2005

Mean stress effects on random fatigue of nonlinear structures

Karl Sweitzer

N.S. Ferguson

ICSV12 2005 Lisbon 11 - 14 July

Twelfth International Congress
on Sound and Vibration

MEAN STRESS EFFECTS ON RANDOM FATIGUE OF NONLINEAR STRUCTURES

Karl A. Sweitzer[†‡] and N. S. Ferguson[‡]

ITT Industries Space Systems Division, Rochester, NY USA[†]
Institute of Sound and Vibration Research, University of Southampton, UK[‡]
email: karl.sweitzer@itt.com

Abstract

Random fatigue analysis techniques have concentrated on linear structures that exhibit symmetric stress range probability density functions (*PDF*). In narrowband response and broadband random response it is assumed that the bending stress range *PDF* is symmetric and has zero mean stress. Structures that can exhibit geometrical nonlinearity, under large deflections, typically introduce membrane effects, which are additional positive tensile stresses that tend to skew the stress range *PDF*. This paper investigates fatigue analysis approaches that include mean stress effects and show how they can be applied to nonlinear random vibration problems.

INTRODUCTION

Recent advances in nonlinear random vibration analysis techniques for structures have identified a need for a fatigue analysis failure criterion that includes the tensile membrane stresses. The tensile membrane stresses combine with the cyclic bending stresses to form a skewed stress range *PDF*. This paper reviews sinusoidal and random fatigue relationships and extends this theory to include nonlinear mean stresses.

Sinusoidal Fatigue Life

The standard sinusoidal stress life power law fatigue model is:

$$\sigma_{ar} = \sigma_f^{'}\left(2N_f\right)^b = AN_f^b$$
$$A = \sigma_f^{'}2^b \tag{1}$$

where S_{ar} is the fully reversed alternating stress and N_f is the median cycles to failure. The power law slope b (fatigue strength exponent) and the stress intercept $S_f^{'}$ at one reversal (fatigue strength coefficient), or A at one cycle, are determined experimentally. The symbol S is used here to denote stress instead of σ, which is used to denote the standard deviation. The life N_f that is expected for a given alternating stress under zero mean stress is:

$$N_f = \left(\frac{S_{ar}}{A} \right)^{1/b} \tag{2}$$

Sinusoidal Fatigue Life with Mean Stress

A recent paper by Dowling [1] describes four methods that have been developed over the past ~100 years to determine fatigue life for alternating stresses combined with mean stresses. This section summarizes the currently accepted methods.

The Morrow equation for the number of cycles to failure with mean stress S_0 is:

$$N_{f\,mi} = \left[S_a \Big/ A \left(1 - \frac{S_0}{S_f^{'}} \right) \right]^{1/b} \tag{3}$$

For nonferrous materials, a modification using the true stress at fracture $\tilde{S}_{fB}$ is:

$$N_{f\,mB} = \left[S_a \Big/ A \left(1 - \frac{S_0}{\tilde{S}_{fB}} \right) \right]^{1/b} \tag{4}$$

The Walker equation is:

$$N_{f\,w} = \left[\frac{S_{\max}}{A} \left(\frac{1-R}{2} \right)^{\gamma} \right]^{1/b} \tag{5}$$

where the ratio R of minimum and maximum stress values at the trough and peak of the cycle is:

$$R = S_{\min} \big/ S_{\max} \tag{6}$$

The fully reversed zero mean case is when R = -1. Note, (5) reduces to (2) when the exponent γ=1.0. The special case of the Walker equation for γ=0.5 gives the Smith-Watson-Topper [5] equation:

$$N_{f\,swt} = \left[\frac{S_{\max}}{A} \left(\frac{1-R}{2} \right)^{0.5} \right]^{1/b} = \left[\frac{S_{\max}}{A} \sqrt{\frac{1-R}{2}} \right]^{1/b} \tag{7}$$

A modified version of the Walker equation is used in MMPDS-01 [2], (the replacement for MIL-HDBK-5), based on the stress ratio R and an equivalent stress S_{eq}, where:

$$S_{eq} = S_{\max}\left(1-R\right)^{A_3}$$
$$\log\left(N_{f5}\right) = A_1 + A_2\log\left(S_{eq} - A_4\right) \tag{8}$$

(Section 9.6.1.4 from [2] gives a thorough discussion of (8)). The term A_4 represents the fatigue limit stress or "endurance limit".

