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The Chromatic Villainy of Complete Multipartite Graphs

by

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School of Mathematical Sciences, College of Science

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Abstract

Suppose the colors in a $\chi(G)$-coloring of a graph G have been rearranged. We will call this rearrangement c^*. The chromatic villainy of the c^* is defined as the minimum number of vertices that need to be recolored in order to return c^* to a proper coloring in which each color appears the same number of times as in the initial coloring. The maximum chromatic villainy when considering all rearrangements of all $\chi(G)$-coloring of G is the chromatic villainy of G. Here, the chromatic villainies of certain families of graphs were investigated and the chromatic villainies of paths and certain classes of complete multipartite graphs were found. Bounds were found for certain classes of odd cycles and complete multipartite graphs as well.

Contents

I. Introduction

I.1 Background

A *graph* G is defined as an ordered pair (V, E) where $V(G)$ represents the vertex set of G and $E(G)$ represents the set of edges connecting the vertices of G. An edge between two vertices x and y is notated as xy or yx. Two vertices of G are *adjacent* or *neighbors* if an edge exists between them. The *neighborhood* of x, denoted $N(x)$, is the set of vertices adjacent to a vertex x. A *proper k-coloring* of G is a labeling of the vertices of G using k colors, usually represented by numbers 1 through k, in such a way that adjacent vertices receive different colors. The *chromatic number* of a graph G, denoted $\chi(G)$, is the smallest k such that a proper k-coloring of G exists. We define $c(v_j)$ as the color of vertex v_j under coloring c.

Many algorithms exist to produce proper colorings. These algorithms also give upper bounds on the chromatic number of a given graph. However, these algorithms do not always result in a $\chi(G)$-coloring of G. One such algorithm is referred to as *greedy coloring*. In a greedy coloring, the vertices of G are given an order $v_1, v_2, \ldots, v_{|V(G)|}$. Colors are assigned by going through the vertices in order and assigning the color with the lowest number that is not given to a previously colored adjacent vertex. The orders given to the vertices of G changes the upper bound on $\chi(G)$. Note that there is an ordering for every graph that will use the chromatic number of colors when fed through the greedy algorithm. However, there is a graph G (in fact there are many) with chromatic number 2 that when put into the greedy coloring *with the wrong ordering* will have a chromatic number that will grow with $\log_2(|V(G)|)$.

For example, consider the graph given in Figure 1. It is properly colored with two colors. If we were to implement a greedy coloring in which we colored the vertices in order from left to right, we would achieve the coloring given in Figure 2. While this coloring is proper, it uses four colors. Note that $4 = log_2(8) + 1$ and this graph has eight vertices.

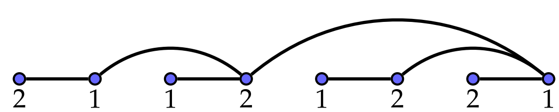


Figure 1: A proper coloring with 2 colors.

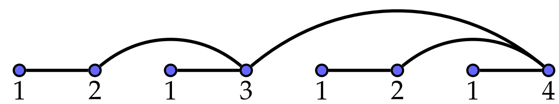


Figure 2: A proper coloring with 4 colors.

Let $\Delta(G)$ be the maximum degree of a vertex in G and $\delta(G)$ be the minimum degree of a vertex in G. Given that a vertex in G has at most $\Delta(G)$ neighbors, a greedy coloring will use at most $\Delta(G)+1$ colors. Thus, $\chi(G) \leq \Delta(G)+1$ [6]. However, $\Delta(G)+1$ is rarely a strict upper bound. It was proven by Brooks in [1] that if G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$. A stricter bound on $\chi(G)$ was given by Welsh and Powell in [5]. Let d_i be the degree of vertex i. Let the vertices in the greedy coloring be ordered in non-increasing order of degree, such that $d_1 \geq d_2 \geq \ldots \geq d_{|V(G)|}$. The color given to the vertex with degree d_i is at most one greater than the number of neighbors of d_i that are already colored. This value is bounded by d_i and $i-1$. Thus, a stricter bound on $\chi(G)$ is given by $\chi(G) \leq 1 + \max_i(\min(d_i, i-1))$. In [4], Szekeres and Wilf found an upper bound on $\chi(G)$ using the degrees of the subgraphs of G to order the vertices. This bound is given by $\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H)$. Subgraphs and complete graphs are defined in section 1.2.

Clark et al. [2] introduced the concept of chromatic villainy. Let us assume that $\chi(G) = k$ and let c be a proper k-coloring of G. Let c^* be a coloring of G that is a rearrangement of the colors in c. The *weak chromatic villainy* of c^*, denoted $B_w(c^*)$, is the minimum number of vertices that must be recolored with the same set of colors as c^* in order to re-obtain a proper coloring. The *villainy* of c^*, denoted $B(c^*)$, is the minimum number of vertices that must be recolored with the same set of colors as c^* with the additional stipulation that each color must appear exactly as many times as it does in c.

The *weak villainy of the graph G* is the the largest number of vertices that need to be recolored over all rearrangements of all $\chi(G)$-colorings of G. Let c be a proper $\chi(G)$-coloring of G and let c^* be a rearrangement of c. The weak villainy of G is given by

$$B_w(G) = \max_c \left(\max_{c^*} \left(B_w(c^*) \right) \right).$$

The *villainy of the graph G* is the largest number of vertices that need to be recolored over all rearrangements of all k-colorings of G where each color in the resulting proper coloring appears exactly as many times as it does in c. Clark et al. [2] established that $B_w(G) \leq B(G)$ given that any recoloring that follows the stipulations of villainy is also valid under the stipulations of weak villainy. Let c be a proper $\chi(G)$-coloring of G and let c^* be a rearrangement of c. The villainy of G is given by

$$B(G) = \max_{c} \left(\max_{c^*}(B(c^*)) \right).$$

Consider the following example. Let G be a path on 7 vertices with optimal coloring c as shown in Figure 3 and let c^* be the rearrangement of c given by Figure 4.

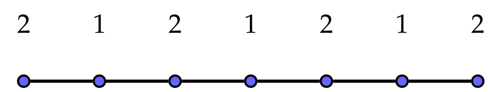


Figure 3: A properly colored path on 7 vertices.

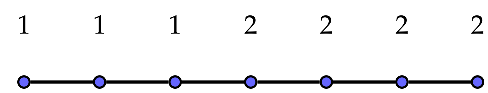


Figure 4: An improperly colored path on 7 vertices.

In order to properly color an odd path, the color that appears most often must be on the outermost two vertices and the colors must alternate. In the case where the number of each color is maintained, the outermost vertices of G must receive color 2. Therefore, to re-obtain a proper coloring of G while maintaining the number of each color that appeared in c, the coloring must be identical to c. This can be obtained by recoloring the first, third, fourth, and sixth vertices. It holds that $B(c^*) = 4$.

In the weak case, recoloring the second, fifth, and seventh vertices restores G to a proper coloring. However, this proper coloring is not equivalent to c. The color 1 appears four times in G while 2 appears three times. Therefore, the two outer-most vertices receive color 1. It holds that $B_w(c^*) = 3$.

In [2], Clark et al. proved several results regarding both chromatic villainy and weak chromatic villainy. It was determined that $B(G) = B_w(G) = 0$ if and only if G is a complete or empty graph. Additionally, the class was determined for graphs with a weak chromatic villainy of 1. Additional results have been found regarding the villainies of uniquely colorable and pseudo-uniquely colorable graphs, as well as other categories of graphs such as connected bipartite graphs, cycles, disjoint unions of graphs, and certain classes of subgraphs. The authors also posed a number of open questions that can be summarized as follows:

- What are the characteristics of graphs with a chromatic villainy of 2?
- Is it the case that the chromatic villainy of a cycle with $2k+1$ vertices is k when $k \geq 2$?
- What are the chromatic villainies of complete multipartite graphs?
- What are the largest possible values of $B(G)$ and $B_w(G)$ when G has n vertices and $\chi(G) = k$?
- Is the weak chromatic villainy of the disjoint union of two graphs greater than or equal to the sum of the two graph's respective weak villainies?

We will focus on the chromatic villainy of complete multipartite graphs, paths, and odd cycles.

I.2 Terms

A graph H is a *subgraph* of G, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. It holds that G *contains* its subgraphs. A graph is *complete* if every pair of vertices are adjacent. Note that a complete graph on n vertices is denoted K_n. A *matching* in G is a set of edges in which no two edges share a vertex. Edges that do not share a vertex are also referred to as *independent edges*. The matching *covers* the vertices in its edges. A matching is *perfect* if it covers every vertex in G.

A graph G is *weighted* if there are numerical values or *weights* assigned to its edges. G is *unweighted* otherwise. Note that an *independent set* is a set of vertices in a graph in which none are adjacent. A graph G is *bipartite* if $V(G)$ consists of two independent sets. The *maximum weighted matching* in a bipartite graph is a matching in which the sum of the edge weights has a maximal value. The value is *maximal* if it cannot be made larger. Let G be a bipartite graph with independent sets X and Y. By Hall's Theorem, a matching exists that covers every vertex in X iff for every S where $S \subseteq X$, $|N(S)| \geq |S|$ [3]. Any undefined terms can be found in [6].

Section II will focus on the chromatic villainy of complete multipartite graphs. A graph is *k-partite* if it is the union of k independent sets. These sets are referred to as *partite sets*. Note that a partite set can be empty. Therefore, a k-partite graph is also j-partite for all $j \geq k$. A k-partite graph can also be referred to as *multipartite*. An example of an incomplete 5-partite graph is given in Figure 5. Let P_i be a partite set in a k-partite graph for all $i \in [k]$. A k-partite graph with partite

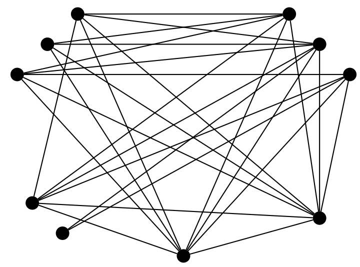


Figure 5: An incomplete 5-partite graph

sets P_1, P_2,...,P_k is considered *complete* if for all $i, j \in \{1, 2, \ldots, k\}$, every vertex in P_i is adjacent to every vertex in P_j iff $i \neq j$. Such a graph will be notated $K_{n_1,n_2,\ldots,n_k}$ where n_i is the size of P_i for $i \in \{1, 2, \ldots, k\}$. Let r_i be the number of partite sets of size n_i. Without loss of generality, we will notate a complete multipartite graph with $\sum_{i=1}^{k} r_i$ partite sets as $K_{\underbrace{n_1,\ldots,n_1}_{r_1},\underbrace{n_2,\ldots,n_2}_{r_2},\ldots,\underbrace{n_k,\ldots,n_k}_{r_k}}$ such that $r_1n_1 \geq r_2n_2 \geq r_3n_3 \geq \ldots \geq r_kn_k > 0$. In the event that $r_in_i = r_{i+1}n_{i+1}$ for some $i \in \{1, 2, \ldots, k\}$, the set of larger partite sets will receive the lower index. That is, $n_i > n_{i+1}$ when $r_in_i = r_{i+1}n_{i+1}$. Note that in general, n_i need not be greater than n_{i+1}. Consider the example given in Figure 6. The graph in Figure 6 is a complete 7-partite graph with five partite sets of size 2 and two partite sets of size 4. Therefore, the graph is denoted $K_{\underbrace{2,2,2,2,2}_{5},\underbrace{4,4}_{2}}$ and $n_1 = 2$ while $n_2 = 4$.

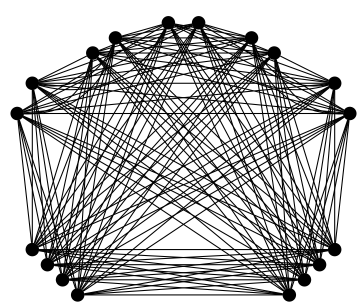


Figure 6: A complete 7-partite graph

A graph G is *uniquely colorable* if proper colorings of G using $\chi(G)$ colors differ only by the names of the colors [2]. Note that a complete multipartite graph is uniquely colorable; there is only one way to color a complete multipartite graph up to permutation of the colors. Given that every vertex in a given partite set P_i is adjacent to every vertex in P_j for all $i \neq j$, each partite set must be colored with a different color. Because each partite set is an independent set, a given partite set can be colored with one color. Note that the chromatic number of a graph G where $G = K_{\underbrace{n_1,\dots,n_1}_{r_1},\underbrace{n_2,\dots,n_2}_{r_2},\dots,\underbrace{n_k,\dots,n_k}_{r_k}}$ is $\sum_{j=1}^{k} r_j$, the total number of partite sets. For example, a proper coloring c of $K_{3,3,2,1,1}$ is given by Figure 7 and $K_{3,3,2,1,1}$ has a chromatic number of 5. Consider a rearrangement of c. To return G to a proper coloring, every color that appears n_i times must be in a partite set of size n_i for $i \in [5]$. Note that in the proper recoloring, the vertices in a partite set might have a different color than they did in c. Let c^* be the rearrangement of c shown in Figure 8.

Under c^*, the partite sets of size 1 contain colors that appear twice, and the partite set of size 2 contains colors that appear once. Therefore, every vertex in these partite sets must be re-colored. All of the vertices in the partite sets of size 3 appear three times. However, the color 2 appears more often in the left partite set of size 3 and the color 1 appears more often in the right partite set of size 3. Therefore, the resulting proper coloring when the smallest possible number of recolorings are performed is given by Figure 9.

