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Colored Saturation Parameters for Bipartite Graphs

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by

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Abstract

Let F and H be fixed graphs and let G be a spanning subgraph of H. G is an F-free subgraph of H if F is not a subgraph of G. We say that G is an F-saturated subgraph of H if G is F-free and for any edge $e \in E(H) - E(G)$, *F is a subgraph of* $G + e$. *The saturation number of F in* $K_{n,n}$, *denoted* $\text{sat}(K_{n,n}, F)$, *is the minimum size of an F-saturated subgraph of* $K_{n,n}$. *A t-edge-coloring of a graph G is a labeling* $f : E(G) \to [t]$, *where* $[t]$ *denotes the set* $\{1, 2, \ldots, t\}$. *The labels assigned to the edges are called colors. A rainbow coloring is a coloring in which all edges have distinct colors. Given a family* $\mathcal{F}$ *of edge-colored graphs, a t-edge-colored graph H is* $(\mathcal{F}, t)$*-saturated if H contains no member of* $\mathcal{F}$ *but the addition of any edge in any color completes a member of* $\mathcal{F}$. *In this thesis we study the minimum size of* $(\mathcal{F}, t)$*-saturated subgraphs of edge-colored complete bipartite graphs. Specifically we provide bounds on the minimum size of these subgraphs for a variety of families of edge-colored bipartite graphs, including monochromatic matchings, rainbow matchings, and rainbow stars.*

Key words: Saturation, matching, rainbow, monochromatic, star

Contents

I. Introduction

I.1 Saturation Numbers

A *graph* G consists two sets: a *vertex set*, denoted $V(G)$ and an *edge set*, denoted $E(G)$, where edges are unordered pairs of vertices. A *simple* graph is an unweighted, undirected graph without loops or multiple edges. All graphs considered in this thesis are simple. Let G and F be graphs. We say that G is *F-free* if F is not a subgraph of G. A graph G is *F-saturated* if G is F-free and F is a subgraph of $G + e$ for any edge $e \in \overline{G}$. The *saturation number*, denoted $\text{sat}(G)$, is the minimum size of a saturated graph. For example, consider C_4, the cycle on four vertices. The complement of C_4, denoted $\overline{C_4}$, contains two edges. C_4 does not contain a triangle, K_3 (see figure 1). C_4 is K_3-saturated since adding either of the two edges in $\overline{C_4}$ to C_4 will force it to contain a triangle (see Figure 2).

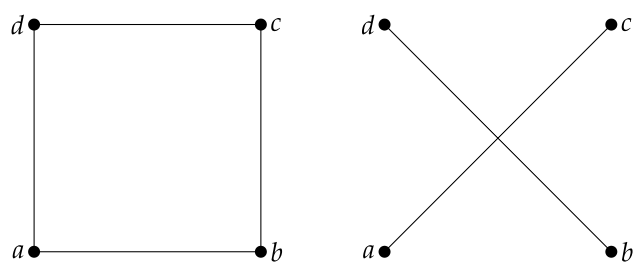

Figure 1: C_4 and its complement $\overline{C_4}$.

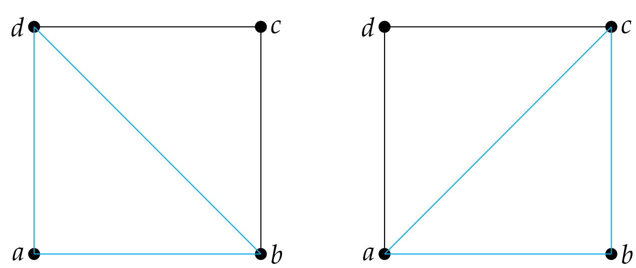

Figure 2: C_4 with the addition of an edge from $\overline{C_4}$.

Let F and H be fixed graphs and let G be a spanning subgraph of H. G is an *F-free subgraph* of H if F is not a subgraph of G. We will refer to H as the host graph. We say that G is an *F-saturated subgraph* of H if G is F-free and for any edge $e \in E(H) - E(G)$, F is a subgraph of $G + e$. The *saturation number* for F in H, denoted $sat(H, F)$, is the minimum size of an F-saturated subgraph of H. For bipartite graphs, we will focus on $\text{sat}(K_{n,n}, F)$.

Erdős, Hajnal, and Moon [5] introduced saturation numbers in 1964. They determined the saturation number of K_k and characterized the n-vertex K_k-saturated graphs of minimum size inside complete graphs. Later, Bollobás [2] and Wessel [22] independently proved a conjecture of Erdős, Hajnal, and Moon, regarding saturation numbers in bipartite graphs. In [11], Kászonyi and Tuza provided a general construction and an upper bound for $sat(n, F)$. They found that the saturation numbers are at most linear in n, the order of the host graph. A year later, Hanson and Toft [8] introduced saturation numbers for edge-colored graphs. Then in 2012, Moshkovitz and Shapira [17] considered saturation in d-partite hypergraphs. If we allow $d = 2$, the problem reduces to saturation in bipartite graphs.

I.2 Rainbow Matchings

A *t-edge coloring* of a graph G is a labeling $f : E(G) \to R$, where $|R| = t$. The labels are colors and the set of edges of one color form a *color class*. A k-edge coloring is *proper* if no two edges of the same color share an endpoint. A *matching* in a graph G is a set of edges with pairwise disjoint sets of endpoints (see Figure 3). A *perfect matching* in a graph G is a matching such that the set of edges is incident to every vertex in G (see Figure 3). In an edge-colored graph, a *rainbow matching* is a matching in which all edges have distinct colors (see Figure 4).

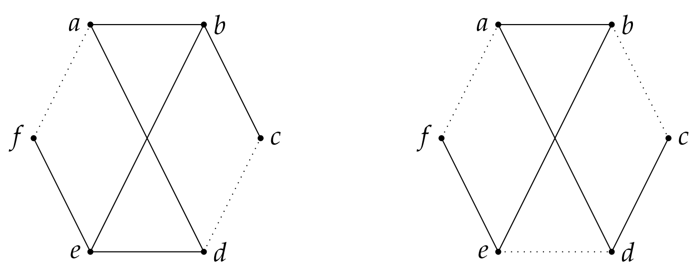

Figure 3: A matching, the set $\{af, cd\}$ and a perfect matching, the set $\{af, bc, de\}$ for a graph G.

