Iterative Methods for Stochastic Variational Inequalities

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by

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Abstract

In this work, we consider stochastic variational inequalities arising from a certain class of equilibrium problems with uncertainties. Uncertainties in the models are introduced through data that are known through their probabilistic distributions. We consider several extragradient methods for the solutions of the variational inequalities and compare their relative efficiency and effectiveness through thorough numerical comparisons. Several applications such as traffic equilibrium, environmental games, and oligopolistic market equilibrium are considered.

Keywords: stochastic linear complementarity problem, stochastic variational inequalities, traffic equilibrium, oligopolistic market equilibrium, environmental games, Cournot oligopology.

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Chapter 1

Introduction

In this chapter, we will introduce linear complementarity problems(LCP). We will give some application examples, and also briefly discuss deterministic and stochastic variational inequalities related to linear complementarity problems.

1.1 Linear Complementarity Problem

The linear complementarity problem is to find a vector in a finite-dimensional real vector space in which the vector needs to satisfy a certain system of inequalities.
Given a pair (q, M) of a vector $q \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times n}$, the *Linear Complementarity Problem* (LCP), is to find a vector $p \in \mathbb{R}^n$ such that

$$p \geq 0 \quad (1.1)$$

$$q + Mp \geq 0 \quad (1.2)$$

$$p^\top(q + Mp) = 0 \quad (1.3)$$

or to show that no such p exists.

1.2 Problems equivalent to LCP

In this section, we will introduce several LCP application problems whose formulations are used to develop computational methods.

1.2.1 Quadratic Programming

Consider the quadratic program (QP)

$$\begin{aligned} \text{minimize } f(x) \quad &= \quad c^\top x + \frac{1}{2} x^\top Q x \\ \text{subject to} \quad & \quad Ax \geq b \\ & \quad x \geq 0 \end{aligned} \tag{1.4}$$

where $Q \in \mathbb{R}^{n \times n}$ is symmetric, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. Note that $Q = 0$ gives a linear program. It is known that if x is a locally optimal solution of (1.4), then there exists a vector $y \in \mathbb{R}^m$ such that the pair (x, y) satisfies the so-called Karush-Kuhn-Tucker (KKT) conditions

$$u = c + Qx - A^\top y \geq 0, \ x \geq 0, \ x^\top u = 0 \tag{1.5}$$

$$v = -b + Ax \geq 0, \ y \geq 0, \ y^\top v = 0. \tag{1.6}$$

Furthermore, if Q is positive semi-definite (i.e. the objective function $f(x)$ is convex), the the conditions (1.5) and (1.6) are sufficient for the vector x to be globally optimal solution of the quadratic program (1.4).
The conditions is (1.5) and (1.6) define the LCP(q, M) where

$$q = \begin{bmatrix} c \\ -b \end{bmatrix} \text{ and } M = \begin{bmatrix} Q & -A^\top \\ A & 0 \end{bmatrix}. \tag{1.7}$$

Notice that M is not symmetric. Also, if Q is positive semi-definite, then so is M.
A special case of (1.4) is

$$\text{minimize } f(x) \quad = \quad c^\top x + \frac{1}{2} x^\top Q x \tag{1.8}$$

$$\text{subject to} \quad x \geq 0. \tag{1.9}$$

If Q is positive semi-definite, them the program (1.9) is equivalent to LCP(c, Q) (with symmetric Q).

1.2.2 Market Equilibrium

A *market equilibrium* is the state of an economy in which demands of consumers and the supplies of producers are balanced at the prevailing price level. Consider a market equilibrium problem where supply side is described by a linear programming model to capture the details of the production activities. The market demand function is generated by models with commodity prices as the primary independent variables. The mathematical problem is to find vectors p^* and r^* such that the constraints stated in (i)- (iii)are satisfied.
(i) supply side

$$\begin{aligned} \text{minimize } f(x) \quad &= \quad c^\top x \\ \text{subject to} \quad & \quad Ax \geq b && (1.10) \\ & \quad Bx \geq r^* && (1.11) \\ & \quad x \geq 0 \end{aligned}$$

where c is the cost vector for supply activities, x is the vector of production activity levels. Condition (1.10) represents the technological constraints on production, and the condition (1.11) is the demand requirement constraints.
(i) demand side

$$\text{minimize } r^* = Q(p^*) = Dp^* + d \tag{1.12}$$

where Q is the market demand function with p^* and r^* representing the vectors of demand prices and quantities respectively.
(iii) equilibrium conditions

$$p^* = \pi^* \tag{1.13}$$

where π^* denotes the vector of shadow prices (i.e. the market supply prices) corresponding to the constraint (1.11).

1.2.3 Nonlinear Complementarity and Variational Inequality Problems

The LCP is a special case of *nonlinear complementarity* problem (NCP) the goal of which is to find a vector p such that

$$p \geq 0,\ \ f(p) \geq 0, \text{and } p^\top f(p) = 0 \tag{1.14}$$

where f is a given mapping from $\mathbb{R}^n$ into itself. LCP is a particular case of NCP where $f(p) = q + Mp$ (a linear function). The nonlinear complementarity problem provides a unified formulation to nonlinear programming and many equilibrium problems such as traffic equilibrium problem and the n-person Nash-Cournot equilibrium problem. One of the solution methods for the nonlinear complementarity problem are linear approximation methods where you solve a sequence of linear complementarity problems of form

$$vi_firstp \geq 0,\ \ w = f(p^{(k)}) + A(p^{(k)})(p - p^{(k)}) \geq 0,\ \ p^\top w = 0 \tag{1.15}$$

where $p^{(k)}$ is the current iterate and $A(p^{(k)})$ is some suitable approximation of the Jacobian matrix $\nabla f(p^{(k)})$. For example, when $A(p^{(k)})$ is the Jacobian matrix, then we have Newton's method for NCP.
Another generalization of the nonlinear complementarity problem is *variational inequality problem*: Given a nonempty subset K of $\mathbb{R}^n$ and a mapping f from $\mathbb{R}^n$ to itself, find a vector x^* such that

$$(y - x^*)^\top f(x^*) \geq 0,\ \text{ for all } y \in K.$$

The problem is denoted by $\mathrm{VI}(K, f)$.

1.3 Stochastic Linear Complementarity Problem

In applications of LCP there is almost always certain types of uncertainties such as weather, material, load, supply demand are involved. Let $(\Omega, \mathscr{F}, \mathcal{P})$

be a probability space with $\Omega \subseteq \mathbb{R}^m$ where probability distribution $\mathcal{P}$ is known. For $\omega \in \Omega$, we consider random quantities $M(\omega) \in \mathbb{R}^{n\times n}$ and $q(\omega) \in \mathbb{R}^n$.
The *Stochastic Linear Complementarity Problem* (SLCP) is to find a vector $x \in \mathbb{R}^n$ such that

$$M(\omega)x + q(\omega) \geq 0,\ x \geq 0,\ x^\top(M(\omega)x + q(\omega)) = 0, \quad \omega \in \Omega. \tag{1.16}$$

1.4 Stochastic Variational Inequality

In this section, we introduce the particular form of stochastic variational inequality that we will consider in the following chapters. Let $(\Omega, \mathscr{F}, \mathcal{P})$ be a probability space. We define a random set

$$M(\omega) := \{x \in \mathbb{R}^k :\ Ax \leq D(\omega)\} \tag{1.17}$$

for $\omega \in \Omega$ by using a given matrix $A \in \mathbb{R}^{m\times k}$ and random vector D in $\mathbb{R}^m$.
Consider the following Stochastic Variational Inequality (SVI): For $\omega \in \Omega$, find $x := x(\omega) \in M(\omega)$ such that

$$\langle S(\omega)\, G(x) + H(x), z - x\rangle \geq \langle R(\omega)\, c + b, z - x\rangle$$

for every $z \in M(\omega)$.
Here, $G, H : \mathbb{R}^k \to \mathbb{R}^k$ are two given maps. Real valued random variables R and S are defined on Ω, and b, c are fixed vectors in $\mathbb{R}^k$.

