8-1-2006

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Recommended Citation

William Basener, Transverse disks, symbolic dynamics, homology direction vectors, and Thurston–Nielson theory, In Topology and its Applications, Volume 153, Issue 14, 2006, Pages 2760-2764, ISSN 0166-8641, https://doi.org/10.1016/j.topol.2006.03.025

TRANSVERSE DISKS, SYMBOLIC DYNAMICS, HOMOLOGY DIRECTION VECTORS, AND THURSTON-NIELSON THEORY

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Abstract. We review some properties of transverse disks and use symbolic dynamics to determine rotation vectors from the return map to a transverse disk. We also prove connections between the symbolic dynamics and Nielson equivalence of orbits.

homology direction; Nielson equivalence; transverse disk; global cross section;

1. Introduction

Let M be a n-dimensional manifold and $\varphi : \mathbb{R} \times M \to M$ be a C^1 nonsingular flow on M. A transverse disk Σ for φ is a compact $(n-1)$-dimensional disk imbedded in M and transverse to the flow. Since Σ is compact, by transverse to the flow we mean that there exists an open $(n-1)$-dimensional disk E containing Σ that is transverse to the flow. We call Σ a global transverse disk if it is a transverse disk and the positive and negative orbits through any $x \in M$ both intersect Σ. It is proven in [2] that every C^1 nonsingular flow on a manifold of dimension greater than 2 has a global transverse disk.

Transverse disks have been used as a tool in various settings. In [8], Carlos Gutierrez uses transverse disks in dimension 3 to prove a result concerning prime knotting of orbits. Based on Gutierrez' work, it is proven in [4] that if a flow has a dense orbit and $H_2(M) = 0$ then there exists a dense open set N such that any periodic orbit intersecting N is a prime knot. Marcy Barge and Bob Williams use transverse disks in torus flows to classify Denjoy Continua and prove results concerning continued fractions. They are a common tool in the study of flows on surfaces, especially regarding Cherry flows and billiards.

We take the point of view that the return map to a transverse disk captures all of the topology of a flow. Specifically, in **??** it is proven that, for flows φ, φ' with transverse disks Σ, Σ' and return maps h, h', the return maps h and h' are conjugate if and only if the flows are topologically equivalent. In this paper we prove that important topological invariants can be "read off" from the symbolic dynamics of the return map to a transverse disk. Specifically, from the symbolic dynamics of the return map one can determine the space of homology directions for the flow, the Albelian Nielson classes of periodic orbits. One can also determine information about the periodic Nielson classes and strong Nielson classes of orbits as described in Section 4.

2. Basic Definitions

For the rest of this paper, assume that M is 3-dimensional and that Σ is a global transverse disk for φ. Associated with Σ is a first return map $h : \Sigma \to \Sigma$. In [1] it is proven that if Σ is chosen correctly then there exists a partition of Σ into points, 1-manifolds, and 2-manifolds such that h is continuous on each submanifold. The images of these manifolds also forms a partition of Σ. We make this rigorous in the following definitions.

A natural structure that we need is an M complex, which is a generalization of a CW complex and is defined as follows.

DEFINITION 1. *An **M complex** is a topological space defined as follows.*

For each $n = 0, 1, \ldots, N$, let $\{\overline{e^n_\alpha}\}$ be a set of compact n-dimensional manifolds with boundary where α runs over some finite indexing set. For each $\overline{e^n_\alpha}$, we denote the interior of $\overline{e^n_\alpha}$ by e^n_α. The e^n_α are called M-cells, being manifolds which play the role of cells in the definition of a CW complex.

1. *Let $X^0 = \{e^0_\alpha\}$ be a discrete set of points.*
2. *Inductively define X^n, called the $n-$skeleton, from X^{n-1} by attaching each e^n_α by maps $\psi_\alpha : \partial\overline{e^n_\alpha} \to X^{n-1}$. That is, X^n is the identification space of $X^{n-1} \coprod_\alpha \overline{e^n_\alpha}$ under $x \sim \psi_\alpha(x)$ for $x \in \partial\overline{e^n_\alpha}$.*

Following the notational conventions in [10] *for CW complexes, if C denotes the set of cells and attaching maps then $|C| = X^N$ denotes the resulting topological space.*

The following is our definition of M-cellwise continuous.