Random Fatigue & Damage

The number of cycles to failure for a narrow band random process (see [6]) is:

$$N_{f\,nr} = \int_0^{\infty}\left(S_a/A\right)^{1/b} p\left(S_a\right)dS_a \tag{9}$$

where $p\left(S_a\right)$ is the *PDF* of stress peaks. For the narrow band response assumption, $p\left(S_a\right)$ is assumed to be Rayleigh. It has been shown [6] that (9) reduces to:

$$N_{f\,nr} = \left(\frac{\sqrt{2}\sigma_S}{A}\right)^{1/b} \Big/ \Gamma\left(1+\frac{\beta}{2}\right) \tag{10}$$

where σ_S is the standard deviation of stress, $\beta = -1/b$, and Γ is the Gamma Function. The Palmgren-Miner linear cumulative damage rule is:

$$D = \sum_{k=1}^{n}\frac{N_k}{N_{fk}} \tag{11}$$

where D is the summation of damage, N_k are the number of cycles for load case k and N_{fk} are the number of cycles to failure for case k. D=1 indicates a probability of failure = 0.5 (i.e. half of population is expected to have failed).

When one considers a narrow band random process the expected value of damage for a given random load case reduces to:

$$E[D_n] = E[0]\,T\int_0^{\infty}\left(S_a/A\right)^{\beta} p\left(S_a\right)dS_a = E[0]\,T\left(\frac{\sqrt{2}\sigma_S}{A}\right)^{\beta}\Gamma\left(1+\frac{\beta}{2}\right) \tag{12}$$

where $E[0]$ is the expected rate of zero crossings and T is the duration of the load:

$$N = E[0]\,T \tag{13}$$

A wideband random process is more interesting; the distribution of peak amplitudes diverges from the Rayleigh *PDF* and the response will include positive peaks with negative magnitude (see [7] and the cited references). The de-facto method for evaluation of a wideband random response is the time domain rainflow range cycle counting method [8].

Dirlik [9] (see also [7]) developed an empirical relationship for estimating the rainflow range *PDF* based on the spectral moments of the stress power spectral density (*PSD)*. It is interesting to note that Dirlik's *PDF* only estimates the rain flow range amplitudes, i.e. the estimates do not include mean stresses.

PROPOSED TWO DIMENSIONAL STRESS & DAMAGE *PDF*

The proposed method to include mean stresses in a random linear or nonlinear damage equation is:

$$E\left[D_{ms}\right] = E\left[P\right] T \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{\sigma_a^{\beta} p\left(\sigma_a, \sigma_0\right)}{k\left(\sigma_a, \sigma_0\right)} d\sigma_a d\sigma_0 \qquad (14)$$

where $p\left(\sigma_a, \sigma_0\right)$ is the joint *PDF* of stress amplitude & mean stress rainflow ranges and $k\left(\sigma_a, \sigma_0\right)$ is a function of material properties and the chosen non-zero mean stress fatigue equation. The problem is that there is no known closed-form or empirical relationship for $p\left(\sigma_a, \sigma_0\right)$. The joint probability density function $p\left(\sigma_a, \sigma_0\right)$ can be estimated numerically as a histogram with cycle counting algorithms (see for example the `dat2rfm` function in WAFO [10]).

WAFO Rain Flow Matrix

The 2D histogram of rainflow ranges calculated by the WAFO `dat2rfm` function is called a rainflow matrix (*RFM*). A *RFM* is a convenient numerical estimation of the joint *PDF*. The *RFM* has minimum and maximum stress axes, which can easily be related to the amplitude and mean stress dimensions proposed for the 2D rainflow *PDF*. Only half of the *RFM* is used because the minimum stress can never be greater than the maximum stress. The WAFO group has chosen to display the *RFM* as shown in Figure 1 with the minimum stress on the x-axis and maximum stress on the y-axis. The figure also has lines drawn along constant values of the stress ratio *R*, which give insight into the distribution of ranges. Constant amplitude conditions are found along diagonals of positive slope and constant mean stress ranges are found along diagonals of negative slope.

