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An FIO Calculus for Marine Seismic Imaging: Folds and Cross Caps

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An FIO calculus for marine seismic imaging: folds and cross caps

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Abstract

We consider a linearized inverse problem arising in offshore seismic imaging. Following Nolan and Symes [27], one wishes to determine a singular perturbation of a smooth background sound speed in the Earth from measurements made at the surface resulting from various seismic experiments; the overdetermined data set considered here corresponds to marine seismic exploration. In the presence of at most fold caustics for the background, we identify the geometry of the canonical relation underlying the linearized forward scattering operator F, which is a Fourier integral operator. We then establish a composition calculus for general FIOs associated with canonical relations having the same structure, which we call *folded cross caps*, sufficient for identifying the normal operator F^*F. In contrast to the case of a single source experiment, treated by Nolan [24] and Felea [5] and for which the normal operator belongs to a similar class, here the resulting artifact is $\frac{1}{2}$ order smoother than the main pseudodifferential part of F^*F.

1 Introduction

This article deals with a linearized inverse scattering problem considered by Nolan and Symes [23]. Acoustic waves are generated at the surface of the earth, scatter off heterogeneities in the subsurface and return to the surface. The full inverse problem would use the pressure field at the surface to reconstruct an image of the subsurface. We instead consider the linearized operator F which maps singular perturbations of a smooth background sound speed in the subsurface, assumed known, to perturbations of the resulting pressure field at the surface. The goal is to left-invert F; standard techniques suggest studying left invertibility of the normal operator $N = F^*F$. To start, we make two assumptions: (i) no single ray connects a source to a receiver; and (ii) no ray originating in the subsurface grazes the surface. Under these assumptions, in the case of a *single* source and receivers ranging over an open subset of the surface, $\{x_3 = 0\}$, Rakesh [28] showed that F is a Fourier integral operator (FIO).

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Beylkin [1] showed that if caustics do not occur for the background soundspeed, F^*F is a pseudodifferential operator (ΨDO).

For more general data acquisition geometries, the canonical relation of F depends on the sets of sources and receivers. Nolan and Symes [27] proved that, if both sources and receivers vary over open, bounded subsets Σ_r and Σ_s of the surface, then under the traveltime injectivity condition (TIC), generalizing the no-caustic assumption, F^*F is still a ΨDO. The same result was stated by ten Kroode, Smit and Verdel [17] and their proof was completed by Stolk [29], who also relaxed the TIC condition in low dimensions.

For applications in three spatial variables, an important problem is to understand the nature of F and F^*F for the *marine* data acquisition geometry [27], where measurements are made on the codimension one submanifold $\Sigma_{r,s} = \{(r_1, r_2; s_1, s_2) \in \Sigma_r \times \Sigma_s : s_2 = r_2\}$. This arises as follows: a seismic vessel trails behind it both an acoustic source and recording instruments. The point source consists of an airgun which sends acoustic waves through the ocean to the subsurface. Reflections occur when the sound waves encounter singularities in the material of the subsurface. The reflected rays are received by a linear array of hydrophones towed behind the vessel. The vessel then makes repeated passes along parallel lines (say, parallel to x_1 axis).

The purpose of this paper is to consider the marine geometry under the assumption that only the simplest, most prevalent type of caustics, namely *fold caustics*, occur for the background soundspeed. Fold caustics are initially defined as follows: A ray departing from a source s in the direction α reaches at time t a point denoted $x(t,\alpha)$ in the subsurface. If there is a source s, such that the spatial projection map $(t,\alpha) \to x(t,\alpha)$ has a fold singularity and only singularities of this type, then we say that the background soundspeed exhibits a fold caustic. By the stability of folds, the maps $(t,\alpha) \to x(t,\alpha)$ also have at most fold singularities for all nearby sources s'. However, it seems that the natural notion of a fold caustic in the context of the overdetermined marine data set considered here is the requirement that the analogous spatial projection be a *submersion with folds*, which is the simplest singularity in the non-equidimensional setting. This will be elaborated on in §2 and §4.

We now introduce the linearized scattering operator F considered in [27],[17]. The model for the scattered waves is given by the wave equation:

$$\begin{aligned} \frac{1}{c^2(x)} \frac{\partial^2 p}{\partial t^2}(x,t) - \triangle p(x,t) &= \delta(t)\delta(x-s) \\ p(x,t) &= 0, \quad t < 0, \end{aligned} \tag{1}$$

where $x \in Y = \mathbb{R}^3_+ = \{x \in \mathbb{R}^3, x_3 \geq 0\}$ represents the Earth, $p(x,t)$ is the pressure field resulting from a pulse at the source s and $c(x)$ is the velocity field. The linearization consists in assuming c to be of the form $c = c_0 + \delta c$, where c_0 is a smooth known background field. The associated pressure field p_0 is also assumed known. The linearization of (1) then becomes

$$\begin{aligned} \frac{1}{c_0^2(x)}\frac{\partial^2 \delta p}{\partial t^2}(x,t) - \triangle \delta p(x,t) &= \frac{2\delta c(x)}{c_0^3(x)}\frac{\partial^2 p_0}{\partial t^2} \\ \delta p &= 0, \quad t<0, \end{aligned} \tag{2}$$

where $p = p_0 + \delta p$. Now, for a given data acquisition submanifold $\Sigma_{r,s} \subset \partial Y \times \partial Y$ and appropriate time interval $(0,T)$, we define the linearized scattering operator $F : \delta c \to \delta p|_{\Sigma_{r,s}\times(0,T)}$. The assumption (1) ensures that F is an FIO ([14],[17],[28],[27]) and the second assumption ensures that the composition F^*F makes sense.

In the case of the single source model, with only fold caustics appearing, Nolan [24] showed that F is an FIO associated to a folding canonical relation in the sense of [20] (also called a two-sided fold), and stated that the Schwartz kernel of the operator F^*F belongs to a class of distributions associated to two cleanly intersecting Lagrangians in $(T^*Y \setminus 0) \times (T^*Y \setminus 0)$. This was fully proved in [5]. The corresponding canonical relations are the diagonal Δ and a folding canonical relation, different from the original one and lying in $T^*Y \times T^*Y$.

In this article we show that, for the the marine geometry, the linearization F is an FIO associated to what we call a *folded cross cap* canonical relation. We then prove that the Schwartz kernel of F^*F belongs to a class of distributions similar to that in the single source geometry, but with the order of the non-pseudodifferential part of F^*F being $\frac{1}{2}$ lower than in the case of a single source. This means that *the artifacts arising in the seismic imaging from the presence of the fold caustics are $\frac{1}{2}$ derivative smoother for the marine geometry than for the single source geometry.*

Composition of FIOs under other singular geometries arising in integral geometry and inverse problems has been previously studied in, e.g., [12],[8],[10], [11],[24] and [5].

The article is organized as follows. In §2 we review some C^∞ singularity theory and define the submersion with folds and cross cap singularities. §3 is a review of the distribution classes associated to two cleanly intersecting Lagrangians, $I^{p,l}(\Lambda_0, \Lambda_1)$ and the operators which have these as their Schwartz kernels. In §4 we show that submersions with folds and cross caps appear microlocally in the marine geometry in the presence of the fold caustics, and we formulate a general class of canonical relations exhibiting these singularities. §5 is dedicated to analyzing a model folded cross cap canonical relation, C_0, in $T^*\mathbb{R}^n \times T^*\mathbb{R}^{n-1}$. In this section, we establish the composition calculus for F^*F and show that $F^*F \in I^{p,l}(\Delta, \tilde{C}_0)$ where $\tilde{C}_0$ is a folding canonical relation. Finally, §6 provides the extension of this to the general class of folded cross caps. We find a weak normal form for any folded cross cap canonical relation $C \subset T^*X \times T^*Y$ which allows us to show that $F^*F \in I^{p,l}(\Delta, \tilde{C})$, with $\tilde{C}$ a folding canonical relation in $T^*Y \times T^*Y$.

We would like to thank Cliff Nolan for explaining the calculations in [24] to us, including helpful discussions at the Institute for Mathematics and its Applications, Minneapolis, in October, 2005.

2 Fourier integral operators and singularity classes

Let X and Y be manifolds and $I^m(X,Y;C)$ denote the class of m-th order Fourier integral operators (FIOs), $F:\mathcal{E}'(Y)\to\mathcal{D}'(X)$, associated to a canonical relation $C\subset(T^*X\setminus 0)\times(T^*Y\setminus 0)$. We will focus on the composition calculus of two FIOs. Let $C_1\subset(T^*X\setminus 0)\times(T^*Y\setminus 0)$ and $C_2\subset(T^*Y\setminus 0)\times(T^*Z\setminus 0)$ be two canonical relations and $F_1\in I^{m_1}(X,Y;C_1)$ and $F_2\in I^{m_2}(Y,Z;C_2)$. If $C_1\times C_2$ intersects $T^*X\times\Delta_{T^*Y}\times T^*Z$ transversally, then Hörmander [15] proved that $F_1\circ F_2\in I^{m_1+m_2}(X,Z;C_1\circ C_2)$ where $C_1\circ C_2$ is the composition of C_1 and C_2 as relations in $T^*X\times T^*Y$ and $T^*Y\times T^*Z$. Duistermaat and Guillemin [3] and Weinstein [30] extended this calculus to the case of clean intersection and showed that if $C_1\times C_2$ and $T^*X\times\Delta_{T^*Y}\times T^*Z$ intersect cleanly with excess e then $A\circ B\in I^{m_1+m_2+e/2}(X,Z;C_1\circ C_2)$. In each of these cases, $C_1\circ C_2$ is again a smooth canonical relation. However, in many interesting problems, these assumptions fail, and it is important to analyze the composition and understand the resulting operators. It turns out that the geometry of each canonical relation and the structure of their projections play an important role.

Let π_L and π_R be the projections, to the left and right, from C to T^*X and T^*Y, respectively. If either one is a local diffeomorphism, so is the other one and then C is a local canonical graph. In the case of two canonical relations, if at least one of C_1 and C_2 is a local canonical graph, then $C_1\times C_2$ intersects $T^*X\times\Delta_{T^*Y}\times T^*Z$ transversally and the transverse composition calculus applies.

Now consider the case when the projections are no longer local diffeomorphisms. When one of the projections is singular, i.e., when the rank of its differential is nonmaximal, then the other one is, too, and C is called a *singular canonical relation.* (Note: C is still assumed to be smooth.)

Although corank$(d\pi_L)$=corank$(d\pi_R)$ at all points, the two projections, π_R and π_L, may have quite different singularities. The singularities considered in this article are folds, submersion with folds and cross caps, which we now briefly describe.

Let f be a smooth function $f:V\to W$, $\dim V=\dim W=N$, and $\mathcal{S}:=\mathcal{S}(f)=\{x\in V:\det(df(x))=0\}$.

Definition 2.1. *f has a (Whitney) fold singularity along $\mathcal{S}$ if $d(det(df))\neq 0$ on $\mathcal{S}$, so that $\mathcal{S}$ is a smooth hypersurface, df drops rank by 1 there, and Ker $df(x)$ intersects $T_x\mathcal{S}$ transversally for every $x\in\mathcal{S}$.*

Any map which has a fold singularity can be put into a local normal form: $f(x_1,x_2,\dots,x_N)=(x_1,x_2,\dots,x_{N-1},x_N^2)$ with respect to suitable local coordinates in the domain and codomain [6].

Whitney folds are the singularities denoted by $S_{1,0}$ (in the Thom theory [6]) and by $\Sigma_{1,0}$ (in the Boardman-Morin theory [22, 23]) in the equidimensional case. The non-equidimensional versions of Whitney folds are submersions with folds and cross caps. We note the difference in notation: when $\dim V\leq\dim W$, the singularity classes $S_{r,0}=\Sigma_{r,0}$, while, if $\dim V>\dim W$, then $S_{r,0}=\Sigma_{r+k,0}$, where $k=\dim V-\dim W$.

Let f be a smooth function $f : V \to W$, $\dim V = N$, $\dim W = M$, $N > M$.

Definition 2.2. The map f is a *submersion with folds* if the only singularities of f are of type $S_{1,0}$, i.e., of type $\Sigma_{N-M+1,0}$.

One checks that f is a submersion with folds as follows. At points where rank $df \geq M - 1$, by [23] we can choose suitable adapted local coordinates on V and W such that f has the form: $f(x_1, x_2, \ldots, x_{M-1}, x_M, \ldots, x_N) = (x_1, x_2, \ldots x_{M-1}, f_1(x))$. The set $\mathcal{S}_1(f)$ where f drops rank by 1 is described by $\mathcal{S}_1(f) = \{x : \frac{\partial f_1}{\partial x_i} = 0,\ M \leq i \leq N\}$. Then f is a submersion with folds if $\mathcal{S}_1(f)$ is a smooth submanifold, i.e., $\left\{d\left(\frac{\partial f}{\partial x_i}\right) : \ M \leq i \leq N)\right\}$, is linearly independent, and if the $(N - M + 1)$-dimensional kernel of df is transversal to the tangent space to $\mathcal{S}_1(f)$ in TV. These conditions can be combined [22] into

$$\det\left[\frac{\partial^2 f_1}{\partial x_i \partial x_j}\right]_{M \leq i,j \leq N} \neq 0. \tag{3}$$

and this is independent of the choice of adapted coordinates.

There are a finite number of local normal forms for a submersion with folds, determined by the signature of the Hessian of f [6]:

$$f(x_1, x_2, \ldots, x_N) = (x_1, x_2, \ldots, x_{M-1}, x_M^2 \pm x_{M+1}^2 \pm \cdots \pm x_N^2).$$

In the case relevant here, $N = M + 1$ and the last entry is a quadratic form in two variables, which is either sign definite or indefinite; we refer to these two possibilities as *elliptic* and *hyperbolic* respectively.

We now define the third singularity class of interest; like the class of submersions with folds, it is stable under small C^2 perturbations. It is now assumed that $\dim V = N$, $\dim W = M$ with $N < M$, and $g : V \to W$ is a smooth function.

Definition 2.3. We say that g is a *cross cap* if the only singularities of g are of type $S_{1,0}$, i.e., of type $\Sigma_{1,0}$.