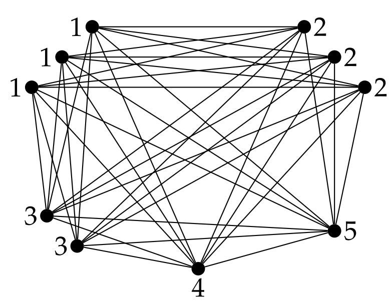


Figure 7: A proper coloring c of $K_{3,3,2,1,1}$

Note that Figure 9 is not the original proper coloring. To obtain the original proper coloring, eight recolorings would be needed while this recoloring was obtained in six recolorings. However, $B(G) \neq 6$. Let c^* be the rearrangement of c given by Figure 10. In Figure 10, only two colors, 1 and 2, are in a partite set of the right size, size 3. They cannot both be correct in the same partite set. Therefore, only one vertex is colored correctly in this rearrangement and the coloring has a

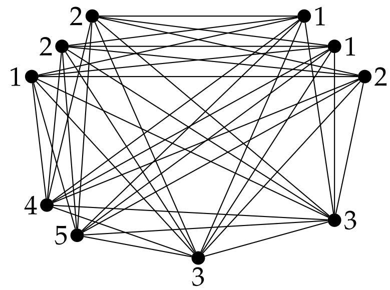


Figure 8: An improper recoloring of c

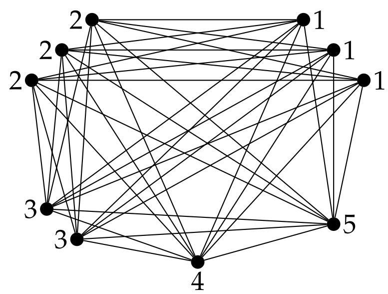


Figure 9: A proper recoloring of $K_{3,3,2,1,1}$

chromatic villainy of 9. Note that this is equivalent to $\sum_{i=1}^{k} r_i n_i - \left\lceil \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$. It holds that there are only four vertices with colors that don't appear in the graph three times and six vertices with colors that appear three times. Therefore, under any recoloring of G, at least two of the vertices in the partite sets of size 3 must have a color that appears in the graph three times. At least one of these vertices will be colored correctly. Therefore, the villainy of the graph cannot be greater than 9.

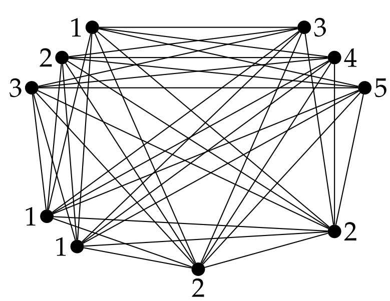


Figure 10: An improper recoloring of c

II. Complete Multipartite Graphs

Let $G = K_{\underbrace{n_1,\ldots,n_1}_{r_1},\underbrace{n_2,\ldots,n_2}_{r_2},\ldots,\underbrace{n_k,\ldots,n_k}_{r_k}}$ be a complete multipartite graph with r_i parts of size n_i for $i \in [k]$, where $r_1 n_1 \geq r_2 n_2 \geq \ldots \geq r_k n_k > 0$. Let P_i be a partite set in G for $i \in [\chi(G)]$ and let $P_1, P_2, \ldots, P_{r_1}$ be the partite sets of size n_1. The villainy of G is dependent on the number of vertices in $\bigcup_{j=r_1+1}^{\chi(G)} P_j$ in relation to the number of vertices in $\bigcup_{i=1}^{r_1} P_i$. It holds that $\bigcup_{j=r_1+1}^{\chi(G)} P_j$ contains $\sum_{i=2}^{k} r_i n_i$ vertices. Multipartite graphs can be sorted into three cases:

1. $r_1 n_1 \leq \sum_{j=2}^{k} r_j n_j$,
2. $r_1 n_1 > \sum_{j=2}^{k} r_j n_j$ and $\sum_{j=2}^{k} r_j n_j < \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$, and
3. $r_1 n_1 > \sum_{j=2}^{k} r_j n_j$ and $\sum_{j=2}^{k} r_j n_j \geq \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$.

Consider case 1 where $r_i n_i \leq \sum_{j \neq i} r_j n_j$ for all i and all $j \in [k]$ where $j \neq i$. It holds that no set of partite sets of the same size contains more than $\left\lfloor \frac{|V(G)|}{2} \right\rfloor$ vertices. Consider a consecutive labeling of the vertices where each vertex receives a label v_ℓ in G such that the vertices in the r_i partite sets of size n_i receive labels that precede those in the r_{i+1} partite sets of size n_{i+1}. For example, in $K_{4,4,3,3,2,2}$,

- P_1 contains v_1, v_2, v_3, and v_4,
- P_2 contains v_5, v_6, v_7, and v_8,
- P_3 contains v_9, v_{10}, and v_{11},
- P_4 contains v_{12}, v_{13}, and v_{14},
- P_5 contains v_{15}, and v_{16}, and
- P_6 contains v_{17}, and v_{18}.

Let P_i get color i under the proper $\chi(G)$-coloring c. Since no set of partite sets of the same size comprises more than half the graph, the color of vertex v_i under c does not have the same color as $v_{i+\left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|}}$ (note that if $i + \left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|} = 0$, then $v_{i+\left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|}} = v_{|V(G)|}$). Furthermore, $c(v_i)$ and $c(v_{i+\left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|}})$ each appear a different number of times in G. Therefore, if the colors of the vertices in G are redistributed such that $v_{i+\left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|}}$

receives color $c(v_i)$, no vertex in a set of size n_i receives a color that appears n_i times in G for all $i \in [k]$.

Consider $G = K_{4,4,3,3,2,2}$. A labeling and a proper coloring of $K_{4,4,3,3,2,2}$ are given in Figure 11. In this case, $\left\lfloor \frac{|V(G)|}{2} \right\rfloor = 9$. The coloring achieved by recoloring every vertex v_i with $c(v_{i+9 \pmod{18}})$ is given in Figure 12. Note that in this coloring, no color that appears four times is in a partite set of size 4, no color that appears three times is in a partite set of size 3, and no color that appears twice is in a partite set of size 2. Therefore, every vertex in Figure 12 must be recolored to achieve a proper coloring, and $B(G) = 18$.

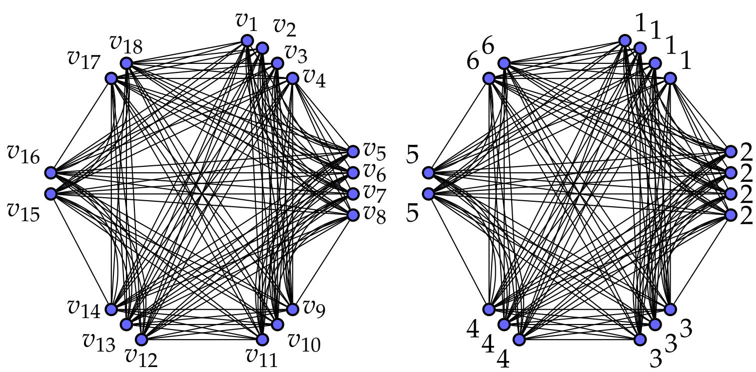


Figure 11: A proper coloring of $K_{4,4,3,3,2,2}$

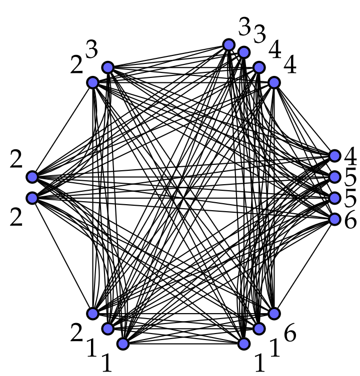


Figure 12: An improper recoloring of $K_{4,4,3,3,2,2}$ with a villainy of $|V(G)|$

Theorem 1. *Let* $G = K_{\underbrace{n_1,\dots,n_1}_{r_1},\underbrace{n_2,\dots,n_2}_{r_2},\dots,\underbrace{n_k,\dots,n_k}_{r_k}}$ *be a complete multipartite graph with* r_i *partite sets of size* n_i *for all* $i \in [k]$ *where* $r_1 n_1 \geq r_2 n_2 \geq r_k n_k > 0$. *If for all* $i \in [k]$ *we have* $r_1 n_1 \leq \sum_{j=2}^{k} r_j n_j$, *then*

$$B(G) = |V(G)|.$$

Proof. For $j \in \sum_{i=1}^{k} r_i$, let P_j be a partite set in G. Note that $\chi(G) = \sum_{i=1}^{k} r_i$. Iterating through the partite sets such that the r_i sets of size n_i precede the r_{i+1} sets of size n_{i+1} we label the vertices $v_1, v_2, \ldots, v_{|V(G)|}$. That is,

$$\begin{aligned} P_1 &= \{v_1, v_2 \ldots v_{n_1}\}, \\ P_2 &= \{v_{n_1+1}, v_{n_1+2}, \ldots, v_{2n_1}\}, \\ &\vdots \\ P_{\chi(G)} &= \left\{ v_{\sum_{d=1}^{k-1} r_d n_d + (r_k-1)n_k+1}, v_{\sum_{d=1}^{k-1} r_d n_d + (r_k-1)n_k+1}, \ldots, v_{\sum_{d=1}^{k} r_d n_d} \right\}. \end{aligned}$$

Let c be a proper coloring of $V(G)$ that uses the smallest number of colors. Therefore all vertices in each partite set get the same color under c. Let us define c such that the vertices in P_i receive color i.

The graph G can be represented as a circle where the vertices are labeled consecutively. It holds that $r_i n_i \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ for all $i \in [k]$. Note that if a vertex has index $0 \pmod{|V(G)|}$, it will recieve label $v_{|V(G)|}$. Therefore, $c\left(v_{\ell + \left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|}}\right) \neq c(v_\ell)$. Furthermore, $c(v_\ell)$ and $c\left(v_{\ell + \left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|}}\right)$ each appear a different number of times in G.

Let c^* be a clockwise rotation of the colors of the vertices under c by $\left\lfloor \frac{|V(G)|}{2} \right\rfloor$ units. We define $c^*(v_\ell)$ to be the color of vertex ℓ under c^*. For all $\ell \in [|V(G)|]$, it holds that $c^*\left(v_{\ell + \left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|}}\right) = c(v_\ell)$. Since $c(v_\ell)$ and $c\left(v_{\ell + \left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|}}\right)$ each appear a different number of times in G, $v_{\ell + \left\lfloor \frac{|V(G)|}{2} \right\rfloor \pmod{|V(G)|}}$ must be recolored to return G to a proper coloring. This holds for all $\ell \in [|V(G)|]$. Thus,

$$B(G) = |V(G)|. \qquad \square$$

Let $G = K_{\underbrace{n_1, \ldots, n_1}_{r_1}, \underbrace{n_2, \ldots, n_2}_{r_2}, \ldots, \underbrace{n_k, \ldots, n_k}_{r_k}}$ be a complete multipartite graph such that $r_1 n_1 > \sum_{j=2}^{k} r_j n_j$. Let c be a proper $\chi(G)$-coloring of G where the vertices in P_i receives color i. In both case 2 and case 3, any rearrangement of c will have at least $r_1 n_1 - \sum_{j=2}^{k} r_j n_j$ vertices in $\bigcup_{i=1}^{r_1} P_i$ that have a color in $\{1, 2, \ldots, r_1\}$.

If $r_1 | n_1$, then $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$ will be equivalent to 0. The value $\sum_{j=2}^{k} r_j n_j$ cannot be negative, and in case 2, $\sum_{j=2}^{k} r_j n_j$ is strictly less than $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$. Therefore, if $r_1 | n_1$ and $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 = 0$ where $r_1 n_1 \geq \sum_{j=2}^{k} r_j n_j$, the graph falls into case 3. Note that the value $\left\lfloor \frac{n_1}{r_1} \right\rfloor$ represents how many times one vertex of each color in $\{1, 2, \dots, r_1\}$ can be placed in each partite set of size n_1. In a given P_i where $i \in [r_1]$, $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right)$ vertices remain uncolored after coloring $\left\lfloor \frac{n_1}{r_1} \right\rfloor$ vertices in P_i with each color in $\{1, 2, \dots, r_1\}$. Thus, the value $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$ is equivalent to how many vertices are left uncolored in G after $\left\lfloor \frac{n_1}{r_1} \right\rfloor$ vertices of each color in $\{1, 2, \dots, r_1\}$ have been placed in each partite set in $\{P_1, P_2, \dots, P_{r_1}\}$. Note that $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right)$ is equivalent to $n_1 \pmod{r_1}$ and is strictly less than r_1. Therefore, if $\left\lfloor \frac{n_1}{r_1} \right\rfloor$ vertices in each partite set in $\bigcup_{i=1}^{r_1}$ are colored with each color in $\{1, 2, \dots, r_1\}$, any remaining uncolored vertices that need to be colored with colors in $\{1, 2, \dots, r_1\}$ can no longer be colored such that each color in $\{1, 2, \dots, r_1\}$ is placed in a single partite set.

In case 2 where $r_1 n_1 > \sum_{j=2}^{k} r_j n_j$ and $\sum_{j=2}^{k} r_j n_j < \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$, we first color the vertices in $\bigcup_{i=1}^{r_1} P_i$ such that $\left\lfloor \frac{n_1}{r_1} \right\rfloor$ vertices in each partite set get each color in $\{1, 2, \dots, r_1\}$. This results in $\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2$ recolored vertices in $\bigcup_{i=1}^{r_1} P_i$. Let us refer to the uncolored portion of P_i as Q_i. Note that each Q_i contains $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ vertices.

It holds that $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 - \sum_{i=2}^{k} r_i n_i$ vertices in G still need to be colored with a color in $\{1, 2, \dots, r_1\}$. By our assumption, $\sum_{j=2}^{k} r_j n_j < \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$. Therefore, $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 - \sum_{i=2}^{k} r_i n_i > 0$. These $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 - \sum_{i=2}^{k} r_i n_i$ colored vertices cannot be distributed such that one of each color in $\{1, 2, \dots, r_1\}$ is in each Q_i. We will distribute these colors such that $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ vertices of each consecutive color are used and are distributed consecutively across Q_is. It holds that at least $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=1}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ total colors in $\{1, 2, \dots, r_1\}$ must be used in $\bigcup_{i=1}^{r_1} Q_i$. Therefore, $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=1}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ partite sets contain $\left\lfloor \frac{n_1}{r_1} \right\rfloor + 1$ vertices of a unique color. Subsequently, $r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=1}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ partite sets must be colored with a color that appears

$\left\lfloor \frac{n_1}{r_1} \right\rfloor$ times. It follows that at most

$$\left(r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=1}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil\right) \left\lfloor \frac{n_1}{r_1} \right\rfloor + \left(\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=1}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil\right) \left(\left\lfloor \frac{n_1}{r_1} \right\rfloor + 1\right)$$
$$= r_1 \left\lfloor \frac{n_1}{r_1} \right\rfloor + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=1}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$$

vertices do not need to be recolored.