DEFINITION 2. *Suppose that C_d and C_r are M complexes and $h : |C_d| \to |C_r|$ is a (not necessarily continuous) map. (The notation is chosen because C_d is the cell complex on the domain of h and C_r is the cell complex on the range of h.) If h restricted to any M-cell of C_d is continuous and the image of any M-cell of C_d under h is an M-cell of C_r then we say h is **M-cellwise continuous**. For us h will be a bijection.*

So an M-cellwise continuous map is essentially a piecewise continuous map where the regions of continuity are M-cells. For the rest of this paper we will use the term cell for short, instead of M-cell.

Consider again the case of our 3-dimensional M. Define $N : \Sigma \to \mathbb{N}$ by

$$N(x) = min\{n > 0 : h^n(x) \in \text{int}\Sigma\}.$$

As is proven in [1] that for a generic global transverse disk Σ, we can choose C_d satisfying the following properties:

1. The union of the two cells of C_d is the set $\{x \in \Sigma : N(x) = 1\}$.
2. The union of the one cells in intΣ is the set $\{x \in \Sigma : N(x) = 2\}$.
3. The union of the zero cells in intΣ is the set $\{x \in \Sigma : N(x) = 3\}$.

In this case, the one cells of C_d on $\partial\Sigma$ are the images under h of the one cells in intΣ and the zero cells in $\partial\Sigma$ are the images of the zero cells in intΣ. It is clear then that C_d and C_r agree on $\partial\Sigma$.

It is sometimes required for Σ to obey further restrictions. It is possible to perturb any global cross section Σ by an arbitrarily small amount so that every intersection between $h(\partial\Sigma)$ and $h^{-1}(\partial\Sigma)$ occurs at the intersection of 1-cells and is transverse. Such a perurbation exists by standard arguments regarding transversality. Assuming Σ is so perturbed, it is possible to reduce the M-complexes C_d and C_r (by subdividing some of the cells) to get new complexes C'_d and C'_r such that every cell of C'_d and C'_r is simply connected and so that $h(A) \cap B$ is path connected for every pair of cells A, B in C'_d. For such a Σ, we say Σ and the complexes C'_d and C'_r are in topologically reduced form. Observe that if Σ and the complexes C_d and C_r are in topologically reduced form then C'_d no longer satisfies properties (1) through (3) above.

3. Symbolic Dynamics in Homology and the Fundamental Group

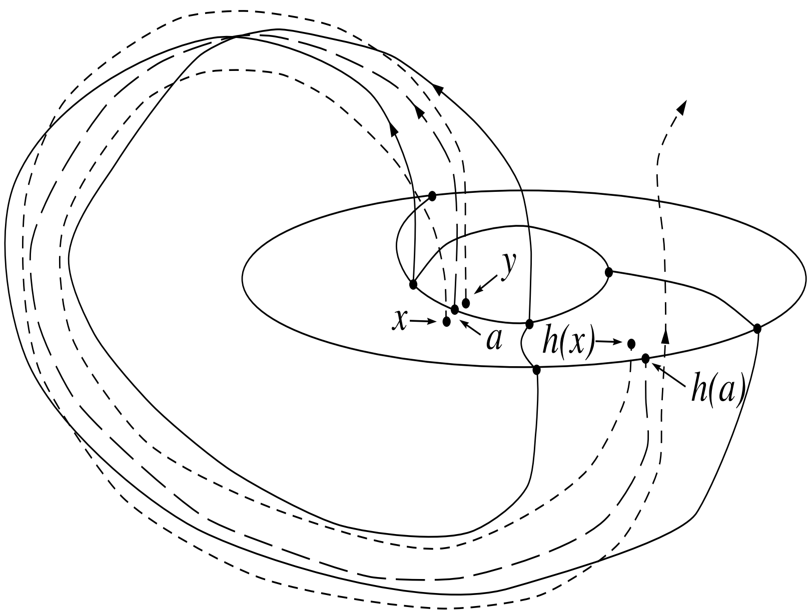

FIGURE 1. For a point a in a 1-dimensional M-cell, orbits beginning to one side of a return close to $h(a)$ while orbits beginning to the other side of a pass by Σ near $h(a)$ and return to Σ away from $h(a)$.