To identify a cross cap, we use the description of [22]. At a point where dg has rank $\geq N - 1$, we can find suitable adapted coordinates such that $g(x_1, x_2, \ldots, x_{N-1}, x_N) = (x_1, x_2, \ldots, x_{N-1}, g_1, g_2, \ldots g_q)$, where $q = M - N + 1$. The set $\mathcal{S}_1(g)$ where g drops rank by 1 is given by $\mathcal{S}_1(g) = \{x : \frac{\partial g_i}{\partial x_N} = 0, \quad 1 \leq i \leq q\}$. Assume that there is an i_0, such that $\frac{\partial^2 g_{i_0}}{\partial x_N^2}(0) \neq 0$. Then, g has a cross cap singularity near 0 if the map $\chi : \mathbb{R}^N \to \mathbb{R}^q$ given by $\chi(x_1, x_2, \ldots x_N) = (\frac{\partial g_1}{\partial x_N}, \frac{\partial g_2}{\partial x_N}, \ldots, \frac{\partial g_q}{\partial x_N})$ satisfies rank $d\chi(0) = q$. (Notice that this forces $N \geq q$, i.e., $M \leq 2N - 1$.) These conditions can be reformulated as: (i) $\mathcal{S}_1(g)$ is smooth and of codimension q; (ii) the $N \times N$ minors of dg generate the ideal of $\mathcal{S}_1(g)$; and (iii) Ker $(dg) \cap T\mathcal{S}_1(g) = (0)$.

As for folds, there is a local normal form for cross caps, due to Whitney [31] and Morin [22]:

$$g(x_1, x_2, \dots, x_N) = (x_1, x_2, \dots, x_{N-1}, x_1 x_N, \dots x_{M-N} x_N, x_N^2). \tag{4}$$

3 Distributions and operators associated to two cleanly intersecting Lagrangians

Classes of distributions associated to two cleanly intersecting Lagrangian manifolds were introduced by Melrose and Uhlmann [21] and Guillemin and Uhlmann[13]. We briefly review their definitions and properties.

First, one proves that any two pairs of cleanly intersecting Lagrangian submanifolds are (micro)locally equivalent. Thus, one can consider the model pair $(\tilde\Lambda_0, \tilde\Lambda_1)$ where $\tilde\Lambda_0 = T_0^*\mathbb{R}^n = \{(x,\xi) : x = 0\}$ and $\tilde\Lambda_1 = N^*\{x'' = 0\} = \{(x,\xi) : x'' = \xi' = 0\}$ with $x' = (x_1, x_2, \dots, x_k)$, and $x'' = (x_{k+1}, x_{k+2}, \dots, x_n)$. One defines a class of distributions given by oscillatory integrals whose amplitudes are called *product-type* symbols. Let $z = (x, s)$ be coordinates in $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^k$ and (ξ, σ) the dual coordinates.

Definition 3.1. $S^{p,l}(m, n, k)$ is the set of all functions $a(z, \xi, \sigma) \in C^\infty(\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^k)$ such that for every $K \subset\subset \mathbb{R}^m$ and every $\alpha \in \mathbb{Z}_+^n, \beta \in \mathbb{Z}_+^k, \gamma \in \mathbb{Z}_+^m$ there is a $c_{\alpha\beta\gamma K} < \infty$ such that

$$|\partial_\xi^\alpha \partial_\sigma^\beta \partial_z^\gamma a(z, \xi, \sigma)| \le c_{\alpha\beta\gamma K}(1 + |\xi|)^{p-|\alpha|}(1 + |\sigma|)^{l-|\beta|}, \forall (z, \xi, \tau) \in K \times \mathbb{R}^n \times \mathbb{R}^k. \tag{5}$$

Definition 3.2. [13] Let $I^{p,l}(\mathbb{R}^n; \tilde\Lambda_0, \tilde\Lambda_1)$ be the set of all distributions u such that $u = u_1 + u_2$ with $u_1 \in C_0^\infty$ and

$$u_2(x) = \int e^{i((x'-s)\cdot\xi' + x''\cdot\xi'' + s\cdot\sigma)} a((x, s), \xi, \sigma) d\xi d\sigma ds$$

with $a \in S^{p',l'}(m, n, k)$ where $p' = p - \frac{n}{4} + \frac{k}{2}$ and $l' = l - \frac{k}{2}$.

At this point, if X is a manifold of dimension n, we can define the class $I^{p,l}(X; \Lambda_0, \Lambda_1)$ for any pair of Lagrangians in $T^*X \setminus 0$ cleanly intersecting in codimension k. The oscillatory integrals we use are oscillatory integrals in sense of Hörmander [15, p.88].

Definition 3.3. [13] $u \in I^{p,l}(X; \Lambda_0, \Lambda_1)$ if $u = u_1 + u_2 + \sum v_i$ where $u_1 \in I^{p+l}(\Lambda_0 \setminus \Lambda_1)$, $u_2 \in I^p(\Lambda_1 \setminus \Lambda_0)$, the sum $\sum v_i$ is locally finite and $v_i = F w_i$ where F is a zero order FIO associated to χ^{-1} where $\chi : T^*X \setminus 0 \to T^*\mathbb{R}^n \setminus 0$ is a canonical transformation such that $\chi(\Lambda_j) \subseteq \tilde\Lambda_j, j = 0, 1$, microlocally, and $w_i \in I^{p,l}(\mathbb{R}^n; \tilde\Lambda_0, \tilde\Lambda_1)$.

We say that a distribution $u \in I^r(X; \Lambda_0 \setminus \Lambda_1)$ if, microlocally away from Λ_1, $u \in I^r(X; \Lambda_0)$, the standard Hörmander class of Fourier integral distributions on X associated with Λ_0.

Remark 3.4. [13] If $u \in I^{p,l}(X; \Lambda_0, \Lambda_1)$ then $u \in I^{p+l}(X; \Lambda_0 \setminus \Lambda_1)$ and also $u \in I^p(X; \Lambda_1 \setminus \Lambda_0)$.

We will also use the notion of nondegenerate phase functions which parametrize two cleanly intersecting Lagrangians, introduced by Mendoza [15]. Let $\lambda_0 \in \Lambda_0 \cap \Lambda_1$ and $\Gamma \subset X \times \mathbb{R}^k \times \left(\mathbb{R}^N \setminus 0\right)$ an open, conic set.

Definition 3.5. [18] A phase function $\phi(x, s, \theta)$ defined on Γ is a *parametrization* for the pair (Λ_0, Λ_1) if

i) $\phi_0(x, \theta) := \phi(x, 0, \theta)$ where ϕ_0 is a nondegenerate phase function parametrizing Λ_0 near λ_0*;* and

ii) $\phi_1(x, (\theta, \sigma)) := \phi(x, \frac{\sigma}{|\theta|}, \theta)$ is a nondegenerate phase function parametrizing Λ_1 near λ_0.

We also refer to $\phi_1(x; \theta; \sigma)$ as a *multi-phase function* for (Λ_0, Λ_1).

For simplicity, we now focus on the case of codimension 1 intersection relevant here, i.e., $k = 1$. Let us consider the following example: If $\tilde{\Lambda}_0 = N^*\{x' = 0\}$ and $\tilde{\Lambda}_1 = N^*\{x = 0\}$, with $x' = (x_2, x_3, \ldots, x_n)$, then $\phi(x, s, \theta') = x' \cdot \theta' + x_1 s \mid \theta' \mid$ is a parametrization for $(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ since $\phi_0(x, \theta') := \phi(x, 0, \theta') = x' \cdot \theta'$, which is a parametrization for $\tilde{\Lambda}_0$, and $\phi_1(x, (\theta', \sigma)) := \phi(x, \frac{\sigma}{|\theta|}, \theta) = x' \cdot \theta' + x_1 \sigma$ is a parametrization for $\tilde{\Lambda}_1$.

Proposition 3.6. [18] Let p_1 be a homogenous function of degree 1 such that $p_1(\lambda_0) = 0$ and H_{p_1} (the Hamiltonian vector field associated to p_1) is not tangent to Λ_0. If ϕ_0 is a parametrization for Λ_0 a*nd* Λ_1 is the flow out from $\Lambda_0 \cap \{p_1 = 0\}$ by H_{p_1}, then there is a parametrization ϕ for (Λ_0, Λ_1) which can be found by solving the initial value problem $\phi(x, 0, \theta) = \phi_0$, $\frac{\partial \phi}{\partial s}(x, s, \theta) = p_1(x, d_x \phi)$.

Remark 3.7. [15] If $\phi(x; \theta; \sigma)$ is a multi-function for (Λ_0, Λ_1) near $\lambda_0 \in \Lambda_0 \cap \Lambda_1$, then, for $u \in \mathcal{D}'(X)$ with $WF(u)$ contained in a small conic neighborhood of λ_0, then $u \in I^{p,l}(X; \Lambda_0, \Lambda_1)$ iff

$$u(x) = \int e^{i\phi(x;\theta;\sigma)} a(x; \theta; \sigma) d\sigma d\theta,$$

where $a \in S^{\tilde{p}, \tilde{l}}(X \times (\mathbb{R}^N \setminus 0) \times \mathbb{R})$, the space of *symbol-valued symbols of order* $\tilde{p}, \tilde{l}$, defined by: for all $K \subset\subset X, \alpha \in \mathbb{Z}_+^N, \beta \in \mathbb{Z}_+, \gamma \in \mathbb{Z}_+^n$,

$$|\partial_\theta^\alpha \partial_\sigma^\beta \partial_x^\gamma a(x; \theta; \sigma)| \leq c_{\alpha\beta\gamma K}(1 + |\theta| + |\sigma|)^{\tilde{p} - |\alpha|}(1 + |\sigma|)^{\tilde{l} - \beta}, \tag{6}$$

with $p = \tilde{p} + \tilde{l} + \frac{N+1}{2} - \frac{n}{4}, l = -\tilde{l} - \frac{1}{2}$.

Finally, we define the classes of generalized (or *paired Lagrangian*) Fourier integral operators, to one of which we will show the normal operator F^*F belongs. Recall that a *canonical relation* $C \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ is a smooth, conic

submanifold such that $C' := \{(x, y; \xi, \eta) : (x, \xi; y, -\eta) \in C\}$ is a Lagrangian submanifold of $\left(T^*(X \times Y), \omega_{T^*(X\times Y)}\right)$, i.e, C is a Lagrangian with respect to $\omega_{T^*X} - \omega_{T^*Y}$.

Definition 3.8. If $C_0, C_1 \subset (T^*X \setminus 0) \times (T^*Y \setminus 0)$ are smooth, conic canonical relations intersecting cleanly, then $I^{p,l}(X, Y; C_0, C_1)$ denotes the set operators $F : \mathcal{E}'(Y) \longrightarrow \mathcal{D}'(X)$ whose Schwartz kernels are in $I^{p,l}(X \times Y; C_0', C_1')$.

4 Fold caustics in the marine geometry

In three spatial dimensions, let s be a fixed source on the surface, $\partial Y = \{x_3 = 0\}$; $H(x, \xi) = \frac{1}{2}(c_0(x)^{-2} - |\xi|^2)$ the Hamiltonian associated to the smooth background sound speed $c_0(x)$ in (2); and Λ_s the image of $T_s^*\mathbb{R}^3 \setminus 0$ under the bicharacteristic flow associated to H, which is a Lagrangian submanifold of $T^*\mathbb{R}^3 \setminus 0$. The assumption of a (point) fold caustic means that the only singularities of the spatial projection $\pi_Y : \Lambda_s \to Y$ are folds. We make use of the description of Λ_s in Nolan[24]. It can be parametrized by t_{inc}, the time travelled by the incident ray, and the takeoff direction $(p_1, p_2, p_3) \in \mathbb{S}^2$. We can change these coordinates to (x_1, x_2, p_3) [24]. Hence on Λ_s, $x_3 = f(x_1, x_2, p_3)$ and $(p_1, p_2) = (g_1(x_1, x_2, p_3), g_2(x_1, x_2, p_3))$. In this new setting, $\det d\pi_S = \frac{\partial f}{\partial p_3}(x_1, x_2, p_3)$ and fold caustics occur where $\frac{\partial f}{\partial p_3} = 0$ and $\frac{\partial^2 f}{\partial p_3^2} \neq 0$.

In the marine geometry, the source $s = (s_1, s_2, 0)$ is subject to the restriction $s_2 = r_2$, but we now consider s_1 as an independent coordinate. Fix an $s_2 \in \mathbb{R}$, i.e., consider a single pass of the vessel, and let Λ_{s_2} be the union of the flowouts $\left\{\Lambda_{(s_1,s_2,0)} : s_1 \in \mathbb{R}\right\}$, so that $\Lambda_{s_2} \subset T^*\mathbb{R}^3 \setminus 0$ is an involutive submanifold. We say that fold caustics (and no worse) appear for the background sound speed if, considering s_1 as a variable, the spatial projection $\pi_Y : \Lambda_{s_2} \longrightarrow \mathbb{R}^3$ is a submersion with folds. By the structural stability of submersions with folds, this condition will then hold for all s_2' close to s_2. The presence of fold caustics may be characterized as follows. The variables x_3 and (p_1, p_2) are functions of the other four: $x_3 = f(x_1, x_2, s_1, p_3)$, $(p_1, p_2) = (g_1(x_1, x_2, s_1, p_3), g_2(x_1, x_2, s_1, p_3))$. The differential $d\pi_Y$ then becomes:

$$d\pi_Y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial s_1} & \frac{\partial f}{\partial p_3} \end{pmatrix}.$$

We have

$$\text{rank } \; d\pi_Y = \begin{cases} 2, & \text{if} \quad \frac{\partial f}{\partial s_1} = \frac{\partial f}{\partial p_3} = 0 \\ 3, & \text{if} \quad \frac{\partial f}{\partial s_1} \neq 0 \text{ or } \frac{\partial f}{\partial p_3} \neq 0. \end{cases}$$