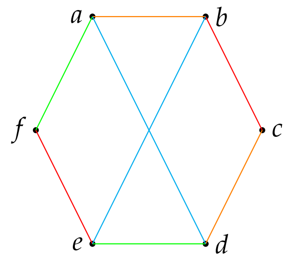

Figure 4: A rainbow matching of graph G, the set $\{af, be, cd\}$.

A *Latin Square* of order n is an $n \times n$ array with entries from $\{1, 2, \ldots, n\}$ arranged such that no row or column contains the same number twice. A *transversal* of such a square is a set of n entries such that no two entries share the same row, column, or symbol. In 1967, Ryser [18] conjectured that every Latin Square of odd order has a transversal.

For example, the following matrices, L_1 and L_2, are Latin Squares of order 4 and 5, respectively.

$$L_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \quad L_2 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Notice that L_1 has even order and does not contain a transversal. Suppose we try to create a transversal with the element, $l_{1,1} = 1$ in L_1. Next we want to look at the second column where we can choose between the elements from the second and third rows. We cannot however choose an element from the fourth row in this example because the element in the fourth row, second column is a 1. We cannot repeat elements in the transversal. Suppose we choose $l_{3,2} = 4$. In the third column, we have to choose $l_{4,3}$ otherwise we would repeat 4 in the transversal. This implies the last choice is $l_{2,4}$. But now the set contains $\{1, 4, 2, 1\}$ and this is not a transversal. Repeating the process with different selection of the elements in the matrix results in the repetition of one element for the transversal. Thus we cannot build a transversal from L_1.

Take the following transversal:

$$L_2 = \begin{bmatrix} \mathbf{1} & 2 & 3 & 4 & 5 \\ 2 & \mathbf{3} & 4 & 5 & 1 \\ 3 & 4 & \mathbf{5} & 1 & 2 \\ 4 & 5 & 1 & \mathbf{2} & 3 \\ 5 & 1 & 2 & 3 & \mathbf{4} \end{bmatrix}.$$

This is a transversal since each element in the set is a unique element from $1-5$ and we have selected one element from each row and one from each column.

A Latin square of order n can be encoded as a properly edge-colored copy of $K_{n,n}$. Let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of vertices corresponding to the rows of matrix and $Y = \{y_1, y_2, \ldots, y_n\}$ be the set of vertices corresponding to the columns. We then color the edge x_iy_j with the color corresponding to the element in the ith row and jth column of the matrix. The transversal appears in the bipartite graph as a rainbow matching of size n. So in our example, L_2 is a matrix of order 5 that has encoded a rainbow matching of size 5 inside $K_{5,5}$ (see Figure 5). In Figure 5, we have translated the elements in the matrix to colors, so 1 =red, 2 =green, 3 =orange, 4 =purple, and 5 =blue.

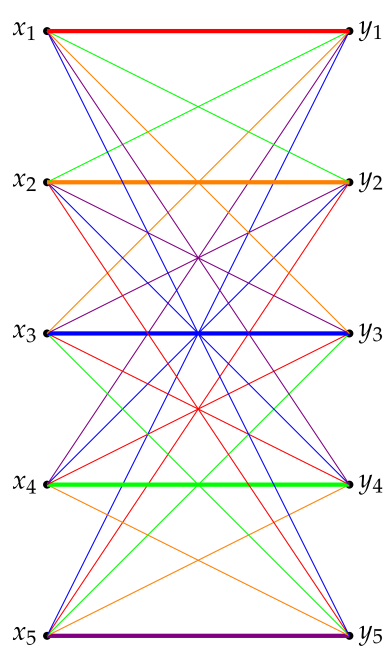

Figure 5: A rainbow matching in $K_{5,5}$ based on L_2.

I.3 Edge-colored Saturation Parameters

In 1987, Hanson and Toft began the exploration of saturation parameters in edge-colored graphs. They determined the saturation number with respect to monochromatic complete subgraphs in t-edge-colored graphs. They conjectured that they could relate edge-colored saturation numbers to Ramsey numbers, leading to new results by Chen [3] and Ferrera et al. [6].

Let $\mathfrak{R}(F)$ denote the set of all rainbow-colored copies of H. A t-edge colored graph is $(\mathfrak{R}(F), t)$-*saturated* if G does not contain a rainbow copy of F but for any edge $e \in \overline{G}$ and any color $i \in [t]$, the addition of e in color i to G produces a rainbow copy of F. For example, C_4 with four distinct edge colors is $(\mathfrak{R}(K_3), 4)$-saturated. Note that before the addition of a new edge, this graph does not contain a rainbow K_3. We can see in Figure 6 the complement of the host graph without specifying edge colors.

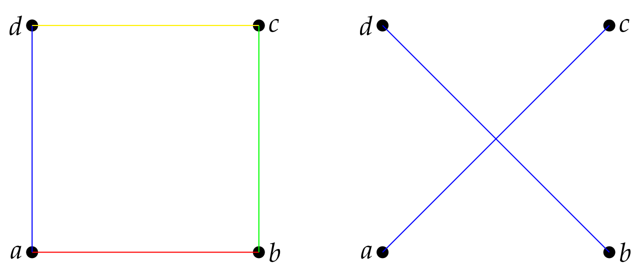

Figure 6: Properly edge-colored C_4 on four colors and its complement $\overline{G}$.

Now in Figure 7, we have identified the edges in $\overline{C_4}$ that can be added to the graph in any of the four colors. If we add the new edge ac in red or blue, then we have a rainbow K_3 using the edges ac, ab and bc. If we add ac in green or yellow, than we have a rainbow K_3 using edges ac, ad and cd. Similarly if we add the edge bd to the C_4 in yellow or red, then we have a rainbow K_3 using edges bd, ab, and ad. If we add bd in green or blue, we obtain a rainbow K_3 using edges bd, bc, and cd.

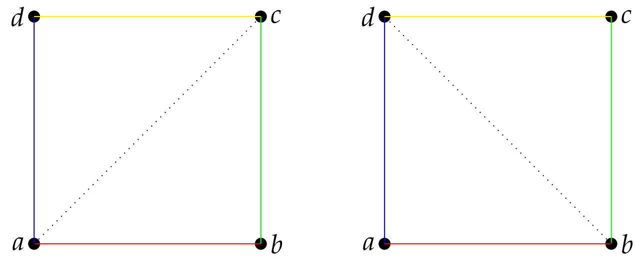

Figure 7: C_4 with the addition of new edge e.

