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HENRY KANDRUP'S IDEAS ABOUT RELAXATION OF STELLAR SYTEMS

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Abstract: Henry Kandrup wrote prolifically on the problem of relaxation of stellar systems. His picture of relaxation was significantly more refined than the standard description in terms of phase mixing and violent relaxation. In this article, I summarize Henry's work in this and related areas.

Henry Kandrup was a leading figure in galactic dynamics, distinguishing himself both as an original thinker and as an educator. His published contributions were prolific and wide-ranging. While I never formally collaborated with Henry, we often discussed dynamics during my visits to the University of Florida or at scientific meetings, and both of us were conscientious readers of the other's papers. Henry's ideas often influenced my own work, and I believe that I can see evidence of the reverse influence in some of Henry's published papers.

Here I will focus on a topic that occupied Henry's attention throughout his career: dynamical chaos and its connection with relaxation and equilibrium of stellar systems. Henry first addressed this topic in his PhD thesis, "Stochastic Processes in Stellar Dynamics" [1], and returned to it in one of his last papers, "Chaos and Chaotic Phase Mixing in Galaxy Evolution and Charged Particle Beams" [2].

Henry's unique contribution was to associate relaxation in stellar systems with the exponential instability of orbits. Prior to Henry's work, discussions in the astronomical literature of dynamical relaxation rarely addressed the mixing properties of the flow, and discussions of stochasticity rarely drew conclusions about relaxation. Henry argued that there was a fundamental connection between the two. He pointed out that the trajectories of stars even in "collisionless" systems like galaxies could often mimic the exponentially unstable trajectories of molecules in a collisional fluid, due either to non-integrability of the steady-state potential, or to time variations in the potential associated with external perturbations or departures from a steady state. Henry demonstrated that this chaos or near-chaos could be very effective at inducing evolution to a steady state, in much the same way that collisions between molecules in a gas erase memory of the initial conditions. In this way, Henry established a new and important link between dynamical chaos, statistical mechanics, and the structure and evolution of stellar systems.

Many stellar and galactic systems are smooth and symmetric in appearance. This is surprising at first sight, since the time required for gravitational encounters to "smear out" trajectories is very long in such low-density systems, often much longer than the age of the universe. One of the first to note this puzzle was Fritz Zwicky [3]. Zwicky was so impressed by the regular appearance of the Coma galaxy cluster that he argued against Hubble's expanding-universe model – on the grounds that the age of the universe in Hubble's model was too short for encounters between galaxies to remove irregularities in their spatial distribution.

At the time that Henry began his career, the smooth appearance of galaxies and galaxy clusters was generally understood to be due to a combination of phase mixing and "violent relaxation". Phase mixing is the gradual shearing of points in a fixed, integrable potential; after many orbital periods, phase mixing results in a coarse-grained density that is constant with respect to angle over the invariant torus. "Violent relaxation" was defined as the more extreme redistribution of particles that occurs when the gravitational potential is rapidly varying. King [4], Hénon [5], Lynden-Bell [6] and others realized that relaxation under conditions of a rapidly varying potential might be very efficient, and the last-named author identified the relaxation rate directly with the rate of change of the potential. Support for this hypothesis was seen in numerical simulations of the collapse of a cold cloud of stars, where a nearly steady state is reached after just a few crossing times.

Henry and I often discussed violent relaxation during my visits to the University of Florida. During one of these conversations, Henry told me that he had been very impressed by a letter written by Richard Miller to Donald Lynden-Bell on July 21, 1966. In the letter, Miller argued that the identification of relaxation with changes in the potential was problematic:

> A counter example is furnished by a potential that depends only upon the time. Consider a stellar dynamical system described by a Hamiltonian H_0, and another described by $H = H_0 + V(t)$ where V depends only upon the time. All motions in the two systems are identical; no relaxation is induced by $V(t)$, contrary to the assertion of your equation 1.