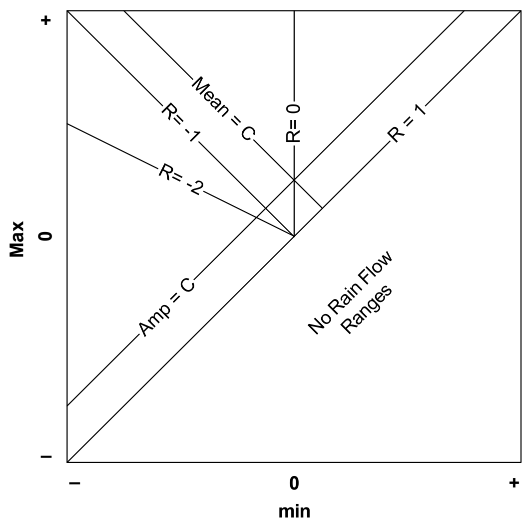

Figure 1 The WAFO convention for Rain Flow Matrix (***RFM***), with special stress range cases.

Random Fatigue Damage using the *RFM*

The proposed discrete random fatigue damage equation based on the *RFM* 2D histogram estimate of the joint *PDF* is:

$$D=\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}\frac{RFM\left(\sigma_{\min},\sigma_{\max}\right)}{N_f\left(\sigma_{\min},\sigma_{\max}\right)}=N_{RF}\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}\frac{RFMn\left(\sigma_{\min},\sigma_{\max}\right)}{N_f\left(\sigma_{\min},\sigma_{\max}\right)} \tag{15}$$

where the normalized rainflow matrix *RFMn* and the number of rain flow cycles N_{RF} are defined by:

$$RFMn=\frac{RFM\left(\sigma_{\min},\sigma_{\max}\right)}{N_{RF}} \qquad N_{RF}=\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}RFM\left(\sigma_{\min},\sigma_{\max}\right) \tag{16}$$

Note that *RFMn* is related to the 2D stress *PDF* by:

$$\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}RFMn\left(\sigma_{\min},\sigma_{\max}\right)\approx\int_{-\infty}^{\infty}\int_{0}^{\infty}p\left(\sigma_a,\sigma_0\right)d\sigma_a d\sigma_0=1 \tag{17}$$

The denominator of (15) is determined based on a choice of fatigue equations. The estimate using the alternating stress equation (2) (with no mean stress effects) is:

$$D_{ar}=\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}\frac{RFM\left(\sigma_{\min},\sigma_{\max}\right)}{N_{f\,ar}\left(\sigma_a,\sigma_0\right)}=N_{RF}\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}RFMn\left(\sigma_{\min},\sigma_{\max}\right)\left[\sigma_a/A\right]^{\beta} \tag{18}$$

The alternating stress $\sigma_a=\left(\sigma_{\max}-\sigma_{\min}\right)/2$ varies over the *RFM* and is calculated for each histogram bin. The estimate using the Morrow equation (3) is:

$$D_M=\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}\frac{RFM\left(\sigma_{\min},\sigma_{\max}\right)}{N_{f\,M}\left(\sigma_a,\sigma_0\right)}=N_{RF}\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}RFMn\left(\sigma_{\min},\sigma_{\max}\right)\left[\sigma_a\Big/A\left(1-\frac{\sigma_0}{\sigma_f'}\right)\right]^{\beta} \tag{19}$$

The mean stress $\sigma_0=\left(\sigma_{\max}+\sigma_{\min}\right)/2$ also varies over the *RFMn* and is calculated for each histogram bin. An alternate form of the Morrow equation (4) follows this form with $\tilde{\sigma}_{fB}$ substituted for σ_f'. The damage estimate using the Walker equation (5) is:

$$D_W=\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}\frac{RFM\left(\sigma_{\min},\sigma_{\max}\right)}{N_{f\,W}\left(\sigma_{\min},\sigma_{\max}\right)}=N_{RF}\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}RFMn\left(\sigma_{\min},\sigma_{\max}\right)\left[\frac{\sigma_{\max}}{A}\left(\frac{1-R}{2}\right)^{\gamma}\right]^{\beta} \tag{20}$$

The *R* value is calculated for each histogram bin using equation (6). The Smith-Watson-Topper equation (7) follows this form with the exponent $\gamma=\frac{1}{2}$. The damage estimate using the modified Walker equation and the equivalent stress equation (8) from [2] is:

$$D_5=\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}\frac{RFM\left(\sigma_{\min},\sigma_{\max}\right)}{N_{f5}\left(\sigma_{\min},\sigma_{\max}\right)}=N_{RF}\sum_{\sigma_{\min}}\sum_{\sigma_{\max}}\frac{RFMn\left(\sigma_{\min},\sigma_{\max}\right)}{10\,\hat{}\left[A_1+A_2\log\left(\sigma_{eq}-A_4\right)\right]} \tag{21}$$

A set of functions has been developed to estimate damage given a *RFM* and a choice of fatigue equations (18) through (21). These will be compared with experimental results in the next section.