The *rainbow saturation number* of a graph H, denoted $\text{sat}_t(\mathfrak{R}(F), n)$, is the minimum number of edges in a t-edge-colored $(\mathfrak{R}(F), t)$-saturated graph of order n.

We define the saturation number for a family of monochromatic edge-colored graphs, denoted $\text{sat}_t((F_1, F_2, \ldots, F_t), K_{n,m})$, inside host graph, $K_{n,m}$, as the minimum number of edges in a t-edge-colored $((F_1, F_2, \ldots, F_t), t)$-saturated graph of order n. This implies that the addition of a new edge e in color i to the host graph, we complete a copy of F_i in color i, where each F_i is monochromatic.

In many of the monochromatic saturation problems, the results are linear in the number of vertices in the graph. In [1], Barrus et al. prove that rainbow saturation numbers can have nonlinear orders of growth.

I.4 Overview

In Section 2, we provide background for the problems considered in Sections 3 and 4. In Section 3, we prove our main results regarding monochromatic saturation numbers in edge-colored bipartite graphs. In Section 4, we present our main results for the rainbow saturation numbers of matchings and stars in a edge-colored complete bipartite graphs. In Section 4.1, we determine an upper and lower bound for $\text{sat}_t(\mathfrak{R}(mK_2), K_{n,n})$. In Section 4.2, we prove asymptotically sharp upper and lower bounds for $\text{sat}_t(\mathfrak{R}(K_{1,k}), K_{n,n}$. Finally, in Section 5 we summarize our results and discuss directions for future work.

I.5 Definitions and Notation

In this section, we provide the definitions and notation used in this thesis. A *graph* G consists two sets: a *vertex set*, denoted $V(G)$ and an *edge set*, denoted $E(G)$, where each edge is an unordered pair of vertices. We associate the two vertices with each edge as the *endpoints* of the edge. We say that two vertices are *adjacent*, or are *neighbors*, if they are endpoints of the same edge. The *neighborhood* of a vertex v, denoted $N(v)$, is the set of all vertices that are adjacent to v. The *degree* of a vertex v, denoted as $d(v)$, is the size of the neighborhood of the v. So, $d(v) = |N(v)|$. The *minimum degree* of a graph G, denoted $\delta(G)$, is the minimum degree of all the vertices in G.

A graph is *simple* if the graph does not contain any loops of multiple edges, where *multiple edges* refers to different edges having the same endpoints. A complete graph on n vertices, denoted K_n, is a graph that contains every possible edge between n vertices. A *path* on n vertices, denoted $P(n)$, is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *tree* is a connected acyclic graph. A *star*, denoted $K_{1,k}$, is a tree consisting of one central vertex that is adjacent to the other k vertices. The *complement* of a simple graph G, denoted $\overline{G}$, is the simple graph with the vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

A graph G is *bipartite* if we can partition the vertex set into two sets $V_1(G)$ and $V_2(G)$, such that every edge in G has one endpoint in $V_1(G)$ and one endpoint in $V_2(G)$. A *complete bipartite graph*, denoted $K_{n,m}$, is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets and $|V_1(G)| = n$ and $|V_2(G)| = m$.

A *t-edge-coloring* of a graph G is a labeling $f : V(G) \to [t]$, where $[t]$ denotes the set $\{1, 2, \ldots, t\}$. The labels assigned to the edges are *colors*. An edge-coloring is *proper* if edges with the same endpoint have distinct colors. The *edge-chromatic number* of a graph G, denoted $\chi^{'}(G)$, is the smallest number of colors required to properly edge-color G. We say that a t-edge-colored graph is $(\mathfrak{R}(F), t)$*-saturated* if G does not contain a rainbow copy of F, but with the addition of any edge $e \in \overline{G}$ in any of the t possible colors creates a rainbow copy of F. The *rainbow saturation number* of F, denoted $\text{sat}_t(n, \mathfrak{R}(F))$ is the minimum number of edges in a $(\mathfrak{R}(F), t)$-saturated graph of order n. Similarly, a graph is (F, t)-saturated if G does not contain a monochromatic copy of F in any of the possible t colors, but with the addition of any edge $e \in \overline{G}$ in any of the t possible colors creates a monochromatic copy of F.

A *matching* is a set of edges with no shared endpoints. A *perfect matching* in a graph G is a matching such that the set of edges is incident to every vertex in G. In an edge-colored graph, a *rainbow matching* is a matching such that no two edges have the same color. A *vertex cover* of a graph G is a set $R \subseteq V(G)$ that contains at least one endpoint of every edge.

Table 1: Reference for Notation

Notation	Brief Description
$\delta(G)$	Minimum degree of G
$\delta^c(G)$	Minimum color degree of G
$\text{sat}(\mathcal{R}(H), n)$	Monochromatic saturation number of graph H
$\text{sat}_t(\mathfrak{R}(H), n)$	Rainbow Saturation number of H with color set $[t]$
$\text{sat}_t((m_1K_2, \ldots, m_kK_2), K_{n,n})$	Saturation number for multiple monochromatic matchings inside $K_{n,n}$
$\text{sat}_t(\mathfrak{R}(H), K_{n,n})$	Rainbow saturation number for a graph, H, inside $K_{n,n}$

II. Prior Work

II.1 Saturation Numbers

In [5], Erdős, Hajnal, and Moon introduced graph saturation problems by defining property-(n,k) on a graph. Let n and k be integers such that $2 \leq k \leq n$. A graph G on n vertices has *property-*(n,k) if the addition of any new edge increases the number of copies of K_k contained in G. In [5], they defined minimal (n,k) graphs as graphs that have property-(n,k) and contain the minimum number of edges. They wanted to know if every bipartite graph with partitions n, m that does not contain a $K_{k,k}$ but will contain this complete subgraph with the addition of any new edge, will have at least $(k-1)(n+m-k+1)$ edges. Using induction on k and l and assigning weights to edges in the graph, Bollobás [2] determined the number of edges necessary in a graph with property(k,n). We present Bollobás' theorem, which influenced our results for the monochromatic matchings.

Theorem 1 (Bollobás [2])**.** *Let $k \leq n$ and $l \leq m$, then $sat(K_{k,l}, K_{n,m}) = (k-1)m + (l-1)n - (k-1)(l-1)$.*

Wessel [22] independently determined the saturation number for bipartite graphs by specifying the coloring of the edges in sets. A survey of saturation problems can be found in [9].