Suppressing s_2, let $\mathcal{S}_1^\Lambda := \mathcal{S}_1(\pi_Y) = \{\frac{\partial f}{\partial p_3} = \frac{\partial f}{\partial s_1} = 0\}$ be the critical set of π_Y, where rank $d\pi_Y$ drops by 1. At points of $\mathcal{S}_1^\Lambda$, Ker $d\pi_Y$ is two-dimensional

and spanned by $\{(0,0,\delta s_1,\delta p_3)\}$. The tangent space to $\mathcal{S}_1^\Lambda$ is

$$T\mathcal{S}_1^\Lambda = \text{ Ker } (d_{x_1,x_2,p_3,s_1}(\frac{\partial f}{\partial p_3})) \cap \text{ Ker } (d_{x_1,x_2,p_3,s_1}(\frac{\partial f}{\partial s_1})). \tag{7}$$

where

$$d_{x_1,x_2,p_3,s_1}(\frac{\partial f}{\partial p_3}) = (\frac{\partial^2 f}{\partial x_1 \partial p_3}, \frac{\partial^2 f}{\partial x_2 \partial p_3}, \frac{\partial^2 f}{\partial p_3^2}, \frac{\partial^2 f}{\partial s_1 \partial p_3}) \tag{8}$$

and

$$d_{x_1,x_2,p_3,s_1}(\frac{\partial f}{\partial s_1}) = (\frac{\partial^2 f}{\partial x_1 \partial s_1}, \frac{\partial^2 f}{\partial x_2 \partial s_1}, \frac{\partial^2 f}{\partial p_3 \partial s_1}, \frac{\partial^2 f}{\partial s_1^2}). \tag{9}$$

Then, π_Y is a submersion with folds if

$$\begin{gathered}\mathcal{S}_1^\Lambda \text{ is smooth, i.e., the gradients in (8) and (9) are} \\ \text{linearly independent, and } T\mathcal{S}_1^\Lambda \text{ is transversal to } \text{Ker } d\pi_S,\end{gathered} \tag{10}$$

i.e., if

$$\begin{vmatrix} \frac{\partial^2 f}{\partial p_3^2} & \frac{\partial^2 f}{\partial s_1 \partial p_3} \\ \frac{\partial^2 f}{\partial p_3 \partial s_1} & \frac{\partial^2 f}{\partial s_1^2} \end{vmatrix} \neq 0. \tag{11}$$

Next, we parametrize the canonical relation C of F in terms of s_1, x_1, x_2 and p_3 ; $(\alpha_1, \alpha_2, \sqrt{1-|\alpha|^2})$, the take off direction of the reflected ray, writing $\alpha = (\alpha_1, \alpha_2)$; and τ, the variable dual to time. Following [24], the canonical relation $C \subset T^*(\Sigma_{r,s} \times (0,T)) \times T^*\mathbb{R}^3_+$ is parametrized as

$$\begin{aligned} C = \big\{(\quad & s_1 \quad, r_1(\cdot), r_2(\cdot), t_{inc}(\cdot) + t_{ref}(\cdot), \sigma(\cdot), \rho_1(\cdot), \rho_2(\cdot), \tau; \\ & x_1 \quad, x_2, f(\cdot), -\tau(c_0^{-1}(\cdot)\alpha_1 + g_1(\cdot)), -\tau(c_0^{-1}(\cdot)\alpha_2 + g_2(\cdot)), \\ & \qquad -\tau(c_0^{-1}(\cdot), \alpha)\sqrt{1-|\alpha|^2} + p_3)\big\} \end{aligned}$$

where

$$\begin{aligned} f(\cdot) &= f(x_1, x_2, s_1, p_3); \\ r_j(\cdot) &= r_j(x_1, x_2, f(x_1, x_2, s_1, p_3), j = 1, 2; \\ t_{inc}(\cdot) &= t_{inc}(x_1, x_2, p_3); \\ t_{ref}(\cdot) &= t_{ref}(x_1, x_2, f(x_1, x_2, s_1, p_3); \\ \sigma(\cdot) &= \sigma(x_1, x_2, f(x_1, x_2, s_1, p_3); \\ \rho_j(\cdot) &= \rho_j(x_1, x_2, f(x_1, x_2, s_1, p_3), \alpha), j = 1, 2; \\ g_j(\cdot) &= g_j(x_1, x_2, s_1, p_3), j = 1, 2; \text{and} \\ c_0^{-1}(\cdot) &= c_0^{-1}(x_1, x_2, f(\cdot)). \end{aligned}$$

It was proven in [27] that $F : \mathcal{E}'\left(\mathbb{R}^3_+\right) \longrightarrow \mathcal{D}'\left(\Sigma_{r.s} \times (0,T)\right)$ is a Fourier integral operator, $F \in I^{\frac{3}{4}}(\Sigma_{r,s} \times (0,T), \mathbb{R}^3_+; C)$; we now show that the presence of caustics of fold type (and no worse) imposes certain conditions on C, namely that $\pi_R : C \to T^*\mathbb{R}^3 \setminus 0$ is a submersion with folds and $\pi_L : C \to T^*(\Sigma_{r,s} \times (0,T)) \setminus 0$ is a cross cap. In fact, with respect with the coordinates above, $\pi_R : \mathbb{R}^7 \to \mathbb{R}^6$ is given by

$$
\begin{aligned}
\pi_R(x_1, x_2, p_3, s_1, \alpha_1, \alpha_2, \tau) = \big(x_1, x_2, f(\cdot); \quad & - \quad \tau(c_0^{-1}(\cdot)\alpha_1 + g_1(\cdot)), \\
& - \quad \tau(c_0^{-1}(\cdot)\alpha_2 + g_2(\cdot)), \\
& - \quad \tau(c_0^{-1}(\cdot)\sqrt{1-|\alpha|^2} + p_3)\big).
\end{aligned}
$$

Thus,

$$
d\pi_R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial p_3} & \frac{\partial f}{\partial s_1} & 0 & 0 & 0 \\
A_1 & A_2 & A_3 & A_4 & -\tau c_0^{-1} & 0 & -(c_0^{-1}\alpha_1 + g_1) \\
B_1 & B_2 & B_3 & B_4 & 0 & -\tau c_0^{-1} & -(c_0^{-1}\alpha_2 + g_2) \\
C_1 & C_2 & C_3 & C_4 & \frac{-\tau c_0^{-1}\alpha_1}{\sqrt{1-|\alpha|^2}} & \frac{-\tau c_0^{-1}\alpha_2}{\sqrt{1-|\alpha|^2}} & -(c_0^{-1}\sqrt{1-|\alpha|^2} + p_3)
\end{pmatrix}
$$

Thus, rank $d\pi_R = \begin{cases} 5, & \text{if} \quad \frac{\partial f}{\partial p_3} = \frac{\partial f}{\partial s_1} = 0 \\ 6, & \text{if} \quad \frac{\partial f}{\partial p_3} \neq 0 \text{ or } \frac{\partial f}{\partial s_1} \neq 0 \end{cases}$ because the matrix

$$
\begin{pmatrix}
-\tau c_0^{-1} & 0 & -(c_0^{-1}\alpha_1 + g_1) \\
0 & -\tau c_0^{-1} & -(c_0^{-1}\alpha_2 + g_2) \\
\frac{-\tau c_0^{-1}\alpha_1}{\sqrt{1-|\alpha|^2}} & \frac{-\tau c_0^{-1}\alpha_2}{\sqrt{1-|\alpha|^2}} & -(c_0^{-1}\sqrt{1-|\alpha|^2} + p_3)
\end{pmatrix}
$$

is nonsingular [24]. Hence, the critical set $\mathcal{S}_1^C := \mathcal{S}_1(\pi_R)$ is a smooth, codimension two submanifold. (Recall that by general considerations [15], this must equal $\mathcal{S}_1(\pi_L)$, and $d\pi_R$ and $d\pi_L$ must drop rank by the same amount at each point.) At these points, Ker $d\pi_R = \{(0, 0, \delta p_3, \delta s_1, \delta\alpha_1, \delta\alpha_2, \delta\tau)\}$ where $\delta\alpha_1, \delta\alpha_2, \delta\tau$ depend on $\delta p_3, \delta s_1$. The tangent space to $\mathcal{S}_1^C$ is

$$
T\mathcal{S}_1^C = \text{ Ker } \left(d_{x_1,x_2,p_3,s_1,\alpha_1,\alpha_2,\tau}(\frac{\partial f}{\partial p_3})\right) \cap \text{ Ker } \left(d_{x_1,x_2,p_3,s_1,\alpha_1,\alpha_2,\tau}(\frac{\partial f}{\partial s_1})\right),
$$

with

$$
d_{x_1,x_2,p_3,s_1,\alpha_1,\alpha_2,\tau}(\frac{\partial f}{\partial p_3}) = (\frac{\partial^2 f}{\partial x_1 \partial p_3}, \frac{\partial^2 f}{\partial x_2 \partial p_3}, \frac{\partial^2 f}{\partial p_3^2}, \frac{\partial^2 f}{\partial s_1 \partial p_3}, 0, 0, 0) \tag{12}
$$

and

$$
d_{x_1,x_2,p_3,s_1,\alpha_1,\alpha_2,\tau}(\frac{\partial f}{\partial s_1}) = (\frac{\partial^2 f}{\partial x_1 \partial s_1}, \frac{\partial^2 f}{\partial x_2 \partial s_1}, \frac{\partial^2 f}{\partial p_3 \partial s_1}, \frac{\partial^2 f}{\partial s_1^2}, 0, 0, 0), \tag{13}
$$

and so we see that Ker $d\pi_R$ is transversal to $T\mathcal{S}_1^C$ because of the condition (10). Hence, π_R is a submersion with folds. Note that without further restrictions on f, the projection π_R can either an elliptic or hyperbolic submersion with folds.

Similarly, with respect to the above coordinates, reordered for ease of display, $\pi_L : \mathbb{R}^7 \to \mathbb{R}^8$ is given by

$$\pi_L(s_1, x_1, x_2, \alpha_1, \alpha_2, p_3, \tau) = \big(s_1, r_1(\cdot), r_2(\cdot), t_{inc}(\cdot) + t_{ref}(\cdot); \sigma(\cdot), \rho_1(\cdot), \rho_2(\cdot), \tau\big),$$

and thus $d\pi_L =$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial r_1}{\partial s_1} & \frac{\partial r_1}{\partial x_1} + \frac{\partial r_1}{\partial x_3}\frac{\partial f}{\partial x_1} & \frac{\partial r_1}{\partial x_2} + \frac{\partial r_1}{\partial x_3}\frac{\partial f}{\partial x_2} & \frac{\partial r_1}{\partial \alpha_1} & \frac{\partial r_1}{\partial \alpha_2} & \frac{\partial r_1}{\partial x_3}\frac{\partial f}{\partial p_3} & 0 \\
\frac{\partial r_2}{\partial s_1} & \frac{\partial r_2}{\partial x_1} + \frac{\partial r_2}{\partial x_3}\frac{\partial f}{\partial x_1} & \frac{\partial r_2}{\partial x_2} + \frac{\partial r_2}{\partial x_3}\frac{\partial f}{\partial x_2} & \frac{\partial r_2}{\partial \alpha_1} & \frac{\partial r_2}{\partial \alpha_2} & \frac{\partial r_2}{\partial x_3}\frac{\partial f}{\partial p_3} & 0 \\
\frac{\partial t_{ref}}{\partial s_1} & \frac{\partial t_{inc}}{\partial x_1} + \frac{\partial t_{ref}}{\partial x_3}\frac{\partial f}{\partial x_1} & \frac{\partial t_{inc}}{\partial x_2} + \frac{\partial t_{ref}}{\partial x_3}\frac{\partial f}{\partial x_1} & \frac{\partial t_{ref}}{\partial \alpha_1} & \frac{\partial t_{ref}}{\partial \alpha_2} & \frac{\partial t_{inc}}{\partial p_3} + \frac{\partial t_{ref}}{\partial x_3}\frac{\partial f}{\partial p_3} & 0 \\
\frac{\partial \sigma}{\partial s_1} & \frac{\partial \sigma}{\partial x_1} + \frac{\partial \sigma}{\partial x_3}\frac{\partial f}{\partial x_1} & \frac{\partial \sigma}{\partial x_2} + \frac{\partial \sigma}{\partial x_3}\frac{\partial f}{\partial x_2} & \frac{\partial \sigma}{\partial \alpha_1} & \frac{\partial \sigma}{\partial \alpha_2} & \frac{\partial \sigma}{\partial x_3}\frac{\partial f}{\partial p_3} & 0 \\
\frac{\partial \rho_1}{\partial s_1} & \frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_3}\frac{\partial f}{\partial x_1} & \frac{\partial \rho_1}{\partial x_2} + \frac{\partial \rho_1}{\partial x_3}\frac{\partial f}{\partial x_2} & \frac{\partial \rho_1}{\partial \alpha_1} & \frac{\partial \rho_1}{\partial \alpha_2} & \frac{\partial \rho_1}{\partial x_3}\frac{\partial \rho_1}{\partial p_3} & 0 \\
\frac{\partial \rho_2}{\partial s_1} & \frac{\partial \rho_2}{\partial x_1} + \frac{\partial \rho_2}{\partial x_3}\frac{\partial f}{\partial x_1} & \frac{\partial \rho_2}{\partial x_2} + \frac{\partial \rho_2}{\partial x_3}\frac{\partial f}{\partial x_2} & \frac{\partial \rho_2}{\partial \alpha_1} & \frac{\partial \rho_2}{\partial \alpha_2} & \frac{\partial \rho_2}{\partial x_3}\frac{\partial \rho_2}{\partial p_3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Since t_{inc} and p_3 are independent coordinates, we have $\frac{\partial t_{inc}}{\partial p_3} = 0$. Also, because of the choice of the coordinates x_1, x_2, x_3, it follows that $\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = 0$ at the caustic points [25].