For our 2-dimensional global transverse disk Σ, the set of top dimensional cells of C_d is $\{e^2_1, ..., e^2_K\}$. For every cell e^n_α with $n \in \{1, 2\}$, there is a unique e^2_k such that $e^n_\alpha \subset \overline{e^2_k}$ and h restricted to $e^n_\alpha \cup e^2_k$ is continuous. This is proven in [1], although the reason is simple. For a point $a \in e^n_\alpha$, $h(a) \in \partial\Sigma$ and for all x in a 2-cell near a, $h(x)$ is near $h(a)$. See Figure 1. For each 2-dimensional cell e^2_k, let A_k denote the union of e^2_k with all lower dimensional cells having this property. Equivalently, A_k is the largest set containing e^2_k such that h restricted to A_k is continuous. So the discontinuity set of h is the boundaries of the regions A_k. Clearly, Σ is the disjoint union

$$\Sigma = \bigcup_{k=1}^{K} A_k.$$

If the cross section Σ and complexes C_d' and Cr' are in reduced form it is still possible to write Σ as the union of nonintersecting regions $A_1, ..., A_K$ where each A_k is the union of a 2-cell with some lower dimensional cells that are contained in its closure. Moreover, the return map h will be continuous on each A_k. The difference is that the discontinuity set of h will be contained in the boundaries of the $A_k' s$, but may not be equal to the union of the boundaries of the A_k.

Define $A : \Sigma \to \{A_1, .., A_k\}$ by letting $A(X)$ be the unique region that contains x. Let

$$a_n(x) = A(h^n(x)).$$

and let

$$\overline{a}(x) = \{..., a_{-1}(x), a_0(x), a_1(x), ...\}$$

Let Σ_K be the space of all bi-infinite sequences of the symbols $\{A_1, ..., A_k\}$, endowed with the usual cylinder topology making Σ_K a Cantor set. Then $\overline{a}$ is a function from Σ to Σ_K that takes a point x in Σ to the itinerary of x. Define the usual shift map $\sigma : \Sigma_K \to \Sigma_K$ by the rule that the i^{th} coordinate of $\sigma(\overline{a})$ is equal to the $(i+1)^{\text{th}}$ coordinate of $\overline{a}$. Then $\overline{a}(\Sigma)$ is a shift invariant subset of Σ_K.

The sequence $\overline{a}(x)$ keeps account of the way $O(x)$ winds around the manifold M. Let b be any point in Σ, and for every $x \in \Sigma$ let $b(x)$ be a path from b to x. For each A_k, choose a point $x_k \in A_k$. For each $k = 1, ..., K$, define the loop γ_k by

$$\gamma(A_k) = b(x_k) \cup \overrightarrow{x_k h(x_k)} \cup b(h(x_k))^{-1},$$

where $\overrightarrow{xy}$ denotes the orbit segment from x to y. Observe that the homotopy equivalence class of $\gamma(A_k)$ does not depend on the choice of the paths $b(x)$ or on the choice of $x_k \in A_k$. Let $[[\gamma(A_k)]]$ denote the equivalence class of $\gamma(A_k)$ in $\pi_1(b, M)$ and let $[\gamma(A_k)]$ denote the equivalence class of $\gamma(A_k)$ in $H_1(M; \mathbb{R})$. For a point $x \in M$, define

$$\begin{aligned}
\alpha(x) &= \gamma(a(x)) \\
\alpha_n(x) &= \gamma(a_n(x)) \\
\overline{\alpha}(x) &= \{..., \alpha_{-1}(x), \alpha_0(x), \alpha_1(x), ...\}
\end{aligned}$$

We consider $\overline{\alpha}(x)$ as a point in Σ_K, where Σ_K is now the space of bi-infinite sequences of the K symbols $\{\alpha(A_1), ..., \alpha(A_K)\}$.

The winding of an orbit the the homology of a manifold can be measured in a number of ways. We use ideas from [6] and track the homology of the orbit in the *space of homology directions*, $D_M = H_1(M; \mathbb{R})/(x \sim rx, r > 0)$, with the topology of a sphere together with a point representing the zero class. Let p be the natural projection from $H_1(M; \mathbb{R})$ to D_M which is continuous everywhere except at zero. Each point in D_M represents a direction vector in the homology space for M. The flow gives rise to a subset $D_\varphi \subset D_M$ as follows. Let R denote the set of all x in M for which there exists a sequence of points x_n and real numbers t_n such that $x_n \to x$, $\varphi(t_n, x_n) \to x_n$ and $t_n \nrightarrow 0$ (all as $n \to \infty$), called the recurrent set of φ. Such a sequence (x_n, t_n) is called a closing sequence at x. The set of nonwandering points of a flow φ, denoted by $NW(\varphi)$, is the set of all points $p \in M$ such that every neighborhood U os p, there is a time $t > 1$ such that $\varphi(t, U) \cap U \neq \varnothing$. It is obvious that a point has a closing sequence if and only if it is a nonwandering point. For each pair (x_n, t_n) in a given closing sequence, let γ_n be a short path from x_n to x, joined to the flowline from x_n to $\varphi(t_n, x_n)$, and then joined to a short path from $\varphi(t_n, x_n)$ to x. For each closing sequence, $p([\gamma_n])$ is a sequence in D_M and hence has accumulation points. Each such accumulation point is called a homology direction, and, as in [6], we denote the space of all homology directions by D_φ.