Chapter 2

Stochastic Variational Inequalities

We will study stochastic variational inequalities in this chapter. Theory of variational inequalities provides an efficient mathematical apparatus for studying a wide range of problems arising in diverse fields such as structural mechanics, elasticity, economics, optimization, financial mathematics, and others. Variational inequalities have been used extensively for various network equilibrium problems and in particular for transportation science models (see, for instance, the book [34] and the cited references therein). Many of the research on the applications of variational inequalities to various aspects of transportation science and others so far have been in connection with deterministic models. However, since the data for the most problems are often affected by uncertainty or randomness in real-world applications, their variational inequality formulations must take into account for this stochasticity. This is a well justified need, and in recent years scientific community have witnessed an acute increase in research where the authors have incorporated stochasticity in the models.

We will first talk about the so-called elliptic regularization technique in the context of stochastic variational inequalities introduced in [23]. The regularization methods have been studied extensively for deterministic variational inequalities and the motivation to study regularization stems from the fact that network problems lead naturally to monotone variational inequalities. In order to use effective numerical techniques designed for strongly monotone variational inequalities, the authors in [23] resorted to regularization strategies. Theoretical results that allow a satisfactory treatment of monotone variational inequalities are provided. The authors performed a comparison of a rigorous L_p approach they considered with a popular sample-path approach for stochastic variational inequalities proposed by Agdeppa, Yamashita, and Fukushima [1] and Chen, Zhang, and Fukushima [6] by using a suite of test problems.

2.0.1 L_p approach

The methodology adopted there is the L_p-approach pioneered by Gwinner [16] in the context of variational inequalities with linear random operators. He gave new existence theorems and discretization schemes and also presented an interesting application of the proposed theory to unilateral boundary value problems. The functional setting introduced in [16] was later strengthened in [18] to include randomness in the underlying constraints set (see also [17]). More recently, in [19] and [20], the authors investigated stochastic variational inequalities with nonlinear monotone maps. Besides presenting a generalization of the existing theory, the nonlinear extension was motivated by the need to cope with the nonlinearity in many equilibrium problems arising in operations research such as the random traffic equilibrium problems which is studied in detail in this article. In these studies, the focus was on functional analytic methods to obtain approximations of the solution (a random vector) together with approximations of statistical quantities such as the mean and variance of the (random) solution.

In contrast to the aforementioned L_p approach, the so-called sample-path approach (SPA), commonly studied in connection to stochastic variational inequalities, aims to associate to the original (stochastic) problem to a deterministic problem which is obtained by sampling/averaging the data of the original problem. Many techniques have been used for the purpose of averaging/sampling. For instance, a Monte Carlo sampling method is available in Patriksson [35], Shapiro [40], Shapiro, Dentcheva, and Ruszczyński [41], and Shapiro and Xu [42], among others. Gürken, Özge, and Robinson [15] initiated the use of sample-path methods for variational inequalities (see also [14]). In these works, the authors focused on stochastic variational inequalities involving a Fréchet differentiable maps defined on polyhedral sets in finite dimensional setting. Since then this methodology has been extensively used by M. Fukushima and his co-workers (see [5], [1], [6]). In these works, the authors proposed an expected residual minimization method for stochastic linear complementarity problems and variational inequalities and gave various applications to equilibrium problems. Recently, many researchers applied the sample-path approach to Stackelberg and Nash games (see De Miguel and Xu [10], and Ravat and Shanbhag [37] and [38]). We would like to also point out an interesting work by Dentcheva and Ruszcynsky [11] where the authors investigated optimization problems with the so-called stochastic dominance constraints.

2.0.2 Existing Methods

In the following, we briefly discuss some of the existing methodologies available in the literature. We begin with a discussion of the expected residual minimization method. The method described in Chen and Fukushima [5] is in the context of the following variational inequality: find $x \in S \subseteq \mathbb{R}^n$ such that

$$F(x,\omega)^T(y-x) \geq 0, \quad \forall\, y \in S,$$

where the set S is closed and convex, $(\Omega, \mathcal{A}, P)$ is the probability space, and $F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$.

In fact, the authors focused on the case when S is the positive orthant. For this particular case, the above variational inequality reduces to the following complementarity problem: Find $x \in \mathbb{R}^n$ such that

$$F(x,\omega) \geq 0, \quad x \geq 0, \quad F(x,\omega,x)^T x = 0.$$

The main issue here is how to construct an *averaged problem* that can replace the stochastic one and how to devise a variant of the approach proposed in [15] and [14]. We recall that the strategy of [15] consists of solving the deterministic complementarity problem that results from replacing the map $F(x,\omega)$ with its expectation $F_\infty = \mathbf{E}[F(x,\omega)]$. In general, this problem is different from the one that is obtained by replacing the random variable ω with its expectation. To approximate F_∞ in an efficient way, a sequence of approximate problems can be considered in which F_∞ is approximated by functions $F_k(x)$ by employing discrete distributions and Monte-Carlo methods. Once the stochastic complementarity has been converted into a deterministic one, the latter can be solved by some suitable method. The authors in the aforementioned studies used the so-called nonlinear complementary functions to solve the deterministic complementarity problem. Recall that a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is called a nonlinear complementary function if an only if

$$\varphi(a,b) = 0 \quad \Longleftrightarrow \quad a \geq 0, \quad b \geq 0, \quad ab = 0.$$

The averaged problem associated to the stochastic complementarity problem is:

$$\min_{x \in \mathbb{R}^n_+} \mathbf{E}\|\Phi(x,\omega\|^2 \tag{2.1}$$

where

$$\Phi(x,\omega) = (\varphi(F_1(x,\omega),x_1),\ldots,\varphi(F_n(x,\omega),x_n))^T.$$

This approach can be interpreted as an average least-squares approach. To compute the expectations, the authors generated observation by using quasi-Monte-Carlo methods. Moreover, they proved that every accumulation point of sample approximation problems is a minimizer for (2.1). We remark that when for every ω, the stochastic problem has the unique solution $x(\omega)$, no relation between the accumulation point of the approximation and of $\mathbf{E}[(x(\omega)]$

has been established. This approach is further developed in [6] where the authors discussed the robustness of their method, gave new error bounds, and proposed a procedure to generate a class of stochastic complementarity problem to which their method can be applied. They also gave a concrete application to stochastic traffic equilibrium problems. In [1], the authors studied a stochastic affine variational inequality by the same approach and proposed a convex expected residual model. They also applied their results to traffic equilibrium problems. The above mentioned papers contain a deep analysis of the structure of the deterministic problem which emerges from the stochastic one. In [1, p.2], the authors state that their approach leads to a *reasonable solution* of the stochastic variational inequality. Clearly, a reasonable solution should be close to the exact mean value solution.
A thorough comparison between the L^p approach and the sample-path approach was done in [23]. The authors introduced randomness into the traffic equilibrium models first used in [1] and [6] and solved them by applying the L^p-approach. The L_p approach allows to compute approximations to the exact mean value of the random solution vector, and the authors were also able to compute the *exact* mean values. Through the examples, they attempted to validate the approximation procedure and compare with the expected residual method.