From this, it follows that the rank of $d\pi_L = \begin{cases} 6, & \text{if} \quad \frac{\partial f}{\partial p_3} = \frac{\partial f}{\partial s_1} = 0 \\ 7, & \text{if} \quad \frac{\partial f}{\partial p_3} \neq 0 \text{ or } \frac{\partial f}{\partial s_1} \neq 0, \end{cases}$ because the matrix

$$\begin{pmatrix}
\frac{\partial r_1}{\partial x_1} & \frac{\partial r_1}{\partial x_2} & \frac{\partial r_1}{\partial \alpha_1} & \frac{\partial r_1}{\partial \alpha_2} \\
\frac{\partial r_2}{\partial x_1} & \frac{\partial r_2}{\partial x_2} & \frac{\partial r_2}{\partial \alpha_1} & \frac{\partial r_2}{\partial \alpha_2} \\
\frac{\partial \rho_1}{\partial x_1} & \frac{\partial \rho_1}{\partial x_2} & \frac{\partial \rho_1}{\partial \alpha_1} & \frac{\partial \rho_1}{\partial \alpha_2} \\
\frac{\partial \rho_2}{\partial x_1} & \frac{\partial \rho_2}{\partial x_2} & \frac{\partial \rho_2}{\partial \alpha_1} & \frac{\partial \rho_2}{\partial \alpha_2}
\end{pmatrix}$$

is nonsingular [24]. Furthermore, $\mathcal{S}_1(\pi_L) = \mathcal{S}_1^C = \{\frac{\partial f}{\partial p_3} = \frac{\partial f}{\partial s_1} = 0\}$ is smooth and the 7×7 minors of $d\pi_L$ generate the ideal of $\mathcal{S}_1^C$. Finally, it also follows that

$$\text{Ker } d\pi_L = \{(0, \delta x_1, \delta x_2, \delta\alpha_1, \delta\alpha_2, \delta p_3, 0)\},$$

where $\delta\alpha_1, \delta\alpha_2, \delta x_1, \delta x_2$ depend on δp_3, and thus Ker $d\pi_L \cap T\mathcal{S}_1^C = (0)$. Hence, π_L is a cross cap.

This leads us to formulate a general class of canonical relations with this structure.

Definition 4.1. Let X and Y be manifolds of dimensions n and $n-1$, respectively, and let C be a canonical relation in $(T^*X \setminus 0) \times (T^*Y \setminus 0)$. C is a *folded cross cap* canonical relation if:

a) $\pi_R : C \to T^*Y \setminus 0$ is a submersion with folds, with singular set $\mathcal{S}_1$, and $\pi_R(\mathcal{S}_1)$ is a nonradial hypersurface in $T^*Y \setminus 0$*;* and

b) $\pi_L : C \to T^*X \setminus 0$ has a cross cap singularity along $\mathcal{S}_1$ and $\pi_L(\mathcal{S}_1)$ is a nonradial submanifold. We say that C is an elliptic, respectively hyperbolic folded cross cap if π_R is an elliptic, respectively hyperbolic submersion with folds.

Remark 4.2. We will see in §6 (see discussion following Prop. 6.2) that $\pi_L(\mathcal{S}_1)$ is necessarily *maximally noninvolutive*, defined after (16) below.

Here, as usual, a conic submanifold $\Gamma \subset T^*Y \setminus 0$ is *nonradial* if for all $(y, \eta) \in \Gamma$, the canonical one-form $\sum \eta_j dy_j$ does not vanish identically on $T_{(y,\eta)}\Gamma$. Since $\pi_R(\mathcal{S}_1)$ is the immersed image of $\mathcal{S}_1$, $\pi_R(\mathcal{S}_1)$ is nonradial at $\pi_R(c)$ iff $\sum(\pi_R^*\eta_j)d\pi_R^*(dy_j) \neq 0$ as an element of T_c^*C. We can understand the significance of this for the canonical relation arising in the marine geometry as follows. Using the fact that $df = 0$ at the caustics, the expressions for π_R and $d\pi_R$ computed above imply that, at $c \in \mathcal{S}_1$,

$$\sum_{j=1}^{3}(\pi_R^*\eta_j)d\pi_R^*(dy_j) = -\tau\left((c_0^{-1}\alpha_1 + g_1)dx_1 + (c_0^{-1}\alpha_2 + g_2)dx_2\right). \tag{14}$$

In order to be 0 on $T_c\mathcal{S}_1$, this must be a linear combination of (12) and (13). By (11), the only possible linear combination is the trivial one; however, by the expression for π_R, this forces $\eta = (0, 0, \eta_3)$ for some $\eta_3 \neq 0$. That is, the fold caustic surface in Y must be horizontal at this point. While there certainly exist background soundspeeds $c_0(\cdot)$ for which this happens, the most basic examples of fold caustics arising from refraction about a low velocity lens (see Nolan and Symes [26]) have fold surfaces which are not horizontal.

Similarly, since $\pi_L|_{\mathcal{S}_1}$ is an immersion, $\pi_L(\mathcal{S}_1) \subset T^*X$ is nonradial iff $\sum(\pi_L^*\xi_j)d\pi_L^*(dx_j) \neq 0$ in T_c^*C. The interpretation of a radial point, i.e., a point $c \in \mathcal{S}_1$ where this fails, for the marine geometry is less clear and, to proceed, we will simply need to assume that such points are absent.

5 Model case

We showed in the previous section that the canonical relation C arising from the marine geometry in the presence of fold caustics is a folded cross cap. To help understand the nature of the normal operator, we first consider a model folded cross cap canonical relation, C_0, in $(T^*\mathbb{R}^n \setminus 0) \times (T^*\mathbb{R}^{n-1} \setminus 0)$, parametrized by the phase function

$$\phi_0(x, y, \theta'') = (x'' - y'') \cdot \theta'' + \left((x_n^2 - x_{n-1}^2)y_{n-1} + x_n y_{n-1}^2\right) \theta_1,$$

where $(x, y, \theta'') \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times (\mathbb{R}^{n-2} \setminus 0)$, in the region $\{|\theta_1| \geq c|\theta''|\}$. Here and at various points below we use the notation $x = (x_1, \ldots, x_n) = (x'', x_{n-1}, x_n) = (x_1, x''', x_n)$, $y = (y'', y_{n-1}) = (y_1, y''', y_{n-1})$ and $\theta'' = (\theta_1, \theta''')$.

For simplicity, in this section and the next one we will make the choice that C is a hyperbolic folded cross cap. This corresponds to the choice of the $(-)$ sign in the $(x_{n-1}^2 - x_n^2)$ term of ϕ_0. There are no significant changes needed in the calculations for the elliptic case.

One easily calculates that

$$\begin{aligned} C_0 = \big\{(x'', \quad & x_{n-1}, x_n, \theta'', -2x_{n-1}y_{n-1}\theta_1, y_{n-1}^2\theta_1 + 2x_n y_{n-1}\theta_1; \\ & y'', y_{n-1}, \theta'', -\left(2x_n y_{n-1} + (x_n^2 - x_{n-1}^2)\right)\theta_1) \\ & : |\theta_1| \geq c|\theta''|, \ x_i = y_i, \ 2 \leq i \leq n-2, \\ & x_1 - y_1 + (x_n^2 - x_{n-1}^2)y_{n-1} + x_n y_{n-1}^2 = 0\big\} \end{aligned}$$

We will verify that the projections of C_0 to the left and to the right have the desired singularities. Note that (x, y_{n-1}, θ'') are coordinates on C_0. Hence,

$$\pi_R(x, y_{n-1}, \theta'') = \left(x_1 + (x_n^2 - x_{n-1}^2)y_{n-1} + x_n y_{n-1}^2, x''', y_{n-1}; \theta'', -\left(2x_n y_{n-1} + (x_n^2 - x_{n-1}^2)\right)\theta_1\right)$$

and

$$d\pi_R = \begin{pmatrix} 1 & 0 & A & 0 & 0 & B & D \\ 0 & I_{n-3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n-3} & 0 & 0 \\ 0 & 0 & E & F & 0 & 2x_{n-1}\theta_1 & -2y_{n-1}\theta_1 - 2x_n\theta_1 \end{pmatrix}$$

where $A = (x_n^2 - x_{n-1}^2) + 2x_n y_{n-1}, B = -2x_{n-1}y_{n-1}, D = 2x_n y_{n-1} + y_{n-1}^2$, $E = -2x_n\theta_1$, and $F = -2x_n y_{n-1} - (x_n^2 - x_{n-1}^2)$. This is a $(2n-2) \times (2n-1)$ matrix, with

$$\text{rank } d\pi_R = \begin{cases} 2n-3, & \text{if} \quad x_{n-1} = x_n + y_{n-1} = 0 \\ 2n-2, & \text{if} \quad x_{n-1} \neq 0 \text{ or } x_n + y_{n-1} \neq 0. \end{cases}$$

Let $\mathcal{S}_1^{C_0} := \mathcal{S}_1(\pi_R) = \{(x, y_{n-1}, \theta'') : x_{n-1} = x_n + y_{n-1} = 0\}$ be the set where $d\pi_R$ drops rank by 1. Off of this smooth, codimension two submanifold, π_R is a submersion. The kernel of π_R at $\mathcal{S}_1^{C_0}$ is spanned by $\{\frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial x_n}\}$ and thus intersects the tangent space $T\mathcal{S}_1^{C_0}$ transversally. We also note that the Hessian, $(x_{n-1}, x_n) \to -(x_n^2 - x_{n-1}^2)\theta_1$, is sign-indefinite. We thus conclude that π_R is a hyperbolic submersion with folds.

As for π_L, we have

$$\pi_L(x, y_{n-1}, \theta'') = \left(x; \theta'', -2x_{n-1}y_{n-1}\theta_1, \left(y_{n-1}^2 + 2x_n y_{n-1}\right)\theta_1\right),$$

so that

$$d\pi_L = \begin{pmatrix} I_{n-2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{n-3} & 0 \\ 0 & A' & 0 & B' & 0 & -2x_{n-1}\theta_1 \\ 0 & 0 & D' & E' & 0 & 2(x_n + y_{n-1})\theta_1 \end{pmatrix}$$

where $A' = -2y_{n-1}\theta_1, B' = -2x_{n-1}y_{n-1}, D' = 2y_{n-1}\theta_1$, and $E' = 2x_n y_{n-1} + y_{n-1}^2$. This is a $2n \times (2n-1)$ matrix and

$$\text{rank } d\pi_L = \begin{cases} 2n-2, & \text{if} \quad x_{n-1} = x_n + y_{n-1} = 0 \\ 2n-1, & \text{if} \quad x_{n-1} \neq 0 \text{ or } x_n + y_{n-1} \neq 0. \end{cases}$$

Thus, $d\pi_L$ drops rank by 1 at $\mathcal{S}_1^{C_0}$, and off of this set is an immersion. The kernel of π_L is spanned by $\{\frac{\partial}{\partial y_{n-1}}\}$ and intersects the tangent space $T\mathcal{S}_1^{C_0}$ transversally. Also, the rank of the differential of $(x, y_{n-1}, \theta'') \to (-2x_{n-1}\theta_1, (2y_{n-1} + 2x_n)\theta_1)$ is 2 at $\mathcal{S}_1^{C_0}$. We conclude that π_L has a cross cap singularity.

We note for future use the images of $\mathcal{S}_1^{C_0}$ under π_L and π_R. Since $\pi_R(\mathcal{S}_1^{C_0}) = \{(x'', -x_n; \theta'', x_n^2\theta_1)\}$ and $\pi_L(\mathcal{S}_1^{C_0}) = \{(x'', 0, x_n; \theta'', 0, -x_n^2\theta_1)\}$, one has

$$\pi_R\left(\mathcal{S}_1^{C_0}\right) \subseteq \left\{\eta_{n-1} - y_{n-1}^2\eta_1 = 0\right\} \tag{15}$$

and

$$\pi_L\left(\mathcal{S}_1^{C_0}\right) \subseteq \left\{x_{n-1} = \xi_{n-1} = \xi_n + x_n^2\xi_1 = 0\right\}. \tag{16}$$

One sees that $\pi_R\left(\mathcal{S}_1^{C_0}\right)$ is a nonradial hypersurface in $T^*\mathbb{R}^{n-1} \setminus 0$, while $\pi_L(\mathcal{S}_1^{C_0})$ is a nonradial, codimension three submanifold of $T^*\mathbb{R}^n \setminus 0$, given by defining functions $p_1 = \xi_{n-1}, p_2 = \xi_1 x_{n-1}, p_3 = \xi_n + x_n^2\xi_1$ with Poisson brackets $\{p_1, p_2\} = 1, \{p_1, p_3\} = 0, \{p_2, p_3\} = 0$. This is *maximally noninvolutive* in the sense that $\omega_{T^*\mathbb{R}^n}|_{\pi_L(\mathcal{S}_1)}$ has the maximal possible rank for a codimension three submanifold of $T^*\mathbb{R}^n \setminus 0$, namely $2n - 4$.

Next, we calculate the composition

$$C_0^t \circ C_0 = \{(x,\xi;y,\eta) \in T^*\mathbb{R}^{n-1} \times T^*\mathbb{R}^{n-1} \quad : \quad \exists (z,\zeta) \in T^*\mathbb{R}^n \text{ such that } (x,\xi;z,\zeta) \in C_0^t \text{ and } (z,\zeta;y,\eta) \in C_0\}.$$

One sees that $(x,\xi;z,\zeta) \in C_0^t$ iff $(z,\zeta;x,\xi) \in C_0$ iff

$$z_i = x_i, \quad 2 \le i \le n-2;$$

$$z_1 - x_1 + (z_n^2 - z_{n-1}^2)y_{n-1} + z_n x_{n-1}^2 = 0;$$

$$\zeta_i = \xi_i \quad 1 \le i \le n-2;$$

$$\zeta_n = 2z_n x_{n-1}\xi_1 + x_{n_1}^2 \xi_{n-1};$$

$$\zeta_{n-1} = -2z_{n-1}x_{n-1}\xi_1; \text{ and}$$

$$\xi_{n-1} = -(z_n^2 - z_{n-1}^2)\xi_1 - 2z_n x_{n-1}\xi_1,$$

with $(z,\zeta;y,\eta) \in C_0$ being determined by the same equations, but where (y,η) replaces (x,ξ). Thus, $C_0^t \circ C_0$ consists of $(x,\xi;y,\eta)$ such that, for some $(z_{n-1},z_n) \in \mathbb{R}^2$,

$$x_i = y_i, \quad 2 \le i \le n-2;$$

$$\xi_i = \eta_i, \quad 1 \le i \le n-2;$$

$$z_{n-1}(x_{n_1} - y_{n-1})\xi_1 = 0;$$

$$(x_{n-1} - y_{n-1})(2z_n + x_{n-1} + y_{n-1}) = 0;$$

$$\xi_{n-1} = -\left((z_n^2 - z_{n-1}^2) + 2z_n x_{n-1}\right)\xi_1;$$

$$\eta_{n-1} = -\left((z_n^2 - z_{n-1}^2) + 2z_n y_{n-1}\right)\eta_1; \quad \text{and}$$

$$y_1 - z_1 + (z_n^2 - z_{n-1}^2)(x_{n-1} - y_{n-1}) + z_n(x_{n-1}^2 - y_{n-1}^2) = 0.$$

If $x_{n-1} = y_{n-1}$, then $x_1 = y_1$ and $\xi_{n-1} = \eta_{n-1}$, so this contribution to $C_0^t \circ C_0$ is contained in Δ, the diagonal canonical relation in $T^*\mathbb{R}^{n-1} \times T^*\mathbb{R}^{n-1}$.