Hanson and Toft [8] applied concepts from [5] to observe edge-colored saturated graphs. In a complete graph K_n colored in such a way that there is no monochromatic complete k_i-subgraph, they determined the number of edges necessary such that the addition of any edge in any color will create a monochromatic complete k_i-subgraph in that respective color. G is $(k_1, k_2, \ldots, k_t)$*-saturated* if there exists a coloring of the edges in G with t colors such that there is no monochromatic copy of K_{k_i} in color i for any i, but the addition of any new edge in color i creates a monochromatic K_{k_i}.

For graphs $G, H_1, \ldots, H_k$ we write that $G \to (H_1, \ldots, H_k)$ if every k-coloring of the edges of G contains a monochromatic copy of H_i in color i for some i. A graph G is $(H_1, \ldots, H_k)$*-Ramsey minimal* if $G \to (H_1, \ldots, H_k)$ but for any edge $e \in G$, $(G-e) \not\to (H_1, \ldots, H_k)$. Let $\mathcal{R}_{\min}(H_1, \ldots, H_k)$ denote the family of $(H_1, \ldots, H_k)$-Ramsey minimal graphs. Given simple graphs $G_1, \ldots, G_k$, the classic Ramsey number is the smallest integer n such that every k-coloring of $E(K_n)$ contains a copy of G_i in color i for some i. Hanson and Toft [8] conjectured that if $r = r(k_1, \ldots, r_k)$ is the classical Ramsey number for complete graphs, then

$$\text{sat}(\mathcal{R}_{\min}(K_{k_1}, \ldots, K_{k_t}), n) = \begin{cases} \binom{n}{2} & \text{if } n < r \\ \binom{r-2}{2} + (r-2)(n-r+2) & \text{if } n \geq r. \end{cases}$$

In [3], Chen et al. considered the first non-trivial case for this conjecture. For sufficiently large n, they showed that $\text{sat}(\mathcal{R}_{\min}(K_3, K_3), n) = 4n - 10$ for $n \geq 56$. Ferrara, Kim, and Yeager [6] showed that $\text{sat}(\mathcal{R}_{\min}(m_1K_2, \ldots, m_kK_2), n) = 3(m_1 + \cdots + m_k - k)$ given that $m_1, \ldots m_k \geq 1$ and $n \geq 3(m_1 + \cdots + m_k - k)$. They prove this result and characterize saturated graphs of minimum size using the approach of iterated recoloring, in which they reassign the colors of various edges to force structures in G to appear.

Moving away from monochromatic saturation numbers, Barrus et al. [1] studied rainbow edge-colorings in an edge-colored graph. A *rainbow* edge-coloring of a graph H is an edge coloring in which every edge in H receives a different color. The set of rainbow-colored copies of H is denoted $\mathfrak{R}(H)$. A graph is $(\mathfrak{R}(K_k), t)$-saturated if G does not contain a rainbow copy of K_k, but with the addition of any edge $e \in \overline{G}$ in any of the t possible colors creates a rainbow copy of H. Thus $\text{sat}_t(n, \mathfrak{R}(K_k))$ is the minimum number of edges in a $(\mathfrak{R}(K_k), t)$-saturated graph of order n. They observed that if $k \geq 3$ and the number of colors t is at least $\binom{k}{2}$, then for large n there exists constants c_1 and c_2 such that:

$$c_1 \frac{n \log n}{\log \log n} \leq \text{sat}_t(\mathfrak{R}(K_k), n) \leq c_2 n \log n$$

They also determined the rainbow saturation number for several classes of graphs and bounds for other graphs. We present their proof for rainbow stars in K_n, which lead to the observations for rainbow stars in $K_{n,n}$

Theorem 2 (Barrus et al. [1]). *If* $n \geq (k+1)(k-1)/t$ *then* $\text{sat}_t(\mathfrak{R}(K_{1,k}), n) = \Theta(\frac{(k-1)}{2t} n^2)$.

Proof. Let G be a $(\mathfrak{R}(K_{1,k}), t)$-saturated graph on n vertices. We have the following observations:

1. No vertex is incident to edges of k or more colors, otherwise G already contains a rainbow $K_{1,k}$.

2. If v is incident to edges of at most $k-2$ colors then v has degree $n-1$.

3. If vertices w and v both see color i, then v must be adjacent to w.

Now, by observations 1 and 2, we can partition the vertex set into two parts: a set X of vertices that see at most $k-2$ colors and a set Y that sees exactly $k-1$ colors. We can use observation 3 to partition the set Y based on the colors each vertex sees. This partitioning results in $\binom{t}{k-1}$ sets. Notice that if two of the partitioned sets in Y correspond to vertices that are incident to edges of a common color, then the sets must be completely joined. Now if we contract each independent set to a vertex and include edges where we have complete bipartite graphs between the independent sets, we have the blow up of Y, $G[Y]$. We want to look at the complement. We can see that $\overline{G[Y]}$ is $(\mathfrak{R}(K_{1,k}), t)$-saturated since no vertex sees k colors and the only missing edge are between sets in Y that correspond to vertices that are incident to disjoint color sets. If we add a new edge vw in color i, where v and w are in different sets, then with out loss of generality, w did not contain a neighbor of color i. Now we want to minimize the edges for the graph presented. Note that minimizing the number of edges present in the graph is equivalent to maximizing the number of edges in the complement of G. Using Turáns Theorem, they conclude that the number of edges is $\frac{k-1}{t}\binom{n}{2}$ edges. □

They concluded by considering the rainbow saturation number for matchings. By presenting lower bounds for the size of G and the number of colors, they showed that for a positive integer m, with $t \geq 5m-5$ and $n \geq \frac{5}{2}m - 1$, that:

$$\frac{11}{4}m \leq \text{sat}_t(\mathfrak{R}(mK_2), n) \leq 5m - \epsilon,$$

where $\epsilon = 5$ if m is even and $\epsilon = 4$ if m is odd.

II.2 Rainbow Saturation

A *rainbow matching* in an edge-colored graph is a matching such that no two edges have the same color. A *Latin Square* of order n is an $n \times n$ array with entries from $\{1, 2, \ldots n\}$ arranged so that no row or column contains the same number twice. A *Latin transversal* is a set of entries in a Latin square that includes exactly one entry from each row and column and one of each element in this set is unique. Ryser [18] conjectured that each Latin Square of odd order contains a Latin transversal. A survey of rainbow matchings and rainbow subgraphs in edge colored graphs can be found in [10].