2.1 Stochastic Variational Inequalities

In this section, we recall some recent results from Jadamba, Khan, and Raciti [23]. Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let $G, H : \mathbb{R}^k \to \mathbb{R}^k$ be two given maps, let $b, c \in \mathbb{R}^k$ be fixed vectors, and let R and S be two real-valued random variables defined on Ω. Let λ be a random vector in $\mathbb{R}^k$, let D be random vector in $\mathbb{R}^m$, and let $A \in \mathbb{R}^{m \times k}$ be a given matrix. For $\omega \in \Omega$, we define a random set

$$M(\omega) := \{x \in \mathbb{R}^k : \; Ax \leq D(\omega)\}.$$

Consider the following stochastic variational inequality: For almost all $\omega \in \Omega$, find $\hat{x} := \hat{x}(\omega) \in M(\omega)$ such that

$$\langle S(\omega)\, G(\hat{x}) + H(\hat{x}), z - \hat{x} \rangle \geq \langle R(\omega)\, c + b, z - \hat{x} \rangle \tag{2.2}$$

for every $z \in M(\omega)$. Variational inequality (2.2) holds pointwise on Ω, except a fixed null set depending on the solution $\hat{x}$.
Now set

$$F(\omega, x) := S(\omega)\, G(x) + H(x).$$

The assumption here is that S, G and H are such that the map $F : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$ is a Carathéodory function. That is, for each fixed $x \in \mathbb{R}^k$, the function $F(\cdot, x)$ is measurable with respect to $\mathcal{A}$ whereas for each $\omega \in \Omega$ the function $F(\omega, \cdot)$ is continuous. We also assume that $F(\omega, \cdot)$ is monotone for every $\omega \in \Omega$.
Let $\Sigma : \Omega \hookrightarrow \mathbb{R}^k$ be the set-valued map that associates to each $\omega \in \Omega$, the set of all solutions $\Sigma(\omega)$ of (2.2). Gwinner and Raciti [18] proved the measurability of the set-valued map Σ for variational inequalities defined via bilinear forms. However, the proof given there can readily be extended to the general case of nonlinear operators. If (2.2) is uniquely solvable, then suitable conditions ensure that the solution belongs to an L^p space for some $p \geq 2$. This observation allows us to compute statistical quantities such as the mean values and the variances of the solution.

2.1.1 Integral Formulation

Now, we proceed to derive the integral formulation of the variational inequality (2.2). For a fixed $p \geq 2$, we define the reflexive Banach space $L^p(\Omega, P, \mathbb{R}^k)$ of random vectors V from Ω to $\mathbb{R}^k$ such that the expectation (p-moment) is given by:

$$E^P \|V\|^p = \int_\Omega \|V(\omega)\|^p dP(\omega) < \infty.$$

For the subsequent development, the following growth condition is needed:

$$\|F(\omega, z)\| \leq \alpha(\omega) + \beta(\omega)\|z\|^{p-1}, \quad \forall z \in \mathbb{R}^k, \quad \text{for some } p \geq 2, \tag{2.3}$$

where $\alpha \in L^p(\Omega, P)$ and $\beta \in L^\infty(\Omega, P)$.
Due to the above growth condition, the Nemitsky operator $\hat{F}$ associated to F, acts from $L^p(\Omega, P, \mathbb{R}^k)$ to $L^q(\Omega, P, \mathbb{R}^k)$, where $p^{-1} + q^{-1} = 1$. Furthermore, we have

$$\hat{F}(V)(\omega) := F(\omega, V(\omega)), \quad \omega \in \Omega.$$

Assuming $D \in L^p_m(\Omega) := L^p(\Omega, P, \mathbb{R}^m)$, we introduce the following nonempty, closed and convex subset of $L^p_k(\Omega)$

$$M^P := \{V \in L^p_k(\Omega) : A\,V(\omega) \leq D(\omega),\ P - a.s.\},$$

which is the L^p analogue of $M(\omega)$ defined above.
Let $S(\omega) \in L^\infty$, $0 < \underline{s} < S(\omega) < \overline{s}$, and $R(\omega) \in L^q$. Equipped with these notations, we consider the following L^p formulation of (2.2). Find $\hat{U} \in M^P$ such that for every $V \in M^P$, we have

$$\begin{aligned}&\int_\Omega \langle S(\omega)\, G(\hat{U}(\omega)) + H(\hat{U}(\omega)), V(\omega) - \hat{U}(\omega)\rangle \, dP(\omega) \geq \\ &\int_\Omega \langle b + R(\omega)\, c, V(\omega) - \hat{U}(\omega)\rangle dP(\omega). \end{aligned} \tag{2.4}$$

If problems (2.2) and (2.4) are uniquely solvable then they are equivalent provided that the solution of (2.2) defines an L^p function. The relation between the two formulations in the general case has been analyzed in [20, Proposition 1].
To get rid of the abstract sample space Ω, we consider the joint distribution $\mathbb{P}$ of the random vector (R, S, D) and work with the special probability space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mathbb{P})$, where the dimension $d := 2 + m$. For simplicity, we assume that R, S and D are independent random vectors. We set

$$\begin{aligned} r &= R(\omega), \\ s &= S(\omega), \\ t &= D(\omega), \\ y &= (r, s, t). \end{aligned}$$

For each $y \in \mathbb{R}^d$, we define the set

$$M(y) := \{x \in \mathbb{R}^k : \ Ax \le t\}.$$

The pointwise formulation of the variational inequality reads: Find $\hat{x}$ such that $\hat{x}(y) \in M(y)$, $\mathbb{P}$ - a.s., and the following inequality holds for $\mathbb{P}$ - almost every $y \in \mathbb{R}^d$ and for every $x \in M(y)$, we have

$$\langle s\, G(\hat{x}(y)) + H(\hat{x}(y)), x - \hat{x}(y)\rangle \ge \langle rc + b, x - \hat{x}(y)\rangle \,. \tag{2.5}$$

In order to obtain the integral formulation of (2.5), consider the space $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ and introduce the closed and convex set

$$M_{\mathbb{P}} := \{v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : \quad Av(r,s,t) \le t, \quad \mathbb{P} - a.s.\}.$$

With this terminology, we consider the variational inequality of finding $\hat{u} \in M_{\mathbb{P}}$ such that for every $v \in M_{\mathbb{P}}$ we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^d} \langle s\, G(\hat{u}(y)) + H(\hat{u}(y)), v(y) - \hat{u}(y)\rangle \, d\mathbb{P}(y) \ge$$
$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^d} \langle b + r\, c, v(y) - \hat{u}(y)\rangle \, d\mathbb{P}(y). \tag{2.6}$$

The equivalence of (2.5) and (2.6) can easily be proven.
It is also observed that this approach and analysis extends readily to more general finite Karhunen-Loève expansions:

$$\lambda(\omega) = b + \sum_{l=1}^{L} R_l(\omega)\ c_l \quad F(\omega, x) = H(x) + \sum_{l=1}^{L_F} S_l(\omega)\, G_l(x).$$

We recall the following general result useful to ensure solvability of an infinite dimensional variational inequality like (2.4), (see [29] for a recent survey on existence results for variational inequalities).

Theorem 2.1.1 *Let E be a reflexive Banach space and let K be a nonempty, closed, and convex subset of E Let $A : K \longrightarrow E^*$ be monotone and continuous* *on finite dimensional subspaces of K. Consider the variational inequality problem of finding $u \in K$ such that*

$$\langle Au, v-u\rangle_{E,E^*} \geq 0, \quad \textit{for every} \;\; v \in K.$$

Then a necessary and sufficient condition for the above problem to be solvable is the existence of $\delta > 0$ such that at least a solution of the variational inequality:

$$u_\delta \in K_\delta, \; \langle Au_\delta, v-u_\delta\rangle_{E,E^*} \geq 0, \forall v \in K_\delta$$

satisfies $\|u_\delta\| < \delta$, where

$$K_\delta = \{v \in K: \; \|v\| \leq \delta\}.$$

2.2 Approximation by Discretization of Distributions

This section contains an introduction to an approximate solution of stochastic variational inequalities by discretization of distributions. The approach was first introduced by Gwinner [16]. Assume, without any loss of generality, that $R \in L^q(\Omega, P)$ and $D \in L^p_m(\Omega, P)$ are nonnegative (otherwise we can use the standard decomposition in the positive part and the negative part). Moreover, we assume that the support (the set of possible outcomes) of $S \in L^\infty(\Omega, P)$ is the interval $[\underline{s}, \overline{s}) \subset (0, \infty)$. Furthermore, we assume that the probability measures P_R, P_S, and P_D are continuous with respect to the Lebesgue measure, so that according to the theorem of Radon-Nikodym, they have the probability densities φ_R, φ_S, and φ_{D_i}, $i = 1, \ldots, m$, respectively. Therefore, for $i = 1, \ldots, m$, we have

$$\begin{aligned}
\mathbb{P} &= P_R \otimes P_S \otimes P_D, \\
dP_R(r) &= \varphi_R(r)\, dr, \\
dP_S(s) &= \varphi_S(s)\, ds \\
dP_{D_i}(t_i) &= \varphi_{D_i}(t_i)\, dt_i.
\end{aligned}$$