If, on the other hand, $x_{n-1} \neq y_{n-1}$, then

$$x_i = y_i, \quad 2 \leq i \leq n-2;$$

$$\xi_i = \eta_i, \quad 1 \leq i \leq n-2;$$

$$\xi_{n-1} = \frac{(x_{n-1} + y_{n-1})(3x_{n-1} - y_{n-1})}{4}\xi_1;$$

$$\eta_{n-1} = \frac{(x_{n-1} + y_{n-1})(3y_{n-1} - x_{n-1})}{4}\xi_1; \text{and}$$

$$x_1 - y_1 + \frac{(x_{n-1} + y_{n-1})^2(x_{n-1} - y_{n-1})}{4} = 0,$$

giving a contribution to $C_0^t \circ C_0$ contained in $\tilde{C}_0$, where $\tilde{C}_0$ is the twisted conormal bundle,

$$\tilde{C}_0 = N^* \left\{ x_1 - y_1 + \frac{(x_{n-1} + y_{n-1})^2(x_{n-1} - y_{n-1})}{4} = 0, x_i - y_i = 0,\ 2 \leq i \leq n-2 \right\}'.$$

In conclusion $C_0^t \circ C_0 \subseteq \Delta \cup \tilde{C}_0$, from which it follows that $\tilde{C}_0$ is symmetric, i.e., $\tilde{C}_0^t = \tilde{C}_0$. It is easy to see that both projections from $\tilde{C}_0$ have Whitney fold singularities. Such canonical relations were introduced in [20] and called *folding canonical relations*; they are also referred to as *two-sided folds* [7] and we will use this latter terminology. One also sees that $\tilde{C}_0$ intersects Δ cleanly in codimension 1, and furthermore, $\Delta \cap \tilde{C}_0$ is in fact the fold surface of $\tilde{C}_0$. As described in §3, one has a well defined class of distributions associated to the two cleanly intersecting Lagrangians $(\Delta', \tilde{C}_0')$, namely $I^{p,l}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Delta', \tilde{C}_0')$, for $p, l \in \mathbb{R}$. A distribution is in this class iff it has an oscillatory representation,

$$u(x,y) = \int_{\mathbb{R}^{n-1}} e^{i\{(x_1 - y_1 - \frac{(x_{n-1} - y_{n-1})(x_{n-1} + y_{n-1})^2}{4})\xi_1 + (x''' - y''')\cdot\xi'''\}} a(x, y; \xi''; \xi_{n-1}) d\xi$$

where a is a symbol-valued symbol, satisfying the estimates:

$$|\partial_{\xi''}^\alpha \partial_{\xi_{n-1}}^\beta \partial_{x,y}^\gamma a(x, y, \xi)| \leq c(1 + |\xi|)^{\tilde{p} - |\alpha|}(1 + |\xi_{n-1}|)^{\tilde{l} - \beta},$$

with $\tilde{p} = p + l + \frac{1}{2}, \tilde{l} = -l - \frac{1}{2}$.

Next we will show that if $F \in I^{m-\frac{1}{4}}(C_0)$ then $F^*F \in I^{p,l}(\Delta, \tilde{C}_0)$, i.e., $K_{F^*F} \in I^{p,l}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Delta', \tilde{C}_0')$, for some $p, l \in \mathbb{R}$. We have:

$$Ff(x) = \int e^{i\{(x'' - y'')\cdot\theta'' + ((x_n^2 - x_{n-1}^2)y_{n-1} + x_n y_{n-1}^2)\theta_1\}} a(x, y, \theta) f(y) d\theta dy,$$

where $a \in S^{m+\frac{1}{2}}$. Thus,

$$K_{F^*F}(x,y) = \int e^{i\{\phi_0(z,y,\theta'')-\phi_0(z,x,\eta'')\}} a(z,y,\theta'')\overline{a}(z,x,\eta'')dzd\theta''d\eta''.$$

After a stationary phase in the variables z'', η'', the phase function becomes:

$$\begin{aligned}\phi(x,y,\theta'',z_{n-1},z_n) &= (x''-y'')\cdot\theta'' + (z_n^2 - z_{n-1}^2)(y_{n-1}-x_{n-1})\theta_1 + z_n(y_{n-1}^2 - x_{n-1}^2)\theta_1 \\ &= (x''-y'')\cdot\theta'' + (y_{n-1}-x_{n-1})[(z_n^2 - z_{n-1}^2) + z_n(x_{n-1}+y_{n-1})]\theta_1\end{aligned}$$

and the amplitude becomes $\tilde{a}(x,y,z_{n-1},z_n;\theta'') \in S^{2m+1}$.

Following an idea from [11],[5], we now make a singular change of variables, $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$, $T(\theta'', z_{n-1}, z_n) = (\xi'', \xi_{n-1})$, given by:

$$\begin{aligned}\xi_i &= \theta_i,\ 1 \le i \le n-2, \\ \xi_{n-1} &= -\left((z_n^2 - z_{n-1}^2) + z_n(x_{n-1}+y_{n-1})\right)\theta_1.\end{aligned}$$

The kernel of F^*F can then be rewritten as

$$K_{F^*F}(x,y) = \int_{\mathbb{R}^{n-1}} e^{i\{(x''-y'')\cdot\xi'' + (x_{n-1}-y_{n-1})\xi_{n-1}\}} b(x,y,\xi)d\xi,$$

where, using the coarea formula [4, p.249],

$$b(x,y;\xi) = \int_{\{((z_n^2 - z_{n-1}^2)\theta_1 + z_n(x_{n-1}+y_{n-1})\theta_1) = -\xi_{n-1}\}} \tilde{a}(x,y,z_{n-1},z_n;\xi'')\frac{d\nu}{J_{n-1}}$$

with J_{n-1} the $(n-1)$-Jacobian of T and $d\nu$ the arc length measure on $\left\{\left((z_n^2 - z_{n-1}^2) + z_n(x_{n-1}+y_{n-1})\right)\theta_1 = -\xi_{n-1}\right\}$. The Jacobian is given by

$$\begin{aligned}J_{n-1} &= |\nabla_z \xi_{n-1}| \\ &= |(2z_{n-1}\theta_1, -(2z_n + x_{n-1} + y_{n-1})\theta_1)| \\ &= 2\left(z_{n-1}^2 + (z_n + \frac{x_{n-1}+y_{n-1}}{2})^2\right)^{\frac{1}{2}} |\theta_1|,\end{aligned}$$

which vanishes to first order at $z_{n-1} = z_n + \frac{x_{n-1}+y_{n-1}}{2} = 0$. Note that, at these points, $\xi_{n-1} = \frac{(x_{n-1}+y_{n-1})^2}{4}\xi_1$. Thus, b is of order $2m$ in ξ'' and has a conormal singularity of order -1 at $\{\xi_{n-1} - \frac{(x_{n-1}+y_{n-1})^2}{4}\xi_1 = 0\}$; thus, it has an oscillatory representation

$$b(x,y,\xi) = \int_{\mathbb{R}} e^{i\{(\xi_{n-1} - \frac{(x_{n-1}+y_{n-1})^2}{4}\xi_1)\frac{\rho}{\xi_1})\}} \tilde{b}(x,y;\xi;\rho)d\rho.$$

Finally, the kernel of F^*F becomes

$$K_{F^*F}(x,y) = \int_{\mathbb{R}^n} e^{i\{(x''-y'')\cdot\xi''+(x_{n-1}-y_{n-1})\xi_{n-1}+(\xi_{n-1}-\frac{(x_{n-1}+y_{n-1})^2}{4}\xi_1)\frac{\rho}{\xi_1}\}} \times \tilde{b}(x,y;\xi;\rho)d\xi d\rho, \quad (17)$$

where one can check that $\tilde{b}$ is a product-type symbol, $\tilde{b} \in S^{2m,-1}(2n-2, n-1, 1)$.

We now introduce a conic partition of unity in (ξ, ρ), with supports in $\{|\rho| \leq 2|\xi|\}$ and $\{|\rho| \geq \frac{1}{2}|\xi|\}$, resulting in a decomposition $K_{F^*F} = K^0 + K^1$. Letting

$$\psi(x,y;\xi;\rho) = (x''-y'')\cdot\xi''+(x_{n-1}-y_{n-1})\xi_{n-1}+(\xi_{n-1}-\frac{(x_{n-1}+y_{n-1})^2}{4}\xi_1)\frac{\rho}{\xi_1},$$

one easily sees that on the region $\{|\rho| \leq c|\xi|\}$, this is a multi-phase function for $(\Delta', \tilde{C}_0')$ in the sense of Def. 3.5, i.e., $\psi_0(x,y;\xi) := \psi(x,y;\xi;0)$ parametrizes the diagonal Lagrangian Δ' and $\psi_1(x,y;(\xi,\rho)) := \psi(x,y;\xi;\rho)$ parametrizes the Lagrangian $\tilde{C}_0'$. Furthermore, on this region, the amplitude is a symbol-valued symbol, belonging to $S^{2m,-1}(\mathbb{R}^{2n-2} \times (\mathbb{R}^{n-1} \setminus 0) \times \mathbb{R})$. In view of Remark 3.7, $K^0 \in I^{p,l}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; \Delta', \tilde{C}_0')$, for some $p, l \in \mathbb{R}$. The orders may also be found by applying Remark 3.7 to (17). We see that

$$p = (2m-1) + \frac{(n-1)+1}{2} - \frac{2n-2}{4} = 2m - \frac{1}{2}$$

and

$$l = -(-1) - \frac{1}{2} = \frac{1}{2}.$$

Hence, this contribution to F^*F is in $I^{2m-\frac{1}{2},\frac{1}{2}}(\Delta, \tilde{C}_0)$, with $\tilde{C}_0$ a two-sided fold.

Next, we show that $K^1 \in I^{2m-\frac{1}{2}}(\tilde{C}_0)$. First, let $s = \xi_{n-1}/\xi_1$. Then we can express $K^1(x,y) = \int L(x,y,s)ds$, where L has the phase function

$$\Psi(x,y,s;\rho;\xi'') = (x''-y'')\cdot\xi'' + (x_{n-1}-y_{n-1})\xi_1 s + (s - \frac{(x_{n-1}+y_{n-1})^2}{4})\rho$$

and amplitude $c \in S^{-1,2m+1}(\mathbb{R}^{2n-1} \times (\mathbb{R}\setminus 0) \times \mathbb{R}^{n-1})$. Note that $\Psi_0(x,y,s;\rho) := \Psi(x,y,s;\rho;0)$ parametrizes

$$\Gamma_0 := N^*\{s - \frac{(x_{n-1}+y_{n-1})^2}{4} = 0\} \subset T^*\mathbb{R}^{2n-1} \setminus 0$$

and $\Psi_1(x,y,s;(\rho,\xi'')) := \Psi(x,y,s;\rho;\xi''))$ parametrizes

$$\Gamma_1 := N^*\left\{x_i - y_i = 0, 2 \leq i \leq n-2,\ x_1 - y_1 + s(x_{n-1}-y_{n-1}) = s - \frac{(x_{n-1}+y_{n-1})^2}{4} = 0\right\}.$$

Thus, Ψ is a multi-phase function for (Γ_0, Γ_1) in the sense of Def 3.5. Introduce a cutoff function, $\chi \in C_0^\infty(R^n), \chi = 1$ near 0, set

$$L^0(x,y,s) = \int e^{i\Psi(x,y,s;\rho,\xi'')}\chi(\frac{|\xi''|}{c|\rho|})c(x,y;\rho;\xi'')d\xi'' d\rho,$$

and let $L^1 = L - L^0$. We have $L^0 \in I^{2m-\frac{1}{4},\frac{1}{2}}(\mathbb{R}^{2n-1};\Gamma_0,\Gamma_1)$, $L^1 \in I^{2m-\frac{1}{4}}(\mathbb{R}^{2n-1};\Gamma_1)$ and $WF(L^1)$ is contained in the complement of a neighborhood of Γ_0.

Since K^2 is simply $\pi_* L$, the pushforward of L by π, the projection $\pi(x,y,s) = (x,y)$, which is a submersion, we compute the pushforwards of the Lagrangians Γ_0, Γ_1. It follows from standard results about pushforwards [15] that, for $u \in \mathcal{E}'\left(\mathbb{R}^{2n-1}\right)$,

$$\begin{aligned} WF(\pi_* u) \subseteq \{(x,y;\xi,\eta) \quad | \quad & \exists(\hat{x},\hat{y},s,\hat{\xi},\hat{\eta},\sigma) \in WF(u) \ \ s.t. \\ & (x,y) = \pi(\hat{x},\hat{y},s), (\hat{\xi},\hat{\eta},\sigma) = d\pi^t(\xi,\eta)\}. \end{aligned}$$

Using $d\pi^t(\xi,\eta) = (\xi,\eta,0)$, it is then an easy calculation that the push forward of Γ_0, and indeed any neighborhood of Γ_0 on which $\sigma \neq 0$, is empty, so that $\pi_*(L^0) \in C^\infty$, while $\pi_*(\Gamma_1) \subseteq \tilde{C}_0'$. In fact, π_* is a FIO of order $-\frac{1}{4}$, and the application of π_* to $I^{\cdot}(\Gamma_1)$ is covered by the transverse intersection calculus. Hence $\pi_* : I^r(\Gamma_1) \to I^{r-\frac{1}{4}}(\tilde{C}_0')$, so that

$$K^1 = \pi_*(L^1) \mod C^\infty \in I^{2m-\frac{1}{2}}(\tilde{C}_0),$$

and thus $K_{F^*F} = K^0 + K^1 \in I^{2m-\frac{1}{2},\frac{1}{2}}(\Delta', \tilde{C}_0')$.

