 59x 69x 89x 79x 56x 569 68x 58x 689 589 67x 57x 679 579 568 567 346 345 146 246 245 145 356 378 278 178 124 23x 13x 239 139 12x 129 238 138 237 137 126 125
20	89x 789 78x 58x 68x 689 589 49x 79x 29x 39x 148 178 568 348 248 138 128 17x 179 16x 15x 169 159 56x 569 34x 24x 349 249 23x 239 467 457 456 156 134 124 367 267 357 257 237 236 235

Table 13. Minimal $T(10, 4, 3)$ Systems (II)

No.	Turán system
1	345 346 347 389 38x 38y 39y 39x 3xy 457 456 467 49y 49x 4xy 578 568 678 589 58y 58x 789 78y 78x 689 68y 68x 567 579 57y 57x 569 56y 56x 679 67y 67x 9xy 135 235 237 137 236 136 248 148 249 149 24y 14y 24x 14x 29y 19y 29x 19x 2xy 1xy 123 128 125 127 126
2	456 457 467 567 678 679 67x 67y 578 579 57y 57x 478 479 47y 47x 568 569 56y 56x 468 469 46y 46x 458 459 45y 45x 89y 89x 8xy 9xy 189 289 389 18y 38y 28y 18x 38x 28x 19y 39y 29y 19x 39x 29x 1xy 3xy 2xy 137 127 237 136 126 236 135 125 235 134 124 234 123
3	167 168 179 178 169 189 145 12x 13y 13x 12y 134 135 125 124 7xy 6xy 679 678 9xy 8xy 89y 89x 57y 47y 57x 47x 56y 46y 56x 46x 5xy 4xy 59y 49y 59x 49x 58y 48y 58x 48x 589 489 457 456 367 267 379 279 378 278 369 269 368 268 345 245 237 236 23y 23x 239 238
4	167 14x 15y 15x 14y 145 126 137 136 127 138 139 129 128 123 7xy 6xy 79y 78y 69y 68y 79x 78x 69x 68x 9xy 8xy 789 689 89y 89x 567 467 579 578 569 568 479 478 469 468 589 489 45y 45x 367 267 3xy 2xy 35y 25y 35x 25x 34y 24y 34x 24x 345 245 237 236 239 238
5	178 15y 157 158 149 14x 13y 136 138 137 126 12x 129 135 124 6xy 69y 9xy 69x 68y 67y 8xy 89y 7xy 79y 68x 689 67x 679 89x 79x 56x 569 59x 578 46y 48y 47y 468 467 478 456 45x 459 39x 378 28y 27y 278 35y 358 357 34x 349 25y 258 257 24x 249 236 23x 239 234
6	139 13x 13y 179 129 12y 12x 123 147 148 178 157 167 168 158 156 9xy 49y 49x 4xy 7xy 89y 89x 8xy 69y 59y 69x 59x 6xy 5xy 379 37y 37x 389 38y 38x 239 23y 23x 469 459 46y 45y 46x 45x 279 27y 27x 346 345 237 356 478 248 246 245 678 578 268 258 567 568 256

Table 14. Minimal $T(11, 4, 3)$ Systems (I)

No.	Turán system
7	179 17x 12y 13y 13x 12x 139 129 178 137 127 138 128 145 146 156 9xy 8xy 89y 89x 6xy 5xy 4xy 69y 59y 49y 69x 59x 49x 37x 27x 379 279 23y 23x 239 67y 57y 47y 68y 58y 48y 68x 58x 48x 689 589 489 378 278 236 235 234 567 467 457 356 346 345 256 246 245 456
8	789 78x 79x 89x 4xy 5xy 6xy 49y 69y 59y 48y 68y 58y 46y 45y 56y 467 457 567 46x 45x 56x 469 459 569 468 458 568 456 17y 27y 37y 17x 37x 27x 179 379 279 178 378 278 19x 39x 29x 18x 38x 28x 189 389 289 13y 12y 23y 134 124 234 136 126 236 135 125 235 123
9	789 7xy 8xy 9xy 479 478 489 57x 67y 67x 57y 69y 69x 59y 59x 68y 68x 58y 58x 46y 46x 45y 45x 567 569 568 56y 56x 456 179 279 379 178 378 278 189 389 289 1xy 3xy 2xy 147 347 247 149 349 249 148 348 248 13y 12y 23y 13x 12x 23x 136 126 236 135 125 235 123
10	13x 13y 129 12y 12x 179 189 123 148 147 178 158 168 167 157 156 9xy 49y 49x 4xy 8xy 7xy 69y 59y 69x 59x 6xy 5xy 349 23y 23x 38y 37y 38x 37x 369 359 469 459 46y 45y 46x 45x 289 279 28y 27y 28x 27x 789 346 345 238 237 356 478 246 245 678 578 256 568 567
11	13x 13y 12y 12x 179 189 149 159 123 168 167 178 158 148 157 147 9xy 39y 39x 69y 69x 6xy 8xy 7xy 5xy 4xy 369 23y 23x 38y 37y 38x 37x 269 28y 27y 28x 27x 789 56y 46y 56x 46x 259 249 589 489 579 479 45y 45x 238 237 356 346 345 678 256 246 578 478 456 245
12	14x 179 12y 13y 13x 12x 139 129 145 146 178 137 127 138 128 156 9xy 49y 7xy 8xy 89y 89x 6xy 5xy 69y 59y 69x 59x 47y 479 48y 489 34x 24x 379 279 23y 23x 239 67y 57y 67x 57x 68y 58y 68x 58x 689 589 478 234 346 345 246 245 456 378 278 236 235 567 356 256
13	19x 14y 179 18x 189 17x 15y 16y 146 145 123 178 126 136 135 125 4xy 49y 3xy 2xy 39y 29y 89x 79x 69x 59x 34x 24x 349 249 46y 45y 23y 23x 239 38y 28y 37y 27y 78y 68x 58x 689 589 67x 57x 679 579 56y 56x 569 348 248 347 247 478 456 378 278 236 235 568 567
14	19x 16y 179 18x 189 17x 14y 15y 136 135 134 178 156 146 125 124 3xy 39y 6xy 69y 2xy 29y 89x 79x 59x 49x 36x 369 23y 23x 239 38y 37y 26x 269 28y 27y 78y 56y 46y 58x 48x 589 489 57x 47x 579 479 45y 45x 459 238 237 378 356 346 268 267 678 278 245 458 457
15	19x 16y 17y 18y 189 18x 17x 179 168 167 178 124 135 134 125 145 9xy 3xy 2xy 39y 29y 89x 79x 5xy 4xy 59y 49y 59x 49x 36x 26x 369 269 68y 67y 23y 23x 239 78y 78x 789 56y 46y 56x 46x 569 469 236 678 238 237 358 348 258 248 357 347 257 247 345 245 458 457
16	19x 17y 179 17x 18y 18x 189 148 145 146 178 126 136 135 125 156 4xy 49y 79x 89x 3xy 2xy 39y 29y 6xy 5xy 69y 59y 69x 59x 34y 24y 34x 24x 349 249 78y 78x 789 23y 23x 239 67y 57y 67x 57x 679 579 347 247 234 468 458 456 237 238 368 268 358 258 568 356 256

