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GEOMETRY OF MINIMAL FLOWS DRAFT (DRAFT)

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Abstract. Our main result is that a minimal flow on a compact manifold is either topologically conjugate to a Riemannian flow or every parametrization of φ is nowhere equicontinuous, defined as follows. A flow is Riemannian if given any points $x, y \in M$, the value of $d(\varphi_t(x), \varphi_t(y))$ is independent of $t \in \mathbb{R}$. A flow is nowhere equicontinuous if there exists an $\epsilon > 0$ such that given any point $x \in M$ and any $\delta > 0$, there exists a point $y \in N_\delta(x)$ and time $t \in \mathbb{R}$ such that $d(\varphi_t(x), \varphi_t(y)) > \epsilon$. Equivalently, a flow φ_t is nowhere equicontinuous if there exists an $\varepsilon > 0$ such that given any x the set $E_\varepsilon(x) = \{y \in M \,|\, d(\varphi_t(y), \varphi_t(x)) > \varepsilon \text{ for some } t \in \mathbb{R}\}$ is open and dense in M. Our results depend heavily on the work of Lopez and Candel.

1. Introduction

Throughout this paper, let M denote a closed Riemannian n-dimensional manifold. A flow $\varphi : M \times \mathbb{R} \to M$ is minimal if every orbit $O(x) = \{\varphi_t(x) \,|\, t \in \mathbb{R}\}$ is dense in M. Our main theorem, Theorem 1, provides a dichotomy for minimal flows. Either the flow is topologically conjugate to a Riemannian flow, in which case the distance between points is preserved by the flow, or the flow is nowhere equicontinuous, in which case the flow tends move points apart. The irrational flows on tori fall into the first category and horocylce flows on unit tangent bundles of surfaces of constant negative curvature fall into the second category. In both of these cases the flow is defined using the geometry of the manifold. This observation, together with our main theorem, suggests that perhaps every minimal flow is related to some geometry on the ambient manifold.

THEOREM 1. *A minimal flow φ on a compact Riemannian manifold M is either topologically conjugate to a Riemannian flow or every parametrization of φ is nowhere equicontinuous.*

Our primary tool for proving Theorem 1 is Theorem 3, proven by Lopez and Candel in [LC1].

A flow is *Riemannian* if given any points $x, y \in M$, the value of $d(\varphi_t(x), \varphi_t(y))$ is independent of $t \in \mathbb{R}$. A flow is *nowhere equicontinuous* if there exists an $\epsilon > 0$ such that given any point $x \in M$ and any $\delta > 0$, there exists a point $y \in N_\delta(x)$ and time $t \in \mathbb{R}$ such that $d(\varphi_t(x), \varphi_t(y)) > \epsilon$. Proposition 1 gives an alternative formulation of nowhere equicontinuous.

In [G], Gottschalk conjectured wether there exists a minimal flow on S^3. By Theorem 1, a minimal flow on S^3 must either be Riemannian or nowhere equicontinuous. The Riemannian flows on S^3 are well-known, and are not minimal. Hence, we get the following corollary.

COROLLARY 1. *If φ is a minimal flow on S^3 then φ is nowhere equicontinuous.*

It is proven in [KR] that a minimal flor on S^3 must be weakly mixing, which implies Corollary 1. However, we include this corollary because of the relationship between to expansive flows. A flow φ on M is said to be expansive if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x, y \in M$ satisfy $d(\varphi_t(x), \varphi_{h(t)}(y)) < \delta$ for some increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ and all $t \in \mathbb{R}$, then $y = \varphi_t(x)$ for some $|t| < \varepsilon$. Proposition 1 suggests that a nowhere equicontinuous flow behaves like an expansive one. It is proven in [IM] and [P] that S^3 does not admit an expansive flow. Hence a minimal flow would have to be nowhere equicontinuous but not expansive. Perhaps the construction of stable and unstable foliations in [IM] for expansive flows can be repeated for nowhere equicontinuous minimal flows to provide an obstruction to a minimal flow on S^3.

PROPOSITION 1. *A flow φ_t on a compact Manifold M is nowhere equicontinuous $\Leftrightarrow$ there exists an $\varepsilon > 0$ such that given any x the set*

$$E_\varepsilon(x) = \{y \in M \mid d(\varphi_t(y), \varphi_t(x)) > \varepsilon \text{ for some } t \in \mathbb{R}\}$$

is open and dense in M. We call $E_\varepsilon(x)$ an expansion set of radius ε about x.

Proof. It is clear that the above property implies that φ is nowhere equicontinuous, so it suffices to prove the forward implication.