Consider $G = K_{3,3,3,2}$. A proper coloring of G is given in Figure 13 such that the vertices in P_i receive color i. In this case,

$$\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 = \left(8 - \left\lfloor \frac{8}{3} \right\rfloor 3\right) 3 = 6$$

while $\sum_{i=2}^{k} r_i n_i = 2$ and it holds that $\sum_{i=2}^{k} r_i n_i < \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$. We first recolor $\left\lfloor \frac{8}{3} \right\rfloor$ vertices in P_1, P_2, and P_3 with each color in $\{1, 2, 3\}$ as shown in Figure 14. It holds that $\left(8 - \left\lfloor \frac{8}{3} \right\rfloor\right) = 2$ and $|V(Q_i)| = 2$ for all $i \in \{1, 2, 3\}$. Note that at least

$$\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=1}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil = \left\lceil \frac{3(8) - \left\lfloor \frac{8}{3} \right\rfloor 3^2 - 2}{8 - \left\lfloor \frac{8}{3} \right\rfloor 3} \right\rceil$$
$$= \left\lceil \frac{4}{2} \right\rceil$$
$$= 2$$

total colors in $\{1, 2, \ldots, r_1\}$ must be used in $\bigcup_{i=1}^{r_1} Q_i$. Thus the remaining four vertices that must be colored with a color in $\{1, 2, 3\}$ will be colored such that Q_1 and Q_2 contain a vertex with color 1 and Q_3 and Q_4 contain a vertex of color 2. The additional two vertices in $\bigcup_{i=1}^{3} P_i$ will receive color 4 and the remaining unused colors are placed in P_4. This recoloring is given in Figure 15.

In the coloring given in Figure 15, there are at most 3 vertices in a partite set that have the same color. However, there are only two distinct colors that appear three times and the vertices in one partite set must be colored with a color that only appears twice. Recoloring the vertices in P_1 with 1 would require five recolorings. Likewise, recoloring the vertices in P_2 with 2 would require five

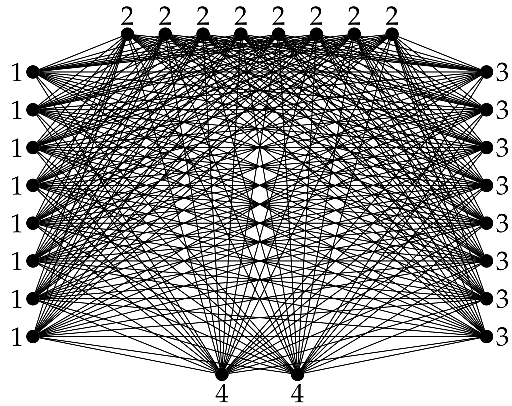


Figure 13: A proper coloring of $K_{8,8,8,2}$

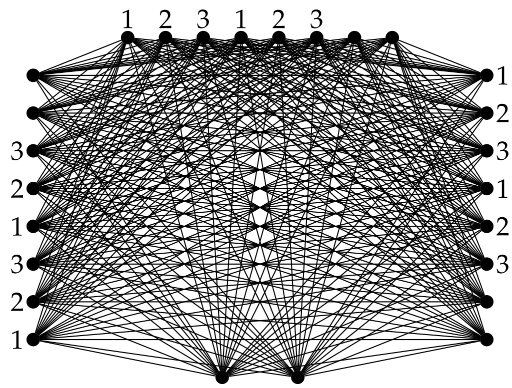


Figure 14: A partial recoloring of $K_{8,8,8,2}$

recolorings. This leaves the vertices in P_3 to be recolored with 3, requiring six recolorings. Thus, to return the graph in Figure 15 to a proper coloring, 18 vertices need to be recolored and 8 are already correctly colored. Note that

$$r_1 \left\lfloor \frac{n_1}{r_1} \right\rfloor + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil = 3 \left\lfloor \frac{8}{3} \right\rfloor + \left\lceil \frac{3(8) - \left\lfloor \frac{8}{3} \right\rfloor 3^2 - 2}{8 - \left\lfloor \frac{8}{3} \right\rfloor 3} \right\rceil = 8.$$

Theorem 2. *Let* $G = K_{\underbrace{n_1, \ldots, n_1}_{r_1}, \underbrace{n_2, \ldots, n_2}_{r_2}, \ldots, \underbrace{n_k, \ldots, n_k}_{r_k}}$ *be a complete multipartite graph with* r_j *partite sets of size* n_j *for all* $1 \le j \le k$*. Note that* $r_1 n_1 \ge r_2 n_2 \ge \ldots \ge r_k n_k > 0$*. If we have*

$$r_1 n_1 > \sum_{j=2}^{k} r_j n_j \text{ and } \sum_{j=2}^{k} r_j n_j < \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) r_1,$$

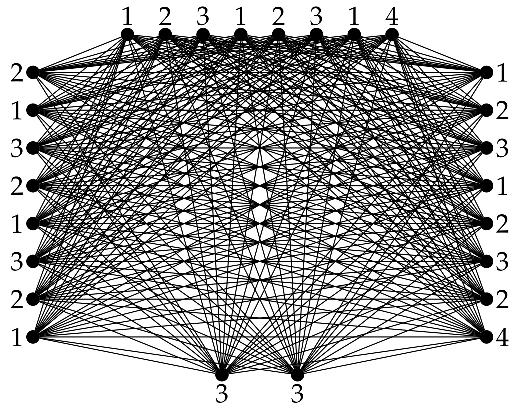


Figure 15: An improper recoloring of $K_{8,8,8,2}$

then

$$B(G) \geq \sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil .$$

Proof. For $j \in \left[\sum_{j=1}^{k} r_j\right]$, let P_j be a partite set in G. Let c be a proper coloring of $V(G)$ that uses $\chi(G)$ colors. Therefore all vertices in each partite set get the same color under c, and $\chi(G) = \sum_{i=1}^{k} r_k$ colors are used. We define c such that $P_1, P_2, \ldots, P_{r_1}$ are the r_1 sets of size n_1 and each vertex in P_i where $i \in [r_1]$ has color i. Note that $\sum_{j=2}^{k} r_j n_j$ is the total number of vertices in $\bigcup_{i=r_1+1}^{\chi(G)} P_i$.

To prove $B(G) \geq \sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$, we define a coloring c^* of G that is a rearrangement of a $\chi(G)$-coloring of G. For each i and j where $i \in \{1, \ldots, r_1\}$ and $j \in \{1, \ldots, r_1\}$ we color $\left\lfloor \frac{n_1}{r_1} \right\rfloor$ vertices in P_i with color j. This results in each P_i having $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ vertices that are not yet colored. It holds that $r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j$ vertices in $\bigcup_{i=1}^{r_1} P_i$ must still receive a color in $\{1, 2, \ldots, r_1\}$. By our assumption, $\sum_{j=2}^{k} r_j n_j < \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$. Thus, the total number of vertices in $\bigcup_{i=1}^{r_1} P_i$ that must receive a color in $\{1, 2, \ldots, r_1\}$ is

$$\begin{aligned} r_1 n_1 - \sum_{j=2}^{k} r_j n_j &> r_1 n_1 - \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 \\ &= \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2. \end{aligned}$$

It follows that $r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j$ is greater than 0.

Let us refer to the uncolored vertices in P_i as Q_i. We aim to color these remaining vertices such that the fewest possible number of colors in $\{1, \ldots, r_1\}$ is used and these colors are distributed such that at most one of each color appears in each Q_i. We will color the vertices in $\bigcup_{i=1}^{r_1} Q_i$ such that $\sum_{j=2}^{k} r_j n_j$ vertices receive colors from the set $\{r_1 + 1, \ldots, \chi(G)\}$ and the remaining $r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j$ vertices, those that are not yet colored, receive colors in $\{1, \ldots, r_1\}$.

So far, each color in $\{1, \ldots, r_1\}$ has been used $\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ times in $\bigcup_{i=1}^{r_1} P_i$. Therefore, since each color in $\{1, \ldots, r_1\}$ appears at most n_1 times in G, each color in $\{1, \ldots, r_1\}$ can appear in $\bigcup_{i=1}^{r_1} Q_i$ at most $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ times. It follows that the minimum number of colors in $\{1, \ldots, r_1\}$ that we need to color the remaining vertices in $\bigcup_{i=1}^{r_1} Q_i$ is $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$; the total number of vertices in $\bigcup_{i=1}^{r_1} Q_i$ that must be colored with a color in $\{1, \ldots, r_1\}$ divided by how many times each color can be used. Given that this expression simplifies to $r_1 + \left\lceil \frac{- \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ and $- \sum_{i=2}^{k} r_i n_i$ is never positive, this value will never be greater than r_1. Note that by our assumption, $\sum_{j=2}^{k} r_j n_j < \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) r_1$. Since $\sum_{j=2}^{k} r_j n_j$ cannot be negative, this implies $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) r_1$ is strictly greater than 0 and likewise, $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ is strictly greater than 0.

Let us distribute the colors in $\left\{ 1, \ldots, \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil - 1 \right\}$ such that each color appears $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ times in $\bigcup_{i=1}^{r_1} Q_i$. We will distribute them such that we place one vertex of color 1 in each consecutive Q_i beginning with Q_1. When all $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ vertices of a color are placed, the color will be increased by 1 and placed in the next Q_i, returning to Q_1 after a color has been placed in Q_{r_1}. After placing these colors, there are

$$r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i - \left(\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil - 1 \right) \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right)$$

vertices uncolored in $\bigcup_{i=1}^{r_1} Q_i$ that still need to be colored with a color in $\{1, 2, \ldots, r_1\}$. Note that

$$\begin{aligned}
r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i - \left(\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil - 1 \right) \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) \\
= r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i - \left(r_1 + \left\lceil \frac{- \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil - 1 \right) \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) \\
= r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i - r_1 n_1 + \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 \\
- \left\lceil \frac{- \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) + \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) \\
= - \sum_{i=2}^{k} r_i n_i - \left\lceil \frac{- \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) + \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) \\
= \left(1 - \left\lceil \frac{- \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil \right) \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) - \sum_{i=2}^{k} r_i n_i \\
= \left(1 + \left\lfloor \frac{\sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rfloor \right) \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) - \sum_{i=2}^{k} r_i n_i.
\end{aligned}$$

The value of $\left\lfloor \frac{\sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rfloor$ is strictly greater than $\frac{\sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} - 1$ and less than or equal to $\frac{\sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1}$. It holds that

$$\begin{aligned}
\left(1 + \left\lfloor \frac{\sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rfloor \right) \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) - \sum_{i=2}^{k} r_i n_i &> \left(1 + \left(\frac{\sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right) - 1 \right) \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) \\
&\quad - \sum_{i=2}^{k} r_i n_i \\
&= \sum_{i=2}^{k} r_i n_i - \sum_{i=2}^{k} r_i n_i \\
&= 0,
\end{aligned}$$

and

$$\left(1+\left\lfloor\frac{\sum_{i=2}^{k} r_i n_i}{n_1-\left\lfloor\frac{n_1}{r_1}\right\rfloor r_1}\right\rfloor\right)\left(n_1-\left\lfloor\frac{n_1}{r_1}\right\rfloor r_1\right)-\sum_{i=2}^{k} r_i n_i \leq \left(1+\left(\frac{\sum_{i=2}^{k} r_i n_i}{n_1-\left\lfloor\frac{n_1}{r_1}\right\rfloor r_1}\right)\right)\left(n_1-\left\lfloor\frac{n_1}{r_1}\right\rfloor r_1\right) - \sum_{i=2}^{k} r_i n_i$$

$$= n_1-\left\lfloor\frac{n_1}{r_1}\right\rfloor r_1+\sum_{i=2}^{k} r_i n_i-\sum_{i=2}^{k} r_i n_i$$

$$= n_1-\left\lfloor\frac{n_1}{r_1}\right\rfloor r_1.$$

Therefore, the number of vertices that still need to be colored with a color in $\{1, 2, \ldots, r_1\}$ is strictly greater than 0 and less than or equal to $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$. Thus, these vertices can be colored with one color. These vertices receive color $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$. We will place this color such that the first vertex to receive color $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ is placed in the next Q_i after the last vertex of color $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil - 1$. The rest of the vertices of color $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ will be placed in consecutive Q_i's such that the color will be placed in Q_1 after a color has been placed in Q_{r_1}. The remaining vertices in $\bigcup_{i=1}^{r_1} Q_i$ will be colored with a color in $\{r_1 + 1, r_1 + 2, \ldots, \chi(G)\}$.

Note that $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ is equivalent to $n_1 \pmod{r_1}$. Therefore, $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 < r_1$. Since each color in $\{1, \ldots, r_1\}$ appears at most $n_1 \pmod{r_1}$ times in $\bigcup_{i=1}^{r_1} Q_i$ and these colors are distributed across partite sets as evenly as possible, the colors in this set are distributed in $\bigcup_{i=1}^{r_1} Q_i$ such that no color appears more than once in a given Q_i. Therefore, each P_i has at most $\left\lfloor \frac{n_1}{r_1} \right\rfloor + 1$ vertices of each color in $\{1, \ldots, r_1\}$.