Miller went on to note:

> I think there is a germ of an idea in your assertion [that relaxation can be identified with changes in the potential], but it wants more complete working-out. Essentially, I think that the kind of term that might replace $\langle\dot{\Phi}^2\rangle$ in your equations is $\langle(G\rho)^{-1/2}\dot{\Phi}(a \cdot \nabla\Phi)\rangle$ or some rather fancy term of that character – displaying both time and space derivatives, but averaged over the cluster and measured in time units characteristic of the cluster. (a is some length characterizing the cluster.)

Henry told me that he shared Miller's reservations about equating relaxation with $\dot{\Phi}$. As he later put it [7]:

> More pragmatically, one infers from N-body simulations that a strongly convulsing mean field potential is not necessary. One observes a comparably efficient approach towards a meta-equilibrium on a time scale $\sim t_{cr}$ [the crossing time] both for "violent" evolution, where Φ exhibits huge changes on a very short time scale, and for "nonviolent" evolution, where Φ exhibits only relatively small changes. Nonviolent relaxation can be just as efficient as violent relaxation.

In this passage, Henry is referring to what he elsewhere called "chaotic mixing": the exponential spreading of an ensemble of initially localized, stochastic trajectories. He continued:

> This means that phase mixing can proceed *much* more efficiently for chaotic flows than for regular flows, where any approach towards a (near-) equilibrum typically proceeds as a power law in time. Chaotic flows should relax much more efficiently than do regular flows. It would thus seem that the phase mixing of chaotic flows ... could serve to provide an explanation of why various systems in nature seem to approach an equilibrium or near-equilibrium as fast as they do. In particular, chaotic mixing could help explain the remarkable efficacy of violent relaxation: Why do galaxies look "so relaxed" when the nominal relaxation time t_R is typically much longer than t_H, the age of the Universe?

Rather than identify relaxation with either phase mixing in a fixed potential, or "violent relaxation" in a time-varying potential, Henry proposed that the proper distinction was between phase mixing and chaotic mixing, and that the time dependence of the potential was secondary. The efficient relaxation observed in simulations of collapse was due, he argued, to the more chaotic nature of the phase-space flow when the potential was rapidly varying, and not simply to the redistribution of energies that takes place when the potential has a time-dependent component.

While the existence of stochastic orbits in galactic potentials had been appreciated since the work of Hénon, Contopoulos, Miller and others in the 1960s, Henry was one of the first to ask what the *consequences* of the chaos might be for the evolution of an ensemble of orbits toward a statistical steady state. Henry began a systematic investigation of this question by looking at the effects of chaos in time-independent potentials; by definition, "violent relaxation" can not occur if the potential is fixed. In a series of papers with M. E. Mahon and other collaborators [8-11], Henry investigated the relation between stochasticity of individual trajectories – as measured, for instance, by Liapunov exponents – and the rate at which an initially compact *ensemble* of stars evolves toward a uniform distribution over the accessible phase space. These papers showed that the coarse-grained distribution function typically exhibits an exponential approach toward equilibrium at a rate that correlates well with the mean Liapunov exponent for the ensemble. When evolved for much longer times, the phase space density slowly changes as orbits diffuse into regions that, although accessible, are avoided over the shorter time interval. Henry coined the term

"near-invariant distribution" to describe the end-point of chaotic mixing of an isoenergetic ensemble of points.

In another paper [12], Henry compared the efficiency of phase mixing and chaotic mixing. He noted that – for initially very localized ensembles – the two processes occur at very different rates: chaotic mixing takes place on the Liapunov, or exponential divergence, time scale while the phase mixing rate falls to zero. But phase mixing of a group of points with a finite extent can be much more rapid. Furthermore the mixing rate of chaotic ensembles typically falls below the Liapunov rate once the trajectories separate; this is especially true for those stochastic orbits that are confined over long periods of time to restricted parts of phase space. The effective rates of phase mixing and chaotic mixing might therefore be comparable in real galaxies. Henry noted also that chaotic mixing in 3D potentials can occur at substantially different rates in different directions.