So suppose that φ is nowhere equicontinuous, and choose an $\epsilon > 0$ as provided by the definition. For any $x \in M$, $\varepsilon > 0$ the set $E_\varepsilon(x)$ is open by continuity with respect to initial conditions.

Choose any point $x \in M$. We wish to show that there exists some $\varepsilon_x > 0$ and such that $E_{\varepsilon_x}(x)$ is dense in M. Then by the compactness of M, we can find an $\varepsilon > 0$ such that $\varepsilon > \varepsilon_x$ for all $x \in M$.

Let $\varepsilon = \epsilon/2$. Suppose, to get a contradiction, that $E_\varepsilon(x)$ is not dense in M. Then there exists an $y_0 \in N_\varepsilon(x)$ and an $r > 0$ such that

$$d(\varphi_t(x), \varphi_t(y)) < \varepsilon \text{ for all } y \in N_r(y_0) \text{ and } t \in \mathbb{R}.$$

Assume that r is small enough that $N_r(y_0) \subset N_\varepsilon(x)$. Then, for all $y \in N_r(y_0)$ and all $t \in \mathbb{R}$,

$$d(\varphi_t(y_0), \varphi_t(y)) \leq d(\varphi_t(y_0), \varphi_t(x)) + d(\varphi_t(x), \varphi_t(y)) < \varepsilon + \varepsilon = \epsilon.$$

This contradicts the assumption that φ is nowhere equicontinuous. □

2. Foliations and Holonomy

In this section we provide some basic notation and a few simple lemmas concerning holonomy pseudogroups and flows. A k-dimensional foliation of an n-dimensional manifold M is a partition of M into k-dimensional immersed submanifolds called the leaves of the foliation. See the excellent book [CC] for details. A nonsingular C^1 flow defines a foliation, the orbits of the flow being the leaves of the foliation. An important tool in studying foliations is the holonomy pseudogroup of the foliation. We want to define the holonomy pseudogroup in the case of a flow. The definitions for the general case are presented in [CC].

We begin with some definitions. Let M be a closed manifold and let $\varphi : \mathbb{R} \times M \to M$ be a nonsingular continuous flow that has local flow box coordinates. (It is is sufficient to assume that φ is C^1, and if M is 3-dimensional then it is sufficient

to just assume φ is continuous.) Since M is compact we can choose a finite cover $\{B_i\}_{i\in I}$ by flow boxes. Choose coordinates on the flow boxes as

$$(s, x_1, ..., x_{n-1}), \text{ where } s, x_k \in (0,1) \text{ for all } k.$$

We can assume that the flow φ restricted to B_i is $\varphi_t(s, x_1, ..., x_{n-1}) = (t+s, x_1, ..., x_{n-1})$ for $t \in (-s, 1-s)$. For each $i \in I$ define the map π_i by

$$\pi_i(s, x_1, ..., x_{n-1}) = s.$$

For each $i \in I$, choose a point $x_i \in B_i$ and define

$$\Sigma_i = \pi_i^{-1} \circ \pi_i(x_i)$$

and

$$\Sigma = \cup_{i\in I}\Sigma_i.$$

Observe that each Σ_i is a cross section to the flow, being transverse to the flow. Moreover, Σ is a global cross section, being transverse to the flow and intersecting the orbit through every $x \in M$ in both positive and negative time. We assume that the Σ_i are chosen as to be pairwise disjoint.

Suppose that $x_0 \in \Sigma$ and $y_0 = \varphi_t(x_0) \in \Sigma$. Then there exists neighborhoods V of x_0 and W of y_0 in Σ and a map $h : V \to W$ such that $h(x_0) = y_0$, and $h(x) = \varphi_{\tau(x)}(x)$ for all $x \in V$ for some continuous function $\tau : V \to \mathbb{R}$. The collection of all such maps h is called the *holonomy pseudogroup* for φ, denoted Γ_φ. The definition for a pseudogroup of local homeomorphisms of a manifold N is given in Definition 1. (See [LC1], [LC2], or [T].) The holonomy pseudogroup of a flow is a pseudogroup of local homeomorphisms of the global cross section Σ that are induced by flow.