Table 15. Minimal $T(11, 4, 3)$ Systems (II)

No.	Turán system
17	19x 19y 179 189 18y 17y 18x 17x 148 147 178 125 136 135 126 156 4xy 3xy 2xy 89y 79y 89x 79x 69y 59y 69x 59x 6xy 5xy 349 249 34y 24y 34x 24x 469 459 46y 45y 46x 45x 239 23y 23x 789 78y 78x 234 478 238 237 368 358 268 258 367 357 267 257 356 256 568 567
18	19x 19y 17y 17x 189 18y 18x 148 145 146 178 126 136 135 125 156 9xy 4xy 89y 89x 3xy 2xy 69y 59y 69x 59x 6xy 5xy 479 349 249 34y 24y 34x 24x 78y 78x 379 279 239 23y 23x 679 579 67y 57y 67x 57x 347 247 234 468 458 456 237 238 368 268 358 258 568 356 256
19	19x 19y 17y 18y 18x 17x 159 169 123 124 134 178 168 158 167 157 4xy 3xy 2xy 89y 79y 89x 79x 6xy 5xy 349 249 239 489 389 289 479 379 279 78y 78x 46y 36y 26y 45y 35y 25y 46x 36x 26x 45x 35x 25x 569 56y 56x 234 348 248 238 347 247 237 678 578 456 356 256
20	79x 89y 8xy 89x 7xy 79y 9xy 678 6xy 69y 69x 478 578 5xy 4xy 59y 49y 59x 49x 56y 46y 56x 46x 569 469 458 457 38y 38x 389 37y 37x 379 178 278 268 168 267 167 258 158 248 148 257 157 247 147 356 346 345 245 145 23y 13y 23x 13x 239 139 12y 12x 129 126 123
21	89x 89y 489 4xy 589 5xy 3xy 78y 79y 79x 78x 69y 68y 69x 68x 459 458 34y 34x 479 478 35y 35x 56y 56x 379 378 369 368 679 678 67y 67x 457 356 367 189 289 2xy 1xy 24y 14y 24x 14x 259 159 258 158 23y 13y 23x 13x 234 134 246 146 257 157 129 128 125 127 126
22	89x 89y 789 78y 79y 79x 78x 3xy 4xy 589 689 6xy 5xy 479 379 478 378 349 348 46y 45y 36y 35y 46x 45x 36x 35x 569 568 56y 56x 347 456 356 29y 28y 29x 28x 189 1xy 27y 27x 249 239 248 238 14y 13y 14x 13x 169 159 168 158 234 267 257 134 167 157 127 126 125
23	89x 89y 48x 49x 489 59x 58x 589 6xy 69y 68y 7xy 79y 78y 3xy 45y 46y 34y 57x 579 578 35y 67x 67y 679 678 36x 369 368 37x 379 378 456 345 346 367 19x 29x 28x 18x 289 189 24x 14x 249 149 248 148 25y 15y 23y 13y 247 147 256 156 235 135 12y 125 126 127 123
24	89x 89y 4xy 59y 59x 5xy 3xy 689 789 79y 69y 79x 69x 78y 68y 78x 68x 459 458 45y 45x 349 348 34y 34x 358 679 678 67y 67x 467 357 356 367 289 29y 29x 28y 28x 2xy 1xy 149 148 14y 14x 158 139 138 13y 13x 247 246 235 257 256 237 236 134 157 156 125 127 126
25	89x 89y 5xy 69y 69x 6xy 789 79y 79x 78y 78x 3xy 4xy 569 568 56y 56x 459 359 458 358 45y 35y 45x 35x 468 368 349 348 34y 34x 345 467 367 347 289 29y 29x 28y 28x 2xy 189 19y 19x 18y 18x 179 178 17y 17x 257 267 246 236 247 237 157 146 136 125 126 124 123
26	89x 89y 9xy 8xy 4xy 589 689 789 59y 79y 69y 59x 79x 69x 58y 78y 68y 58x 78x 68x 5xy 7xy 6xy 348 34y 34x 359 379 369 457 456 467 357 356 367 567 249 248 24y 24x 238 23y 23x 149 148 138 13y 13x 234 257 256 267 145 147 146 157 156 167 129 128 12y 12x 123