Note that for all i and j where $i, j \in \{1, \ldots, r_1\}$, partite set P_i contains exactly $\left\lfloor \frac{n_1}{r_1} \right\rfloor$ or $\left\lfloor \frac{n_1}{r_1} \right\rfloor + 1$ vertices with color j. Moreover, at most $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ colors in $\{1, \ldots r_1\}$ appear $\left\lfloor \frac{n_1}{r_1} \right\rfloor + 1$ times in some partite set P_j. The vertices in partite sets $\{P_{r+1}, \ldots, P_{\chi(G)}\}$ are colored with colors in the set $\{1, \ldots, r_1\}$. Therefore, in reconstructing a proper coloring of G, there are at most

$r_1 \left\lfloor \frac{n_1}{r_1} \right\rfloor + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ vertices that do not need to be recolored, and

$$B(G) \geq \sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil. \qquad \square$$

We believe $B(G) \leq \sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ as well, but this has yet to be shown. A less strict upper bound of $B(G) \leq \sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$ has been proven. Note that since $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ is equivalent to $n_1 \pmod{r_1}$, it holds that $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1$ is strictly less than r_1. Thus,

$$\sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil \leq \sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil.$$

Theorem 3. *Let* $G = K_{\underbrace{n_1,\ldots,n_1}_{r_1},\underbrace{n_2,\ldots,n_2}_{r_2},\ldots,\underbrace{n_k,\ldots,n_k}_{r_k}}$ *be a complete multipartite graph with* r_j *partite sets of size* n_j *for all* $1 \leq j \leq k$*. Note that* $r_1 n_1 \geq r_2 n_2 \geq \ldots \geq r_k n_k > 0$*. If we have*

$$r_1 n_1 > \sum_{j=2}^{k} r_j n_j \text{ and } \sum_{j=2}^{k} r_j n_j < \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) r_1,$$

then

$$B(G) \leq \sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil.$$

Proof. For $j \in \left[\sum_{j=1}^{k} r_j\right]$, let P_j be a partite set in G. Let c be a proper coloring of $V(G)$ that uses $\chi(G)$ colors. Therefore all vertices in each partite set get the same color under c and $\chi(G) = \sum_{i=1}^{k} r_k$ colors are used. We define c such that $P_1, P_2, \ldots, P_{r_1}$ are the r_1 sets of size n_1 and each vertex in P_i where $i \in [r_1]$ has color i. Note that $\sum_{j=2}^{k} r_j n_j$ is the total number of vertices in $\bigcup_{i=r_1+1}^{\chi(G)} P_i$.

Let c^* be the rearrangement of c that has the highest villainy. It holds that under any recoloring, at least $r_1 n_1 - \sum_{i=2}^{k} r_i n_i$ vertices in $\bigcup_{i=1}^{r_1} P_i$ must have a color in $\{1, 2, \ldots, r_1\}$. Any vertex in

$\bigcup_{i=1}^{r_1} P_i$ with a color in $\{r_1+1, r_1+2, \ldots, \chi(G)\}$ is colored incorrectly. Likewise, any vertex in $\bigcup_{j=r_1+1}^{\chi(G)} P_j$ with a color in $\{1, 2, \ldots, r_1\}$ is colored incorrectly. Thus, in c^*, as many vertices as possible in $\bigcup_{i=1}^{r_1} P_i$ are recolored with a color in $\{r_1+1,\ r_1+2, \ldots, \chi(G)\}$. It follows that exactly $r_1 n_1 - \sum_{i=2}^{k} r_i n_i$ vertices in $\bigcup_{i=2}^{k} r_i n_i$ are colored with a color in $\{1, 2, \ldots, r_1\}$ under c^*.

Let us represent the coloring under c^* of $\bigcup_{i=1}^{r_1} P_i$ as a weighted bipartite graph with partite sets A and B where the vertices in A represent the set of colors in $\{1, 2, \ldots, r_1\}$ and the vertices in B represent the partite sets in $\{P_1, \ldots, P_{r_1}\}$. An edge between a vertex f in A and g in B with weight k indicates that there are k vertices with color f in P_g. Let the weight of such an edge be denoted $w_{f,g}$. The sum of these weights is equivalent to $r_1 n_1 - \sum_{j=2}^{k} r_j n_j$, the total number of vertices in $\bigcup_{i=1}^{r_1} P_i$ with colors in $\{1, 2, \ldots, r_1\}$. Consider the maximum weighted matching between A and B. Note that the bipartite graph is complete, thus the maximum weighted matching is also a perfect matching. By choosing to recolor each partite set such that the vertices in P_i receive the color in A that was matched with i in B, we choose the rearrangement of c^* that achieves the proper coloring of $\bigcup_{i=1}^{r_1} P_i$ that requires the fewest number of recolorings.

Without loss of generality, let us assume that in each partite set P_i, the vertices of color i do not need to be recolored when reconstructing the proper coloring of G that requires the minimum number of recolorings. Therefore, the sum of the edges in the maximum weighted matching is $w_{1,1} + w_{2,2} + \ldots + w_{r_1,r_1}$. For each fixed integer s, with $s \in \{1, \ldots, r_1\}$, if we recolor the vertices in $\bigcup_{i=1}^{r_1} P_i$ in such a way that all vertices in P_i get final color $i+s$ modulo r_1 (note that if $i+s \equiv 0 \pmod{r_1}$, the vertices in P_i will receive color r_1), then each element $w_{i,j}$ where $i, j \in \{1, \ldots, r_1\}$ will be included in a sum exactly once in the set $\{w_{1,1} + w_{2,2} + \ldots + w_{r_1,r_1},\ w_{1,2} + w_{2,3} + \ldots + w_{r_1,1}, \ldots, w_{1,r_1} + w_{2,1} + \ldots + w_{r_1,r_1-1}\}$. The sum of the elements in this set can be expressed as $\sum_{i=1}^{r_1} \sum_{j=1}^{r_1} w_{i,j}$ and will be the sum of r_1^2 different values of $w_{i,j}$. Note that this value represents the sum of the weights of all edges in the weighted bipartite graph between A and B. Therefore, this sum is the total number of vertices with colors $\{1, \ldots, r_1\}$ in $\bigcup_{i=1}^{r_1} P_i$.

Let us assume that an improper coloring of G exists such that the number of vertices that do not need to be recolored is less than $\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$. This implies that the maximum weighted matching between A and B is less than $\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$. If the sum of the weights in the maximum weighted matching, $w_{1,1} + w_{2,2} + \ldots + w_{r_1,r_1}$, is less than $\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 +$

$\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$, then each element in the set $\{w_{1,1} + w_{2,2} + \ldots + w_{r_1,r_1}, w_{1,2} + w_{2,3} + \ldots + w_{r_1,1}, \ldots, w_{1,r_1} + w_{2,1} + \ldots + w_{r_1,r_1-1}\}$ must also be less than $\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$. Given that there are r_1 elements in this set, the maximum possible value of $\sum_{i=1}^{r_1} \sum_{j=1}^{r_1} w_{i,j}$ is given by $r_1 \left(\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil - 1 \right)$. Note that if x or y is an integer, it holds that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$. It follows that:

$$
\begin{aligned}
r_1 \left(\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil - 1 \right) &= r_1 \left(\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + \left\lceil n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + \frac{-\sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil - 1 \right) \\
&= r_1 \left(\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + \left\lceil \frac{-\sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil - 1 \right) \\
&= r_1 \left(n_1 + \left\lceil \frac{-\sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil - 1 \right) \\
&= r_1 n_1 + r_1 \left\lceil \frac{-\sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil - r_1.
\end{aligned}
$$

The value of $\left\lceil -\frac{\sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$ is strictly less than $\frac{-\sum_{j=2}^{k} r_j n_j}{r_1} + 1$ and greater than or equal to $\frac{-\sum_{j=2}^{k} r_j n_j}{r_1}$. Therefore,

$$
\begin{aligned}
r_1 n_1 + r_1 \left\lceil \frac{-\sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil - r_1 &< r_1 n_1 + r_1 \left(\frac{-\sum_{j=2}^{k} r_j n_j}{r_1} + 1 \right) - r_1 \\
&= r_1 n_1 - \sum_{j=2}^{k} r_j n_j + r_1 - r_1 \\
&= r_1 n_1 - \sum_{j=2}^{k} r_j n_j
\end{aligned}
$$

and $\sum_{i=1}^{r_1} \sum_{j=1}^{r_1} w_{i,j}$ is strictly less than $r_1 n_1 - \sum_{j=2}^{k} r_j n_j$.

This is a contradiction given that the sum of weighted edges of the multipartite graph between A and B must be equal to $r_1 n_1 - \sum_{j=2}^{k} r_j n_j$. Therefore, the number of vertices that do not need to be recolored is at least $\left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$. Note that all vertices in $\bigcup_{r_1+1}^{\chi(G)} P_i$ are colored with a color in the set $\{1, 2, \ldots, r_1\}$ and are thus colored incorrectly. Therefore,

$$
B(G) \leq \sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil . \qquad \square
$$

Consider case 3 where $r_1n_1 \geq \sum_{j=2}^{k} r_jn_j$ and $\sum_{j=2}^{k} r_jn_j \geq \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$. Let

$$G = K_{\underbrace{n_1,\ldots,n_1}_{r_1},\underbrace{n_2,\ldots,n_2}_{r_2},\ldots,\underbrace{n_k,\ldots,n_k}_{r_k}}.$$

Let G be properly colored such that P_i receives color i and $\{P_1, P_2, \ldots, P_{r_1}\}$ are the parts of size n_1. We aim to improperly recolor $\bigcup_{i=1}^{r_1} P_i$ such that the colors in $\{1, 2, \ldots, r_1\}$ are distributed as evenly as possible. We first want to distribute as many of the $r_1n_1 - \sum_{i=2}^{k} r_in_i$ vertices with a color in $\{1, 2, \ldots, r_1\}$ such that each color is in a partite set the same number of times and there is the same number of vertices with each color in each partite set. Thus, each color appears in each partite set $\left\lfloor \frac{r_1n_1 - \sum_{i=2}^{k} r_in_i}{r_1^2} \right\rfloor$ times. This leaves $r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{i=2}^{k} r_in_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_in_i$ vertices in $\bigcup_{i=1}^{r_1} P_i$ left to be recolored with a color in $\{1, 2, \ldots, r_1\}$. Let Q_i be the uncolored vertices in P_i for $i \in [r_1]$. We aim to recolor these vertices in $\bigcup_{i=1}^{r_1} Q_i$ such that at most one vertex of each color in $\{1, 2, \ldots, r_1\}$ is in a given Q_i and the colors are distributed across as few partite sets as possible. The minimum number of partite sets these colors can be distributed over is given by

$$\left\lceil \frac{r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{i=2}^{k} r_in_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_in_i}{r_1} \right\rceil.$$

However, partite set $\left\lceil \frac{r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{i=2}^{k} r_in_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_in_i}{r_1} \right\rceil$ may not contain r_1 additional colors in $\{1, 2, \ldots, r_1\}$. We color r_1 uncolored vertices in the first $\left\lceil \frac{r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{i=2}^{k} r_in_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_in_i}{r_1} \right\rceil - 1$ partite sets with each color in $\{1, 2, \ldots, r_1\}$. This leaves

$$r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{i=2}^{k} r_in_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_in_i - \left(\left\lceil \frac{r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{i=2}^{k} r_in_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_in_i}{r_1} \right\rceil - 1\right) r_1$$

vertices that need to be recolored with a color in $\{1, 2, \ldots, r_1\}$. We color

$$r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i - \left(\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{r_1} \right\rceil - 1 \right) r_1$$

vertices in partite set $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{r_1} \right\rceil$ such that there is a vertex of each color i where

$$i \in \left[r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i - \left(\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{r_1} \right\rceil - 1 \right) r_1 \right].$$

The first $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{r_1} \right\rceil$ partite sets will contain $\left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor + 1$ of a distinct color while the rest contain at most $\left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor$ vertices of the same color. If we recolor the vertices in the first $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{r_1} \right\rceil$ sets with a color that appears $\left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor + 1$ times and the rest of the partite sets with a color that appears $\left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor$ times, at most

$$\begin{aligned}
&\left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1 + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1^2 - \sum_{i=2}^{k} r_i n_i}{r_1} \right\rceil \\
&\qquad = \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1 + n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1^2} \right\rfloor r_1 + \left\lceil \frac{- \sum_{i=2}^{k} r_i n_i}{r_1} \right\rceil \\
&\qquad = n_1 + \left\lceil \frac{- \sum_{i=2}^{k} r_i n_i}{r_1} \right\rceil \\
&\qquad = \left\lceil \frac{r_1 n_1 - \sum_{i=2}^{k} r_i n_i}{r_1} \right\rceil
\end{aligned}$$

vertices in G will not need to be recolored.

Consider $G = K_{8,8,8,10}$. A proper coloring of $K_{8,8,8,10}$ is given in Figure 16. In this case,

$$\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 = \left(8 - \left\lfloor \frac{8}{3} \right\rfloor 3\right) 3 = 6$$

while $\sum_{i=2}^{k} r_i n_i = 10$ and it holds that $\sum_{i=2}^{k} r_i n_i \geq \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1$. Let us define P_i as the partite set that contains vertices with color i under the proper coloring given in Figure 16. We first recolor $\left\lfloor \frac{14}{9} \right\rfloor$ vertices in each partite set of size 8 with each color in $\{1, 2, 3\}$ as shown in Figure 17. There are five vertices left in $\bigcup_{i=1}^{3} P_i$ that need to be recolored with a color in $\{1, 2, 3\}$ and these vertices will be in P_1 and P_2. We will color an additional vertex in P_1 with each color in $\{1, 2, 3\}$ and one vertex in P_2 with each color in $\{1, 2\}$. The remaining vertices in $\bigcup_{i=1}^{3} P_i$ receive color 4. This recoloring is given in Figure 18.