Notice that $v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ means that $(r,s,t) \mapsto \varphi_R(r)\varphi_S(s)\varphi_D(t)v(r,s,t)$ belongs to the Lebesgue space $L^p(\mathbb{R}^d, \mathbb{R}^k)$ with respect to the Lebesgue measure where

$$\varphi_D(t) := \prod_i \varphi_{D_i}(t_i).$$

Therefore, we can define the probabilistic integral variational inequality: Find $\hat{u} := \hat{u}(y) \in M_{\mathbb{P}}$ such that for every $v \in M_{\mathbb{P}}$, we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} \langle s\, G(\hat{u}) + H(\hat{u}), v - \hat{u} \rangle\, \varphi_R(r)\varphi_S(s)\varphi_D(t)\, dy \geq \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} \langle b + r\, c, v - \hat{u} \rangle\, \varphi_R(r)\varphi_S(s)\varphi_D(t)\, dy\,.$$

For numerical approximation of the solution $\hat{u}$, we begin with a discretization of the space $X := L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$. For this, we introduce a sequence $\{\pi_n\}_n$ of partitions of the support

$$\Upsilon := [0, \infty) \times [\underline{s}, \overline{s}) \times \mathbb{R}^m_+$$

of the probability measure $\mathbb{P}$ induced by the random elements R, S, and D. For this, we set

$$\pi_n = (\pi_n^R, \pi_n^S, \pi_n^D),$$

where

$$\begin{aligned}
\pi_n^R &:= (r_n^0, \ldots, r_n^{N_n^R}),\\
\pi_n^S &:= (s_n^0, \ldots, s_n^{N_n^S}),\\
\pi_n^{D_i} &:= (t_{n,i}^0, \ldots, t_{n,i}^{N_n^{D_i}})\\
0 &= r_n^0 < r_n^1 < \ldots r_n^{N_n^R} = n\\
\underline{s} &= s_n^0 < s_n^1 < \ldots s_n^{N_n^S} = \overline{s}\\
0 &= t_{n,i}^0 < t_{n,i}^1 < \ldots t_{n,i}^{N_n^{D_i}} = n \;\; (i = 1, \ldots, m)\\
|\pi_n^R| &:= \max\{r_n^j - r_n^{j-1} : j = 1, \ldots, N_n^R\} \to 0 \qquad (n \to \infty)\\
|\pi_n^S| &:= \max\{s_n^k - s_n^{k-1} : k = 1, \ldots, N_n^S\} \to 0 \qquad (n \to \infty)\\
|\pi_n^{D_i}| &:= \max\{t_{n,i}^{h_i} - t_{n,i}^{h_i - 1} : h_i = 1, \ldots, N_n^{D_i}\} \to 0 \qquad (i = 1, \ldots, m;\, n \to \infty)\,.
\end{aligned}$$

These partitions give rise to the exhausting sequence $\{\Upsilon_n\}$ of subsets of Υ, where each Υ_n is given by the finite disjoint union of the intervals:

$$I^n_{jkh} := [r^{j-1}_n, r^j_n) \times [s^{k-1}_n, s^k_n) \times I^n_h \,,$$

where we use the multi-index $h = (h_1, \cdots, h_m)$ and

$$I^n_h := \Pi^m_{i=1} \, [t^{h_i-1}_{n,i}, t^{h_i}_{n,i}).$$

For each $n \in \mathbb{N}$, we consider the space of the $\mathbb{R}^l$-valued step functions ($l \in \mathbb{N}$) on Υ_n, extended by 0 outside of Υ_n:

$$X^l_n := \{v_n : v_n(r,s,t) = \sum_j \sum_k \sum_h v^n_{jkh} 1_{I^n_{jkh}}(r,s,t) \,, v^n_{jkh} \in \mathbb{R}^l\}$$

where 1_I denotes the $\{0,1\}$-valued characteristic function of a subset I.
To approximate an arbitrary function $w \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R})$, we employ the mean value truncation operator μ^n_0 associated to the partition π_n given by

$$\mu^n_0 w := \sum_{j=1}^{N^R_n} \sum_{k=1}^{N^S_n} \sum_h (\mu^n_{jkh} w) \, 1_{I^n_{jkh}} \,, \tag{2.7}$$

where

$$\mu^n_{jkh} w := \begin{cases} \dfrac{1}{\mathbb{P}(I_{jkh})} \displaystyle\int_{I^n_{jkh}} w(y) \, d\mathbb{P}(y) & \text{if} \quad \mathbb{P}(I^n_{jkh}) > 0 \,; \\ 0 & \text{otherwise.} \end{cases}$$

Analogously, for a L^p vector function $v = (v_1, \ldots, v_l)$, we define

$$\mu^n_0 v := (\mu^n_0 v_1, \ldots, \mu^n_0 v_l).$$

From [16, Lemma 2.5], and the remarks therein, we obtain the following result.

Lemma 2.2.1 *For any fixed $l \in \mathbb{N}$, the linear operator $\mu^n_0 : L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l) \to L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$ is bounded with $\|\mu^n_0\| = 1$ and for $n \to \infty$, μ^n_0 converges pointwise in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^l)$ to the identity.*

To construct approximations for

$$M_{\mathbb{P}} = \{v \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) : Av(r,s,t) \leq t\,,\ \mathbb{P} - a.s.\},$$

we introduce the orthogonal projector $q : (r,s,t) \in \mathbb{R}^d \mapsto t \in \mathbb{R}^m$ and define for each elementary cell I^n_{jkh},

$$\begin{aligned}\overline{q}^n_{jkh} &= (\mu^n_{jkh} q) \in \mathbb{R}^m,\\ (\mu^n_0 q) &= \sum_{jkh} \overline{q}^n_{jkh}\, 1_{I^n_{jkh}} \in X^m_n.\end{aligned}$$

This leads to the following sequence of convex and closed sets of the polyhedral type:

$$M^n_{\mathbb{P}} := \{v \in X^k_n : \quad Av^n_{jkh} \leq \overline{q}^n_{jkh}\,,\ \forall j,k,h\}.$$

It is known (see [19]) that the sequence $\{M^n_{\mathbb{P}}\}$ approximate the set $M_{\mathbb{P}}$ in the sense of Mosco (see [31]). That is, we have

$$\text{weak-limsup}_{n\to\infty} M^n_{\mathbb{P}} \subset M_{\mathbb{P}} \subset \text{strong-liminf}_{n\to\infty} M^n_{\mathbb{P}}. \tag{2.8}$$

Since our objective is to approximate the random variables R and S, we introduce

$$\begin{aligned}\rho_n &= \sum_{j=1}^{N^R_n} r^{j-1}_n\, 1_{[r^{j-1}_n, r^j_n)} \in X_n\\ \sigma_n &= \sum_{k=1}^{N^S_n} s^{k-1}_n\, 1_{[s^{k-1}_n, s^k_n)} \in X_n.\end{aligned}$$

Notice that

$$\begin{aligned}\sigma_n(r,s,t) &\to \sigma(r,s,t) = s, \quad \text{in } L^\infty(\mathbb{R}^d, \mathbb{P})\\ \rho_n(r,s,t) &\to \rho(r,s,t) = r, \quad \text{in } L^p(\mathbb{R}^d, \mathbb{P}),\end{aligned}$$

where the second convergence is a consequence of the Chebyshev inequality.