APPLICATION OF TWO DIMENSIONAL STRESS & DAMAGE HISTOGRAMS

WPAFB Experimental Data

The results here are from a series of broadband base excited (input 20-500 Hz) clamped-clamped beam experiments, conducted at WPAFB [11]. The large deflection response of the beam resulted in significant nonlinear membrane stresses, resulting in skewed surface stress range *PDF*s. The results of a numerical analysis of this response data (88 s at 4096 Hz sample rate for each input) using the WAFO *RFM* functions and the proposed damage functions are presented below.

Figure 2 shows the normalized *RFM* and *RFD* matrices from the WPAFB data at the lowest test level of 0.5 g RMS input. The data was normalized using the sample mean $\mu_{\mathcal{S}}$ and standard deviation $\sigma_{\mathcal{S}}$ of the stress response by:

$$\mathcal{S}_n = \frac{\mathcal{S} - \mu_{\mathcal{S}}}{\sigma_{\mathcal{S}}} \tag{22}$$

The *RFM* shows a nearly symmetric distribution about the *R* = -1 diagonal. There are also many low amplitude cycles in the data from the wideband multi-modal response of the continuous beam.

(a) (b)

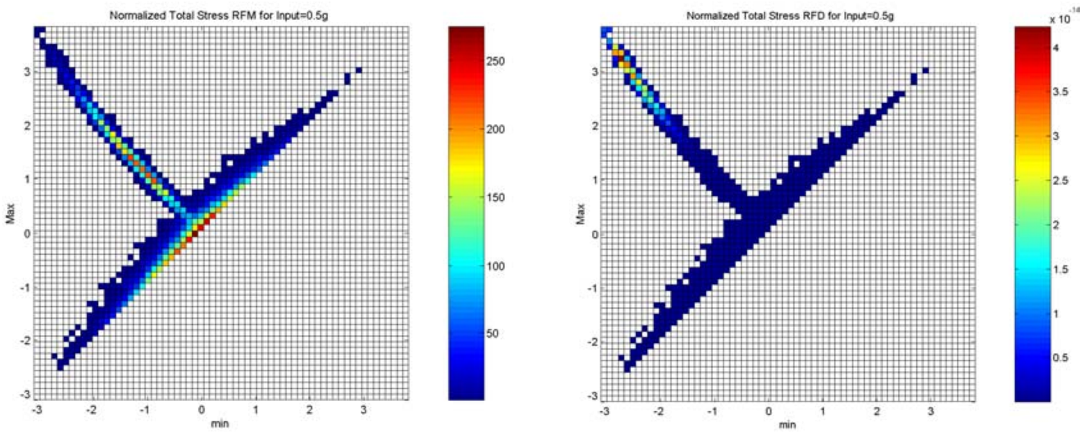

Figure 2 Normalized Total Stress (a) and Damage (b) Rain Flow Matrices: $\sigma_{\mathcal{S}}$ = 5.543 Mpa (0.804 ksi), $\mu_{\mathcal{S}}$ = 0.215 Mpa (0.031 ksi), E[Dm]= 6.29e-13, Input = 0.5 g

The *RFD* matrix was calculated using the Morrow equation (19) and SAE 1015 material properties from [1]. Note that most of the damage is a result of the large amplitude cycles that occur between ~ 2 and 4 sigma (see Figure 2(b)); the large quantity of low amplitude cycles cause very little damage. As the input level was increased, the stress response of the beam became much more nonlinear, resulting in skewed *RFM*s as shown in Figure 3(a) and Figure 4(a). As with the low level case, the majority of the damage occurs between ~ 2 and 4 sigma. A comparison of the damage calculated using different fatigue equations is shown using the 8 g input data in Figure 4(b) (Morrow equation (19)), Figure 5(a) (Walker equation (20)) and Figure 5(b) (alternating stress equation (18)). As expected, the alternating stress equation estimates the smallest damage (5.05e-06) while the Morrow and Walker estimate greater damage (7.38e-06 and 1.31e-05 respectively) due to the nonlinear tensile mean stresses.