DEFINITION 1. *Let N be a q-manifold. A C^r pseudogroup Γ on N is a collection of C^r diffeomorphisms $h : D(h) \to R(h)$ between open subsets of N satisfying the following axioms:*

(1) *If $g, h \in \Gamma$ and $R(h) \subset D(g)$ then $g \circ h \in \Gamma$.*

(2) *If $h \in \Gamma$ then $h^{-1} \in \Gamma$.*

(3) *$Id_N \in \Gamma$.*

(4) *If $h \in \Gamma$ and $W \subset D(h)$ is an open set then $h|_W \in \Gamma$.*

(5) *If $h : D(h) \to R(h)$ is a C^r diffeomorphism between open subsets of N and if for each $w \in D(h)$, there is a neighborhood W of w in $D(h)$ such that $h|_W \in \Gamma$, then $h \in \Gamma$.*

A set of generators of Γ is a subset $G \subset \Gamma$ such that every element can be constructed out of elements of G be the above pseudogroup operations of composition (where defined), inversion, restriction, and combination of maps.

A holonomy pseudogroup Γ_φ is said to be *Riemannian* if each $h \in \Gamma_\varphi$ is a local isometry for some Riemannian metric on Σ. If the flow is C^1, then the holonomy pseudogrouop is Riemannian if and only if the flow is topologically conjugate to a Riemannian flow.

A holonomy pseudogroup is said to be *quasi-analytic* if each $h \in \Gamma_\varphi$ is the identity around some x in the domain of h whenever h is the identity on some open set whose

closure contains x. (See [LC1], [H].) A foliation is said to be quasi-analytic if its holonomy pseudogroup is.

LEMMA 1. *The holonomy pseudogroup of a minimal flow is quasi-analytic.*

Proof. Suppose that Γ_φ is the holonomy pseudogroup for a minimal flow φ. Suppose that $h \in \Gamma_\varphi$ is a map $h : D(h) \to R(h)$ and O is an open set in $D(h)$ such that h is the identity on O. Since a minimal flow has no periodic orbits, h is the time-zero map $h(x) = \varphi_0(x)$. Thus, if $x \in D(h)$ is any point in the closure of O then h is the identity on some neighborhood of x. □

A holonomy pseudogroup Γ_φ is said to be *strongly equicontinuous* if there exists a symetric set S of generators of Γ_φ that is closed under compositions such that, for every $\varepsilon > 0$, there is some $\delta(\varepsilon) > 0$ such that

$$d(x, y) < \delta(\varepsilon) \Rightarrow d(h(x), h(y)) < \varepsilon$$

for all $h \in S$, and $x, y \in \Sigma_i \cap h^{-1}(\Sigma_j \cap \mathrm{Im} h)$. A foliation is said to be strongly equitcontinuous if its holonomy pseudogroup is.

Remark 1: In [LC2], the case of general pseudogroups of local homeomorphisms of a quasi-local metric space are considered. In our case the metric on each Σ_i is induced from the metric on M.

Remark 2: We say that a flow is strongly equicontinuous if its holonomy pseudogroup is strongly equicontinuous. By equivelance of holonomy pseudogroups, (see Lemma 8.8 of [LC2]) the holonomy pseudogroup generated by the return maps for some cross section Σ is strongly equicontinuous if and only if the the holonomy pseudogroup generated by the return maps for every cross section is strongly equicontinuous.

Remark 3: Observe that Γ_φ is not strongly equicontinuous if and only if every parametrization of φ is nowhere equicontinuous.

For spaces Y and Z, let $C(Y, Z)$ denote the family of continuous maps from Y into Z, which will be denoted by $C_{c-o}(Y, Z)$ when it is endowed with the compact-open topology. For open subspaces O and P of a space Z, the space $C_{c-o}(O, P)$ will be considered as an open subspace of $C_{c-o}(O, Z)$ in the canonical way. The following theorem appears as Theorem 2.4 in [LC1] and as Theorem 12.1 in [LC2].

THEOREM 2. *Let $\mathcal{H}$ be a quasi-efective, compactly generated and strongly equicontinuous pseudogroup of local transformations of a locally compact polish space Z. Let S be a symmetric set of generators of $\mathcal{H}$ that is closed under compositions and restrictions to open subsets, and satisfies the conditions of strong equicontinuity and quasi-effectiveness. Let $\tilde{\mathcal{H}}$ be the set of maps h between open subsets of Z that satisfy the following property: for every x in the domain of h, there exists a neighborhood O_x of x in the domain of h so that the restriction $h|_{O_x}$ is in the closure of $C(O_x, Z) \cap S$ in $C_{c-o}(O_x, Z)$. Then:*

1. *$\tilde{\mathcal{H}}$ is closed under composition, combination and restriction to open sets;*
2. *every map in $\tilde{\mathcal{H}}$ is a homeomorphism around every point of its domain;*
3. *the maps of $\tilde{\mathcal{H}}$ that are homeomorphisms form a pseudogroup $\overline{\mathcal{H}}$ that contains $\mathcal{H}$;*
4. *$\overline{\mathcal{H}}$ is strongly equicontinuous;*
5. *the orbits of $\overline{\mathcal{H}}$ are equal to the closures of the orbits of $\mathcal{H}$; and*
6. *$\tilde{\mathcal{H}}$ and $\overline{\mathcal{H}}$ are independent of the choice of S.*

If a pseudogroup $\mathcal{H}$ satisfies the conditions of Theorem 2, then the pseudogroup $\overline{\mathcal{H}}$ is called the closure of $\mathcal{H}$.