In the coloring given in Figure 18, there are at most two vertices in a partite set that appear in G eight times and are the same color. However, only P_1 and P_2 contain a color that appears two times. Recoloring the vertices in P_2 with color 1 and the vertices in P_1 with color 2 requires 12 recolorings. Recoloring P_3 with color 3 requires 7 recolorings. Thus, to return the graph in Figure 18 to a proper coloring, 19 vertices need to be recolored and 5 vertices are already correctly colored. Note that

$$\left\lceil \frac{r_1 n_1 - \sum_{i=2}^{k} r_k n_k}{r_1} \right\rceil = \left\lceil \frac{8(3) - 10}{3} \right\rceil = 5.$$

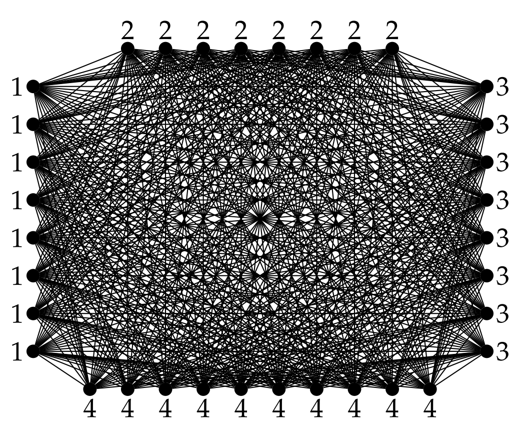


Figure 16: A proper coloring of $K_{8,8,8,10}$

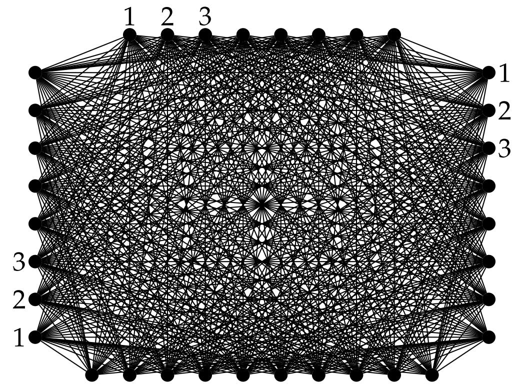


Figure 17: A partial recoloring of $K_{8,8,8,10}$

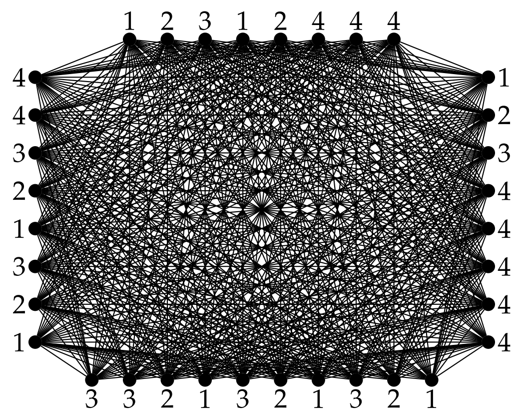


Figure 18: An improper recoloring of $K_{8,8,8,10}$

Theorem 4. *Let* $G = K_{\underbrace{n_1,\dots,n_1}_{r_1},\underbrace{n_2,\dots,n_2}_{r_2},\dots,\underbrace{n_k,\dots,n_k}_{r_k}}$ *be a complete multipartite graph with* r_j *partite sets of size* n_j *for all* $j \in [k]$*, with* $r_1n_1 \geq r_2n_2 \geq \ldots \geq r_kn_k > 0$*. If we have*

$$r_1n_1 > \sum_{j=2}^{k} r_jn_j \text{ and } \sum_{j=2}^{k} r_jn_j \geq \left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1,$$

then

$$B(G) = \sum_{j=1}^{k} r_jn_j - \left\lceil \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1} \right\rceil.$$

Proof. For $\ell \in \left[\sum_{j=1}^{k} r_j\right]$, let P_ℓ be a partite set in G. Let c be a proper coloring of G that uses $\chi(G)$ colors; we define c such that $P_1, P_2, \ldots, P_{r_1}$ are the r_1 sets of size n_1 and each vertex in P_ℓ has color ℓ under c. Note that $\chi(G) = \sum_{j=1}^{k} r_j$.

Let c^* be the rearrangement of c that has the highest villainy. It holds that under any recoloring, at least $r_1n_1 - \sum_{i=2}^{k} r_in_i$ vertices in $\bigcup_{i=1}^{r_1} P_i$ must have a color in $\{1, 2, \ldots, r_1\}$. Any vertex in $\bigcup_{i=1}^{r_1} P_i$ with a color in $\{r_1 + 1, r_1 + 2, \ldots, \chi(G)\}$ is colored incorrectly. Likewise, any vertex in $\bigcup_{j=r_1+1}^{\chi(G)} P_j$ with a color in $\{1, 2, \ldots, r_1\}$ is colored incorrectly. Thus, in c^*, as many vertices as possible in $\bigcup_{i=1}^{r_1} P_i$ are recolored with a color in $\{r_1 + 1,\ r_1 + 2, \ldots, \chi(G)\}$. It follows that exactly $r_1n_1 - \sum_{i=2}^{k} r_in_i$ vertices in $\bigcup_{i=2}^{k} r_in_i$ are colored with a color in $\{1, 2, \ldots, r_1\}$ under c^*.

We need to recolor the vertices of G in such a way that a proper coloring of G using the same set of colors is obtained. For integers a and b with $a, b \in \{1, \ldots, r_1\}$, let $f_{a,b}$ be the total number of vertices in P_a having color b. Thus, $\sum_{a=1}^{r_1} \sum_{b=1}^{r_1} f_{a,b} = r_1n_1 - \sum_{j=2}^{k} r_jn_j$.

Note that for the vertices in partite sets $P_1, P_2, \ldots, P_{r_1}$ to be properly colored, the vertices in each partite set must be colored with a unique color in the set $\{1, 2, \ldots, r_1\}$. Let c_s be a proper coloring of G where the vertices in P_i receive color $i + s \pmod{r_1}$ for all $i \in [k]$ where $s \in \{0, 1, \ldots, r_1 - 1\}$. If $i + s \pmod{r_1}$ is 0, the vertices in partite set i receive color r_1. Note that there are r_1 colorings in the set $\{c_0, c_1, \ldots, c_{r_1-1}\}$. Let us define F_s as the number of vertices in c^* that do not need to be recolored to achieve coloring c_s. It holds that

$$F_s = f_{1,1+s \pmod{r_1}} + f_{2,2+s \pmod{r_1}} + \ldots + f_{r_1,r_1+s \pmod{r_1}}$$

for a given s. In c^*, it holds that

$$\begin{aligned}
r_1n_1 - \sum_{j=2}^{k} r_jn_j &= \underbrace{(f_{1,1} + f_{1,2} + \ldots + f_{1,r_1})}_{P_1} + \underbrace{(f_{2,1} + f_{2,2} + \ldots + f_{2,r_1})}_{P_2} \\
&\quad + \ldots + \underbrace{(f_{r_1,1} + f_{r_1,2} + \ldots + f_{r_1,r_1})}_{P_{r_1}} \\
&= (f_{1,1} + f_{2,2} + \ldots + f_{r_1,r_1}) + (f_{1,2} + f_{2,3} + \ldots + f_{r_1,1}) \\
&\quad + \ldots + (f_{1,r_1} + f_{2,1} + \ldots + f_{r_1,r_1-1}) \\
&= F_0 + F_1 + \ldots + F_{r_1-1}.
\end{aligned}$$

Therefore,

$$\max_{0 \le s \le r_1-1} (F_s) \ge \left\lceil \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1} \right\rceil .$$

Thus, the upper bound on the smallest number of recolorings needed to properly recolor the vertices in $\bigcup_{i=1}^{r_1} P_i$ is given by $r_1n_1 - \left\lceil \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1} \right\rceil$. Since there are $\sum_{j=2}^{k} r_jn_j$ vertices in $\bigcup_{i=r_1+1}^{\chi(G)} P_i$ with a color in the set $\{1, 2, \ldots, r_1\}$, we have

$$B(G) \leq r_1n_1 - \left\lceil \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1} \right\rceil + \sum_{j=2}^{k} r_jn_j$$
$$= \sum_{i=1}^{k} r_in_i - \left\lceil \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1} \right\rceil.$$

Let c be a proper $\chi(G)$-coloring of G. To prove $B(G) \geq \sum_{i=1}^{k} r_in_i - \left\lceil \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1} \right\rceil$, we define a coloring c^* on G that is a rearrangement of c. For each $a \in \{1, 2, \ldots, r_1\}$ and $b \in \{1, 2, \ldots, r_1\}$ we color $\left\lfloor \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1^2} \right\rfloor$ vertices in P_a with color b. We will recolor an additional $\sum_{j=2}^{r_1} r_jn_j$ vertices in $\bigcup_{i=1}^{r_1} P_i$ with colors in $\{r_1+1, r_1+2, \ldots, \chi(G)\}$. This results in $\bigcup_{j=1}^{r_1} P_j$ having $r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_jn_j$ vertices that are not yet colored. These vertices must be colored with a color from the set $\{1, 2, \ldots, r_1\}$. We aim to color these remaining vertices in $\bigcup_{i=1}^{r_1} P_i$ such that they are placed in the fewest possible number of partite sets and there are no more than $\left\lceil \frac{r_1n_1 - \sum_{j=2}^{r_1} r_jn_j}{r_1} \right\rceil$ of a color in a given partite set. It holds that the fewest number of partite sets the remaining $r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_jn_j$ uncolored vertices can be placed in is $\left\lceil \frac{r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_jn_j}{r_1} \right\rceil$. Let us color a vertex that has not yet been recolored in each partite set P_j where $j \in \left\{1, 2, \ldots, \left\lceil \frac{r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_jn_j}{r_1} \right\rceil - 1 \right\}$ with each color in the set $\{1, 2, \ldots, r_1\}$. This leaves

$$r_1n_1 - \left\lfloor \frac{r_1n_1 - \sum_{j=2}^{k} r_jn_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_jn_j - \left(\left\lceil \frac{r_1n_1 - \left\lfloor \frac{n_1r_1 - \sum_{j=2}^{k} r_jn_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_jn_j}{r_1} \right\rceil - 1 \right) r_1$$

vertices uncolored in $\bigcup_{j=1}^{r_1} P_j$. Note that

$$
\begin{aligned}
& r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^k r_j n_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^k r_j n_j - \left(\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^k r_j n_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^k r_j n_j}{r_1} \right\rceil - 1 \right) r_1 \\
&= r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^k r_j n_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^k r_j n_j - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^k r_j n_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^k r_j n_j}{r_1} \right\rceil r_1 + r_1 \\
&= r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^k r_j n_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^k r_j n_j - r_1 n_1 + \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^k r_j n_j}{r_1^2} \right\rfloor r_1^2 \\
&\qquad - \left\lceil \frac{- \sum_{j=2}^k r_j n_j}{r_1} \right\rceil r_1 + r_1 \\
&= r_1 - \left\lceil \frac{- \sum_{j=2}^k r_j n_j}{r_1} \right\rceil r_1 - \sum_{j=2}^k r_j n_j \\
&= r_1 + \left\lfloor \frac{\sum_{j=2}^k r_j n_j}{r_1} \right\rfloor r_1 - \sum_{j=2}^k r_j n_j .
\end{aligned}
$$

The value of $\left\lfloor \frac{\sum_{j=2}^k r_j n_j}{r_1} \right\rfloor$ is less than or equal to $\frac{\sum_{j=2}^k r_j n_j}{r_1}$ and strictly greater than $\frac{\sum_{j=2}^k r_j n_j}{r_1} - 1$. It holds that

$$
\begin{aligned}
r_1 + \left\lfloor \frac{\sum_{j=2}^k r_j n_j}{r_1} \right\rfloor r_1 - \sum_{j=2}^k r_j n_j &\le r_1 + \left(\frac{\sum_{j=2}^k r_j n_j}{r_1} \right) r_1 - \sum_{j=2}^k r_j n_j \\
&= r_1,
\end{aligned}
$$

and

$$
\begin{aligned}
r_1 + \left\lfloor \frac{\sum_{j=2}^k r_j n_j}{r_1} \right\rfloor r_1 - \sum_{j=2}^k r_j n_j &> r_1 + \left(\frac{\sum_{j=2}^k r_j n_j}{r_1} - 1 \right) r_1 - \sum_{j=2}^k r_j n_j \\
&= r_1 + \sum_{j=2}^k r_j n_j - r_1 - \sum_{j=2}^k r_j n_j \\
&= 0.
\end{aligned}
$$

Therefore, the number of uncolored vertices in $\bigcup_{j=1}^{r_1} P_j$ that need to receive a color in $\{1, 2, \ldots, r_1\}$

is in $\{1,2,\ldots,r_1\}$. We then color $r_1 + \left\lfloor \frac{\sum_{j=2}^{k} r_j n_j}{r_1} \right\rfloor r_1 - \sum_{j=2}^{k} r_j n_j$ vertices in partite set $\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1^2} \right\rfloor - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$ with each color in the set

$$\left\{1,2,\ldots,r_1 + \left\lfloor \frac{\sum_{j=2}^{k} r_j n_j}{r_1} \right\rfloor r_1 - \sum_{j=2}^{k} r_j n_j \right\}.$$

It holds that for all a and b where $a,b \in \{1,2,\ldots,r_1\}$, partite set P_a contains exactly

$$\left\lfloor \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1^2} \right\rfloor \text{ or } \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1^2} \right\rfloor + 1$$

vertices with color b. Moreover,

$$\left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1 r_1 - \sum_{j=1}^{k} r_j n_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$$

partite sets contain a color in $\{1,2,\ldots,r_1\}$ that appears $\left\lfloor \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1^2} \right\rfloor + 1$ times. The vertices in partite sets $\{P_{r_1+1},\ldots,P_{\chi(G)}\}$ are colored with colors in the set $\{1,2,\ldots,r_1\}$. It follows that, in reconstructing a proper coloring of G, the number of vertices that do not need to be recolored is

$$\begin{aligned}
&\left\lfloor \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1^2} \right\rfloor r_1 + \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1^2} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil \\
&= \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1^2} \right\rfloor r_1 + n_1 - \left\lfloor \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1^2} \right\rfloor r_1 + \left\lceil \frac{-\sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil \\
&= n_1 + \left\lceil \frac{-\sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil \\
&= \left\lceil \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil.
\end{aligned}$$

Therefore,

$$B(G) \geq \sum_{j=1}^{k} r_j n_j - \left\lceil \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil. \qquad \square$$

Let $G = K_{n,\ldots,n}$ be a complete k-partite graph with k partite sets of size n. In this case, $\sum_{i=2}^{k} r_i n_i = 0$ and $\sum_{i=2}^{k} r_i n_i$ will always be less than or equal to $kn - \left\lfloor \frac{n}{k} \right\rfloor k^2$. When $\sum_{i=2}^{k} r_i n_i < kn - \left\lfloor \frac{n}{k} \right\rfloor k^2$, G is a subcase of case 2 and will be incorrectly colored using the same method. It follows that the villainy of G is greater than or equal to $\sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$ and less than or equal to $r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2}{r_1} \right\rceil$. However, given that $\sum_{i=2}^{k} r_i n_i = 0$, it holds that

$$r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil = kn - \left\lceil \frac{n}{k} \right\rceil k$$

and

$$r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2}{r_1} \right\rceil = kn - n.$$

Consider $K_{8,8,8}$. Let c be a proper 3-coloring of $K_{8,8,8}$ as shown in Figure 19.