(a) (b)

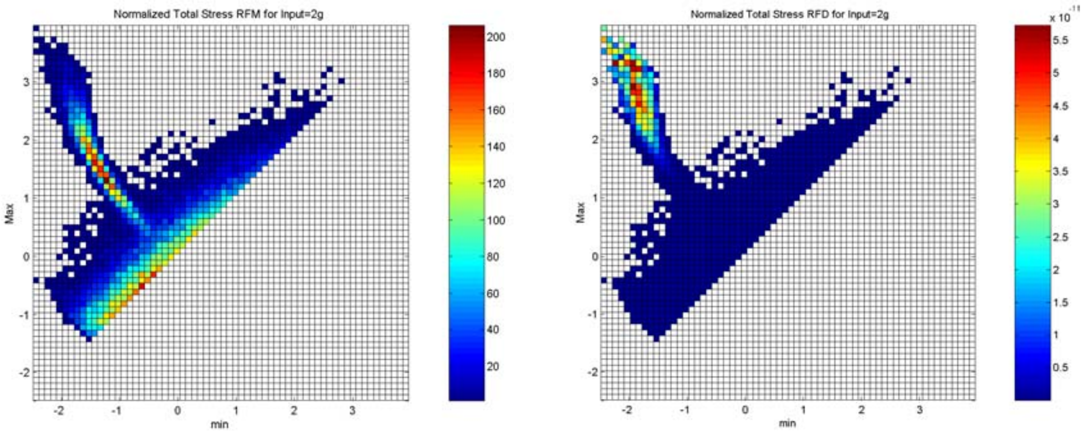

Figure 3 Normalized Total Stress (a) and Damage (b) Rain Flow Matrices: σ_{γ} *= 14.70 Mpa (2.13 ksi),* μ_{γ} *= 0.978 Mpa (0.142 ksi), E[Dm]= 2.77e-09, Input = 2 g*

(a) (b)

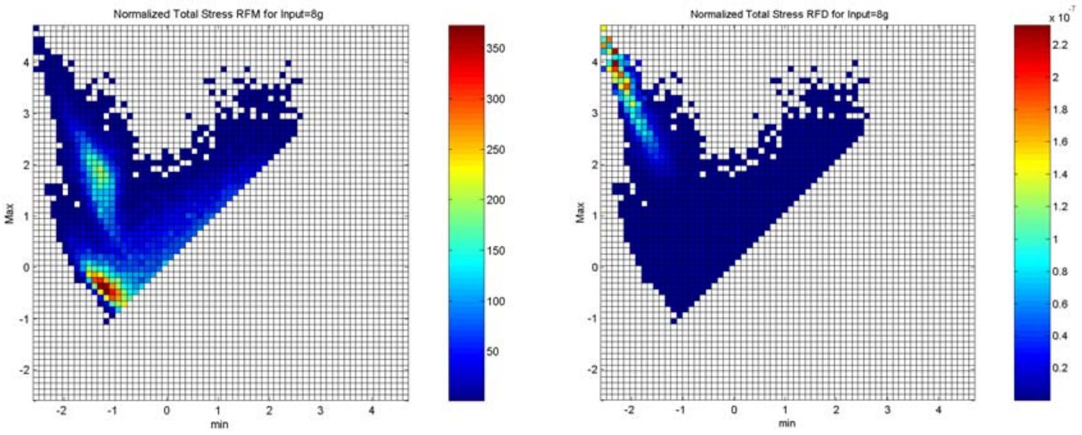

Figure 4 Normalized Total Stress (a) and Damage (b) Rain Flow Matrices: σ_{γ} *= 32.48 Mpa (4.71 ksi),* μ_{γ} *= 9.402 Mpa (1.36 ksi), E[Dm]= 7.38e-06, Input = 8 g*

SUMMARY

A method to include tensile mean stresses in a fatigue damage analysis for nonlinear random response has been developed based on several commonly accepted sinusoidal fatigue equations. Experimental nonlinear results were used to illustrate the usefulness of the proposed method. Further examples and theoretical discussion of the findings will be published in the future.

The primary author thanks WPAFB Structural Mechanics Branch for providing the experimental data and ITT Industries for their educational support at the ISVR.