In conclusion, $F^*F \in I^{2m-\frac{1}{2},\frac{1}{2}}(\Delta, \tilde{C}_0)$. For the single source data acquisition geometry, in Nolan[24] and Felea[5] it was shown that $F^*F \in I^{2m,0}(\Delta,\tilde{C})$, with $\tilde{C}$ a two-sided fold. In that situation the artifact, i.e., the part of F^*F on $\tilde{C} \setminus \Delta$, has, by Remark 3.4, the same strength as on $\Delta \setminus \tilde{C}$. However, it follows from what we have shown here that for the microlocal model C_0 of the marine geometry, the artifact is $\frac{1}{2}$ order smoother then the pseudodifferential operator part. In the next section, we show this in full generality for FIOs associated with folded cross caps.

6 Composition calculus in the general case

By a well known result of Melrose and Taylor[20], any two-sided fold can be conjugated, via canonical transformations on the left and right, to a normal form. Unfortunately, even if folded cross caps could be conjugated to a normal form, such a result would presumably be difficult to prove. However, we will be able to avoid this problem by finding a weak normal form for the canonical relation and thus for a phase function parametrizing it, adapting a method originally developed by Greenleaf and Uhlmann [10, 11] for some canonical relations arising in integral geometry, for which π_R belongs to a class containing the submersions with folds and π_L is maximally degenerate. This will be sufficient for establishing the composition calculus for general folded cross cap canonical relations.

Theorem 6.1. If $F \in I^{m-\frac{1}{4}}(X,Y;C)$ is properly supported and C is a folded cross cap canonical relation, then $F^*F \in I^{2m-\frac{1}{2},\frac{1}{2}}(\Delta,\tilde{C})$. Furthermore, $\tilde{C}$ is a symmetric, two-sided fold in $(T^*Y \setminus 0) \times (T^*Y \setminus 0)$, and $\Delta \cap \tilde{C}$ equals the fold surface in $\tilde{C}$.

Before establishing a weak normal form for a general folded cross cap, we first need to find separate weak normal forms for each of the two projections, π_R and π_L. For the next two propositions, we write $\sigma \in \mathbb{R}^n$ as $\sigma = (\sigma'', \sigma^{(iv)}) = (\sigma'', \sigma_{n-1}, \sigma_n)$, etc..

Proposition 6.2. *Let $f : V \longrightarrow W$ be a smooth, conic map, with V a smooth, conic manifold of dimension $2n-1$ and (W, ω_W) a conic symplectic manifold of dimension $2n-2$. Assume that f is a submersion with folds at $v_0 \in V$ and $f(V)$ is nonradial in W. Then, there exist local homogeneous coordinates $(s'', s_{n-1}, \sigma'', \sigma^{(iv)}) \in \mathbb{R}^{n-2} \times \mathbb{R} \times (\mathbb{R}^{n-2} \setminus 0) \times \mathbb{R}^2$ on V near v_0, and local canonical coordinates $(y', \eta') \in T^*\mathbb{R}^{n-1} \setminus 0$ on W near $w_0 := f(v_0)$, such that $v_0 = (0, 0, e_1^*, 0)$, $w_0 = (0; e_1^*)$, and*

$$f(v) = f(s'', s_{n-1}, \sigma'', \sigma^{(iv)}) = \left(s'', s_{n-1}; \sigma'', A^{s,\sigma}(\sigma^{(iv)}, \sigma^{(iv)})\right), \tag{18}$$

where $A^{\cdot} \in C^\infty(\mathbb{R}^{2n-1}, S^2\mathbb{R}^{2^*})$ is homogeneous of degree -1 in $\cdot$ and takes values in the nonsingular quadratic forms of the same signature as Hess $f(v_0)$.

Proof. In the terminology of [11], a submersion with folds has *clean folds of multiplicity one*. Prop.6.2 is then a special case of [11, Lem.3.A.1], with slight change of notation, and (N, m, n) in [11] being $(n-2, 1, 2)$ here. □

Applying (18) to $\pi_R : C \longrightarrow T^*Y \setminus 0$, one sees that $\mathcal{S}_1^C = \mathcal{S}_1(\pi_R) =$

$\{\sigma^{(iv)} = 0\}$ and $\pi_R(\mathcal{S}_1^C) = \{\eta_{n-1} = 0\}$. Furthermore,

$$\begin{aligned} \omega_C &= \pi_R^*(\sum_{j=1}^{n-1} d\eta_j \wedge dy_j) \\ &= \sum_{j=1}^{n-2} d\sigma_j \wedge ds_j + d\left(A^{s,\sigma}(\sigma^{(iv)}, \sigma^{(iv)})\right) \wedge ds_{n-1}. \end{aligned} \tag{19}$$

Hence,

$$Ker\,(\omega_C|_{T\mathcal{S}_1}) = \mathbb{R} \cdot \left(\frac{\partial}{\partial s_{n-1}} + \dots\right). \tag{20}$$

Also, as in the case of the model canonical relation, C_0, we see that ω_C has rank $2n-2$ on $C \setminus \mathcal{S}_1^C$, and has rank $2n-4$ both at $\mathcal{S}_1^C$ and on $\mathcal{S}_1^C$, i.e., restricted to $T\mathcal{S}_1^C$. Thus, since $\omega_C = \pi_L^*\omega_{T^*X}$ as well, the image $\pi_L(\mathcal{S}_1) \subset T^*X \setminus 0$, which is smooth, conic, nonradial and codimension 3, must also be maximally noninvolutive. Finally, recall that, as a general fact about canonical relations, for all $c_0 = (x_0, \xi_0, y_0, \eta_0) \in C$, the subspace $d\pi_L(T_{c_0}C) \leq T_{(x_0,\xi_0)}(T^*X)$ is involutive; in particular, for $c_0 \in \mathcal{S}_1$, $d\pi_L(T_{c_0}C)$ is a codimension 2, involutive subspace. We are thus led to establishing a (very) weak normal form for cross cap maps into symplectic manifolds reflecting these extra conditions. Note that this would apply to a more general class of maps than cross caps, since at this point we are only using information concerning the first derivatives.

Proposition 6.3. *Let $g : V \longrightarrow U$ be a smooth conic map, with V a smooth, conic manifold of dimension $2n-1$ and (U, ω_U) a conic symplectic manifold of dimension $2n$. Assume that g has a cross cap singularity, with critical set $\mathcal{S}_1$. Let $v_0 \in \mathcal{S}_1, u_0 = g(v_0)$. Assume that $dg(T_{v_0}V) \leq T_{u_0}(U)$ is involutive and $dg\,(T_{v_0}\mathcal{S}_1) = T_{u_0}\,(g(\mathcal{S}_1) \leq T_{u_0}U$ is maximally noninvolutive for all $v_0 \in \mathcal{S}_1$. Then, there exist local homogeneous coordinates*

$$(t, \tau) = (t'', t_n, \tau'', \tau^{(iv)}) \in \mathbb{R}^{n-2} \times \mathbb{R} \times (\mathbb{R}^{n-2} \setminus 0) \times \mathbb{R}^2$$

on V near v_0, and local canonical coordinates $(x, \xi) \in T^*\mathbb{R}^n \setminus 0$ on U near $u_0 := g(v_0)$, such that $v_0 = (0, 0, e_1^*, 0)$, $u_0 = (0; e_1^*)$, and, writing $\tau^{(iv)} = (\tau_{n-1}, \tau_n)$,

$$g(t, \tau) = (t'', g_{n-1}(t, \tau), t_n; \tau'', \gamma_{n-1}(t, \tau), \gamma_n(t, \tau))\,; \tag{21}$$

$$\frac{\partial g_{n-1}}{\partial \tau_{n-1}} \neq 0; \quad \textit{and} \tag{22}$$

$$\gamma_i|_{\tau^{(iv)}=0} = \frac{\partial \gamma_i}{\partial \tau_j}|_{\tau^{(iv)}=0} = 0, \quad n-1 \leq i, j \leq n. \tag{23}$$

Note that, with respect to these coordinates, $\mathcal{S}_1 \subseteq \{\tau^{(iv)} = 0\}$, $\Sigma^{2n-3} := g(\mathcal{S}_1) \subseteq \{x_{n-1} = \xi_{n-1} = \xi_n = 0\}$ and

$$Ker\,(\omega_C|_{T\mathcal{S}_1}) = \mathbb{R} \cdot \left(\frac{\partial}{\partial \tau_{n-1}} + \dots \right). \tag{24}$$

Proof. By assumption, $\Sigma^{2n-3} \subset U$ is nonradial, codimension 3 and maximally noninvolutive. By an application of Darboux's Theorem ([16, Thm. 21.2.4]), there exist local canonical coordinates $(x, \xi) \in T^*\mathbb{R}^n \setminus 0$ such that $\Sigma^{2n-3} \subseteq \{x_{n-1} = \xi_{n-1} = \xi_n = 0\}$ and $u_0 = (0, e_1^*)$. Letting $t_j = g^*(x_j), \quad \tau_j = g^*(\xi_j)$ for $1 \leq j \leq n-2$, and $t_n = g^*(x_n)$, these functions have linearly independent gradients at v_0. If τ_{n-1}, τ_n are any two defining functions for $\mathcal{S}_1$, homogeneous of degree 1, then $(t, \tau) := (t'', t_n, \tau'', \tau_{n-1}, \tau_n)$ are local homogeneous coordinates on V, with $\mathcal{S}_1 \subseteq \{\tau_{n-1} = \tau_n = 0\}$. With respect to these coordinates,

$$g(t, \tau) = (t'', x_{n-1}(t, \tau), t_n; \tau'', \xi_{n-1}(t, \tau), \xi_n(t, \tau))$$

and rank $dg(v) = (2n-3) +$ rank $\mathbf{D}(v)$, where

$$\mathbf{D}(v) = \begin{pmatrix} \frac{\partial x_{n-1}}{\partial \tau_{n-1}} & \frac{\partial x_{n-1}}{\partial \tau_n} \\ \\ \frac{\partial \xi_{n-1}}{\partial \tau_{n-1}} & \frac{\partial \xi_{n-1}}{\partial \tau_n} \\ \\ \frac{\partial x_n}{\partial \tau_{n-1}} & \frac{\partial x_n}{\partial \tau_n} \end{pmatrix} = (D_{n-1}, D_n), \quad D_{n-1}, D_n \in T_u U.$$

Since g is a cross cap, rank $\mathbf{D}(v) = 1, \forall v \in \mathcal{S}_1$, so that by interchanging τ_{n-1} and τ_n, if necessary, we can assume that $D_{n-1} \neq 0$ and $D_n \in \mathbb{R} \cdot D_{n-1}$ for $v \in \mathcal{S}_1$ near v_0. Furthermore, rotating in the x_{n-1}, ξ_{n-1} plane, if necessary, we can likewise assume that $\frac{\partial x_{n-1}}{\partial \tau_{n-1}}(v_0) \neq 0$ and $\frac{\partial \xi_{n-1}}{\partial \tau_{n-1}}(v_0) = 0$, so that

$$\left| \frac{\partial x_{n-1}}{\partial \tau_{n-1}}(v) \right| >> \left| \frac{\partial \xi_{n-1}}{\partial \tau_{n-1}}(v) \right|, \quad \forall v \text{ near } v_0.$$

Let

$$\Pi(v) = dg(T_v V) = span \left\{ \{ \frac{\partial}{\partial x_j} \}_{j=1}^{n-2}, \frac{\partial}{\partial x_n}, \{ \frac{\partial}{\partial \xi_j} \}_{j=1}^{n-2}, D_{n-1} \right\};$$

by assumption, $\mathbf{\Pi}(v) \leq T_{g(v)}U$ is a codimension 2, involutive subspace. We have

$$D_{n-1}(v) = a(v) \frac{\partial}{\partial x_{n-1}} + b(v) \frac{\partial}{\partial \xi_{n-1}} + c(v) \frac{\partial}{\partial \xi_n},$$

with $|a| >> |b|$ near v_0. A simple calculation shows that a vector $\mathbf{X} = \sum_{j=1}^n \alpha_j \frac{\partial}{\partial x_j} + \beta_j \frac{\partial}{\partial \xi_j} \in T_{g(v)}U$ is in the symplectic annihilator $dg(T_v V)^\omega$ iff

$$\alpha_j = \beta_j = 0, \quad 1 \leq j \leq n-2, \beta_n = 0, \quad a\beta_{n-1} - b\alpha_{n-1} - c\alpha_n = 0.$$

Thus, $\left\{b\frac{\partial}{\partial x_{n-1}}+a\frac{\partial}{\partial \xi_{n-1}}, c\frac{\partial}{\partial x_{n-1}}+a\frac{\partial}{\partial \xi_n}\right\}$ forms a basis for $dg(T_vV)^\omega$. Since $dg(T_vV)$ is involutive iff $dg(T_vV)^\omega \leq dg(T_vV)$, we must have $b(v)=c(v)=0$. Hence, $\frac{\partial \xi_i}{\partial \tau_j}|_{\tau^{(iv)}=0}=0, \quad n-1\leq i,j\leq n$. Relabeling, this finishes the proof of Prop.6.3. $\square$

Now, let $C\subset (T^*X\setminus 0)\times(T^*Y\setminus 0)$ be a folded cross cap canonical relation, and apply both Prop.6.2 to $\pi_R: C\longrightarrow T^*Y\setminus 0$ and Prop.6.3 to $\pi_L: C\longrightarrow T^*X\setminus 0$ near $c_0\in C$. As in [10, 11], we now attempt to reconcile the two resulting coordinate systems on C, $(s'', s_{n-1}, \sigma'', \sigma^{(iv)})$ and $(t'', t_n, \tau'', \tau^{(iv)})$. On $T\mathcal{S}_1$ (which is the same for both projections), ω_C has rank $2n-4$. By (20) and (24), since the hypersurface $\{\tau_{n-1}=0\}$ is transverse to Ker (ω_C), it must be locally expressible as $\{s_{n-1}=\tilde{F}(s'',\sigma'')\}$ for some smooth $\tilde{F}$. Now set $F(s,\sigma)=-A^{s,\sigma}(\sigma^{(iv)},\sigma^{(iv)})\tilde{F}(s'',\sigma'')$; then $F=\pi_R^*f$, where $f(y,\eta)=-\eta_{n-1}\tilde{F}(y'',\eta'')$. Using (19), we can find a vector field $\mathbf{Y}$ on C which satisfies

$$i(\mathbf{Y})\omega_C = dF = -\tilde{F}(s'',\sigma'')\frac{\partial}{\partial s_{n-1}} + O(|\sigma^{(iv)}|^2)\cdot(ds'',d\sigma'').$$

Thus, $exp(H_f)$ is a canonical transformation of $T^*Y\setminus 0$, while $exp(\mathbf{Y})$ is an ω_C-morphism of C, mapping $\{\tau_{n-1}=0,\tau^{(iv)}=0\}$ into $\{s_{n-1}=0,\sigma^{(iv)}=0\}$. Applying these simultaneously on T^*Y and C, respectively, we see that one can assume that $L:=\{\tau_{n-1}=0,\tau^{(iv)}=0\}=\{s_{n-1}=0,\sigma^{(iv)}=0\}$. Restricted to this $(2n-4)$-dimensional submanifold, ω_C is symplectic, so by Darboux there exists a canonical transformation $\Phi(y'',\eta'')=(\Phi_{y''},\Phi_{\eta''})$ of $T^*\mathbb{R}^{n-2}\setminus 0$ such that

$$\Phi^*(s''|_L)=t''|_L \quad \text{and } \Phi^*(\sigma''|_L)=\tau''|_L.$$

Extend Φ to $\tilde{\Phi}: T^*Y\setminus 0\longrightarrow T^*Y\setminus 0$ by

$$\tilde{\Phi}(y'',y_{n-1};\eta'',\eta_{n-1})=(\Phi_{y''}(y'',\eta''),y_{n-1};\Phi_{\eta''}(y'',\eta''),\eta_{n-1})$$

and compose C on the right with the graph of $\tilde{\Phi}$. Then,

$$\pi_L^*(x'')=t''=s'', \pi_L^*(x_{n-1}), \pi_L^*(x_n)=t_{n-1}, \pi_R^*(y_{n-1})=s_{n-1} \text{ and } \pi_R^*(\eta'')=\sigma''$$

form coordinates on C near c_0.