Henry recognized that the existence of invariant or near-invariant distributions, in regions of phase space that were not characterized by three isolating integrals of motion, implied the existence of a new class of equilibrium or near-equilibrium states for galaxies. The classical Jeans theorem requires that the phase-space density of a stationary stellar system be expressed solely in terms of the isolating integrals in that potential. Henry pointed out that there existed a far larger class of systems that could be in a stationary state. In the abstract of "Invariant Distributions and Collisionless Equilibria" [13], he wrote:

> This paper discusses the possibility of constructing time-independent solutions to the collisionless Boltzmann equation which depend on quantities other than global isolating integrals such as energy and angular momentum. The key point is that, at least in principle, a self-consistent equilibrium can be constructed from *any* set of time-independent phase-space building blocks which, when combined, generate the mass distribution associated with an assumed time-independent potential.

While noting that strictly "time-independent" phase-space building blocks were mathematical idealizations, Henry pointed out that his "near-invariant distributions" were effectively time independent, and could in principle be used as building blocks in the construction of stationary galaxies. Indeed he argued that chaotic orbits were in a sense more natural components of steady-state galaxies than regular orbits, since an efficient mechanism (chaotic mixing) exists that can convert a generic distribution of points in chaotic phase space into a time-independent one. By contrast, an ensemble of points on a regular torus does not evolve toward a coarse-grained steady state: it simply translates, unchanged, around the torus. Jeans's theorem *postulates* a uniform distribution over each torus but says nothing about how this unlikely distribution is to be achieved.

Henry's insight constituted the most significant updating of Jeans's theorem since at least the 1960s, when various authors pointed out the distinction be-

tween isolating and non-isolating integrals. I propose that a "generalized Jeans theorem" be attributed to Henry:

> **Generalized Jeans Theorem:** The phase-space density of a stationary stellar system is constant within every connected region.

A "connected region" is one that can not be decomposed into two finite regions such that all trajectories remain for all time in either one or the other. Invariant tori are such regions, but so are the more complex parts of phase space associated with stochastic orbits.

As Henry once pointed out to me (with some amusement), people have actually been invoking the generalized version of Jeans's theorem for years without realizing it! A textbook example of a system satisfying the classical Jeans theorem (and one that was discussed by Jeans himself [14]) is an axisymmetric galaxy in which f is a function of the two classical integrals of motion, the energy E and the angular momentum J_z. Writing $f = f(E, J_z)$ implies that the phase space density is constant on hypersurfaces of constant E and J_z. But not all orbits in axisymmetric potentials are characterized by a third isolating integral, hence parts of these hypersurfaces are associated with chaotic trajectories. The two-integral approach to axisymmetric modelling – which assigns a constant density to these regions – thus depends on the generalized form of Jeans's theorem for its justification. Henry went on to note [13] that one could in principle construct novel axisymmetric models, in which the surfaces of constant E and J_z are *not* sampled uniformly; for instance, one could exclude all the chaotic orbits, or assign different densities to different chaotic regions on the same (E, J_z) hypersurface. As far as I know, no one has yet attempted to construct models of this form, although it would be relatively straightforward to do so via orbital superposition. Nevertheless, it has become clear in the last few years that chaotic orbits can be major components of steady-state galaxies, demonstrating that Henry's generalized theorem is potentially very significant for our understanding galactic structure. For instance, one recent study [15] found that 50% or more of the mass in steady-state triaxial nuclei could be associated with chaotic orbits.

Smooth potentials are idealizations of real galaxies. As seen by a single star, any lumpiness or distortions in the stellar density would add small-amplitude perturbing forces to the mean field. Such perturbations would not be expected to have much consequence for either strongly chaotic or precisely regular orbits, but Henry realized that they might have an appreciable effect on the evolution of weakly stochastic orbits, by scattering trajectories away from a trapped region into a region where the mixing is more rapid. Henry, S. Habib and M. E. Mahon [16-18] investigated this idea, adding random noise to otherwise smooth potentials and observing the effects on the mixing rate. They found that even very weak noise, with a characteristic time scale $|\frac{1}{v}\frac{\delta v}{\delta t}|^{-1}$ of order 10^6 crossing times, could induce substantial changes in the motion of trapped stochastic orbits over just $\sim 10^2$ orbital periods.