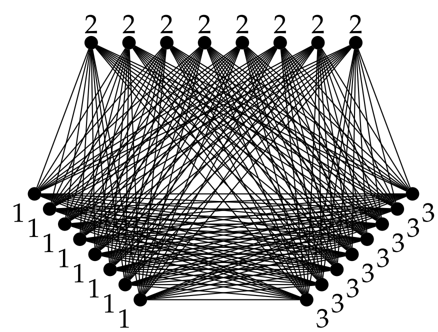


Figure 19: A proper coloring of $K_{8,8,8}$

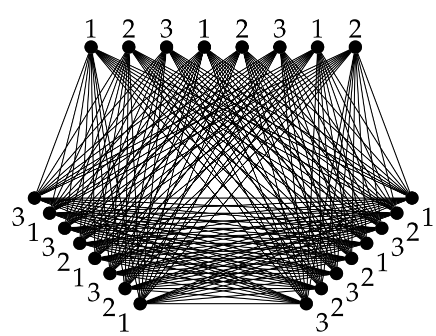


Figure 20: An improper recoloring of $K_{8,8,8}$

Let us define P_i as the partite set that receives color i under c. Following the recoloring procedure given for case 2, $K_{8,8,8}$ is improperly colored as shown in Figure 20. Every partite set contains three vertices with a distinct color. If we recolor the vertices in P_1 with 1, the vertices in P_2 with 2 and the vertices in P_3 with 3, 15 recolorings will be performed to achieve a proper coloring. Note that

$$kn - k\left\lceil \frac{n}{k} \right\rceil = 3(8) - 3\left\lceil \frac{8}{3} \right\rceil = 15$$

and

$$kn - n = 3(8) - 8 = 16.$$

Theorem 5. *Let $G = K_{n,\ldots,n}$ be a k-partite graph with k partite sets of size n where k does not divide n. It holds that*

$$kn - n \geq B(G) \geq kn - k\left\lceil \frac{n}{k} \right\rceil.$$

Proof. Let $G = K_{n,\ldots,n}$ and let P_1, P_2,...,P_k be the partite sets of G. By our assumption, k does not divide n. Given that each partite set must receive a unique color, $\chi(G) = k$. Let c be a proper k-coloring of G in which the vertices in P_i receive color i and let c^* be a rearrangement of c. By Theorem 3, less than or equal to $\sum_{j=1}^{k} r_j n_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$ vertices in c^* need to be recolored to restore G to a proper coloring. However, given that $\sum_{i=2}^{k} r_i n_i = 0$, it holds that

$$\begin{aligned} r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2}{r_1} \right\rceil &= kn - \left\lfloor \frac{n}{k} \right\rfloor k - \left\lceil n - \left\lfloor \frac{n}{k} k \right\rfloor \right\rceil \\ &= kn - \left\lfloor \frac{n}{k} \right\rfloor k - n + \left\lfloor \frac{n}{k} \right\rfloor k \\ &= kn - n \end{aligned}$$

and

$$B(G) \leq kn - n.$$

According to Theorem 2, it holds that $B(G) \geq r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$. However, given that $\sum_{i=2}^{k} r_i n_i = 0$, it holds that

$$r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil = kn - \left\lfloor \frac{n}{k} \right\rfloor k - \left\lceil \frac{kn - \left\lfloor \frac{n}{k} \right\rfloor k^2}{n - \left\lfloor \frac{n}{k} \right\rfloor k} \right\rceil$$
$$= kn - \left\lfloor \frac{n}{k} \right\rfloor k - k.$$

It holds that $k \nmid n$ in case 2. Therefore, $\left\lfloor \frac{n}{k} \right\rfloor + 1 = \left\lceil \frac{n}{k} \right\rceil$. It follows that:

$$kn - \left\lfloor \frac{n}{k} \right\rfloor k - k = kn - \left(\left\lfloor \frac{n}{k} \right\rfloor + 1 \right) k$$
$$= kn - \left\lceil \frac{n}{k} \right\rceil k.$$

Thus,

$$B(G) \geq kn - k \left\lceil \frac{n}{k} \right\rceil . \qquad \square$$

When $\sum_{i=2}^{k} r_i n_i = kn - \left\lfloor \frac{n}{k} \right\rfloor k^2 = 0$, G is a subcase of case 3 and will be incorrectly colored using the same method. Thus, G has a villainy of $\sum_{j=1}^{k} r_j n_j - \left\lceil \frac{r_1 n_1 - \sum_{j=2}^{k} r_j n_j}{r_1} \right\rceil$. However, given that $\sum_{i=2}^{k} r_i n_i = 0$, this can be simplified to

$$\sum_{j=1}^{k} r_j n_j - \left\lceil \frac{r_1 n_1}{r_1} \right\rceil = kn - \left\lceil \frac{kn}{k} \right\rceil$$
$$= kn - n.$$

Given that $k|n$ in this case,

$$kn - n = kn - k \left\lceil \frac{n}{k} \right\rceil .$$

Consider $G = K_{8,8,8,8}$. Let c be a proper 4-coloring of $K_{8,8,8,8}$ as shown in Figure 21.

Let us define P_i as the partite set that receives color i under c. Following the recoloring procedure given for case 3, $K_{8,8,8,8}$ is improperly colored as shown in Figure 22. Every partite set contains two vertices with a distinct color. If we recolor the vertices in P_1 with 1, the vertices in P_2 with 2 the vertices in P_3 with 3, and the vertices in P_4 with 4, 24 recolorings will be performed to acheive

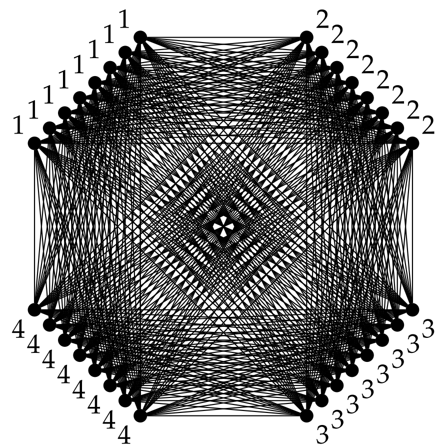


Figure 21: A proper coloring of $K_{8,8,8,8}$

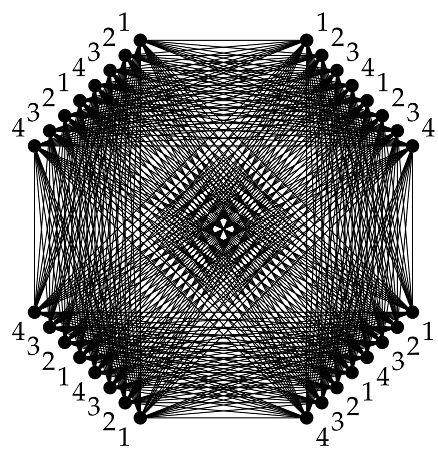


Figure 22: An improper coloring of $K_{8,8,8,8}$

a proper coloring. Note that

$$kn - k\left\lceil \frac{n}{k} \right\rceil = 4(8) - 4\left\lceil \frac{8}{4} \right\rceil = 24.$$

Theorem 6. *Let $G = K_{n,\ldots,n}$ be a k-partite graph with k partite sets of size n where k divides n. It holds that*

$$B(G) = kn - k\left\lceil \frac{n}{k} \right\rceil.$$

Proof. Let $G = K_{n,\ldots,n}$ and let P_1, P_2,...,P_k be the partite sets of G. Given that each partite set must receive a unique color, $\chi(G) = k$. Let c be a proper k-coloring of G in which the vertices in P_i

receive color i and let c^* be a rearrangement of c. Our aim is to recolor at most $kn - k\left\lceil \frac{n}{k} \right\rceil$ vertices of G in c^* in order to obtain a proper coloring.

Let us represent the coloring under c^* of $\bigcup_{i=1}^{k} P_i$ as a weighted bipartite graph with partite sets A and B where the vertices in A represent the set of colors in $\{1, 2, \ldots, k\}$ and the vertices in B represent the partite sets in $\{P_1, \ldots, P_k\}$. An edge between a vertex f in A and g in B with weight h indicates that there are h vertices with color f in P_g. Let the weight of such an edge be denoted $w_{f,g}$. The sum of these weights is equivalent to kn, the total number of vertices in $\bigcup_{i=1}^{k} P_i$. Consider the maximum weighted matching between A and B. Note that the bipartite graph is complete, thus the maximum weighted matching is also a perfect matching. Without loss of generality, let us assume the maximum weighted matching is given by $w_{1,1} + w_{2,2} + \ldots + w_{k,k}$ By choosing to recolor each partite set such that the vertices in P_i receive the color in A that was matched with i in B, we choose the rearrangement of c^* that achieves the proper coloring of $\bigcup_{i=1}^{k} P_i$ that requires the fewest number of recolorings.

Let us assume that an improper coloring of G exists such that the number of vertices that do not need to be recolored is less than n. This implies that the maximum weighted matching between A and B is less than n. If the sum of the weights in the maximum weighted matching, $w_{1,1} + w_{2,2} + \ldots + w_{k,k}$, is less than n, then each element in the set $\{w_{1,1} + w_{2,2} + \ldots + w_{k,k}$, $w_{1,2} + w_{2,3} + \ldots + w_{k,1}, \ldots, w_{1,k} + w_{2,1} + \ldots + w_{k,k-1}\}$ must also be less than n. Given that there are k elements in this set, the maximum possible value of $\sum_{i=1}^{r_1} \sum_{j=1}^{k} w_{i,j}$ is given by $kn - k$ which is strictly less than kn. This is a contradiction given that the sum of weighted edges of the multipartite graph between A and B must be equal to kn. Therefore, the number of vertices that do not need to be recolored is at least n. Therefore,$B(G) \leq kn - n$. Given that k divides n, it holds that

$$B(G) \leq kn - k\left\lceil \frac{n}{k} \right\rceil.$$

To prove that $B(G) \geq kn - k\left\lceil \frac{n}{k} \right\rceil$, we define a coloring c^* on the vertices of G where c^* is a rearrangement of c. For each $i \in [n]$ and $j \in [k]$, we color $\left\lfloor \frac{n}{k} \right\rfloor$ vertices in P_i with color j. This results in each P_i having $n - k\left\lfloor \frac{n}{k} \right\rfloor$ vertices that are not yet colored.

In the case where k divides n, $\left\lfloor \frac{n}{k} \right\rfloor = \left\lceil \frac{n}{k} \right\rceil$ and each P_i has $n - k\left(\frac{n}{k}\right) = 0$ vertices that are uncolored. For each $i \in [n]$ and $j \in [k]$, each color j appears in P_i exactly $\left\lceil \frac{n}{k} \right\rceil$ times and at most $k\left\lceil \frac{n}{k} \right\rceil$ vertices are colored correctly.

In the case where, k does not divide n, it holds that $\left\lfloor \frac{n}{k} \right\rfloor = \left\lceil \frac{n}{k} \right\rceil - 1$. Let us refer to the uncolored vertices in P_i as Q_i. We aim to recolor $\bigcup_{i=1}^{k} Q_i$ such that each color appears at most once in a given Q_i. So far each color has been used $k \left\lfloor \frac{n}{k} \right\rfloor$ times and can therefore be used at most $n - k \left\lfloor \frac{n}{k} \right\rfloor$ times in $\bigcup_{i=1}^{k} Q_i$. Note that $n - k \left\lfloor \frac{n}{k} \right\rfloor$ is equivalent to $n \pmod{k}$ and can never be greater than k. Therefore, the colors can be distributed over $\bigcup_{i=1}^{k} Q_i$ such that there is at most one vertex of each color in a given Q_i. Therefore, each partite set has at least one color that appears $\left\lceil \frac{n}{k} \right\rceil$ times. Given that $\max_i f_{i,j}$ is $\left\lceil \frac{n}{k} \right\rceil$ in each partite set,

$$B(G) \geq kn - k \left\lceil \frac{n}{k} \right\rceil . \qquad \square$$

We believe that the villainy of $K_{n,\dots,n}$ is $kn - k \left\lceil \frac{n}{k} \right\rceil$ for all values of n and k where $k > 0$, but a strict upper bound has not yet been found in the case where k does not divide n. Clark et al. showed that in a uniquely colorable graph G in which every color appears the same number of times in its $\chi(G)$-coloring, the villainy is equivalent to the weak villainy [2]. Therefore, it also holds that $B_w(K_{n,\dots,n}) = B(K_{n,\dots,n})$.

In [2], Clark et al. gave results on the chromatic villainy of connected bipartite graphs and the chromatic villainy of complete multipartite graphs in which every partite set is a different size.