Table 16. Minimal $T(11, 4, 3)$ Systems (III)

No.	Turán system
27	78y 789 78x 79x 89x 4xy 49y 58y 68y 67y 57y 6xy 5xy 69y 59y 468 458 467 457 46x 45x 469 459 56y 568 567 56x 569 456 178 278 378 18x 38x 28x 17x 37x 27x 189 389 289 179 379 279 19x 39x 29x 14y 34y 24y 13y 12y 23y 134 124 234 136 126 236 135 125 235 123
28	78y 789 78x 89x 79x 4xy 5xy 6xy 49y 69y 59y 468 458 568 46y 45y 56y 467 457 567 46x 45x 56x 469 459 569 456 18y 18x 189 1xy 19y 19x 28y 38y 378 278 38x 28x 389 289 37y 27y 37x 27x 379 279 39x 29x 147 167 157 23y 134 124 136 126 135 125 234 236 235 123
29	78y 89y 8xy 78x 789 7xy 79y 58y 58x 589 5xy 59y 59x 68y 678 68x 689 67y 6xy 69y 67x 679 47y 49x 45y 457 46y 467 456 38x 389 39x 19x 29x 357 348 34x 349 257 157 248 148 24x 14x 249 149 356 256 156 23y 13y 237 137 128 12y 127 12x 129 235 135 236 136 124
30	19y 19x 17y 18y 18x 17x 159 169 168 158 167 157 156 123 124 134 2xy 89y 79y 69x 59x 6xy 5xy 4xy 3xy 289 279 789 78y 78x 26y 25y 26x 25x 569 249 239 489 389 479 379 46y 36y 45y 35y 46x 36x 45x 35x 34y 34x 278 256 568 567 248 238 247 237 478 378 346 345
31	89y 8xy 9xy 79x 78x 789 89x 4xy 5xy 6xy 49y 69y 59y 48y 68y 58y 49x 69x 59x 48x 68x 58x 489 689 589 467 457 567 456 37y 37x 379 378 17y 27y 247 147 267 167 257 157 346 345 356 246 146 245 145 256 156 23y 13y 23x 13x 239 139 238 138 12y 12x 129 128 123
32	89y 8xy 9xy 89x 48y 5xy 59y 59x 39x 68y 78y 7xy 6xy 79y 69y 78x 68x 789 689 79x 69x 45y 45x 459 34x 349 47y 46y 478 468 358 467 357 356 567 367 24x 249 258 23y 238 23x 239 14x 149 158 13y 138 234 257 256 267 135 157 156 137 136 167 12y 128 12x 129 124
33	1xy 2xy 7xy 6xy 28y 18y 89x 27y 17y 27x 17x 26y 16y 26x 16x 68y 67y 67x 39y 39x 38x 49y 59y 59x 49x 58x 48x 37x 35y 34y 45y 45x 129 128 269 169 268 168 789 267 167 123 125 124 389 589 489 379 378 579 479 578 478 235 234 135 134 245 145 356 346 456 345
34	2xy 3xy 1xy 23x 12y 12x 13x 29x 26y 27y 38y 39y 34y 35y 18y 17y 16y 89y 58x 48x 59y 49y 45y 45x 78x 68x 79x 69x 57x 47x 56x 46x 67y 237 236 127 126 138 139 135 134 258 248 245 279 269 267 389 359 349 367 189 159 149 167 458 578 478 568 468 457 456 679
35	4xy 8xy 1xy 2xy 3xy 48x 48y 34y 34x 24y 24x 18y 18x 38y 38x 28y 28x 19y 19x 79y 79x 59y 69y 69x 59x 67y 57y 67x 57x 56y 56x 348 248 349 249 147 146 145 347 247 456 189 239 123 789 689 589 137 127 136 135 126 125 237 236 235 678 578 679 579 156 356 256
36	7xy 78x 79y 79x 78y 9xy 8xy 67y 67x 6xy 679 678 69y 69x 68y 68x 4xy 5xy 59y 49y 59x 49x 58y 48y 58x 48x 589 489 457 456 389 189 289 367 357 347 257 157 247 147 356 346 345 256 156 246 146 245 145 23y 13y 23x 13x 239 139 238 138 127 12y 12x 129 128 123

Table 17. Minimal $T(11, 4, 3)$ Systems (IV)

No.	Turán system
37	7xy 8xy 9xy 789 479 478 489 5xy 6xy 67x 57x 69x 59x 68x 58x 67y 57y 69y 59y 68y 58y 56x 56y 567 569 568 456 37y 39y 38y 379 378 389 179 279 278 178 289 189 34x 24x 14x 24y 14y 247 147 249 149 248 148 346 345 23x 13x 12x 12y 236 136 235 135 126 125 123
38	7xy 8xy 9xy 89x 89y 479 478 5xy 6xy 67x 57x 67y 57y 69x 59x 68x 58x 69y 59y 68y 58y 689 589 46y 45y 567 456 37x 39x 38x 379 378 389 179 279 278 178 34y 24x 14x 247 147 249 149 248 148 346 345 356 256 156 23y 13y 12x 12y 127 129 128 236 136 235 135 124
39	8xy 9xy 789 47x 479 478 489 5xy 6xy 67y 57y 69x 59x 68x 58x 69y 59y 68y 58y 56x 56y 567 569 568 456 3xy 37y 39x 38x 39y 38y 389 17x 27x 279 179 278 178 289 189 367 357 24x 14x 24y 14y 247 147 249 149 248 148 346 345 12x 12y 236 136 235 135 126 125 123
40	8xy 9xy 69x 69y 68y 68x 79y 79x 78y 78x 7xy 389 3xy 489 589 5xy 4xy 679 678 36y 36x 57y 47y 57x 47x 359 349 358 348 45y 45x 567 467 456 457 289 189 26y 26x 16y 16x 239 238 13y 13x 259 249 258 248 159 149 158 148 136 237 137 235 234 145 12y 12x 126 127