Li and Wang [20] examined rainbow matchings in edge-colored graphs. The *color degree* of a vertex v is the number of distinct colors on the edges incident to v and is denoted $d^c(v)$. The *minimum color degree* of a graph G, denoted $\delta^c(G)$, is the smallest number of distinct colors on the edges incident with a vertex in G. Li and Wang showed that if $\delta^c(G) \geq k$, then G will contain a rainbow matching of size $\left\lceil \frac{5k-3}{12} \right\rceil$. If $\delta^c(G) \geq k \geq 4$, then they conjectured that the graph must contain a rainbow matching of size $\left\lceil \frac{k}{2} \right\rceil$.

LeSaulnier et al. [14] proved that if $\delta^c(G) = k$ G contains a rainbow matching of size at least $\left\lfloor \frac{k}{2} \right\rfloor$. They also conjectured three sufficient conditions for a graph to contain a rainbow matching of size $\left\lceil \frac{k}{2} \right\rceil$: G must be triangle-free, it must be properly edge-colored for $G \neq K_4$ and $n \neq k+2$, and $|V(G)| > \frac{3(k-1)}{2}$. Kostochka and Yancey [13] proved that if G is not a properly edge-colored K_4 and $\delta^c(G) \geq k$, then G contains a rainbow matching of size at least $\left\lceil \frac{k}{2} \right\rceil$.

An alternative approach to finding and guaranteeing a rainbow matching in a graph is by using the minimum degree of the vertices of a properly edge-colored graph G, denoted $\delta(G)$. Wang [19] showed that if the number of vertices in a graph G is greater than $\frac{8\delta(G)}{5}$, then G contains a rainbow matching with size at least $\left\lfloor \frac{2\delta(G)}{3} \right\rfloor$. Wang asked if there exists a function $f(n)$ such that if a properly edge colored graph with the property that $|V(G)| \geq f(\delta(G))$, then G must contain a rainbow matching of size $\delta(G)$. Diemunsch et al. [4] answered this question, proving that if the number of vertices in a properly edge-colored graph G is larger than $\frac{98\delta(G)}{23}$, then G contains a rainbow matching of size $\delta(G)$. Thus $f(\delta(G)) < 4.27\delta(G)$ suffices. Using a greedy algorithm, they efficiently constructed a rainbow matching of size $\delta(G)$ in a properly edge-colored graph with order $6.5\delta(G)$. Independently, Gyárfás and Sárkőzy [7] improved this bound to $f(\delta(G)) \leq 4\delta(G) - 3$.

Kostochka, Pfender, and Yancey [12] showed that in every edge colored graph, not necessarily properly colored, that as long as G contains at least $\frac{17k^2}{4}$ vertices and $\delta^c(G) \geq k$, then G contains a rainbow matching of size k. In [16], Lo and Tan showed that every edge-colored graph on n vertices with $\delta^c(G) \geq k$ contains a rainbow matching of size k provided that $n \geq 4n - 4$ for $k \geq 4$. So, $f(k) \leq 4k - 4$ for $k \geq 4$. Lo [15] improved this bound to show that if a graph contains $n \geq \frac{7k}{2} + 2$ vertices for $k \geq 4$ then G contains rainbow matching of size at least k. Moreover, if the graph G is bipartite, he improved the bound to $n \geq (3+\epsilon)k + \epsilon^{-2}$, where $0 < \epsilon \leq \frac{1}{2}$.

The previous results all relate to general graphs G. Now we want to consider bipartite graphs. In [20], Wang and Li consider the color degree of a neighborhood of a set of vertices. Let S be a set of vertices and let $|N^c(S)|$ denote the color degree of the neighborhood of S. They showed that if G is an edge colored bipartite graph with bipartition X, Y and $|N^c(S)| \geq |S|$ for all $S \subseteq X$, then G has a rainbow matching of size $\left\lceil \frac{|X|}{2} \right\rceil$. Wang and Liu [21] showed that if G is a properly edge-colored bipartite graph with partite sets X and Y such that $\delta(G) = k \geq 3$ and $\max\{|X|, |Y|\} \geq \frac{7k}{4}$, then G contains a rainbow matching of size at least $\left\lfloor \frac{3k}{4} \right\rfloor$.

III. Monochromatic Matchings

A *matching* in a graph G is a set of edges with pairwise disjoint sets of end points. A *vertex cover* of a graph is a set of vertices that contains at least one endpoint of every edge. In the proof of the following theorem, we use the Kőnig-Egerváry theorem.

Theorem 3 (Kőnig-Egerváry [23])**.** *If G is a bipartite graph, then the maximum size of a matching in G equals the size of a minimum vertex cover.*

Since our graphs are edge-colored and we seek monochromatic matchings, we introduce a colored version of vertex covers. By constructing vertex covers of the appropriate size for each of the matchings, we are able to bound the number of edges in the graph. Suppose we want the host graph to contain a fixed number of different sized matchings, where the sum of the sizes of matchings is sufficiently less that n and we order the matchings such that $m_1 \leq m_2 \leq \cdots \leq m_t$. We want to know how many edges are necessary in this graph to ensure that the addition of a new edge increases the size of one of the matchings. Our results have an interesting conclusion: the number of edges is solely dependent on the size of the largest in the graph. There can be multiple matchings of the largest size in G but the number of matchings does not dictate the number of edges necessary. Consider a bipartite graph with color set $[t]$. Then,

Theorem 4. *If $m_1, \ldots, m_t$ be positive integers such that $m_1 \leq m_2 \leq \cdots \leq m_t$ and $n \geq \sum_{i=1}^{t} m_i$, then $sat(m_1K_2, K_{n,n}) = n(m_1 - 1) + \left\lfloor \frac{m_1-1}{2} \right\rfloor^2 - \left\lfloor \frac{m_1-1}{2} \right\rfloor (m_1 - 1)$.*