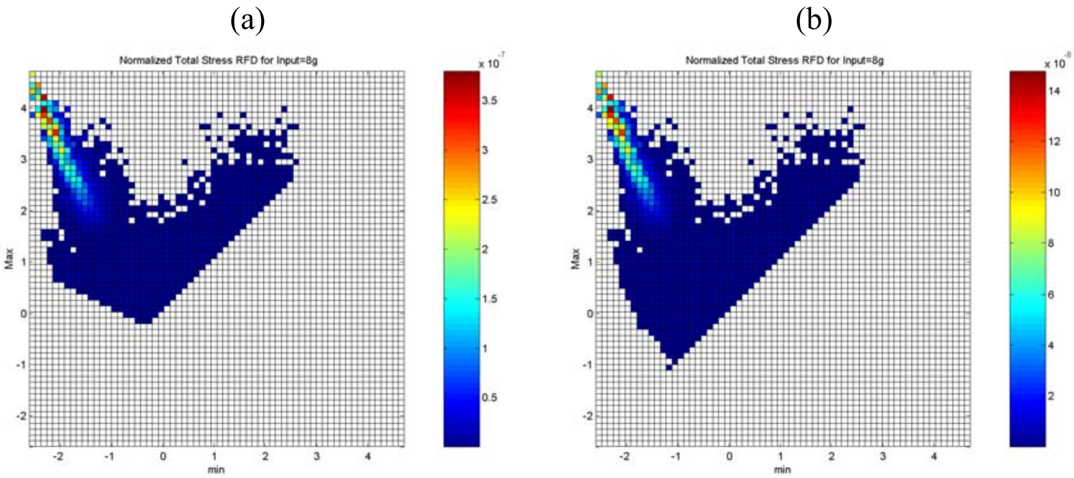

Figure 5 Normalized Damage Rain Flow Matrices using Walker Equation(a) and Alternating Stress Equation (b): E[Dw]= 1.31e-05, E[Da]= 5.05e-06, input = 8 g

REFERENCES

[1] N. E. Dowling, *Mean Stress Effects in Stress-Life and Strain-Life Fatigue*, SAE Paper # 2004-01-2227, in Proceedings of Fatigue 2004: Second SAE Brasil International Conference on Fatigue, São Paulo, Brasil, June 2004.

[2] DOT/FAA/AR-MMPDS-01, Metallic Materials Properties Development and Standardization (replacement to MIL-HDBK-5), January 2003.

[3] J. Morrow, *Fatigue Properties of Metals*, in Proceedings of Division 4 of the SAE Iron and Steel Technical Committee, Nov. 4, 1964.

[4] K. Walker, The Effect of Stress Ratio During Crack Propagation and Fatigue for 2024-T3 and 7075-T6 Aluminum, Effects of Environment and Complex Load History on Fatigue Life, ASTM STP 462, Am. Soc. for Testing and Materials, West Conshohocken, PA, 1970.

[5] K. N. Smith, P. Watson, and T. H. Topper, *A Stress-Strain Function for the Fatigue of Metals*, in ASTM Journal of Materials, Vol. 5, No. 4, December 1970.

[6] R. G. Lambert, *Analysis of Fatigue Under Random Vibration*, The Shock and Vibration Bulletin No 46, Naval Research Laboratory, Washington, DC, May 1976.

[7] K. A. Sweitzer, N. W. M. Bishop, V. L. Genberg, *Efficient Computation of Spectral Moments for Determination of Random Response Statistics*, in Proceedings of International Conference on Noise & Vibration Engineering, Leuven Belgium, September, 2004.

[8] M. Matsuishi, and T. Endo, *Fatigue of metals subjected to varying stress*, Japan Society of Mechanical Engineers, Fukuoka, Japan, March, 1968.

[9] T. Dirlik, *Application of Computers in Fatigue Analysis,* University of Warwick Thesis, 1985.

[10] WAFO Toolbox - Wave Analysis for Fatigue and Oceanography, Version 2.1.0, 07-Apr-2004, Lund Inst. of Tech., Lund University, Sweden. http://www.maths.lth.se/matstat/wafo/index.html

[11] R. W. Gordon, J. J. Hollkamp, and S. M. Spottswood, *Nonlinear Response of a Clamped - Clamped Beam to Random Base Excitation*, in Proceedings of VIII International Conference on Recent Advances in Structural Dynamics Southampton, UK, July 2003.