Table 18. Minimal $T(11, 4, 3)$ Systems (IIV)

No.	Turán system
1	123 134 235 234 135 125 124 12x 12y 12z 345 3xz 3xy 3yz 36x 37x 38x 39x 36z 37z 39z 38z 36y 37y 39y 38y 245 145 25x 25z 25y 24x 24z 24y 15x 15z 15y 14x 14z 14y 267 269 268 279 278 289 167 169 168 179 178 189 45x 45z 45y xyz 6xz 7xz 9xz 8xz 6xy 7xy 9xy 8xy 6yz 7yz 9yz 8yz 567 569 568 579 578 589 467 469 468 479 478 489 679 678 689 789
2	123 134 235 234 135 128 125 124 12y 12z 38z 38y 389 38x 345 3xz 39z 3xy 39y 39x 368 378 37z 36z 37y 36y 258 248 158 148 25z 25y 24z 24y 15z 15y 14z 14y 2yz 1yz 29x 19x 27x 26x 279 269 17x 16x 179 169 267 167 458 8xz 89z 8xy 89y 89x 5yz 4yz 45x 459 9xz 9xy 78z 68z 78y 68y 457 456 7yz 6yz 57x 56x 579 569 47x 46x 479 469 567 467 67x 679
3	123 137 12z 134 124 135 136 126 125 234 236 235 17y 17z 1xy 14z 18x 19x 19y 18y 189 16z 15z 156 27x 37y 37z 279 278 3xy 3xz 3yz 24y 34z 24z 39x 38x 29x 28x 39y 38y 39z 38z 289 26y 25y 26z 25z 356 256 47x 7yz 479 478 789 xyz 9xy 8xy 9xz 8xz 9yz 8yz 489 67x 57x 467 457 679 678 579 578 46x 45x 46y 45y 469 468 459 458 689 589 56x 56y 56z
4	123 13y 23y 234 235 236 134 136 135 127 12y 124 126 125 378 379 37x 34y 36y 35y 34z 36z 35z 38x 389 39x 27y 17y 27z 17z 24y 26y 25y 14y 16y 15y 28z 2xz 29z 18z 1xz 19z 28x 289 29x 18x 189 19x 7yz 467 457 567 478 47x 479 678 67x 679 578 57x 579 46z 45z 56z 456 8yz xyz 9yz 468 46x 469 458 45x 459 568 56x 569 8xy 89y 9xy 8xz 89z 9xz 89x

Table 19. Minimal $T(12, 4, 3)$ Systems

3.2.2 Calculations of T(n, 6, 5)

By the algorithm, we have generated all of the $T(n, 6, 5)$ systems with the following parameters.

n	blocks	number of systems
7	4	1
8	11	5
9	25	77

Table 20.

All minimal $T(n, 5, 3)$ $(8 \le n \le 12)$ systems are given in the following tables.

No.	Turán system
1	12345 12367 24567 34567

Table 21. Minimal $T(7, 6, 5)$ Systems

No.	Turán system
1	12345 12346 12478 13478 13568 12567 34567 24568 23678 23578 45678
2	12345 12368 12478 13458 13467 12567 23467 24568 23578 45678 35678
3	12345 12478 12368 13468 13457 12567 23467 23578 24568 45678 35678
4	12356 12456 13457 13468 12478 12378 23467 23458 45678 35678 25678
5	12356 12457 13467 13458 12468 12378 23456 23478 45678 35678 25678