Proof. Let G be a t-edge-colored bipartite graph with parts $X = \{x_1, x_2, x_3, \ldots, x_n\}$ and $Y = \{y_1, y_2, y_3, \ldots, y_n\}$. Consider a bipartite subgraph in G with parts $X' = \{x_1, x_2, \ldots, x_{\lfloor \frac{m_t}{2} \rfloor}\}$ and $Y' = \{y_1, y_2, \ldots, y_{\lceil \frac{m_t}{2} \rceil - 1}\}$. Partition the set $X - X'$ into t sets such that $|X_i| = m_i$ for $t \in [t-1]$ and $X_t = X - \left(X' \cup X_1 \cup \cdots \cup X_{t-1}\right)$. We partition $Y - Y'$ in the same manner by creating t sets $Y_1, \ldots, Y_t$ such that $|Y_i| = m_i$ for $t \in [t-1]$ and $Y_t = Y - \left(Y' \cup Y_1 \cup \cdots \cup Y_{t-1}\right)$. Let the set K_1 have $\left\lfloor \frac{k_1}{2} \right\rfloor$ vertices in X' and $\left\lceil \frac{k_1}{2} \right\rceil - 1$ vertices in Y'. Connect the set of k_1 in X' to Y and y' to X with edges in color 1. Similarly, let the set K_2 have $\left\lfloor \frac{k_2}{2} \right\rfloor$ vertices in X' and $\left\lceil \frac{k_2}{2} \right\rceil - 1$ vertices in Y'. We want to connect these two sets with edges in color 2. Then connect the set of K_2 in X' to Y and y' to X with edges in color 2, without reassigning the edges already in color 1. We will continue coloring the sets X' and Y' by dividing it into sets of K_i with edges in color i up to the largest set $K_{\lfloor \frac{m}{2} \rfloor}$. Without loss of generality, we order the sets so that $k_1 \leq k_2 \cdots \leq k_{\lfloor \frac{m}{2} \rfloor}$. Then we connect the remaining edges between X' and Y and Y' and X with color $\frac{m}{2}$ so that the vertices in X' and Y' have degree n. This can bee seen in Figure 8, where each solid edge between the K_i sets represents a collection of edges.

The total number of edges in G is:

$$= \left\lfloor \frac{m_t}{2} \right\rfloor \left(n - \left\lceil \frac{m_t}{2} \right\rceil + 1\right) + \left(\left\lceil \frac{m_t}{2} \right\rceil - 1\right)\left(n - \left\lfloor \frac{m_t}{2} \right\rfloor\right) + \left(\left\lfloor \frac{m_t}{2} \right\rfloor\right)\left(\left\lceil \frac{m_t}{2} \right\rceil - 1\right)$$
$$= n(m_t - 1) + \left\lfloor \frac{m_t - 1}{2} \right\rfloor^2 - \left\lfloor \frac{m_t - 1}{2} \right\rfloor (m_t - 1)$$

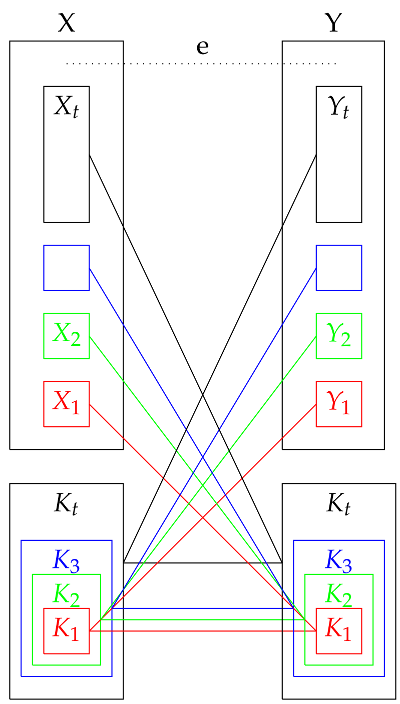

Figure 8: Construction 1: Monochromatic Matching.

Consider the graph G presented in the construction. The size of the largest vertex cover for color i in this graph is $m_i - 1$ by construction. Thus by Kőnig-Egerváry Theorem, the size of the largest matching in this graph is $m_i - 1$ for $i \in [t]$. Thus this graph does not contain a $m_i K_2$. Suppose we add a new edge to G in color i. Since all the vertices in X' and Y' have degree n, we must add this new edge between $Y - Y'$ and $X - X'$. With the addition of e in color i, the size of the minimum vertex cover for color i increased by 1. By the Kőnig-Egerváry Theorem, since our minimum vertex cover has size m_i, our matching in color i will also increase by one. Thus

$$sat((m_1K_2, ..m_kK_2), K_{n,n}) \leq n(M-1) + \left\lfloor \frac{M-1}{2} \right\rfloor^2 - \left\lfloor \frac{M-1}{2} \right\rfloor (M-1).$$

For the lower bound, let G be a t-edge-colored bipartite graph that is $(m_1K_2, \ldots, m_tK_2)$-saturated. Let $M = \max\{m_1, \ldots, m_t\}$. Without loss of generality, let m_t be the largest. The largest possible monochromatic matching in G is size $m_t - 1$, since the addition of an edge in color t would not increase the size of the matching in color t and the graph would not be $(m_1K_2, \ldots, m_tK_2)$-saturated. By the Kőnig-Egerváry Theorem, we know there must be a minimum vertex cover of color t of size $m_t - 1$. For G to be $(m_1K_2, \ldots, m_kK_2)$-saturated, each vertex in the vertex cover must have degree n otherwise the addition of an edge in color t that will not yield a matching m_tK_2 in color t. Let S be the set of vertices that make up the vertex cover of color t. Let $|S \cap X| = r$ and $|S \cap Y| = m_t - 1 - r$. Then the graph has at least $n(m_t - 1) - s$ edges, where s is the number of edges that are between $S \cap X$ and $S \cap Y$. The the total edges between these two parts is $r((m_t - 1) - r) = -r^2 - r(m_t - 1) = s$. Thus the total number of edges in this graph is given by:

$$\begin{aligned} |E(G)| &= (n-r)((m_t-1)-r) + r((m_t-1)-r) + r(n-(m_t-1)+r) \\ &= n(m_t-1) - r(m_t-1) + r^2. \end{aligned}$$

If follows that $E(G)$ is minimized when $r = \left\lfloor \frac{m_t-1}{2} \right\rfloor$, and thus G must contain at least $n(m_t - 1) - \left\lfloor \frac{m_t-1}{2} \right\rfloor (m_t - 1) + \left\lfloor \frac{m_t-1}{2} \right\rfloor^2$ edges.

□

IV. Rainbow

IV.1 Rainbow Matchings

In [1], Barrus et al. determined bounds for the saturation number of rainbow matchings. We want to consider the bounds for the saturation number for a rainbow matching, mK_2, inside $K_{n,n}$.