Combining the above ingredients, for $n \in \mathbb{N}$, we consider the following discretized variational inequality: Find $\hat{u}_n := \hat{u}_n(y) \in M_{\mathbb{P}}^n$ such that for every $v_n \in M_{\mathbb{P}}^n$, we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^d} \langle \sigma_n(y)\, G(\hat{u}_n) + H(\hat{u}_n), v_n - \hat{u}_n \rangle \, d\mathbb{P}(y) \geq \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^d} \langle b + \rho_n(y)\, c, v_n - \hat{u}_n \rangle \, d\mathbb{P}(y)\,. \tag{2.9}$$

It turns out that (2.9) can be split in a finite number of finite dimensional variational inequalities: For every $n \in \mathbb{N}$, and for every j, k, h, find $\hat{u}_{jkh}^n \in M_{jkh}^n$ such that

$$\langle \tilde{F}_k^n(\hat{u}_{jkh}^n), v_{jkh}^n - \hat{u}_{jkh}^n \rangle \geq \langle \tilde{c}_j^n, v_{jkh}^n - \hat{u}_{jkh}^n \rangle, \quad \text{for every } v_{jkh}^n \in M_{jkh}^n, \tag{2.10}$$

where

$$\begin{aligned} M_{jkh}^n &:= \{v_{jkh}^n \in \mathbb{R}^k : A v_{jkh}^n \leq \overline{q}_{jkh}^n\}\,, \\ \tilde{F}_k^n &:= s_n^{k-1}\, G + H \\ \tilde{c}_j^n &:= b + r_n^{j-1}\, c. \end{aligned}$$

Clearly, we have

$$\hat{u}_n = \sum_j \sum_k \sum_h \hat{u}_{jkh}^n \, 1_{I_{jkh}^n} \in X_n^k.$$

We recall the following convergence result from [19].

Theorem 2.2.2 *Assume that $F(\omega, \cdot)$ is strongly monotone, uniformly with respect to $\omega \in \Omega$, that is*

$$\langle F(\omega, x) - F(\omega, y), x - y \rangle \geq \alpha \|x - y\|^2 \quad \forall x,\, y, \;\; a.e.\ \omega \in \Omega,$$

where $\alpha > 0$ and that the growth condition (2.3) holds. Then the sequence $(\hat{u}_n)$, where $\hat{u}_n$ is the unique solution of (2.9)*, converges strongly in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ to the unique solution $\hat{u}$ of* (2.6)*.*

2.3 Regularization

In this section, we detail a regularization approach for stochastic variational inequalities introduced in [23]. Recall that the probabilistic integral formulation seeks $\hat{u} \in M_{\mathbb{P}}$ such that for every $v \in M_{\mathbb{P}}$, we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle s\, G(\hat{u}) + H(\hat{u}), v - \hat{u} \rangle \, \varphi_R(r)\varphi_S(s)\varphi_D(t)\, dy \geq$$
$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle b + r\, c, v - \hat{u} \rangle \, \varphi_R(r)\varphi_S(s)\varphi_D(t)\, dy\,. \tag{2.11}$$

Furthermore, the discretized analogue of the above variational inequality reads: For $n \in \mathbb{N}$, find $\hat{u}_n = \hat{u}_n(y) \in M_{\mathbb{P}}^n$ such that for every $v_n \in M_{\mathbb{P}}^n$, we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle \sigma_n(y)\, G(\hat{u}_n) + H(\hat{u}_n), v_n - \hat{u}_n \rangle \, d\mathbb{P}(y) \geq$$
$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle b + \rho_n(y)\, c, v_n - \hat{u}_n \rangle \, d\mathbb{P}(y)\,. \tag{2.12}$$

The above discrete variational inequality will be regularized and it is shown that its continuous analogue is recovered by the limiting process. First, a sequence $\{\epsilon_n\}$ of regularization parameters is chosen. Also, choose the regularization map to be the duality map $J : L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) \to L^q(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$. Assume that $\epsilon_n > 0$ for every $n \in \mathbb{N}$ and that $\epsilon_n \downarrow 0$ as $n \to \infty$.
Consider the following regularized stochastic variational inequality: For $n \in \mathbb{N}$, find $w_n = w_n^{\epsilon_n}(y) \in M_{\mathbb{P}}^n$ such that for every $v_n \in M_{\mathbb{P}}^n$, we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle \sigma_n(y)\, G(w_n) + H(w_n) + \epsilon_n J(w_n), v_n - w_n \rangle \, d\mathbb{P}(y) \geq$$
$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle b + \rho_n(y)\, c, v_n - w_n \rangle \, d\mathbb{P}(y). \tag{2.13}$$

The solution w_n will be referred to as the regularized solution. The following theorem highlights some of the features of the regularized solutions:

Theorem 2.3.1 *For every $n \in \mathbb{N}$, the regularized stochastic variational inequality* (2.13) *has the unique solution w_n. Any weak limit of the sequence $\{w_n\}$ of the regularized solutions is a solution of* (2.11)*. Furthermore, the sequence of the regularized solutions $\{w_n\}$ is bounded provided that the following coercivity condition holds: There exists a bounded sequence $\{\delta_n\}$ with $\delta_n \in M_{\mathbb{P}}^n$ such that*

$$\frac{\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle \sigma_n(y)\, G(u_n) + H(u_n), u_n(y) - \delta_n \rangle \, d\mathbb{P}(y)}{\|u_n\|} \to \infty \ \textit{as} \ \|u_n\| \to \infty. \tag{2.14}$$

For the sake of completion, we recall the proof presented in [23].

Proof. Notice that the map $\sigma_n(y)\, G(w_n) + H(w_n) + \epsilon_n J(w_n)$ is strongly monotone and consequently the regularized problem in uniquely solvable.

We begin by the assumption that the sequence of the regularized solutions $\{w_n\}$ is bounded. By employing the reflexivity of the space, we can extract a weakly convergent subsequence. Using the same notation for the subsequences as well, let $\{w_n\}$ be the subsequence that converges weakly to some $\bar{u}$. We claim that $\bar{u}$ solves the original problem. In view of the Mosco convergence, the weak convergence ensures that $\bar{u} \in M_{\mathbb{P}}$ (see (2.8)). Let $z \in M_{\mathbb{P}}$ be arbitrary. By employing the Mosco convergence once again, we ensure that there exists a sequence $\{z_n\}$ such that $z_n \in M_{\mathbb{P}}^n$ and $z_n \to z$. By substituting this $z_n = v_n$ in (2.13), we obtain

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle \sigma_n(y)\, G(w_n) + H(w_n) + \epsilon_n J(w_n), z_n - w_n \rangle \, d\mathbb{P}(y) \geq$$
$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle b + \rho_n(y)\, c, z_n - w_n \rangle \, d\mathbb{P}(y)\,.$$

The Minty formulation of the above variational inequality reads:

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle \sigma_n(y)\, G(z_n) + H(z_n) + \epsilon_n J(z_n), z_n - w_n \rangle \, d\mathbb{P}(y) \geq$$
$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \langle b + \rho_n(y)\, c, z_n - w_n \rangle \, d\mathbb{P}(y)\,.$$

Since $\sigma_n \to \sigma$ in $L^\infty(\mathbb{R}^d, \mathbb{P})$, we obtain that $\sigma_n \to \sigma$ in $L^p(\mathbb{R}^d, \mathbb{P})$. Furthermore, since $z_n \to z$ in $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$, by passing to limit and using $\epsilon_n \to 0$, from the above inequality, we obtain

$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle \sigma(y)\, G(z) + H(z), z - \bar{u}(y) \rangle \, d\mathbb{P}(y) \geq$$
$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle b + \rho(y)\, c, z - \bar{u}(y) \rangle \, d\mathbb{P}(y).$$

Using the Minty formulation once again, we obtain

$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle \sigma(y)\, G(\bar{u}(y)) + H(\bar{u}(y)), z - \bar{u}(y) \rangle \, d\mathbb{P}(y) \geq$$
$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle b + \rho(y)\, c, z - \bar{u}(y) \rangle \, d\mathbb{P}(y).$$

Since $z \in M_{\mathbb{P}}$ is any arbitrary element, we obtain that $\bar{u}(y)$ solves the variational inequality. It remains to show that the sequence of regularized solutions $\{w_n\}$ remains bounded.