In summary, we have so far shown that if C is a folded cross cap, we may assume that (x,y_{n-1},η'') form (micro)local coordinates on C. Therefore there is a generating function $S(x,y_{n-1},\theta'')$ for C, where S is C^∞ and homogeneous of degree 1 in θ'' ([16, Thm.21.2.18]). Hence, C can be parametrized as

$$C=\left\{(x,d_xS;d_{\theta''}S,y_{n-1},-\theta'',-d_{y_{n-1}}S)\right\} \tag{25}$$

and the phase function $\chi(x,y,\theta'')=S(x,y_{n-1},\theta'')-y''\cdot\theta''$ is a parametrization for the Lagrangian C'. We now show that the properties of π_L and π_R impose several conditions on S and its derivatives, forcing the phase function to be very similar to the model phase ϕ_0 considered in §5.

To prepare the canonical relation C, we first replicate (15) and (16). Since $\pi_R(\mathcal{S}_1^C)$ is a nonradial hypersurface in $T^*Y \setminus 0$, by Darboux's theorem, microlocally there is a canonical transformation from $T^*Y \setminus 0$ to $T^*\mathbb{R}^{n-1} \setminus 0$ taking $\pi_R(\mathcal{S}_1^C)$ to $\pi_R(\mathcal{S}_1^{C_0})$, given by (15). Similarly, $\pi_L(\mathcal{S}_1^C)$ is a codimension three submanifold of $T^*X \setminus 0$, which is maximally noninvolutive in the sense that $\omega_{T^*X}|_{\pi_L(\mathcal{S}_1^C)}$ has rank $2n-4$. Thus, there exist defining functions p_1, p_2, p_3 for $\pi_L(\mathcal{S}_1^C)$ with $\{p_1, p_2\} = 1, \{p_1, p_3\} = 0, \{p_2, p_3\} = 0$. By Darboux's theorem, we can find a canonical transformation from $T^*X \setminus 0$ to $T^*\mathbb{R}^n \setminus 0$ mapping $\pi_L(\mathcal{S}_1^C)$ to $\pi_L(\mathcal{S}_1^{C_0})$ given by (16).

Using (25), $\pi_R(x, y_{n-1}, \theta'') = (d''_\theta S, y_{n-1}, -\theta'', -d_{y_{n-1}}S)$, so (15) implies

$$-d_{y_{n-1}}S|_{\{x_{n-1}=0=x_n+y_{n-1}\}} = x_n^2\theta_1 \tag{26}$$

From (25), $\pi_L(x, y_{n-1}, \theta'') = (x, d_x S)$ and (16) implies

$$d_{x_{n-1}}S|_{\{x_{n-1}=0=x_n+y_{n-1}\}} = 0 \tag{27}$$

and

$$d_{x_n}S|_{\{x_{n-1}=0=x_n+y_{n-1}\}} = -x_n^2 d_{x_1}S|_{\{x_{n-1}=0=x_n+y_{n-1}\}} \tag{28}$$

Relation (27) means that

$$S(x, y_{n-1}, \theta'') = S_0(x'', \theta'') + x_{n-1}^2 S_1(x, y_{n-1}, \theta'') + (x_n + y_{n-1})S_2(x, y_{n-1}, \theta'') \tag{29}$$

From (26) we obtain

$$S_2(x, y_{n-1}, \theta'')|_{\{x_{n-1}=0=x_n+y_{n-1}\}} = -x_n^2\theta_1, \tag{30}$$

and hence

$$S_2(x, y_{n-1}, \theta'') = -x_n^2\theta_1 + x_{n-1}S_3(x, y_{n-1}, \theta'') + (x_n + y_{n-1})S_4(x, y_{n-1}, \theta''). \tag{31}$$

Identity (31) implies that

$$S_2(x, y_{n-1}, \theta'')|_{\{x_{n-1}=0=x_n+y_{n-1}\}} = -x_n^2 d_{x_1}S|_{\{x_{n-1}=0=x_n+y_{n-1}\}}. \tag{32}$$

Now, (30) and (32) imply $d_{x_1}S|_{\{x_{n-1}=0=x_n+y_{n-1}\}} = \theta_1$, and from (29) we have that $d_{x_1}S_0 = \theta_1$, so $S_0 = x_1\theta_1 + \tilde{S}_0(x''', \theta''')$. At this point, our analysis shows that the generating function has the form

$$\begin{aligned} S(x, y_{n-1}, \theta'') = x_1\theta_1 \quad &+ \quad \tilde{S}_0(x''', \theta''') + x_{n-1}^2 S_1(x', y_{n-1}, \theta'') \\ &+ \quad (x_n + y_{n-1})[-x_n^2\theta_1 + x_{n-1}S_3(x', y_{n-1}, \theta'') \\ &\qquad + (x_n + y_{n-1})S_4(x', y_{n-1}, \theta'')]. \end{aligned} \tag{33}$$

Next we consider the differentials $d\pi_R$ and $d\pi_L$. One computes

$$d\pi_R = \begin{pmatrix} d_{x''\theta''}S & d_{x_{n-1}\theta''}S & d_{x_n\theta''}S & d_{y_{n-1}\theta''}S & d_{\theta''\theta''}S \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -I_{n-2} \\ -d_{x''y_{n-1}}S & -d_{x_{n-1}y_{n-1}}S & -d_{x_ny_{n-1}}S & -d_{y_{n-1}y_{n-1}}S & -d_{\theta''y_{n-1}}S \end{pmatrix}$$

Evaluating at $\mathcal{S}_1^C$, by (33), this becomes

$$d\pi_R|_{S_1} = \begin{pmatrix} d_{x''\theta''}S_0 & 0 & 0 & 0 & d_{\theta''\theta''}S_0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -I_{n-2} \\ 0 & S_3 & -2x_n\theta_1 + 2S_4 & 2S_4 & 0 \end{pmatrix}$$

By assumption, $d\pi_R$ has rank $2n-3$ at $\mathcal{S}_1^C$. Thus, $S_0(x'', \theta'')$ is nondegenerate, $S_3|_{\{x_{n-1}=0=x_n+y_{n-1}\}}$=0 and $S_4|_{\{x_{n-1}=0=x_n+y_{n-1}\}} = x_n\theta_1$. We thus have

$$S_3(x', y_{n-1}, \theta'') = x_{n-1}S_5(x', y_{n-1}, \theta'') + (x_n + y_{n-1})S_6(x', y_{n-1}, \theta''), \quad (34)$$

and

$$S_4(x', y_{n-1}, \theta'') = x_n\theta_1 + x_{n-1}S_7(x', y_{n-1}, \theta'') + (x_n + y_{n-1})S_8(x', y_{n-1}, \theta''). \quad (35)$$

Similarly,

$$d\pi_L = \begin{pmatrix} I_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ d_{x''x''}S & d_{x_{n_1}x''}S & d_{x_nx''}S & d_{y_{n-1}x''}S & d_{\theta''x''}S \\ d_{x''x_{n-1}}S & d_{x_{n-1}x_{n-1}}S & d_{x_nx_{n-1}}S & d_{y_{n-1}x_{n-1}}S & d_{\theta''x_{n-1}}S \\ d_{x''x_n}S & d_{x_{n-1}x_n}S & d_{x_nx_n}S & d_{y_{n-1}x_n}S & d_{\theta''x_n}S \end{pmatrix}$$

Using relation (33) again, we have

$$d\pi_L|_{S_1} = \begin{pmatrix} I_{n-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ d_{x''x''}S_0 & 0 & 0 & 0 & d_{\theta''x''}S_0 \\ 0 & 2S_1 & N & S_3 & 0 \\ 0 & N & 2S_4 - 2x_n\theta_1 & 2S_4 - 2x_n\theta_1 & 0 \end{pmatrix},$$

where $N = S_3 + \partial_{x_{n-1}}S_4$.

By the cross cap condition, the rank of $d\pi_L|_{\{x_{n-1}=0=x_n+y_{n-1}\}}$ is $2n-2$. Thus, $S_3|_{\{x_{n-1}=0=x_n+y_{n-1}\}}=0$ and $S_4|_{\{x_{n-1}=0=x_n+y_{n-1}\}}=x_n\theta_1$, and so we obtain again relations (34) and (35). Putting together all the previous relations, the generating function becomes:

$$\begin{aligned} S(x,y_{n-1},\theta'')=x_1\theta_1 \quad &+ \quad \tilde{S}_0(x''',\theta''')+x_{n-1}^2 S_1(x,y_{n-1},\theta'')+(x_n+y_{n-1})x_n y_{n-1}\theta_1 \\ &+ \quad (x_n+y_{n-1})[x_{n-1}^2 S_5(x,y_{n-1},\theta'') \\ &\qquad +x_{n-1}(x_n+y_{n-1})S_6(x,y_{n-1},\theta'') \\ &\qquad +(x_n+y_{n-1})^2 S_7(x,y_{n-1},\theta'')]. \end{aligned} \tag{36}$$

Since S_0 is nondegenerate, $C\cap\{x_{n-1}=x_n=y_{n-1}=0\}$ is the graph of a canonical transformation on T^*R^{n-2}; we may thus assume that $C\cap\{x_{n-1}=x_n=y_{n-1}=0\}=\{(x'',0,\theta'',0;x'',0,\theta'',0)\}$, so that $S_0(x'',\theta'')=x''\cdot\theta''$, and (36) becomes:

$$\begin{aligned} S(x,y_{n-1},\theta'')=x''\cdot\theta'' \quad &+ \quad x_{n-1}^2 S_1(x,y_{n-1},\theta'')+(x_n+y_{n-1})x_n y_{n-1}\theta_1 \\ &+ \quad (x_n+y_{n-1})[x_{n-1}^2 S_5(x,y_{n-1},\theta'') \\ &\qquad x_{n-1}(x_n+y_{n-1})S_6(x,y_{n-1},\theta'') \\ &\qquad (x_n+y_{n-1})^2 S_7(x,y_{n-1},\theta'')] \end{aligned}$$

Letting

$$\begin{aligned} M(x,y_{n-1},\theta'')=x_{n-1}^2 S_5(x,y_{n-1},\theta'')+x_{n-1}(x_n+y_{n-1})S_6(x,y_{n-1},\theta'') \\ +(x_n+y_{n-1})^2 S_7(x,y_{n-1},\theta''), \end{aligned}$$

we now see that the Lagrangian C' is parametrized by the phase function

$$\begin{aligned} \chi(x,y,\theta'') &= S(x,y_{n-1},\theta'')-y''\cdot\theta'' \\ &= (x''-y'')\cdot\theta''+\left(x_n^2 y_{n-1}+x_n y_{n-1}^2\right)\theta_1 \\ &\quad +x_{n-1}^2 S_1(x,y_{n-1},\theta'')+(x_n+y_{n-1})M(x,y_{n-1},\theta''). \end{aligned} \tag{37}$$

We are now ready to prove Thm.6.1. Composing F on the left and right with elliptic FIOs of order 0 associated with all of the canonical transformations of $T^*X \setminus 0$ and $T^*Y \setminus 0$ used above, we can assume that the Schwartz kernel of F is represented by an oscillatory integral with the phase function (37) and an amplitude $a \in S^{m+\frac{1}{2}}$.