Table 22. Minimal $T(8, 6, 5)$ Systems

No.	Turán system
1	12345 12346 12569 12578 13579 14589 14567 13568 12479 12389 13469 13478 12678 24579 23589 24568 23567 34569 34578 23478 24689 23679 56789 46789 36789
2	12345 12346 12569 12578 13579 14589 14567 13568 12489 12379 13469 13478 12678 24579 23589 24568 23567 34569 34578 23478 24679 23689 56789 46789 36789
3	12345 12346 12569 12578 14589 13579 13568 14567 12389 12479 13479 13468 12678 23589 24579 24568 23567 34569 34578 23478 24689 23679 56789 46789 36789
4	12345 12346 12569 12578 14589 13579 13568 14567 12489 12379 13479 13468 12678 23589 24579 24568 23567 34569 34578 23478 23689 24679 56789 46789 36789
5	12345 12346 12589 12567 13569 14579 14568 13578 12479 12389 13469 13478 12678 24569 23579 23568 24578 34589 34567 23478 24689 23679 56789 46789 36789
6	12345 12346 12589 12567 13569 14579 14568 13578 12479 12389 13479 13468 12678 24569 23579 23568 24578 34589 34567 23478 24689 23679 56789 46789 36789
7	12345 12346 12589 12567 13569 14579 14568 13578 12479 12389 13489 13467 12678 24569 23579 23568 24578 34589 34567 23478 24689 23679 56789 46789 36789
8	12345 12346 12589 12567 13569 14579 14568 13578 12489 12379 13469 13478 12678 24569 23579 23568 24578 34589 34567 23478 24679 23689 56789 46789 36789
9	12345 12346 12589 12567 13569 14579 14568 13578 12489 12379 13479 13468 12678 24569 23579 23568 24578 34589 34567 23478 24679 23689 56789 46789 36789
10	12345 12346 12589 12567 13569 14579 14568 13578 12489 12379 13489 13467 12678 24569 23579 23568 24578 34589 34567 23478 24679 23689 56789 46789 36789
11	12345 12367 12389 12467 13568 13579 13478 13469 12578 12569 12489 14579 14568 23578 23569 23479 23468 34567 24579 24568 34589 36789 26789 56789 46789
12	12345 12367 12389 12467 13568 13579 13478 13469 12579 12568 12489 14578 14569 23578 23569 23479 23468 34567 24578 24569 34589 36789 26789 56789 46789
13	12345 12367 12389 13567 12467 13468 13479 12578 12569 13589 12489 14579 14568 23579 23568 23478 23469 24567 34578 34569 24589 36789 26789 56789 46789

Table 23. Minimal $T(9, 6, 5)$ Systems (I)

No.	Turán system
14	12345 12367 12389 13567 12467 13468 13479 12579 12568 13589 12489 14578 14569 23578 23569 23478 23469 24567 34579 34568 24589 36789 26789 56789 46789
15	12345 12367 12389 12468 13578 13569 13478 13469 12579 12568 12479 14567 14589 23567 23467 23589 23489 34579 34568 24578 24569 36789 26789 56789 46789
16	12345 12367 12389 12468 13578 13569 13479 13468 12578 12569 12479 14578 14569 23579 23568 23478 23469 34567 24567 34589 24589 36789 26789 56789 46789
17	12345 12367 12389 12468 13579 13568 13478 13469 12578 12569 12479 14567 14589 23567 23467 23589 23489 34578 34569 24579 24568 36789 26789 56789 46789
18	12345 12367 12389 13568 13579 13478 13469 12578 12569 12479 12468 14567 14589 23467 23578 23569 23489 24567 34579 34568 24589 36789 26789 56789 46789
19	12345 13458 13459 12567 12589 13567 14567 12479 12369 12468 12378 14689 13789 24569 23579 24578 23568 23467 23489 45789 35689 34679 34678 56789 26789
20	12345 13458 13459 12567 12589 13567 14567 12479 12369 12478 12368 14689 13789 24569 23579 24568 23578 23467 23489 45789 35689 34679 34678 56789 26789
21	12345 13459 12458 13468 13567 13589 12379 12378 14567 12469 12567 14789 12689 34578 23467 23489 23569 23568 24579 34679 45689 24678 25789 36789 56789
22	12345 13459 12458 13478 13567 13589 12379 12368 14567 12469 12567 14689 12789 34568 23467 23489 23578 23569 24579 34679 45789 24678 25689 36789 56789
23	12345 13459 13458 12579 12568 14567 13567 12367 12478 12469 12389 14789 13689 24567 23578 23569 24589 23479 23468 45689 35789 34679 34678 56789 26789
24	12346 12357 12458 13459 12389 13568 13478 12569 12479 14589 14567 12678 13679 23458 23569 23479 34567 24567 23678 35789 34689 25789 24689 56789 46789
25	12346 12357 12458 13459 12389 13568 13478 12579 12469 14589 14567 12678 13679 23458 23569 23479 34567 24567 23678 35789 34689 25689 24789 56789 46789
26	12346 12357 12458 12569 12479 12389 13469 14579 13589 14568 13567 13478 12678 23459 24567 23568 23478 34569 34578 24689 23679 25789 56789 46789 36789

Table 24. Minimal $T(9, 6, 5)$ Systems (II)

No.	Turán system
27	12346 12357 12458 12569 12479 12389 13469 14589 13579 14567 13568 13478 12678 23459 24567 23568 23478 34569 34578 24689 23679 25789 56789 46789 36789
28	12346 12458 12357 12569 12479 12389 13569 14579 13489 14568 13467 13578 12678 23459 24567 23568 23478 34569 34578 24689 23679 25789 46789 56789 36789
29	12346 12458 12357 12569 12479 12389 13569 14589 13479 14567 13468 13578 12678 23459 24567 23568 23478 34569 34578 24689 23679 25789 46789 56789 36789
30	12356 12347 12456 13458 12389 13469 13579 12579 12489 14569 14578 13678 12678 23459 23468 23578 34567 24578 23679 35689 25689 24679 34789 56789 46789
31	12356 12347 12456 13458 12389 13469 13579 12579 12489 14579 14568 13678 12678 23459 23468 23578 34567 24578 23679 35689 25689 24679 34789 56789 46789
32	12356 12347 12456 13458 12389 13469 13579 12579 12489 14589 14567 13678 12678 23459 23468 23578 34567 24578 23679 35689 25689 24679 34789 56789 46789
33	12356 12347 12456 13458 12389 13469 13579 12589 12479 14569 14578 13678 12678 23459 23468 23578 34567 24578 23679 35689 25679 24689 34789 56789 46789
34	12356 12347 12456 13458 12389 13469 13579 12589 12479 14579 14568 13678 12678 23459 23468 23578 34567 24578 23679 35689 25679 24689 34789 56789 46789
35	12356 12347 12456 13458 12389 13469 13579 12589 12479 14589 14567 13678 12678 23459 23468 23578 34567 24578 23679 35689 25679 24689 34789 56789 46789
36	13456 12458 12347 12346 12359 14579 13489 12489 12567 13578 15689 13678 12679 23457 24569 34589 23568 45678 34679 24678 35679 25789 23789 23689 46789
37	13456 12458 12347 12346 12359 14579 13489 12489 12567 13578 15689 13679 12678 23457 24569 34589 23568 45678 34678 24679 35679 25789 23789 23689 46789
38	12347 12346 12358 12456 13489 13578 13569 14567 14589 12489 12579 13679 12678 23459 34579 34568 23567 24578 34678 23789 23689 24679 25689 46789 56789
39	12347 12346 12358 12456 13489 13579 13569 14567 14589 12489 12578 13678 12679 23459 34578 34568 23567 24579 34679 23789 23689 24678 25689 46789 56789