In rainbow matchings, we lose the ability to use the Kőnig-Egerváry Theorem that produced concrete bounds in the monochromatic case. Since we cannot look at rainbow vertex covers to count the number of edges, we use an elementary graph, a $K_{2,2}$ that is properly egde-colored with two colors. The structure of $K_{2,2}$ allows us to chose only one edge to be part of the matching. Since we want a matching of size m in this graph, we will need $m-1$ copies of the $K_{2,2}$. Disjoint copies with disjoint color sets allows us to add any new edge between the $K_{2,2}$'s to increase the size of the matching. As we will see, increasing the lower bound for the number of edges for the rainbow matching saturation is difficult. We know that initially we will need at least $m-1$ edges in the graph. Since we cannot use vertex covers, we start by building up the degree on the vertices in the graph.

Theorem 5. *If $n > m$, then $2m \leq \text{sat}_t(\mathfrak{R}(mK_2), K_{n,n}) \leq 4(m-1)$.*

Let G be a bipartite graph with parts $X = \{x_1, x_2, x_3, \ldots, x_n\}$ and $Y = \{y_1, y_2, y_3, \ldots, y_n\}$. Let the edge set of G consist of $m-1$ disjoint copies of $K_{2,2}$. Properly edge color each $K_{2,2}$ with pairwise disjoint sets of two colors. The number of edges in G is $4(m-1)$. Let H be the induced subgraph of G that contains all vertices of with positive degree (see Figure 9). This graph does not contain a rainbow matching of size m. In each copy of $K_{2,2}$, we can take one edge to be included in any rainbow matching. If we select two edges from the same $K_{2,2}$, then both of the edges would be the same color.

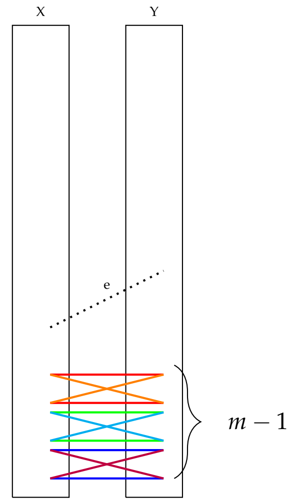

Figure 9: Construction 2: Rainbow Matching.

Consider the addition of a new edge xy to this graph $x, y \in G - H$. Then, regardless of the color of xy this new edge will increase the size of the largest rainbow matching by one. The next case is $x \in H$ and $y \in G - H$. If an edge in the matching was incident to x before the addition of the new edge, then when we add xy we take the other edge of that same color in the $K_{2,2}$. In the third case, we add xy in H such that x is in one $K_{2,2}$ and y is in a different $K_{2,2}$. Regardless of the color of xy we will be able to chose three edges, xy and one edge from each disjoint $K_{2,2}$. As a result, the addition of any new edge e will force H to have a rainbow matching of size m.

For the lower bound, we know that we need at least $m - 1$ edges since that is the size of the largest possible rainbow matching without xy.

Observation 6. *G contains no vertices of degree 1.*

Proof. Suppose that x_1 is a vertex of degree 1 in X with edge x_1y_1 has color 1. Suppose the vertex y_1 has other neighbors in X. Add new edge e to the graph. If this new edge is added such that is has the form x_ry_1, for $x_r \in X$, in color 1, then the size of the rainbow matching will not increase since we can swap out this new edge for the edge between x_1y_1. Therefore, x_1 can not have degree one as claimed. Since every vertex of positive degree has degree at least 2, and we have at least $m - 1$ vertices of positive degree in X, then we have $2m$ as our lower bound. □

Increasing the lower bound beyond this point requires that we increase the overall degree of the vertices incident to the $m - 1$ edges in the matching. However this gets difficult as each time we add a new edge to increase the vertices, we will have to consider the degree of the vertices neighboring the endpoint of the new edge. The goal is to increase the average degree across the vertices to be closer to 3.

IV.2 Rainbow Stars

In [1], Barrus et al. proved the bounds for the saturation number of a rainbow star in a graph on n vertices. We want consider a similar problem with the host graph $K_{n,n}$. In this proof, we make use of the Pigeonhole Principle, by taking a large number of vertices and partitioning them by the color of their edges. In order to achieve asymptotic bounds, we go through a process of duplicating vertices to mimic the vertex of smallest degree in G. Duplication will keep the number of edges in the graph the same, or it will reduce the number of edges needed. There will be some edges in the graph that we can not remove by trying to duplicate the adjacent vertex without losing being $\mathcal{R}(K_{1,k})$-saturated. We refer to these edges as "special" edges. We then proceed to use the pigeonhole principle to group these special edges based on their color and use these sets to count the number of edges necessary in the graph to increase the size of the rainbow star by one.

Consider a bipartite graph $K_{n,n}$ with color set $[t]$.

Theorem 7. *For $t > 2k$, $2n(k-1) - O(t^2) \leq \mathrm{sat}_t(\mathfrak{R}(K_{1,k}), K_{n,n}) \leq 2n(k-1) - O(t^2)$.*

Proof. Let G be a bipartite graph with parts $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Consider a set $X^{'} = \{x_1, x_2, \ldots, x_{k-1}\}$ in X and a set $Y^{'} = \{y_1, y_2, \ldots, y_{k-1}\}$ in the saturation number for a rainbow star. For each $i \in [k-1]$, completely join x_i from $X^{'}$ to the vertices in $Y - Y^{'}$ in color i. Next, completely join y_i from Y' to $X - X'$ with edges of color $k-1+i$. Finally let $X^{'}$ and $Y^{'}$ be completely joined with edges in color 1. The total number of edges in G is:

$$|E(G)| = 2n(k-1) - (k-1)^2$$

Consider the graph G presented in the construction (see Figure 10). From the construction, we know that each vertex can see at most $k-1$ colors, thus the largest rainbow star in G has size $k-1$. Thus this graph does not contain a $\mathfrak{R}(K_{1,k})$. Suppose we add a new edge $e = x_j y_l$ to G in color i. Since all the vertices in $X^{'}$ and $Y^{'}$ have degree n, e must be added between $X - X^{'}$ and $Y - Y^{'}$. With the addition of e in color i, we have increased the rainbow star at x_j unless the vertices in $X^{'}_X$ already have an edge in color i. In which case, since the color sets between X and Y are disjoint, we have increased the size of the rainbow star in y_s. Thus,

$\text{sat}_t(\mathfrak{R}(K_{1,k}), K_{n,n}) \leq 2n(k-1) - (k-1)^2$.