Let $\tilde{\chi}$ be the phase function of F^*F:

$$\begin{aligned}
\tilde{\chi} &= \chi(z,y,\eta'') - \chi(z,x,\xi'') \\
&= (z''-y'')\cdot\eta'' + z_n^2 y_{n-1}\eta_1 + z_n y_{n-1}^2\eta_1 + z_{n-1}^2 S_1(z,y_{n-1},\eta'') \\
&\quad +(z_n+y_{n-1})M(z,y_{n-1},\eta'') - (z''-x'')\xi'' - z_n^2 x_{n-1}\xi_1 \\
&\quad -z_n x_{n-1}^2\xi_1 - z_{n-1}^2 S_1(z,x_{n-1},\xi'') - (z_n+x_{n-1})M(z,x_{n-1},\xi'').
\end{aligned}$$

We will use stationary phase in z'' and η'': set $d_{z''}\tilde{\chi}=0$ and $d_{\eta''}\tilde{\chi}=0$, where

$$\begin{aligned}
d_{z''}\tilde{\chi} &= \eta''-\xi''+(z_n+y_{n-1})\partial_{z''}M(z,y_{n-1},\eta'')-(z_n+x_{n-1})\partial_{z''}M(z,x_{n-1},\xi'') \\
&\quad +z_{n-1}^2(\partial_{z''}S_1(z,y_{n-1},\eta'')-\partial_{z''}S_1(z,x_{n-1},\xi'')) \\
d_{\eta_1}\tilde{\chi} &= z_1-y_1+z_n^2y_{n-1}+z_ny_{n-1}^2+(z_n+y_{n-1})\partial_{\eta_1}M(z,y_{n-1},\eta'')+z_{n-1}^2\partial_{\eta_1}S_1(z,y_{n-1},\eta'') \\
d_{\eta_i}\tilde{\chi} &= z_i-y_i+(z_n+y_{n-1})\partial_{\eta_i}M(z,y_{n-1},\eta'')+z_{n-1}^2\partial_{\eta_i}S_1(z,y_{n-1},\eta''), \quad 2\le i\le n-2.
\end{aligned}$$

Notice that $d^2_{z''\eta''}\tilde{\chi}$ is nondegenerate. We may solve these equations implicitly for z'' and η'' in terms of the other variables:

$$\begin{aligned}
\eta'' &= \xi''+(z_n+x_{n-1})\partial_{z''}M(z,x_{n-1},\xi'')-(z_n+y_{n-1})\partial_{z''}M(z,y_{n-1},\eta'') \\
&\quad -z_{n-1}^2(\partial_{z''}S_1(z,y_{n-1},\eta'')-\partial_{z''}S_1(z,x_{n-1},\xi'')) \\
z_i &= y_i-(z_n+y_{n-1})\partial_{\eta_i}M(z,y_{n-1},\eta'')-z_{n-1}^2\partial_{\eta_i}S_1(z,y_{n-1},\eta''), \quad 2\le i\le n-2; \\
z_1 &= y_1-z_n^2y_{n-1}-z_ny_{n-1}^2-(z_n+y_{n-1})\partial_{z_1}M(z,y_{n-1},\eta'')-z_{n-1}^2\partial_{\eta_1}S_1(z,y_{n-1},\eta'').
\end{aligned}$$

The phase $\tilde{\chi}$ then becomes:

$$\begin{aligned}
\tilde{\tilde{\chi}} &= (x''-y'')\cdot\xi''+z_n^2(y_{n-1}-x_{n-1})\xi_1+z_n(y_{n-1}^2-x_{n-1}^2)\xi_1 \\
&\quad +(z_n+y_{n-1})M(z,y_{n-1},\eta'')-(z_n+x_{n-1})M(z,x_{n-1},\xi'') \\
&\quad +z_{n-1}^2(S_1(z,y_{n-1},\eta'')-S_1(z,x_{n-1},\xi''))+C(x,y,z,\xi'',\eta'') \\
&:= (x''-y'')\cdot\xi''+(y_{n-1}-x_{n-1})[\big(z_n^2+z_n(y_{n-1}+x_{n-1})\big)\xi_1 \\
&\quad +z_{n-1}^2\xi_1U(z_{n-1},z_n,x_{n-1},y_{n-1},\xi'')+N(z_{n-1},z_n,x_{n-1},y_{n-1},\xi'')]
\end{aligned}$$

and the amplitude becomes $\tilde{a}\in S^{2m+1}$. Note for use below that, in the stationary phase calculation, $\eta''=\xi''$ at $\{x_{n-1}=y_{n-1}\}$. Thus, both U and N vanish at $\{x_{n-1}=y_{n-1}\}$, and so do any derivatives of U and N tangential to $\{x_{n-1}=y_{n-1}\}$.

Repeating the argument from §5, we make a singular change of variables,

$$\begin{aligned}
\theta_i &= \xi_i, \quad 1 \le i \le n-2, \\
\theta_{n-1} &= -(z_n^2\xi_1 + z_n(y_{n-1}+x_{n-1})\xi_1 + z_{n-1}^2\xi_1 U(z, x_{n-1}, y_{n-1}, \xi'', \eta'') \\
&\quad +N(z, x_{n-1}, y_{n-1}, \xi'', \eta'')).
\end{aligned}$$

We have

$$\begin{aligned}
\nabla_z\theta_{n-1} &= -(2z_{n-1}\theta_1 U + z_{n-1}^2\theta_1\partial_{z_{n-1}}U + \partial_{z_{n-1}}N, \\
&\qquad 2z_n\theta_1 + (x_{n-1}+y_{n-1})\theta_1 - \partial_{z_n}N + z_{n-1}^2\partial_{z_n}U),
\end{aligned}$$

so that $|\nabla_z\theta_{n-1}| = 0$ iff $z_{n-1} = -\frac{\partial_{z_{n-1}}N + z_{n-1}^2\theta_1\partial_{z_{n-1}}U}{2\theta_1 U}$ and $z_n = -\frac{x_{n-1}+y_{n-1}}{2} - \frac{\partial_{z_n}N + z_{n-1}^2\theta_1\partial_{z_n}U}{2\theta_1}$. At these points,

$$\begin{aligned}
\theta_{n-1} &= \frac{(x_{n-1}+y_{n-1})^2}{4}\theta_1 - N - \frac{1}{4\theta_1 U}(\partial_{z_{n-1}}N + z_{n-1}^2\theta_1\partial_{z_{n-1}}U)^2 \\
&\quad -\frac{1}{4\theta_1}(\partial_{z_n}N + z_{n-1}^2\theta_1\partial_{z_n}U)^2 \\
&:= \frac{(x_{n-1}+y_{n-1})^2}{4}\theta_1 + P.
\end{aligned}$$

By the comment above, P and its tangential derivatives vanish at $\{x_{n-1} = y_{n-1}\}$.

As in the model case, it follows that $K_{F^*F}(x,y)$ has an oscillatory integral representation,

$$\int_{\mathbb{R}^n} e^{i\{(x''-y'')\cdot\theta'' + (x_{n-1}-y_{n-1})\theta_{n-1} + \frac{\rho}{\theta_1}(\theta_{n-1} - \frac{(x_{n-1}+y_{n-1})^2}{4}\theta_1 - P)\}} a(x,y,\theta,\rho)d\theta d\rho,$$

where $a \in S^{2m,-1}(2n-2, n-1, 1)$. On the region $\{|\rho| \le c|\xi|\}$, the new phase function, $\psi(x,y;\theta;\rho)$ is a multi-phase for a pair $(\Delta', \tilde{C}')$ in the sense of Def.3.5: $\psi(x,y;\theta;0)$ parametrizes the diagonal Lagrangian Δ' and $\psi(x,y;\theta;\rho)$ parametrizes a Lagrangian $\tilde{C}'$. Hence, the contribution to F^*F from this region is in $I^{p,l}(\Delta, \tilde{C})$ for some $p, l \in \mathbb{R}$, and the orders of F^*F are computed using Remark 3.7 in the same way as for K^0 in the model case in §5, so that $p = 2m - \frac{1}{2}$ and $l = \frac{1}{2}$. On the other hand, the contribution from $\{|\rho| \ge c|\xi|\}$ is handled in the same way as for K^1 for the model case in §5, giving an element of $I^{2m-\frac{1}{2}}(\tilde{C}) \subset I^{2m-\frac{1}{2},\frac{1}{2}}(\Delta, \tilde{C})$.

Next, we show that $\tilde{C}$ is a two-sided fold; the fact that $\tilde{C}$ is symmetric just follows from $\Delta \cup \tilde{C} = C^t \circ C$. We have:

$$\begin{aligned}\tilde{C} = \big\{\big(\quad x'' \quad & , x_{n-1}, \theta'', \theta_{n-1} - \frac{x_{n-1}+y_{n-1}}{2}\rho - \frac{\rho}{\theta_1}\partial_{x_{n-1}}P; \\ y'' \quad & , y_{n-1}, \theta'', \theta_{n-1} + \frac{x_{n-1}+y_{n-1}}{2}\rho + \frac{\rho}{\theta_1}\partial_{y_{n-1}}P\big) : \\ & \theta_{n-1} - \frac{(x_{n-1}+y_{n-1})^2}{4}\theta_1 - P = 0, \quad x_{n-1} - y_{n-1} + \frac{\rho}{\theta_1} = 0, \\ & x_i - y_i + \frac{\rho}{\theta_1}\partial_{\theta_i}P = 0, \quad 2 \leq i \leq n-2, \\ & x_1 - y_1 - \frac{\rho}{\theta_1^2}(\theta_{n-1} - \frac{(x_{n-1}+y_{n-1})^2}{4}\theta_1 - P) - \frac{\rho}{\theta_1}\frac{(x_{n-1}+y_{n-1})^2}{4} - \frac{\rho}{\theta_1}\partial_{\theta_1}P = 0\big\}.\end{aligned}$$

The coordinates on $\tilde{C}$ are $(x'', x_{n-1}, y_{n-1}, \theta'')$. Note that $\rho = -(x_{n-1}-y_{n-1})\theta_1$ and $\theta_{n-1} = \frac{(x_{n-1}+y_{n-1})^2}{4}\theta_1 + P$; thus, y_i and y_1 become:

$$y_i = x_i - (x_{n-1} - y_{n-1})\partial_{\theta_i}P, \quad 2 \leq i \leq n-2,$$

and

$$y_1 = x_1 + \frac{(x_{n-1} - y_{n-1})(x_{n-1} + y_{n-1})^2}{4} + (x_{n-1} - y_{n-1})\partial_{\theta_1}P.$$

We now consider the projections π_R and π_L.

$$\begin{aligned}\pi_R(x'', x_{n-1}, y_{n-1}, \quad \theta'' \quad) & \\ = \quad & (x_1 + \frac{(x_{n-1} - y_{n-1})(x_{n-1} + y_{n-1})^2}{4} + (x_{n-1} + y_{n-1})\partial_{\theta_1}P, \\ & x''' - (x_{n-1} - y_{n-1})\partial_{\theta'''}P, y_{n-1}; \\ & \theta'', \frac{(x_{n-1} + y_{n-1})(3y_{n-1} - x_{n-1})}{4}\theta_1 + P - (x_{n-1} - y_{n-1})\partial_{y_{n-1}}P)\end{aligned}$$

and

$$\begin{aligned}\pi_L(\quad x'' \quad & , x_{n-1}, \theta'', y_{n-1}) \\ = \quad & \left(x'', x_{n-1}; \theta'', \quad \frac{(x_{n-1} + y_{n-1})(3x_{n-1} - y_{n-1})}{4}\theta_1 + P + (x_{n-1} - y_{n-1})\partial_{x_{n-1}}P\right).\end{aligned}$$

One easily computes

$$d\pi_L = \begin{pmatrix} I_{n-2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_{n-2} & 0 \\ 0 & \cdot & \cdot & D \end{pmatrix}$$

where $D = \frac{x_{n-1}-y_{n-1}}{2}\theta_1 + \partial_{y_{n-1}}P - \partial_{x_{n-1}}P + (x_{n-1} - y_{n-1})\partial^2_{x_{n-1}y_{n-1}}P$, and thus det $d\pi_L = D$. By the earlier comments, we see that D is a defining function for $\{x_{n-1} = y_{n-1}\}$, and thus the critical set for π_L (and hence π_R) is

$$\mathcal{S}_1^{\tilde{C}} := \mathcal{S}_1(\pi_L) = \{D(x'', x_{n-1}, y_{n-1}, \theta'') = 0\} = \{x_{n-1} = y_{n-1}\}\,.$$

The kernel of $d\pi_L$ is spanned by $\frac{\partial}{\partial y_{n-1}}$. Since Ker $d\pi_L$ is transversal to $T\mathcal{S}_1^{\tilde{C}}$, π_L has a Whitney fold singularity.

Similarly, with respect to the coordinates $(x_1, x''', y_{n-1}, \theta'', x_{n-1})$ on $\tilde{C}$ (reordered for convenience) and $(y_1, y''', y_{n-1}; \eta'', \eta_{n-1})$ on T^*Y,

$$d\pi_R = \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ A & B & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & I_{n-2} & 0 \\ \cdot & \cdot & \cdot & \cdot & -D \end{pmatrix}$$

where $A = -(x_{n-1} - y_{n-1})\partial^2_{\theta''' x_1}P$ and $B = I_{n-3} - (x_{n-1} - y_{n-1})\partial^2_{\theta''' x'''}P$. Hence, det $d\pi_R = D$, as well, and Ker $d\pi_R$ is spanned by $\frac{\partial}{\partial x_{n-1}}$, which is again transversal to $T\mathcal{S}_1^{\tilde{C}}$. Hence, π_R is also a Whitney fold, and thus $\tilde{C}$ is a symmetric, two-sided fold, with $\Delta \cap \tilde{C} = \mathcal{S}_1^{\tilde{C}}$.

Remark 6.4. If F is the linearized forward scattering operator for the marine geometry, under the assumptions of (i) at most fold caustics; (ii) the caustic surface is nowhere horizontal; and (iii) $\pi_L(\mathcal{S}_1^C)$ is nonradial, then combining [27] with the analysis in §4 yields that $F \in I^{1-\frac{1}{4}}(\Sigma_{r,s} \times (0,T), Y; C)$, with C a folded cross cap. By Thm. 6.1, we thus obtain that the normal operator $F^*F \in I^{\frac{3}{2},\frac{1}{2}}\left(Y, Y; \Delta, \tilde{C}\right)$ with $\tilde{C} \subset (T^*Y \setminus 0) \times (T^*Y \setminus 0)$ a two-sided fold.

Remark 6.5. An interesting question, particularly relevant for the marine seismic imaging problem, is whether (under an ellipticity assumption on F), F^*F can be inverted, at least modulo operators mapping $H^s \to H^{s-2m+\delta}$ for some $\delta > 0$. Note that, by Remark 3.4, the order of F^*F on $\tilde{C} \setminus \Delta$ is $\frac{1}{2}$ less than on $\Delta \setminus \tilde{C}$. However, the standard technique of parabolic decomposition (see [9]) only results in a decomposition $F^*F = T_1 + T_2$ with $T_1 \in I^{2m}_{\frac{1}{2},\frac{1}{2}}(\Delta)$ and $T_2 \in I^{2m}_{\frac{1}{2},\frac{1}{2}}(\tilde{C})$. Since $\tilde{C}$ is a two-sided fold, there is further loss of $\frac{1}{6}$ derivatives in terms of Sobolev mapping properties [20]. We hope to return to this question in the future.

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