Table 25. Minimal $T(9, 6, 5)$ Systems (III)

No.	Turán system
40	12347 12358 12456 13456 12369 13579 12579 13489 12489 14569 14578 13678 12678 23459 23567 23468 34578 24578 23789 35689 25689 34679 24679 56789 46789
41	12347 12358 13456 12456 12369 13589 12579 12489 13479 14569 14578 13678 12678 23459 23567 23468 34578 24578 23789 25689 35679 34689 24679 56789 46789
42	12357 12346 12458 13456 12389 13478 12479 13589 12569 14567 14589 12678 13679 23459 23478 23568 34579 24567 23679 25789 35678 34689 24689 56789 46789
43	12357 12346 12458 13456 12389 13478 12479 13589 12569 14578 14569 12678 13679 23459 23478 23568 34579 24567 23679 25789 35678 34689 24689 56789 46789
44	12357 12346 12458 13456 12389 13478 12479 13589 12569 14579 14568 12678 13679 23459 23478 23568 34579 24567 23679 25789 35678 34689 24689 56789 46789
45	12357 12346 12458 13456 12389 13578 12479 13489 12569 14567 14589 12678 13679 23459 23478 23568 34579 24567 23679 25789 34678 35689 24689 56789 46789
46	12357 12346 12458 13456 12389 13578 12479 13489 12569 14578 14569 12678 13679 23459 23478 23568 34579 24567 23679 25789 34678 35689 24689 56789 46789
47	12357 12346 12458 13456 12389 13578 12479 13489 12569 14579 14568 12678 13679 23459 23478 23568 34579 24567 23679 25789 34678 35689 24689 56789 46789
48	12357 12346 12458 13459 12389 13478 13568 12479 12569 14568 14579 12678 13679 23458 23479 23569 34567 24567 23678 35789 34689 25789 24689 56789 46789
49	12357 12346 12458 13459 12389 13478 13568 12579 12469 14568 14579 12678 13679 23458 23479 23569 34567 24567 23678 35789 34689 24789 25689 56789 46789
50	12357 12348 13456 12456 12369 12478 13479 13589 12589 14567 14589 13678 12679 23459 23467 23568 34578 24579 23789 25678 35679 34689 24689 56789 46789
51	12357 12348 13456 12456 12369 12478 13479 13589 12589 14578 14569 13678 12679 23459 23467 23568 34578 24579 23789 25678 35679 34689 24689 56789 46789
52	12357 12348 13456 12456 12369 12478 13479 13589 12589 14579 14568 13678 12679 23459 23467 23568 34578 24579 23789 25678 35679 34689 24689 56789 46789

Table 26. Minimal $T(9, 6, 5)$ Systems (VI)

No.	Turán system
53	12357 12348 13456 12456 12369 12578 13479 13589 12489 14567 14589 13678 12679 23459 23467 23568 34578 24579 23789 24678 35679 34689 25689 56789 46789
54	12357 12348 13456 12456 12369 12578 13479 13589 12489 14578 14569 13678 12679 23459 23467 23568 34578 24579 23789 24678 35679 34689 25689 56789 46789
55	12357 12348 13456 12456 12369 12578 13479 13589 12489 14579 14568 13678 12679 23459 23467 23568 34578 24579 23789 24678 35679 34689 25689 56789 46789
56	12457 12348 12356 12379 12589 12469 13479 14589 13589 14567 13567 13468 12678 23459 23578 23467 24568 34569 34578 24789 25679 23689 46789 56789 36789
57	12457 12348 12356 12379 12589 12469 13579 14589 13489 14567 13467 13568 12678 23459 23578 23467 24568 34569 34578 24789 25679 23689 46789 56789 36789
58	12457 12348 12356 12379 12589 12469 14579 13489 13589 13467 13567 14568 12678 23459 23578 23467 24568 34569 34578 24789 25679 23689 46789 56789 36789
59	13457 12456 12347 12346 12358 14589 13489 12489 13569 12579 15678 12678 13679 23459 34568 24578 23567 45679 34678 24679 35789 25689 23789 23689 46789
60	12358 12346 12457 13456 12379 13478 13589 12489 12569 14567 14589 12678 13679 23459 23478 23567 34579 24568 23689 25789 35678 24679 34689 56789 46789
61	12358 12346 12457 13456 12379 13478 13589 12489 12569 14578 14569 12678 13679 23459 23478 23567 34579 24568 23689 25789 35678 24679 34689 56789 46789
62	12358 12346 12457 13456 12379 13478 13589 12489 12569 14579 14568 12678 13679 23459 23478 23567 34579 24568 23689 25789 35678 24679 34689 56789 46789
63	12358 12346 12457 13456 12379 13578 13489 12489 12569 14567 14589 12678 13679 23459 23478 23567 34579 24568 23689 25789 34678 24679 35689 56789 46789
64	12358 12346 12457 13456 12379 13578 13489 12489 12569 14578 14569 12678 13679 23459 23478 23567 34579 24568 23689 25789 34678 24679 35689 56789 46789
65	12358 12346 12457 13456 12379 13578 13489 12489 12569 14579 14568 12678 13679 23459 23478 23567 34579 24568 23689 25789 34678 24679 35689 56789 46789