Let $G = K_{x,y}$. Without loss of generality, in a complete bipartite graph with two partite sets X and Y where $|X| = x$ and $|Y| = y$, either $x = y$ or $x > y$. According to Proposition 3.3 in [2],

- if $x = y$, then $B(G) = 2 \left\lfloor \frac{x+y}{4} \right\rfloor$ and
- if $x > y$, then $B(G) = 2y$.

If $x = y$, then $\sum_{j=2}^{k} r_j n_j = 0$ and the villainy of G is either bounded by Theorem 2 and Theorem 3 or is given by Theorem 4. When $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 \right) r_1 > 0$, G falls under Theorems 2 and 3. Note that since $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 > 0$, r_1 does not divide n_1. Given that $r_1 = 2$, it holds that n_1 must be odd.

Without loss of generality, let $n_1 = 2\ell + 1$. Thus,

$$\begin{aligned} n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 &= 2\ell + 1 - \left\lfloor \frac{2\ell+1}{2} \right\rfloor 2 \\ &= 2\ell + 1 - 2\left(\ell - \left\lfloor \frac{1}{2} \right\rfloor\right) \\ &= 2\ell + 1 - 2\ell - 2(0) \\ &= 1, \end{aligned}$$

and $n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 = 1$. It follows that

$$\begin{aligned} B(G) &\geq r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil \\ &= 2n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor 2 - \left\lceil \frac{2n_1 - \lfloor \frac{n_1}{2} \rfloor 4}{n_1 - \lfloor \frac{n_1}{2} \rfloor 2} \right\rceil \\ &= 2n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor 2 - 2 \\ &= 2\left(n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor - 1\right) \\ &= 2\left(n_1 - \left(\left\lfloor \frac{n_1}{2} \right\rfloor + 1\right)\right) \\ &= 2\left(n_1 - \left\lceil \frac{n_1}{2} \right\rceil\right) \\ &= 2\left\lfloor \frac{n_1}{2} \right\rfloor \\ &= 2\left\lfloor \frac{2n_1}{4} \right\rfloor, \end{aligned}$$

and

$$\begin{aligned}
B(G) &\leq r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1 n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2}{r_1} \right\rceil \\
&= 2n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor 2 - \left\lceil \frac{2n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor 4}{2} \right\rceil \\
&= 2n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor 2 - \left\lceil n_1 - \left\lfloor \frac{n_1}{2} \right\rfloor 2 \right\rceil \\
&= 2n - \left\lfloor \frac{n_1}{2} \right\rfloor 2 - 1 \\
&= 2\left(n - \left\lfloor \frac{n_1}{2} \right\rfloor\right) - 1 \\
&= 2\left\lceil \frac{n_1}{2} \right\rceil - 1 \\
&= 2\left\lceil \frac{2n_1}{4} \right\rceil - 1 \\
&= 2\left(\left\lfloor \frac{2n_1}{4} \right\rfloor + 1\right) - 1 \\
&= 2\left\lfloor \frac{2n_1}{4} \right\rfloor + 1.
\end{aligned}$$

Therefore, Theorems 2 and 3 give results that do not contradict those given in Proposition 3.3 in [2]. If $x = y$ and $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 = 0$, the villainy of G is given by Theorem 4. Therefore,

$$\begin{aligned}
B(G) &= 2n_1 - \left\lceil \frac{2n_1}{2} \right\rceil \\
&= n_1 \\
&= 2\left(\frac{2n_1}{4}\right) \\
&= 2\left\lfloor \frac{2n_1}{4} \right\rfloor
\end{aligned}$$

and Theorem 4 confirms the result given in Proposition 3.3 in [2]. Note that the villainy of G is also given by Theorems 5 and 6 when $x = y$. When k does not divide n, G falls under Theorem 5.

In this case, $k = 2$. It follows that

$$\begin{aligned} B(G) &\geq kn - k\left\lceil \frac{n}{k} \right\rceil \\ &= 2n - 2\left\lceil \frac{n}{2} \right\rceil \\ &= 2\left(n - \left\lceil \frac{n}{2} \right\rceil\right) \\ &= 2\left\lfloor \frac{n}{2} \right\rfloor \\ &= 2\left\lfloor \frac{2n}{4} \right\rfloor, \end{aligned}$$

and

$$\begin{aligned} B(G) &\leq kn - n \\ &= 2n - n \\ &= n. \end{aligned}$$

Therefore, Theorem 5 gives results that do not contradict those given in Proposition 3.3 in [2].

When k divides n, G falls under Theorem 6. In this case, $k = 2$ and

$$\begin{aligned} B(G) &= kn - k\left\lceil \frac{n}{k} \right\rceil \\ &= 2n - 2\left\lceil \frac{n}{2} \right\rceil \\ &= 2\left(n - \left\lceil \frac{n}{2} \right\rceil\right) \\ &= 2\left\lfloor \frac{n}{2} \right\rfloor \\ &= 2\left\lfloor \frac{2n}{4} \right\rfloor. \end{aligned}$$

Therefore, Theorem 6 confirms the results given in Proposition 3.3 in [2].

If $x > y$, then $r_1 = 1$. Thus, it holds that

$$\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 = (n_1 - n_1)r_1 = 0$$

and the villainy of G is given by Theorem 4. Therefore,

$$\begin{aligned} B(G) &= n_1 + n_2 - \left\lceil \frac{n_1 - n_2}{1} \right\rceil \\ &= n_1 + n_2 - n_1 + n_2 \\ &= 2n_2, \end{aligned}$$

and Theorem 4 confirms the result given in Proposition 3.3 in [2].

Let $G = K_{n_1,n_2,\ldots,n_k}$ where $n_1 > n_2 > \ldots > n_k$. According to Proposition 4.1 in [2],

- if $n_1 \geq n_2 + n_3 + \ldots + n_k$, then $B(G) = 2\sum_{i=2}^{k} n_i$, and
- $B(G) = |V(G)|$ otherwise.

When $n_1 \geq n_2 + n_3 + \ldots + n_k$, it holds that $r_1 = 1$ and $\left(n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1\right) r_1 = (n_1 - n_1)\, r_1 = 0$. Assuming $n_2 > 0$, the villainy of G is given by Theorem 4. Thus, the villainy of G is

$$\begin{aligned} \sum_{i=1}^{k} n_i - \left\lceil \frac{n_1 - \sum_{j=2}^{k} n_j}{1} \right\rceil &= \sum_{i=1}^{k} n_i - n_1 + \sum_{j=2}^{k} n_j \\ &= 2\sum_{j=2}^{k} n_j \end{aligned}$$

and Theorem 4 gives a result equivalent to that given in Proposition 4.1 in [2]. Theorem 1 proved that $B(G) = |V(G)|$ for all $G = K_{\underbrace{n_1,\ldots,n_1}_{r_1},\underbrace{n_2,\ldots,n_2}_{r_2},\ldots,\underbrace{n_k,\ldots,n_k}_{r_k}}$ where $r_i n_i \leq \sum_{j \neq i} r_j n_j$ for all $i, j \in [k]$ where $i \neq j$. Therefore, our results do not contradict those given in by Clark et al. in [2].

III. Paths

A *path* on n vertices consists of a set of vertices notated $v_1, v_2, \ldots, v_n$, with the only edges being $v_i v_{i+1}$ for $i \in [n-1]$ [6]. A path consisting of the vertices $v_1, v_2, \ldots, v_n$ can be referred to as $\langle v_1, v_2, \ldots, v_n \rangle$ or P_n. Paths have a chromatic number of 2; the vertices are colored such that the colors alternate.

A path with an even length that is properly colored with two colors will have alternating vertices of color 1 and 2. Therefore, in a path on $2k$ vertices, k vertices are colored 1, and k vertices are colored 2. There are two ways to properly color the path with this set of colors. In one coloring, the vertices with even indices will have color 1 while the vertices with odd indices have color 2. In the other, the vertices with even indices will have color 2 while the vertices with odd indices will have color 1. It holds that, in a path on $2k$ vertices, if it requires m recolorings to achieve one proper coloring, it will require $2k - m$ recolorings to achieve the other proper coloring. It was shown by Clark et al. in [2] that in an even path on $2k$ vertices, the villainy is equivalent to $2\left\lfloor \frac{k}{2} \right\rfloor$.

Let $P_{2k+1} = \langle v_1, v_2, \ldots, v_{2k+1} \rangle$. A properly colored odd path on $2k+1$ vertices will have $k+1$ vertices of one color and k vertices of the second color. Without loss of generality, let these colors be 1 and 2 respectively. In order for P_{2k+1} to be properly colored, the colors must alternate. Given that there are $k+1$ vertices of color 1 and only k vertices with even indices, every vertex of an odd index must receive color 1 in a proper coloring. Additionally, every vertex with an even index must receive color 2. Therefore, there is only one proper 2-coloring of an odd path up to permutation of the colors. In [2], Clark et al. proved that the villainy of an odd path on $2k+1$ vertices is given by $2k$.

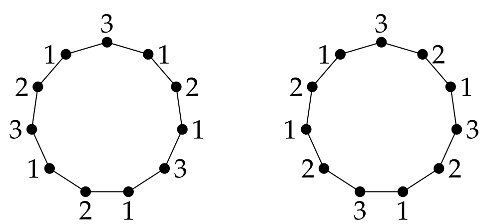
In an odd path, an improper coloring can only be recolored such that the odd indices receive color 1 in the case where the number of each color needs to be maintained. However, if we are considering the weak villainy of an odd path, there are two colorings to compare an improper coloring to; one where the vertices with odd indices receive color 1 and one where the vertices with odd indices receive color 2. In this case, if m vertices need to be recolored to return an improper coloring to a proper coloring where the vertices with odd indices receive color 1, it will take $2k + 1 - m$ recolorings to achieve a proper coloring in which the vertices with odd indices receive color 2.

IV. Odd Cycles

A *cycle* on n vertices is a path with an additional edge between v_n and v_1 and is denoted C_n. A cycle on vertices $v_1, v_2, \ldots, v_n$ is denoted $[v_1, v_2, \ldots, v_n]$. Clark et al. proved that the weak villainy of odd cycle C_{2k+1} is k for all $k \geq 2$ [2]. Thus, $B(C_{2k+1}) \geq k$. It was proposed but not proven that if $k \geq 2$, the villainy of C_{2k+1} is equivalent to k. To prove that $B(C_{2k+1}) = k$, we must show that $B(C_{2k+1}) \leq k$. We found results for odd cycles with one vertex of color 3 and two vertices of color 3, but we could not generalize the proof to show that $B(C_{2k+1}) \leq k$ for all odd cycles.

An odd cycle C_{2k+1} has a chromatic number of 3. Let us define 1 as the color that appears most often and 3 as the color that appears least often. Therefore, the number of vertices of color 3 is between 1 and $\frac{|V(C_{2k+1})|}{3}$. To find the chromatic villainy of an odd cycle, one must consider the rearrangements of all proper colorings using all $\frac{|V(C_{2k+1})|}{3}$ possible numbers of vertices colored with 3. Note that the vertices colored 3 partition the cycle into paths of vertices colored 1 and 2.

The number of vertices with color 3 determinines how many of these paths must be in the proper coloring and whether they can be of odd or even length. For example, consider $G = C_{2k+1}$ where $2k + 1 \geq 11$. Let G have three vertices of color 3 and $2k - 2$ vertices of colors 1 and 2. Let c be a proper coloring of G. No vertices of color 3 can be adjacent in c. Therefore, there must be a path between each vertex of color 3. Given that $2k - 2$ is even and must be split into three paths, these $2k - 2$ vertices can be split amongst three even paths or one even path and two odd paths. For example, two proper colorings of C_{11} are as follows:



.

Note that one graph contains five vertices with color 1 and three vertices of color 2 while the other contains four vertices with color 1 and four vertices of color 2. It holds that every properly colored even path contains the same number of vertices with color 1 as vertices of color 2. Thus, for every extra 1 in G, there must be an odd path in the proper coloring. However, the converse does not necessarily hold. If there are multiple odd paths in a proper coloring of C_{2k+1}, some may be properly colored such that 2 appears more often than 1.

IV.1 Odd Cycles with one vertex of color 3

Let $C_{2k+1} = [v_1, v_2, \ldots, v_{2k+1}]$ be an odd cycle with one vertex of color 3. Therefore, there must be at least one vertex color 1 and one vertex colored 2 in C_{2k+1} and the minimum number of vertices in an odd cycle with one vertex of color 3 is three. However, $C_3 = K_3$, a complete graph on 3 vertices, and $B(C_3) = 0$. Since C_{2k+1} only has one vertex of color 3, any coloring of C_{2k+1} consists of a path of length $2k$. Since this path is of even length, color 1 and 2 both appear in C_{2k+1} k times.

Theorem 7. *Let C_{2k+1} be a cycle on $2k+1$ vertices where $2k+1 \geq 5$ that has k vertices of color 1, k vertices of color 2, and one vertex of color 3. It holds that $B(C_{2k+1}) \leq k$.*

Proof. Let $C_{2k+1} = [v_1, v_2, \ldots, v_{2k+1}]$. Let c be a proper 3-coloring of C_{2k+1} such that one vertex receives color 3, k vertices receive color 1, and k vertices receive color 2. Let c^* be a rearrangement of c. Without loss of generality, let us assume $c^*(v_1) = 3$. Therefore, the path $\langle v_2, v_3, \ldots, v_{2k+1} \rangle$ consisting of vertices of colors 1 and 2 has length $2k$. Clark et al. showed that at most $2\left\lfloor \frac{k}{2} \right\rfloor$ vertices in $\langle v_2, v_3, \ldots, v_{2k+1} \rangle$ need to be recolored to return the path to a proper coloring [2]. It holds that $2\left\lfloor \frac{k}{2} \right\rfloor \leq k$ and $B(C_{2k+1}) \leq k$. □

IV.2 Odd cycles with two vertices of color 3

Let $C_{2k+1} = [v_1, v_2, \ldots, v_{2k+1}]$ be an odd cycle with two vertices of color 3. We define 3 as the color that appears least often in the graph. Therefore, there must be at least two vertices colored 1 and two vertices colored 2. Given that the cycle is odd, the minimum number of vertices an odd cycle can have with two vertices of color 3 is 7.