In a strongly time-dependent potential, chaos should be even more preva-

lent, if only because time-dependent potentials lack at least one isolating integral of the motion (the energy) that is always present in stationary potentials. In three studies [19-21], Henry and collaborators considered the effects of two sorts of strongly time-dependent perturbations on the structure of orbits in 2D and 3D potentials: a time-dependent scale factor $R(t) = t^p$, mimicking expansion or collapse; and strictly periodic driving, a crude model of the oscillations that accompany the final stages of relaxation. By computing the values of short-time Liapunov exponents, Henry and co-workers found that trajectories in the expanding/contracting potential could mimic regular orbits part of the time and stochastic orbits at other times. Contraction tended to make the effects of chaos stronger, and expansion tended to make the chaos weaker. In the oscillatory model, orbits appeared to remain either regular or chaotic for all times, although the periodicity in the global potential seemed to induce some orbits to become what Henry termed "wildy chaotic," exhibiting substantial changes in energy as they chaotically diffused.

Having demonstrated the importance of chaotic mixing in stationary, weakly time dependent, and strongly time dependent potentials, it remained only for Henry only to make explicit the link between chaotic mixing and "violent relaxation." Henry did not get quite this far before his death, but he had a clear idea of how to proceed. In this passage, he describes a simple and beautiful scheme for establishing the connection:

> But what, if anything, might these conclusions imply about violent relaxation? At least crudely, one can visualize an evolution described by the collisinless Boltzmann equation as involving a collection of characteristics corresponding to orbits evolved in a specified time-dependent potential, ignoring the fact that that potential is generated self consistently ... to the extent that this picture is valid, one might then anticipate that the efficacy with which an initial ensemble appproaches an equilibrium or near-equilibrium ... will depend on the degree to which the flow in the specified potential is chaotic. In particular, to the extent that the flow is chaotic, one would expect a rapid and efficient approach towards a near-equilibrium.

In Henry's scheme, one would first carry out a fully self-consistent simulation of collapse and virialization, via an N-body code say, recording the gravitational potential on a grid in both space and time. Returning to the initial conditions, one would then select out initially localized ensembles of phase-space points and evolve them forward in the previously-recorded potential, this time ignoring the self-gravity of the ensemble. The sum total of all such integrations would be a reproduction of the self-consistent collapse, and by analyzing the rate of approach of each ensemble to its near-invariant distribution, one could generate a potentially complete picture of the way in which the properties of the phase-space flow were related to the rate of "violent relaxation." Shortly before his death, Henry told me that he was hoping to carry out this program in collaboration with a student, but apparently this wish never came to fruition.

Like most dynamicists, Henry was fascinated by the concept of entropy. Henry's ideas about entropy were complex, and I'm not sure that I understood them completely, but overall I felt that Henry was skeptical about the relevance of entropy arguments to galactic dynamics. Here I can not resist quoting from V. A. Antonov [22], whose skeptical view of entropy was similar, I think, to Henry's:

> True diffusion is well described by differential equations. On the contrary, mixing is not represented in terms of differential operations. There is the phase density before the mixing, and the phase density after the mixing, but it is difficult to define when and where the transmutation occurs. We could not work out general and utilizable equations of the mixing.

Henry understood that chaotic mixing is irreversible, in the sense that an infinitely precise fine-tuning of the velocities would be required in order to undo its effects. This is a sort of entropy increase, and it implies an evolution toward a state whose properties are in some ways predictable. But like Antonov, Henry realized that it is difficult to establish very general rules that link the initial and final states of a stellar system that evolves via collisionless relaxation, in particular, rules that would allow one to make statements about which final states are preferred. Henry was critical [23], for instance, about a purported demonstration [24] of an "H-theorem" for collisionless systems:

> It is, moreover, clear physically that there exist 'reasonable' choices of initial data, such as those leading to nearly homologous collapse, which exhibit nearly periodic motion; and for such data, one might anticipate that, after one approximate period, H will have returned very nearly to its initial value ... Because the H-functions ... need not increase monotonically, they cannot be used to provide a useful characterization of the continuous dynamics.