Table 27. Minimal $T(9, 6, 5)$ Systems (VII)

No.	Turán system
66	12358 12347 12456 13456 12369 13489 12489 13579 12579 14579 14568 13678 12678 23459 23468 23567 34578 24578 23789 35689 25689 34679 24679 56789 46789
67	12358 12347 13456 12456 12369 13589 12489 12579 13479 14579 14568 13678 12678 23459 23468 23567 34578 24578 23789 25689 34689 35679 24679 56789 46789
68	12458 13459 12347 12346 12356 14567 13489 12489 13578 12579 15689 13678 12679 23457 34568 24569 23589 45789 34679 24678 35679 25678 23789 23689 46789
69	12458 13459 12347 12346 12356 14567 13489 12489 13578 12579 15689 13679 12678 23457 34568 24569 23589 45789 34678 24679 35679 25678 23789 23689 46789
70	13458 12456 12347 12346 12357 14579 13489 12489 12589 13569 15678 12678 13679 23459 24578 34567 23568 45689 34678 24679 35789 25679 23789 23689 46789
71	13458 12456 12347 12346 12357 14579 13489 12489 12589 13569 15678 13678 12679 23459 24578 34567 23568 45689 24678 34679 35789 25679 23789 23689 46789
72	12359 12346 12458 13457 12378 13489 13568 12569 12479 14568 14579 12678 13679 23458 23479 23567 34569 24567 23689 35789 25789 24689 34678 56789 46789
73	12359 12346 12458 13457 12378 13489 13568 12569 12479 14578 14569 12678 13679 23458 23479 23567 34569 24567 23689 35789 25789 24689 34678 56789 46789
74	12359 12346 12458 13457 12378 13489 13568 12569 12479 14589 14567 12678 13679 23458 23479 23567 34569 24567 23689 35789 25789 24689 34678 56789 46789
75	12359 12346 12458 13457 12378 13489 13568 12579 12469 14568 14579 12678 13679 23458 23479 23567 34569 24567 23689 35789 25689 24789 34678 56789 46789
76	12359 12346 12458 13457 12378 13489 13568 12579 12469 14578 14569 12678 13679 23458 23479 23567 34569 24567 23689 35789 25689 24789 34678 56789 46789
77	12359 12346 12458 13457 12378 13489 13568 12579 12469 14589 14567 12678 13679 23458 23479 23567 34569 24567 23689 35789 25689 24789 34678 56789 46789

Table 27. Minimal $T(9, 6, 5)$ Systems (VIII)

3.2.3 Calculations of T(n, 5, 3)

Our last goal is searching for the minimal $T(n, 5, 3)$ systems. By theorem 1, the parameters for maximum degree of extending systems are listed in the following table:

n	5	6	7	8	9	10	11	12	13
maxdeg	1	2,3	5	8,9	12~14	20,21	29,30	40	52

Table 28.

By the algorithm, we have generated all of the $T(n, 5, 3)$ systems with the following parameters.

n	blocks	number of systems
10	20	5
10	21	95
11	29	1
11	30	166
12	39	0
12	40	16
13	52	1
14	67	0

Table 29.

By the Schonheim bound and the table, we have

Theorem 21: $T(11, 5, 3) = 29$.

Theorem 22: $T(12, 5, 3) = 40$.

Theorem 23: $T(13, 5, 3) = 52$.

Theorem 24: $T(14, 5, 3) \geq 68$.

All minimal $T(n, 5, 3)$ $(8 \leq n \leq 12)$ systems are given in the following tables.

No.	Turán system
1	123 125 124 35x 358 359 346 347 19x 29x 28x 18x 267 167 457 456 67x 489 789 689
2	123 137 237 127 12x 34x 35x 36x 27x 17x 289 189 789 456 468 469 459 458 569 568
3	123 129 13x 23x 239 139 12x 39x 29x 19x 456 457 468 568 567 467 458 678 578 478
4	123 129 13x 19x 156 178 349 24x 359 369 27x 28x 356 278 468 467 458 457 78x 569
5	59x 69x 79x 89x 569 56x 78x 789 258 157 468 367 23x 239 14x 149 345 126 138 247

Table 30. Minimal $T(10, 5, 3)$ Systems

No.	Turán system
1	689 789 7xy 6xy 58x 59y 59x 58y 289 189 3xy 4xy 179 178 47y 47x 269 268 36y 36x 567 349 348 12y 12x 237 146 245 135