We now assume that the coercivity condition (2.14) holds. By substituting $v_n = \delta_n$, we obtain

$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle \sigma_n(y)\, G(w_n) + H(w_n) + \epsilon_n J(w_n), \delta_n - w_n \rangle \, d\mathbb{P}(y) \geq$$
$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle b + \rho_n(y)\, c, \delta_n - w_n \rangle \, d\mathbb{P}(y). \tag{2.15}$$

After a rearrangement of terms, we obtain

$$\begin{aligned}
&\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle \sigma_n(y)\, G(w_n) + H(w_n), w_n - \delta_n \rangle \, d\mathbb{P}(y) \\
&\leq \int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} \langle \epsilon_n J(w_n) - b - \rho_n(y)\, c, \delta_n - w_n \rangle \, d\mathbb{P}(y) \\
&\leq \epsilon_n \|w_n\| \|\delta_n\| + \|b + \rho_n(y)\, c\| \|w_n - \delta_n\| \\
&\leq \epsilon_n \|w_n\| \|\delta_n\| + \|w_n\| \|b + \rho_n(y)\, c\| \left[1 + \frac{\|\delta_n\|}{\|w_n\|} \right].
\end{aligned}$$

Therefore,

$$\frac{\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}_+^m} \langle \sigma_n(y)\, G(w_n) + H(w_n), w_n - \delta_n \rangle \, d\mathbb{P}(y)}{\|w_n\|} \leq \epsilon_n \|\delta_n\| + \|b + \rho_n(y)\, c\| \left[1 + \frac{\|\delta_n\|}{\|w_n\|} \right].$$

By passing to the limit $\|w_n\| \to \infty$, we obtain a contradiction to (2.14). Therefore $\{w_n\}$ must be bounded. This completes the proof.

Chapter 3

Iterative Methods for Stochastic Variational Inequalities

In this chapter, we describe several variants of extragradient methods for solving the stochastic variational inequality problem

$$(y - x^*)^\top f(x^*) \geq 0, \text{ for all } y \in K. \tag{3.1}$$

Among many methods for the problem, the simplest one is a projection method which iteratively updates the solution as

$$x^{k+1} = P_K(x^k - \alpha f(x^k))x$$

where α is a steplength and P_K is the orthogonal projection map onto K. Projection $P_K(x^{(k)} - \alpha f(x^{(k)}))$ is the solution of the quadratic programming problem

$$\min_{x \in K} \frac{1}{2} x^\top x - (x^k - \alpha f(x^k))^\top x.$$

Observe that x^* is the solution of (3.1) if and only if $x^* = P_K(x^* - \alpha f(x^*))$. It is known that convergence of the method depends on the contractive properties of the operator $x \to x - \alpha f(x)$. Strong monotonicity, Lipschitz continuity of f, and suitable choice of α depending on the monotonicity and Lipschitz constants guarantee convergence of the method.

3.1 Extragradient Methods

Extragradient methods which require double projections were proposed to relax the strong hypotheses of the projection method. Korpelevich [26] introduced the extragradient method in the context of saddle point problem studied through a variational inequality formulation. The methods require two projections per iteration and takes the following form:

$$\begin{aligned} \bar{x}^k &= P_K(x^k - \alpha f(x^k)) \\ x^{k+1} &= P_K(x^k - \alpha f(\bar{x}^k)). \end{aligned}$$

Convergence can be proven under the conditions that the solution set is nonempty, f is monotone and Lipschitz (with constant L) and $\alpha \in (0, 1/L)$. In the context of variational inequalities, these methods do not require the strong monotonicity of the map f. Extragradient methods are quite attractive for variational inequalities where strong monotonicity is attained through regularization, and these methods demand relaxed conditions on the regularization parameters. In cases of some application problems that we consider in this work, computing the projection is quite inexpensive due to simple constraints as these constraints do not add much additional computational cost.

Clearly, when the constant L is unknown, we may have difficulties choosing an appropriate steplength α. If α is too small, then the algorithm will converge slowly and if α is too big, then it may not converge at all.

3.2 Khobotov Extragradient Method

We will now consider extragradient methods where the steplength α is chosen adaptively. The adaptive steplength was first introduced in [25] to remove the constraint that f must be Lipschitz continuous. The adaptive algorithm is of the form:

$$\begin{aligned} \bar{x}^k &= P_K(x^k - \alpha_k f(x^k)) \\ x^{k+1} &= P_K(x^k - \alpha_k f(\bar{x}^k)). \end{aligned}$$

Better (speedier) convergence is usually achieved when α gets smaller between iterations, however, it is clear that we need to also control how the sequence of $\{\alpha_k\}$ shrinks.
We use the following reduction rule for α_k given in [25]:

$$\alpha_k > \beta \frac{x^k - \bar{x}^k}{f(x^k) - f(\bar{x}^k)},$$

where $\beta \in (0, 1)$. Results from [47] and [25] show that the choice of β as 0.8 or 0.9 performs best, an observation that is also supported by the results we obtained.
The Khobotov extragradient method has the following general form:

Algorithm: Khobotov Extragradient
Choose α_0, x^0, and $\beta \in (0, 1)$
While $\|x^{k+1} - x^k\| > \text{TOL}$
Step 1: Compute $f(x^k)$
Step 2: Compute $\bar{x}^k = P_K(x^k - \alpha_k f(x^k))$
Step 3: Compute $f(\bar{x}^k)$
If $f(\bar{x}^k) = 0$, Stop
Step 4: If $\alpha_k > \beta \frac{\|x^k - \bar{x}^k\|}{\|f(x^k) - f(\bar{x}^k)\|}$
then reduce α_k by a certain rule and go to Step 5
Step 5: Compute $x^{k+1} = P_K(x^k - \alpha_k f(\bar{x}^k))$
End.

3.2.1 Marcotte Choices for Steplength

Marcotte developed a new rule for reducing α_k along with closely related variants [28, 47]. The first Marcotte rule is based on the sequence $a_k =$

$\frac{1}{2}a_{k-1}$ and forces α_k to satisfy Step 5 of Khobotov's algorithm by additionally taking:

$$\alpha_k = \min\left\{\frac{\alpha_{k-1}}{2}, \frac{\|x^k - \bar{x}^k\|}{\sqrt{2}\|f(x^k) - f(\bar{x}^k)\|}\right\}.$$

Marcotte reduction rule still has the risk of choosing an initial α small enough so that α_k is never reduced, resulting in slow convergence. Ideally, α_k should then have the ability to increase if α_{k-1} is smaller than some optimal value. This leads to a modified version of Marcotte's rule where an initial α is selected using the rule

$$\alpha = \alpha_{k-1} + \gamma\left(\beta\frac{\|x^{k-1} - \bar{x}^{k-1}\|}{\|f(x^{k-1}) - f(\bar{x}^{k-1})\|} - \alpha_{k-1}\right)$$

where $\gamma \in (0,1)$.

The reduction rule in Step 5 of Khobotov's algorithm is then replaced with

$$\alpha_k = \max\left\{\widehat{\alpha}, \min\left\{\xi \cdot \alpha, \beta\frac{\|x^{k-1} - \bar{x}^{k-1}\|}{\|f(x^{k-1}) - f(\bar{x}^{k-1})\|}\right\}\right\}$$

where $\xi \in (0,1)$, and $\widehat{\alpha}$ is some lower limit for α_k (generally taken as no less than 10^{-4}).

3.3 Scaled Extragradient Method

We now consider a projection-contraction type extragradient method where the second projection is a more general operator. It was presented by Solodov and Tseng [45] and involves a symmetric positive definite scaling matrix M to accelerate convergence. The main steps read:

$$\begin{aligned} \bar{x}^k &= P_K(x^k - \alpha_k f(x^k)) \\ x^{k+1} &= x^k - \gamma M^{-1}(T_\alpha(x^k) - T_\alpha(P_K(\bar{x}^k)) \end{aligned}$$

where $\gamma \in \mathbb{R}^+$ and $T_\alpha = (I - \alpha f)$. Here, I is the identity matrix, and α is chosen such that T_α is strongly monotone.