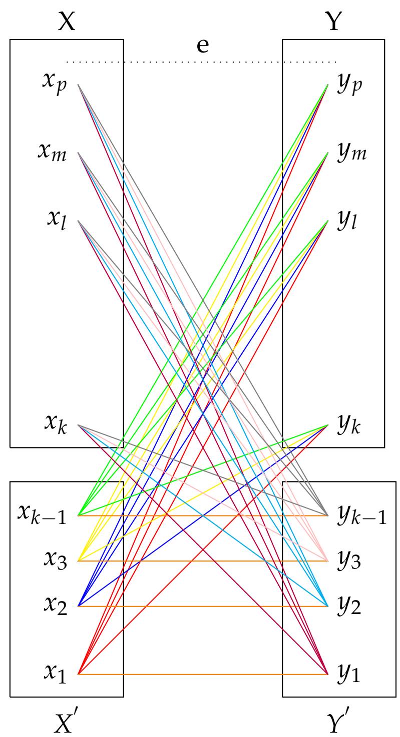

Figure 10: Construction 3: Rainbow Star.

Let G be a $\mathfrak{R}(K_{1,k})$-saturated graph. For the lower bound, we know that we need at least $k-1$ edges since that is the size of the largest possible rainbow star without xy. We begin by presenting some observations about $\mathfrak{R}(K_{1,k})$-saturated graphs.

Observation 8. Every vertex can see at most $k-1$ colors. If a single vertex sees k or more colors, then G would already contain a rainbow $K_{1,k}$.

Observation 9. If a vertex v is incident to edges of at most $k-2$ colors, then it must have degree n. Otherwise suppose that u and v are not adjacent.Let c_i be the color of an edge incident to u. If we add the new edge uv in color c_i , then we have not created a rainbow $K_{1,k}$ at u, nor have we created a rainbow $K_{1,k}$ at v since v sees at most $k-1$ colors. Hence G is not $\mathfrak{R}(K_{1,k})$-saturated.

Observation 10. If two vertices see the same color, they must share an edge. If u and v both see color i, then u and v must be adjacent; otherwise we can add the edge between u and v in color i without creating a rainbow $K_{1,k}$.

Observation 11. Since x_j sees $k-1$ colors, we know $k-1 \leq \delta \leq 2(k-1)$. Otherwise there are $2n(k-1)$ edges in G.

Now we consider the remaining vertices in X. In order to reduce the number of edges from being a complete bipartite graph, we want to try to duplicate the vertex x_j. By duplication, we refer to the process of taking a vertex x_i and removing its edges to Y and then reconnecting it to Y so that it has the edges to the same vertices in Y as x_j in the same color. So if $x_j y_j$ is color j, then the edge $x_i y_j$ is color j. Consider the vertex x_1 and its edges into Y. If all of x_1's edges can be removed without loosing the property of being $\mathfrak{R}(K_{1,k})$-saturated then we can remove these edges and recreate the edges between x_1 and Y so that it is identical to x_j. The edges that we must have to remain $\mathfrak{R}(K, K_{1,k})$-saturated we will call "special," which means that we have vertices we are not able to duplicate. We can repeat this process of duplicating vertices in X and collect the vertices that cannot be repeated.

Observation 12. The duplication process will not increase the number of edges. Since x_j has minimum degree, each duplication will result in either the same number of edges previously adjacent to the vertex, or it will decrease the number of edges. Therefore we may assume that it is not possible to duplicate any vertices in G.

Let X' be the set of vertices in X that are incident to special edges. Since G is edge colored with t colors, then by the Pigeonhole Principle, we know that there are $\frac{x}{t}$ special edges with the same color. Those $\frac{x}{t}$ special vertices in X' that are incident to special edges of the same color. By Observation 10, this implies that there is a complete bipartite graph, $K_{\frac{x}{t},\frac{x}{t}}$. Counting the number of edges, we obtain $\left(\frac{x}{t}\right)^2$ from the complete bipartite graph. Then we count the number of edges that we were able to duplicate, $(n-x)(k-1)$. And then the remaining vertices, the number of edges that are incident to the vertices in Y that are not neighbors of x_j and are also not in the matching in X are $\left(n-2(k-1)-\frac{x}{t}\right)(k-1)$. Therfore:

$$|E(G)| \geq \left(\frac{x}{t}\right)^2 + (n-x)(k-1) + \left(n-2(k-1)-\frac{x}{t}\right)(k-1)$$

Minimizing this with respect to x, we obtain $x = \frac{(k-1)(t+1)}{2}$ and,

$$|E(G)| \geq 2n(k-1) - \left[\frac{(k-1)^2(t+1}{2} + \frac{t(k-1)(t+1)}{2} - \frac{(k-1)(t+1)}{t} + 2(k-1)^2\right].$$

□

V. Conclusion

In this thesis, we have examined the saturation number for a bipartite graph containing multiple matchings. In our problem, we looked at the host graph $K_{n,n}$ with k matchings. Using the Kőnig-Egerváry Theorem, we found that if the number of vertices is sufficiently larger than the number of monochromatic edge matchings in the graph, we have exact results for the number of edges in the graph.

Question 1. How many matchings can we fit into this graph?

Question 2. What will happen to the saturation number if the partitions of the graph G are no longer the same size?

We conjecture that when the partitions of the graph become unbalanced, the construction will change to have more of the vertex covers on the bigger side. This will decrease the number of edges needed in the graph.

As an extension of the monochromatic matchings in the bipartite graph, in Section 4 we consider rainbow matchings. In this case, the lower bounds seem difficult to obtain. We presented a construction for an upper bound in order to obtain the saturation number for the rainbow matching. We then presented some observations for the lower bound. The difficulty with the lower bound lies in trying to increase the average degree of the vertices with positive degree in the graph. Since we observed that every vertex cannot have degree one, to obtain lower bounds closer to our construction, we need to increase the number of vertices with positive degree to get closer to our construction.

Following the ideas from [1], we determined asymptotic results for the rainbow saturation number of a star in $K_{n,n}$. A family of constructions can account for the upper bound on the saturation number of the rainbow star. By duplicating the smallest degree vertex, we were able to isolate the edges that were necessary to keep the graph saturated, the 'special' edges. The Pigeonhole Principle allowed us to group the special edges in order to count the number of edges in the graph. An interesting comparison would be to look at the saturation for monochromatic stars in $K_{n,n}$.

Question 3. We viewed the saturation number for rainbow stars, what is the saturation for monochromatic stars?

For future work, it would be interesting to explore the previously stated questions, as well as look for other subgraphs inside $K_{n,n}$ and then see how these results compare when the partitions become unbalanced.

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