Table 31. Minimal $T(11, 5, 3)$ System

No.	Turán system
1	123 124 134 234 235 236 136 135 126 125 3xy 3xz 3yz 2yz 2xz 2xy 1yz 1xz 1xy 456 xyz 467 468 469 457 459 458 567 569 568 79z 78z 89z 79y 78y 89y 79x 78x 89x 789
2	123 124 134 234 35y 36z 38z 37y 26y 25z 17z 18y 2xz 29y 19z 1xy 368 357 39x 278 156 269 25x 18x 179 4yz 467 458 46x 459 489 47x 68z 57y 69y 5xz 79z 8xy 67x 589
3	123 124 134 234 35y 36z 38z 37y 26z 25y 18z 17y 2xz 29y 1xz 19y 367 358 39x 278 156 269 25x 189 17x 4yz 467 458 469 45x 489 47x 68y 57z 6xy 59z 79z 8xy 68x 579
4	123 134 234 124 125 126 34y 34z 39x 367 368 358 357 29z 2xz 2xy 29y 1xz 19z 1xy 19y 278 178 49x 89x 79x 8yz 7yz 78x 789 6yz 5yz 468 467 458 457 56x 569 56z 56y
5	123 134 234 124 127 128 34y 34z 39x 358 368 367 357 29z 2xz 2xy 29y 1xz 19z 1xy 19y 278 178 49x 8yz 7yz 78x 789 69x 59x 6yz 5yz 468 458 467 457 56x 569 56z 56y
6	123 134 234 124 127 128 34y 34z 3yz 358 368 367 357 29z 2xz 2xy 29y 1xz 19z 1xy 19y 278 178 4yz 89x 79x 78x 789 6yz 5yz 69x 59x 468 458 467 457 56z 56y 56x 569
7	123 134 234 124 129 12x 34z 34y 3yz 35x 36x 379 389 29x 19x 26y 25y 28z 27z 16y 15y 18z 17z 4yz 46x 45x 489 479 8xy 7xy 69z 59z 56z 78y 78x 569 568 567 678 578
8	123 134 235 234 135 128 125 124 35z 34z 3xy 368 378 29z 19z 2xz 2yz 1yz 1xz 2xy 1xy 89y 89x 458 459 xyz 578 568 478 468 79y 79x 69y 69x 457 456 679 67z 67y 67x
9	123 126 124 125 367 368 35x 35y 35z 34x 34z 34y 378 169 269 2xz 1xz 2xy 1xy 2yz 1yz 289 189 279 179 456 568 567 468 467 459 9xz 9xy 9yz xyz 458 457 78x 78z 78y
10	123 126 124 125 36y 36z 369 36x 345 357 358 348 347 1yz 2yz 2xz 1xz 29z 19z 2xy 1xy 29y 19y 278 178 456 568 567 468 467 xyz 9yz 59x 49x 458 457 89x 79x 78z 78y
11	123 136 237 236 137 12z 127 126 34z 35z 3xz 3yz 3xy 2xy 1xy 289 189 27z 26z 17z 16z 45y 45x xyz 458 459 59y 58y 59x 58x 49y 48y 49x 48x 789 689 567 467 679 678
12	123 128 127 13z 24z 29y 2xy 27z 25z 26z 256 39x 348 347 38y 38x 389 367 357 1xy 19y 19x 18z 17z 156 xyz 9yz 89x 79x 478 46y 45y 46x 45x 469 459 678 578 56z 56y
13	123 12x 13x 23x 23y 23z 13z 13y 12z 12y 3xz 3xy 3yz 2xz 2xy 1xz 1xy 2yz 1yz 456 457 467 567 568 569 469 468 459 458 679 678 689 579 578 479 478 589 489 789 xyz

Table 32. Minimal $T(12, 5, 3)$ Systems (I)

No.	Turán system
14	123 13x 23y 23x 13y 126 12y 12x 346 356 3yz 3xz 3xy 2xy 1xy 27z 28z 29z 17z 19z 18z 56y 46y 56x 46x 679 678 689 45z xyz 457 459 458 579 578 589 479 478 489 789
15	123 13x 23y 23x 13y 12z 12y 12x 34z 35z 36z 37z 3xy 2yz 2xz 1yz 1xz 2xy 1xy 289 189 457 456 467 567 xyz 458 459 479 478 469 468 579 578 569 568 679 678 89y 89x
16	789 78x 79y 89z 9yz 8xz 7xy xyz 19x 29x 49x 39x 28y 18y 47z 37z 68y 58y 67z 57z 249 139 569 348 127 268 158 457 367 24x 13x 56x 12z 34y 45z 36z 26y 15y 235 146

Table 32. Minimal $T(12, 5, 3)$ Systems (II)

No.	Turán system
1	123 124 134 234 356 357 367 3yz 3xt 39t 38z 39x 38y 26y 25x 269 258 27t 27z 2zt 29y 28x 16t 15z 168 159 17y 17x 1xy 19z 18t 46z 45t 46x 45y 4yt 4xz 479 478 489 567 6xz 5yt 68t 59z 69y 58x 7zt 7xy 9xt 8yz 789

Table 33. Minimal $T(13, 5, 3)$ Systems

4. Future Work

Like other combinatorial areas, computer science methods involved in the search for Turán systems and number have increased dramatically in importance in recent years. Many numbers, which were open for a long time, were found by new algorithmic methods originated from computer science techniques. The results we have obtained show that algorithm *turexp* and procedures *nauty* are efficient tools for searching Turán numbers. More Turán systems and numbers may be obtained by the algorithms after analysis the properties of the Turán systems. For example, we have obtained all minimum $T(n, 6, 5)$ systems for $n \leq 9$ and we have noticed that all $T(n, 6, 5)$ numbers can be calculated by Theorem 3 and the existence of the systems in Table 21-23. By analyzing the properties of $T(n, 6, 5)$ systems, we may cut more branches in search tree. The number $T(10, 6, 5)$ which is the last open parameter situation on

at most 10 points may be potentially calculated by refined versions *turexp* and *nauty*.

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