Returning to our global cross section Σ, for each point $x \in \Sigma$ and $n \in \mathbb{N}$, let

$$d_n(x) = p([\alpha_0(x)] + \cdots + [\alpha_n(x)]).$$

For a fixed $x \in \Sigma$, we define the ω-limit set of the sequence $\{d_n(x)\}_{n\in\mathbb{N}}$ to be

$$\omega_d(x) = \bigcap_{N\in\mathbb{N}} \overline{\bigcup_{n>N} d_n(x)}.$$

set of points in this omega limit set are the homology directions for x. We call the set $D_\Sigma = \overline{\{\omega_d(x) : x \in \Sigma\}}$ *the collection of homology directions for* Σ. Clearly D_Σ is a compact nonempty subset of D_M. Let $NW(\varphi)$ denote the nonwandering set of φ and let $D_{\Sigma,NW} = \overline{\{\omega_d(x) : x \in \Sigma \cap NW(\varphi)\}}$ which we call *the collection of nonwandering homology directions for* Σ. Observe that for each $d \in D_\Sigma$ there is an obit segment beginning and ending in Σ that, when closed by a segment in Σ, represents a class close to d.

THEOREM 1. *For a smooth flow on a manifold M, $D_\varphi = D_{\Sigma,NW}$.*

Proof. Suppose $d \in D_\varphi$. Then for some closing sequence (t_n, x_n), d can be approximated by a long flowline from x_n to $\varphi(t_n, x_n)$. By pushing x, x_n and $\varphi(t_n, x_n)$ each a bounded amount along the flow until they are all in Σ, one obtains a loop $d_n(x)$. Since the amount that we push x, x_n and $\varphi(t_n, x_n)$ along the flow to get them in Σ is bounded above by the maximum first return time for Σ, the homology class of $d_n(x)$ can be made arbitraraly close to the class of γ_n. Hence, $d \in D_{\Sigma,NW}$.

Suppose $d \in D_{\Sigma,NW}$. Then d can be approximated by loops generated from a closing sequence consisting of points in Σ. Thus, $d \in D_\varphi$. □

Observe that D_Σ determines how orbits wind around the homology of M but, in contrast to the homological rotation vectors of Franks [7], it does not indicate how quickly the orbits go around these directions. This is the most that can be determined from the return map $h : \Sigma \to \Sigma$ because h determines the flow up to topological equivalence and hence does not determine the velocity of orbits.

4. Nielson Equivalence of Orbits

Let γ and γ' be periodic orbits in M and x and x' be points in $\gamma \cap \Sigma$ and $\gamma' \cap \Sigma$ respectively. For a periodic point $y \in \Sigma$, let $\mathrm{per}(y)$ denote the period of y under the return map $h : \Sigma \to \Sigma$. For a thorough treatment of Thurston-Neilson Theory, see [5] and [11]. The following are trivial.

(1) γ and γ' are Abelien Nielson equivalent (they are homologous) if and only if
$$\Sigma_{i=0}^{\mathrm{per}(x)}[\alpha_i(x)] = \Sigma_{j=0}^{\mathrm{per}(x')}[\alpha_j(x')].$$

(2) γ and γ' are periodic Nielson equivalent (they are freely homotopic) if
$$\Sigma_{i=0}^{\mathrm{per}(x)}[[\alpha_i(x)]] = \Sigma_{j=0}^{\mathrm{per}(x')}[[\alpha_j(x')]].$$

(3) Suppose that Σ and the complexes C'_d and C'_r are in reduced form. Then γ and γ' are strongly Nielson equivalent (they are isotopic) if

$$\overline{\alpha}(x) = \sigma^n(\overline{\alpha}(x')) \text{ for some } n \in \mathbb{N}.$$

It is an interesting question under what conditions is item (2) an if and only if. That is, under what conditions is it true that γ and γ' are periodic Nielson equivalent (they are freely homotopic) if and only if

$$\Sigma_{i=0}^{\mathrm{per}(x)}[[\alpha_i(x)]] = \Sigma_{j=0}^{\mathrm{per}(x')}[[\alpha_j(x')]].$$

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