Additional discussion of the scaling matrix is given in [47], however, in both [47] and [45], test problems take M equal to the identity matrix. In our numerical experiments, we consider the scaling matrix as the identity matrix.

Algorithm: Solodov-Tseng

Choose $x^0, \alpha_{-1}, \theta \in (0,2), \rho \in (0,1), \beta \in (0,1), M \in \mathbb{R}^{m\times m}$
Initialize: $\bar{x}^0 = 0, k = 0, rx = \text{ones}(m,1)$
While $\|rx\| > \text{TOL}$
  Step 1: if $\|rx\| < \text{TOL}$ then Stop
    else $\alpha = \alpha_{k-1},\ flag = 0$
  Step 2: if $f(x^k) = 0$ then Stop
  Step 3: While $\alpha(x^k - \bar{x}^k)^T(f(x^k) - f(\bar{x}^k)) > (1-\rho)\|x^k - \bar{x}^k\|^2$ or $flag = 0$
    If $flag \neq 0$ Then $\alpha = \alpha_{k-1}\beta$ endif
      update $\bar{x}^k = P_K(x^k - \alpha f(x^k))$, compute $f(\bar{x}^k)$
      $flag = flag + 1$
    endwhile
  Step 4: update $\alpha_k = \alpha$
  Step 5: compute $\gamma = \theta\rho\|x^k - \bar{x}^k\|^2 / \|M^{1/2}(x^k - \bar{x}^k - \alpha_k f(x^k) + \alpha_k f(\bar{x}^k))\|^2$
  Step 6: compute $x^{k+1} = x^k - \gamma M^{-1}(x^k - \bar{x}^k - \alpha_k f(x^k) + \alpha_k f(\bar{x}^k))$
  Step 7: $rx = x^{k+1} - A^k,\ k = k+1$ go to Step 3
End

The Solodov-Tseng method suggests a more general form for the advanced extragradient methods:

$$\begin{aligned} \bar{x}^k &= P_x(x^k - \alpha_k f(x^k)) \\ x^{k+1} &= P_x(x^k - \eta_k f(\bar{x}^k)), \end{aligned}$$

where α_k and η_k are chosen using different rules.

3.4 Solodov-Svaiter Method

This algorithm was proposed by Solodov and Svaiter in [44]. The idea is to compute the point $P_K(x^k - \mu_k f(x^k))$ and then search the line segment between x^k and $P_K(x^k - \mu_k f(x^k))$ for a poitn z^k such that the hyperplane

$$\{x \in \mathbb{R}^n | \langle f(z^k), x - z^k \rangle = 0\}$$

strictly separates x^k from the solution of the VI x^*. We will use a slightly modified version of this method (see Section 3.7 for the algorithm).

3.5 Goldstein-Type Methods

The classical Goldstein projection method presented in [27] is of the form:

$$x^{k+1} = P_K(x^k - \beta_k f(x^k))$$

The He-Goldstein method, an extragradient method that requires Lipschitz continuity and strong monotonicity of f is of the form:

$$\begin{aligned} \bar{x}^k &= P_K(f(x^k) - \beta_k x^k) \\ x^{k+1} &= x^k - \frac{1}{\beta_k}\{f(x^k) - \bar{x}^k\}. \end{aligned}$$

It can also be expressed:

$$\begin{aligned} r(x^k, \beta_k) &= \frac{1}{\beta_k}\{f(x^k) - P_K[f(x^k) - \beta_k x^k]\} \\ x^{k+1} &= x^k - r(x^k, \beta_k). \end{aligned}$$

A more general version of the above algorithm presented in [27], and it allows to control the second projection (i.e. choosing η_k).

Algorithm: Improved He-Goldstein

Initialize: choose $\beta_U > \beta_L > \frac{1}{(4\tau)}, \gamma \in (0,2), \epsilon > 0, x^0, \beta_0 \in [\beta_L, \beta_U], k = 0$

Step 1: Compute:

$r(x^k, \beta_k) = \frac{1}{\beta_k}\{f(x^k) - P_K[f(x^k) - \beta_k x^k]\}$

If $\|r(x^k, \beta_k)\| \leq \epsilon$ then Stop

Step 2: $x^{k+1} = x^k - \gamma\alpha_k r(x^k, \beta_k)$ where $\alpha_k := 1 - \frac{1}{4\beta_k\tau}$

Step 3: Update β_k

$\omega_k = \frac{\|f(x^{k+1} - f(x^k)\|}{\beta_k\|x^{k+1} - x^k\|}$

If $\omega_k < \frac{1}{2}$ Then $\beta_{k+1} = \max\{\beta_L, \frac{1}{2}\beta_k\}$

Else if $\omega_k > \frac{3}{2}$ Then $\beta_{k+1} = \min\{\beta_U, \frac{6}{5}\beta_k\}$

Step 4: $k = k + 1$, go to Step 1

3.6 Two-step Extragradient Method

Zykina and Melenchuk in [48] consider a three step projection method which they called a two-step extragradient method and investigated its various aspects in [49]. Numerical experiments with mixed variational problem for bilinear function given in [48] shows that the convergent of this method is faster compared to the standard extragradient method. The adaptive algorithm is of the form:

$$\begin{aligned} \bar{x}^k &= P_K(x^k - \alpha_k f(x^k)), \\ \widetilde{x}^k &= P_K(\bar{x}^k - \eta_k f(\bar{x}^k)), \\ x^{k+1} &= P_K(x^k - \xi_k f(\widetilde{x}^k)). \end{aligned}$$

3.7 Hyperplane Extragradient Method

In this method , η_k is chosen using the following rule from [47]:

$$\eta_k = \frac{\langle f(\bar{x}^k), x^k - \bar{x}^k\rangle}{\|f(\bar{x}^k)\|^2}$$

The idea here is that the hyperplane of all solutions x such that

$$\langle f(\bar{x}^k), \bar{x}^k - x\rangle = 0,$$

separates all the solutions onto one side of the hyperplane. Looking at the variational inequality, we know which side the solutions fall onto:

$$\langle f(x), \bar{x}^k - x\rangle \geq 0.$$

Consequently, if f is monotone, then we also have

$$\langle f(\bar{x}^k), \bar{x}^k - x\rangle \geq 0.$$

Thus if

$$\langle f(\bar{x}^k), \bar{x}^k - x^k\rangle < 0,$$

then we know that we have to look for the solution on the other side of the hyperplane.

This method, presented by Iusem, requires three constants, $\epsilon \in (0, 1)$ and $\tilde{\alpha} \geq \hat{\alpha} > 0$ such that the sequence α_k is computed such that

$$\langle f(\bar{x}^k), \bar{x}^k - x^k\rangle \leq 0,$$

when $\alpha_k \in [\hat{\alpha}, \tilde{\alpha}]$.

Algorithm: Hyperplane (Iusem)

Choose: $x^0, \epsilon, \hat{\alpha}, \tilde{\alpha}$

Initialize: $k = 0, rx = \text{ones}(m, 1)$

While $\|rx\| > \text{TOL}$

Step 1: Choose $\tilde{\alpha}_k$ using a finite bracketing procedure

Step 2: Compute $K^k = P_K(x^k - \tilde{\alpha}_k f(x^k))$ and $f(K^k)$

Step 3: If $f(K^k) = 0$ then Stop

Step 4: If $\|f(\tilde{x}^k) - f(x^k)\| \leq \frac{\|K^k - x^k\|^2}{2\tilde{\alpha}_k^2 \|f(x^k)\|}$

Then $\bar{x}^k = K^k$

Else find $\alpha_k \in (0, \tilde{\alpha}_k)$ such that

$\epsilon \frac{\|K^k - x^k\|^2}{2\tilde{\alpha}_k^2 \|f(x^k)\|} \leq \|f(P_K(x^k - \alpha_k f(x^k))) - f(x^k)\| \leq \frac{\|K^k - x^k\|^2}{2\tilde{\alpha}_k^2 \|f(x^k)\|}$

Step 5: Compute $\bar{x}^k = P_K(x^k - \alpha_k f(x^k))$

Step 6: If $f(\bar{x}^k) = 0$ then Stop

Step 7: Compute η_k

Step 8: Compute $x^{k+1} = P_K(x^k - \eta_k f(\bar{x}^k))$

Step 9: $rx = x^{k+1} - f^k, k = k + 1$; go to Step 3;

End

Chapter 4

Applications and Test Problems

In this chapter, we consider several applications of stochastic variational inequalities. We describe models of traffic equilibrium, market equilibrium, and environmental games. In each case we choose a test problem. The problems are discretized, and the regularization is incorporated whenever necessary. We compare the accuracy and efficiency of the extragradient methods described in Chapter 3 for all test problems.