Given that there are two vertices of color 3, any proper coloring of C_{2k+1} is partitioned into two paths. It holds that $(2k+1) - 2 = 2k - 1$ is odd. Therefore, one path is odd and one path is even. Thus, there is one more vertex colored 1 in C_{2k+1} than is colored 2. Therefore, for an odd cycle on

$2k+1$ vertices where $2k+1 \geq 7$ to have two vertices colored 3, it must also have k vertices colored 1 and $k-1$ vertices colored 2.

Theorem 8. *Let k be an integer such that $k \geq 3$. If C_{2k+1} is a cycle on $2k+1$ vertices that is colored with k vertices of color 1, $k-1$ vertices of color 2, and two vertices of color 3, then it holds that*

$$B(C_{2k+1}) \leq k.$$

Proof. Let $C_{2k+1} = [v_1, \ldots, v_{2k+1}]$. Let c be a proper 3-coloring of C_{2k+1} such that two vertices receive color 3 and let c^* be a rearrangement of c. Assume without loss of generality that $c^*(v_1) = 3$ and $c^*(v_i) = 3$ where $i \in \{2, 3, \ldots, 2k+1\}$. Given that $2k+1-2 = 2k-1$ is odd, either the set $\{v_2, \ldots, v_{i-1}\}$ or $\{v_{i+1}, \ldots, v_{2k+1}\}$ must have an odd number of vertices and the other must be even. Without loss of generality, let $\{v_2, \ldots, v_{i-1}\}$ contain $2\ell+1$ vertices. It follows that v_i has an odd index. Let us define c^{**} as the proper coloring that results from recoloring c^*. There are three cases:

Case 1: $\ell+1 < k$ and at most ℓ vertices need to be recolored to properly color $\langle v_2, \ldots, v_{i-1} \rangle$ such that $c^{**}(v_2) = c^{**}(v_{i-1}) = 1$,

Case 2: $\ell+1 < k$ and at least $\ell+1$ vertices need to be recolored to properly color $\langle v_2, \ldots, v_{i-1} \rangle$ such that $c^{**}(v_2) = c^{**}(v_{i-1}) = 1$, and

Case 3: $\ell+1 = k$.

Note that if $\ell+1 = k$, then $2\ell+1 = 2k-1$ which is the maximum possible length of $\langle v_2, \ldots, v_{i-1} \rangle$. Thus, the three cases are exhaustive.

Case 1: Let $\langle v_2, \ldots, v_{i-1} \rangle$ contain at most ℓ vertices that need to be recolored in order to obtain a proper coloring in which $c^{**}(v_2) = c^{**}(v_{i-1}) = 1$. The path $\langle v_{i+1}, \ldots, v_{2k+1} \rangle$ contains $2(k-\ell-1)$ vertices. At most $2\left\lfloor \frac{k-\ell-1}{2} \right\rfloor$ vertices in $\langle v_{i+1}, \ldots, v_{2k+1} \rangle$ need to be recolored to return this path to a proper coloring. It holds that $2\left\lfloor \frac{k-\ell-1}{2} \right\rfloor \leq k-\ell-1$. Therefore, the villainy of the cycle is at most $(k-\ell-1) + (\ell) = k-1$.

Case 2: Let $\langle v_2, \ldots, v_{i-1} \rangle$ contain at least $\ell+1$ vertices that need to be recolored in order to obtain a proper coloring of $\langle v_2, \ldots, v_{i-1} \rangle$ in which $c^{**}(v_2) = c^{**}(v_{i-1}) = 1$. It follows that at most ℓ vertices would need to be recolored in order to obtain a proper coloring of $\langle v_2, \ldots, v_{i-1} \rangle$ in which

$c^{**}(v_2) = c^{**}(v_{i-1}) = 2$.

Let us first properly recolor $\langle v_2, v_3, \ldots, v_{i-1} \rangle$ such that $c^{**}(v_2) = c^{**}(v_{i-1}) = 2$; requiring at most ℓ recolorings. Then, let us recolor $\langle v_{i+1}, \ldots, v_{2k+1} \rangle$ so that at most $k - \ell - 1$ colorings are performed. This requires at most $k-1$ recolorings. The resulting coloring has k vertices with color 2 and $k-1$ vertices with color 1. Thus, this recoloring has not yet met the stipulation that each color must appear as often in c^{**} as in c and additional recolorings are required.

In order for the number of vertices of color 2 to become k, one more vertex with color 1 under c^* receives color 2 under c^{**} than vertices with color 2 under c^* receive color 1 under c^{**}. Otherwise, for every vertex that is recolored with 2, a vertex is recolored with 1. Therefore, in order to achieve this coloring, an odd number of recolorings must be performed. It holds that one vertex with color 3 is adjacent to two vertices of color 2 while the other is adjacent to a vertex of color 1 and a vertex of color 2. Let them be denoted u_1 and u_i respectively.

Consider the case where fewer than $k-1$ vertices have been recolored. Given that $k \geq 3$ and k vertices have color 2, at least one vertex of color 2 that is adjacent to u_1 is not adjacent to u_i. Let u_2 be a vertex with color 2 under c^{**} that is adjacent to u_1 and not adjacent to u_i. Recoloring u_1 and u_2 such that $c^{**}(u_1) = 1$ and $c^{**}(u_2) = 3$ results in a proper coloring with k vertices of color 1 and $k-1$ vertices of color 2. This requires at most 2 additional recolorings and at most k recolorings have been performed. Therefore, the villainy of the cycle is at most k.

Consider the case where $k-1$ vertices have been recolored. The number of vertices that has been recolored is odd. Thus, k must be even. Given that $k \geq 3$ and k is even, $k-1$ is at least 3 and at least two vertices with color 1 under c^* received color 2 under c^{**}. Thus, at least one vertex v_j where $c^*(v_j) = 1$ and $c^{**}(v_j) = 2$ is not adjacent to u_i. Recoloring v_j and u_1 such that $c^{**}(u_1) = 1$ and $c^{**}(v_j) = 3$, results in a proper recoloring with $k-1$ vertices with color 2 and k vertices with color 1. This requires one additional recoloring. Therefore, the villainy of the cycle is at most k.

Case 3: Let $\ell = k-1$. Therefore, $c^*(v_{2k+1}) = 3$. The set $\{v_2, \ldots, v_{2k}\}$ contains $(2k+1) - 2 = 2k-1$ vertices where $k-1$ vertices have color 2 and k vertices have color 1. The path $\langle v_2, v_3, \ldots, v_{2k+1} \rangle$ has length $2k$ and one vertex of color 3.

Given that $\langle v_2, v_3, \ldots, v_{2k} \rangle$ has an odd number of vertices, $c^{**}(v_2) = c^{**}(v_{2k})$ when properly colored with colors 1 and 2. It will require at most $k-1$ recolorings to properly recolor $\langle v_2, v_3, \ldots, v_{2k} \rangle$.

Consider the case where at most $k-2$ recolorings were required to properly color $\langle v_2, v_3, \ldots, v_{2k}\rangle$ such that $c^{**}(v_2) = c^{**}(v_{2k}) = 1$. After recoloring $\langle v_2, v_3, \ldots, v_{2k}\rangle$ such that $c^{**}(v_2) = c^{**}(v_{2k}) = 1$, it holds that no vertices with color 2 are adjacent to a vertex of color 3. By recoloring v_{2k+1} with 1 and v_{2k} with 3 results in a proper coloring. This requires at most k recolorings.

Consider the case where $k-1$ recolorings were required to properly color $\langle v_2, v_3, \ldots, v_{2k}\rangle$ such that $c^{**}(v_2) = c^{**}(v_{2k}) = 1$. Given that $k \geq 3$, at least 2 recolorings were performed. At least one vertex v_j where $j \in \{3, 4, \ldots, 2k\}$ with color 1 under c^* was recolored with color 2 under c^{**}. This vertex is not adjacent to a vertex with color 3. Recoloring v_1 with 2 and v_j with 3 requires one additional recoloring and at most k recolorings are performed.

Consider the case where k recolorings were required to properly color $\langle v_2, v_3, \ldots, v_{2k}\rangle$ such that $c^{**}(v_2) = c^{**}(v_{2k}) = 1$. Note that k must be even. It follows that $k-1$ recolorings were required to properly color $\langle v_2, v_3, \ldots, v_{2k}\rangle$ such that $c^{**}(v_2) = c^{**}(v_{2k}) = 2$. One more vertex with color 1 under c^* received color 2 under c^{**}. The resulting coloring has k vertices with color 2 and $k-1$ vertices with color 1. Given that $k \geq 3$ and $k-1$ must be odd, at least three recolorings were performed and at least two vertices with color 1 under c^* were given color 2 under c^{**}. Without loss of generality let v_j be a vertex such that $c^*(v_j) = 1$ and $c^{**}(v_j) = 2$. If $j = 2$, recoloring v_1 with 1 and v_2 with 3 results in a proper coloring with $k-1$ vertices with color 2 and k vertices with color 1. Similarly, if $j = 2k$, recoloring v_{2k} with 3 and v_{2k+1} with 1 results in a proper coloring with $k-1$ vertices with color 2 and k vertices with color 1. In both cases, at most k recolorings are performed. If j is not 2 or $2k$, either v_1 or v_{2k+1} can be recolored 1 while v_j is recolored with 3, resulting in a proper coloring with $k-1$ vertices with color 2 and k vertices with color 1. Thus, the villainy of this cycle is at most k.

Consider the case where at least $k+1$ recolorings were required to properly color $\langle v_2, v_3, \ldots, v_{2k}\rangle$ such that $c^{**}(v_2) = c^{**}(v_{2k}) = 1$. It follows that at most $k-2$ recolorings were required to properly color $\langle v_2, v_3, \ldots, v_{2k}\rangle$ such that $c^{**}(v_2) = c^{**}(v_{2k}) = 2$. If fewer than $k-1$ recolorings were required to properly color $\langle v_2, v_3, \ldots, v_{2k}\rangle$ such that $c^{**}(v_2) = c^{**}(v_{2k}) = 2$, then v_1 can be recolored with 1 and v_2 can be recolored with 3 resulting in a proper coloring with $k-1$ vertices with color 2 and k vertices with color 1. Thus, the villainy of this cycle is at most k.

Therefore, odd cycles with two vertices of color 3 have a villainy that is less than or equal to k. □

V. Conclusions and Open Questions

In a complete multipartite graph $G = K_{\underbrace{n_1,\ldots,n_1}_{r_1},\underbrace{n_2,\ldots,n_2}_{r_2},\ldots,\underbrace{n_k,\ldots,n_k}_{r_k}}$ where $r_1n_1 \geq r_2n_2 \geq \ldots \geq r_kn_k > 0$, the relationship between r_1n_1 and $\sum_{i=2}^{k} r_in_i$ gives bounds on the minimum number of vertices that can be colored correctly in c^*. The villainy was found for the case where the $r_1n_l \leq \sum_{i=2}^{k} r_in_i$ as well as the case where $r_1n_l > \sum_{i=2}^{k} r_in_i$ and $\sum_{i=2}^{k} r_in_i \geq (n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1)r_1$. An upper and lower bound was found for the villainy of the case where $r_1n_l > \sum_{i=2}^{k} r_in_i$ and $\sum_{i=2}^{k} r_in_i < (n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1)r_1$. However, the upper bound is not strict. It has yet to be proven that the strict upper bound is equivalent to the lower bound. Thus, we leave the following open question.

Question 1: Is the villainy of $K_{\underbrace{n_1,\ldots,n_1}_{r_1},\underbrace{n_2,\ldots,n_2}_{r_2},\ldots,\underbrace{n_k,\ldots,n_k}_{r_k}}$ equivalent to

$$\sum_{j=1}^{k} r_jn_j - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1 - \left\lceil \frac{r_1n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1^2 - \sum_{j=2}^{k} r_jn_j}{n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1} \right\rceil$$

when $r_1n_l > \sum_{i=2}^{k} r_in_i$ and $\sum_{i=2}^{k} r_in_i < (n_1 - \left\lfloor \frac{n_1}{r_1} \right\rfloor r_1)r_1$?

In the case where $\sum_{i=2}^{k} r_in_i = 0$, G can be expressed as $K_{n,\ldots,n}$ where G has k parts of size n. The villainy was found to be $kn - \left\lceil \frac{n}{k} \right\rceil k$ when k divides n. When k does not divide n, the villainy was found to be greater than or equal to $kn - \left\lceil \frac{n}{k} \right\rceil k$ and less than or equal to $kn - n$. We propose that the villainy of $K_{n,\ldots,n}$ is equivalent to $kn - \left\lceil \frac{n}{k} \right\rceil k$ for all values of n and k.

Question 2: Is the villainy of $G = K_{n,\ldots,n}$, where G has k partite sets of size n, equivalent to $kn - \left\lceil \frac{n}{k} \right\rceil k$ for all values of n and k where k is not 0?

In [2], Clark et al. found the villainy of paths. The villainy of paths was in turn used to find an upper bound on certain classes of odd cycles. A cycle C_{2k+1} colored with three colors where 3 is the color that appears least often can have up to $\left\lfloor \frac{|V(C_{2k+1})|}{3} \right\rfloor$ vertices of color 3. The vertices of color 3 partition the cycle into paths of vertices with colors 1 and 2. Clark et al. gave a lower bound of k for the villainy of odd cycles in [2]. An upper bound of k was found for the villainy of odd cycles with one vertex of color 3 and two vertices of color 3. However, it has not yet been proven for the general case. Thus, we propose the same question as Clark et al. under Corollary 3.6 in [2].

Question 3 from [2]: Is the villainy of C_{2k+1} equivalent to k when $k > 1$?

VI. Bibliography

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