LEMMA 2. *The closure of the holonomy pseudogroup of a minimal flow is quasi-analytic. That is, if φ is a minimal flow then $\overline{\Gamma_\varphi}$ is quasi-analytic.*

Proof. Suppose that Γ_φ is the holonomy pseudogroup for a minimal flow φ. Choose Γ_φ itself as the set of generators for use with Theorem 2. If h is in Γ_φ then h satisfies the criteria for quasi-analyticity since Γ_φ is quasi-analytic. If $h \notin \Gamma_\varphi$, then for every x in the domain of h, there exists a neighborhood O_x of x in the domain of h so that the restriction $h|_{O_x}$ is in the closure of $C(O_x, \Sigma) \cap S$ in $C_{c-o}(O_x, \Sigma)$.

Let O be an open set in $D(h)$ such that $h|_O$ is the identity and suppose that $q \in D(h)$ is a point in the closure of O. It suffices to show that there is a neighborhood N_q of q such that h is the identity on N_q.

Since the compact-open topology is the topology of uniform convergence on compact subsets, there exists a compact set $C \subset D(h)$ such that q is an interior point of C, $O \subset C$ (possibly shrinking O) and there exists a sequence of maps $h_i \in \Gamma_\varphi$ such that $h_i \to h$ uniformly on C.

Since φ is minimal, there is a point $p \in O$ and a time $\tau \in \mathbb{R}^+$ such that $\varphi_\tau(p) = q$. Choose a neighborhood N'_p of p and N'_q of q in C such that the first return map $f : N'_p \to N'_q$ is a homeomorphism taking p to q. If N'_p and N'_q are small enough, then we can rescale time so the $f(x) = \varphi_1(x)$ for $x \in N'_p$.

Since h is continuous, $h(q) = q$. Thus, there is a neighborhood N_q of q in $D(h)$ such that $h(N_q) \subset N'_q$. Also let $N_p = f^{-1}(N_q)$. Extend $f : N'_p \to N'_q$ to a homeomorphism of M by

$$f(x) = \varphi_1(x) \text{ for } x \in M.$$

Then $f : M \to M$ is a congugacy of the flow, taking the cross section N'_p to the cross section N'_q. Thus, f commutes with return maps for the cross sections N_p and N_q. For any point $y = f(x) \in N_q$,

$$h(f(x)) = \lim_{i\to\infty} h_i(f(x)) = \lim_{i\to\infty} f(h_i(x)) = f\left(\lim_{i\to\infty} h_i(x)\right) = f(x)$$

and hence h is the identity on N_q.

$\square$

Recall that a foliation is said to be transitive if it has a dense leaf. The main theorem from [LC1] is the following.

THEOREM 3. *Let $(X, \mathcal{F})$ be a transitive compact foliated space. Then $\mathcal{F}$ is a Riemannian foliation if and only if X is locally connected and finite dimensional, $\mathcal{F}$ is strongly equicontinuous and quasi-analytic, and the closure of its holonomy pseudogroup is quasi-analytic.*

Theorem 3 implies the following more simple theorem for flows on closed finite dimensional manifolds (which are always locally connected).

THEOREM 4. *Let φ be a transitive flow on a compact n-dimensional manifold. Then φ is topologically conjugate to a Riemannian flow if and only the holonomy pseudogroup Γ_φ is strongly equicontinuous and quasi-analytic, and the closure of Γ_φ is quasi-analytic.*

3. Proof of Theorem 1

Suppose that φ is a minimal flow on M. Assume that φ is not nowhere equicontinuous and it suffices to prove that φ is topologically conjugate to a Riemannian flow. Then by Remark 3, φ is strongly equicontinuous. By Lemma 1, Γ_φ is quasi-analytic. By Lemma 2, $\overline{\Gamma_\varphi}$ is quasi-analytic. Thus, by Theorem 4, φ is topologically conjugate to a Riemannian flow.

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