Henry also made fundamental contributions to our understanding of another sort of chaos characteristic of stellar systems. Already in 1964 [25], Richard Miller had shown that the trajectories of stars in small-N-body integrations were generically chaotic, in the sense that the $6N$-dimensional phase path was exponentially unstable to small changes in the initial conditions. In a series of papers [26-29], Henry and collaborators carried out a systematic numerical study of this instability. They found that the time scale for growth of perturbations tended to *decrease* with increasing N, remaining of order the crossing time for values of N as large as 4000. This result was consistent with Henry's earlier theoretical arguments [30-32] and in contradiction with a prediction of Gurzadyan & Savvidy [33] that the growth rate should fall as $1/N^{1/3}$. (Henry's prediction that the growth rate should increase with increasing N has recently been verified for values of N as large as 10^5 [34].) Henry recognized that this generic instability of the N-body equations did not necessarily imply that mixing or relaxation would be efficient, since the exponential instability often seemed to "saturate" on scales much smaller than the scale of the system. However he made the interesting

point [7] that the existence of the instability made it difficult to reconcile the N-body equations of motion with the collisionless Boltzmann equation:

> This leads, however, to an important question of principle. The N-body problem appears to be chaotic on a time scale $\sim t_{cr}$ [the crossing time], but the flow associated with the CBE is often integrable or near-integrable in the sense that many or all of the characteristics are regular, i.e., nonchaotic. So what do the (often near-integrable) CBE characteristics have to do with the true (chaotic) N-body problem?

Henry asked: In what sense do the N-body equations of motion "go over" to the collisionless Boltzmann equation as $N \to \infty$? Henry considered several possible ways in which this might happen, and concluded

> Given the fact that the N-body problem is chaotic on a time scale $\sim t_{cr}$, it would seem reasonable to conjecture that the orbits generated in two different N-body realizations will diverge exponentially on a time scale $\sim t_{cr}$ ... However, one might nevertheless expect that, for sufficiently large N, the ensemble average of the different N-body orbits generated from the same (x_0, v_0) will closely track the CBE characteristic for some finite time. In particular, one might conjecture that the rms configuration space deviation between the N-body orbits and the CBE characteristics will scale as
>
> $$\delta r_{rms}(t) \approx F(N) \exp(t/\tau), \tag{1}$$
>
> where $\tau \approx t_{cr}$, roughly independent of the total particle number N, and where the prefactor $F(N) \to 0$ for $N \to \infty$.

Henry was proposing here a "weak" correspondence between the N-body and CBE descriptions, in the sense that an ensemble average of the N-body trajectories might mimic the orbit in the smoothed-out potential. This suggestion was characteristically cautious, and soon after, Henry and I. Sideris [35] numerically demonstrated a stronger sense in which the N-body trajectories approach the smooth-potential orbits:

> ... there is a clear, quantifiable sense in which, as N increases, chaotic orbits in the frozen-N systems remain "close to" integrable characteristics in the smooth potential for progressively longer times. When viewed in configuration or velocity space, or as probed by collisionless invariants like angular momentum, frozen-N orbits typically diverge from smooth potential characteristics as a power law in time, rather than exponentially, on a time scale $\sim N^p t_D$, with $p \sim 1/2$ and t_D a characteristic dynamical, or crossing, time.

By "frozen-N" orbits, Henry meant trajectories computed in a potential where the smooth density had been replaced by a matrix of fixed point masses, the gravitational analog of the Lorentz gas [36]. For large N, these experiments showed that almost all trajectories, and not just their ensemble averages, tended

in their finite behavior toward the behavior of regular orbits, even though the growth of infinitesimal perturbations (as measured by Liapunov exponents) remained large. In one of his last papers [37], Henry and I. Sideris showed that the chaos associated with a smooth potential could often be distinguished from the N-body chaos, in spite of the latter having a much shorter growth time, by comparing initially nearby orbits in a single N-body system, or by tracking orbits with the same initial conditions evolved in two different N-body realizations of the same smooth density.

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