4.1 Stochastic Traffic Equilibrium Problem

In this section, we apply the general theory of stochastic variational inequalities to network equilibrium problems. For the considered problem, we first present the exact solution. This procedure leads to an approximate solution that is very close to the exact one. The test problem chosen for the comparison of extragradient methods is taken from Chen et al. [6].
A common characteristic of many network problems is that they admit two different formulations based either on link variables or on path variables. These two formulations are related to each other through a linear transformation. In general, in the path variables approach, the strong monotonicity assumption is not reasonable. In order to overcome this problem, in [4] a

Mosco convergence result for the transformed sequence of sets was presented. This allows to work in the space of variables where strong monotonicity framework is natural. We circumvent the last of strong monotonicity by means of regularization in this work.

4.1.1 Introduction to Traffic Equilibrium Problem

A traffic network consists of a triple (N, A, W) where $N = \{N_1, \ldots, N_p\}$ with $p \in \mathbb{N}$, is the set of nodes, $A = (A_1, \ldots, A_n)$, $n \in \mathbb{N}$, represents the set of the directed arcs connecting pairs of nodes and $W = \{W_1, \ldots, W_m\} \subset N \times N$, $m \in \mathbb{N}$ is the set of the origin– destination (O, D) pairs. The flow on the arc A_i is denoted by f_i, $f = (f_1, \ldots, f_n)$. For simplicity, we consider arcs with infinite capacity. A set of consecutive arcs is called a path and assume that each (O_j, D_j) pair W_j is connected by r_j, $r_j \in \mathbb{N}$, paths whose set is denoted by P_j, $j = 1, \ldots, m$. All the paths in the network are grouped in a vector $(R_1, \ldots, R_k)$, $k \in \mathbb{N}$. The arc structure of the paths is described by using the arc–path incidence matrix $\Delta = \left(\delta_{ir}\right)_{\substack{i=1,\ldots,n \\ r=1,\ldots,k}}$, whose entries take the value

$$\delta_{ir} = \begin{cases} 1 & \text{if } A_i \in R_r \\ 0 & \text{if } A_i \notin R_r. \end{cases} \tag{4.1}$$

To each path R_r, there corresponds a flow F_r. The path flows are grouped in a vector $(F_1, \ldots, F_k)$ which is called the path (network) flow. The flow f_i on the arc A_i is equal to the sum of the flows on the paths which contain A_i, so that $f = \Delta F$. Let us now introduce the unit cost of going through A_i as a real function $t_i(f) \geq 0$ of the flows on the network, so that $t(f) = (t_1(f), \ldots, t_n(f))$ denotes the arc cost vector on the network. The meaning of the cost is usually that of travel time. Analogously, one can define a cost on the paths as $C(F) = (C_1(F), \ldots, C_k(F))$. Usually $C_r(F)$ is just the sum of the costs on the arcs which build that path:

$$C_r(F) = \sum_{i=1}^{n} \delta_{ir} t_i(f)$$

or in compact form,

$$C(F) = \Delta^T \, t(\Delta \, F) \,. \tag{4.2}$$

For each pair W_j there is a given traffic demand $D_j \geq 0$, so that $(D_1, \ldots, D_m)$ is the demand vector. Feasible flows are nonnegative flows which satisfy the demands, that is, which belong to the set

$$K = \big\{ F \in \mathbb{R}^k : \; F_r \geq 0 \text{ for any } r = 1, \ldots, k \text{ and } \Phi F = D \big\},$$

where Φ is the pair–path incidence matrix whose elements, say φ_{jr}, $j = 1, \ldots, m$, $r = 1, \ldots, k$, are

$$\varphi_{jr} = \begin{cases} 1 & \text{if the path } R_r \text{ connects the pair } W_j \\ 0 & \text{elsewhere.} \end{cases}$$

A path flow H is called an equilibrium flow or *Wardrop Equilibrium*, if and only if $H \in K$ and for any $W_j \in W$ and any $R_q, R_s \in P_j$ there holds

$$C_q(H) < C_s(H) \quad \Longrightarrow \quad H_s = 0. \tag{4.3}$$

This statement is equivalent (see [9] and [43]) to finding $H \in K$ such that

$$\langle C(H), F - H \rangle \geq 0, \quad \forall F \in K. \tag{4.4}$$

Roughly speaking, the meaning of Wardrop Equilibrium is that the road users choose minimum cost paths. Let us note that condition (4.3) implies that all the used paths of a given O-D pair have the same cost.

Although the Wardrop equilibrium principle is expressed in the path variables, it is clear that the "physical" (and measured) quantities are expressed in the link variables. Moreover, the strong monotonicity hypothesis on $c(f)$ is quite common, but as noticed, for instance, in [2] this does not imply the strong monotonicity of $C(F)$ in (4.2), unless the matrix $\Delta^T \Delta$ is nonsingular. Although one can give a procedure for buildings networks preserving the strong monotonicity property (see for instance [36]), the condition fails for a generic network, even for a very simple one as we shall illustrate in the sequel. Thus, it is useful to consider the following variational inequality problem:

$$h \in \Delta K \text{ and } \langle t(h), f - h\rangle \geq 0 \quad \forall f \in \Delta K. \tag{4.5}$$

If t is strongly monotone, one can prove that for each solution H of (4.4), $C(H)$ is constant. In other words, all possibly nonunique solutions of (4.4) share the same cost. From an algorithmic point of view it is worth noting that one advantage in working in the path variables is the simplicity of the corresponding convex set but the price to be paid is that the number of paths grows exponentially with the size of the network.
The random version of (4.4) and (4.5) reads: Find $H(\omega) \in K(\omega)$ such that

$$\langle C(\omega, H(\omega)), F(\omega) - H(\omega)\rangle \geq 0, \quad \forall F(\omega) \in K(\omega), \tag{4.6}$$

where, for any $\omega \in \Omega$,

$$K(\omega) = \left\{F(\omega) \in \mathbb{R}^k : \ F_r \geq 0 \text{ for any } r = 1, \dots, k \text{ and } \Phi F = D(\omega)\right\},$$

Moreover, the random variational inequality in the link-flow variables reads: Find $h(\omega) \in \Delta K(\omega)$ such that

$$\langle t(\omega, h(\omega)), f(\omega) - h(\omega)\rangle \geq 0, \quad \forall f(\omega) \in \Delta K(\omega). \tag{4.7}$$

Furthermore, (4.6) is equivalent to the random Wardrop principle: for any $\omega \in \Omega$ for any $H(\omega) \in K(\omega)$, and for any $W_j \in W$, $R_q, R_s \in P_j$, we have

$$C_q(\omega, H(\omega)) < C_s(\omega, H(\omega)) \quad \Longrightarrow \quad H_s(\omega) = 0.$$

In order to use our approximation scheme, we require the assumption that the deterministic and random variables are separated. However this assumption is very natural in many applications where the random perturbation is treated as a *modulation* of a deterministic process. Under the above mentioned assumptions, (4.6) assumes the particular form:

$$S(\omega)\langle A(H(\omega)), F - H(\omega)\rangle \geq R(\omega)\langle b, F - H(\omega)\rangle, \ \forall F \in K(\omega) \tag{4.8}$$

In equation (4.8), both the left hand side and the right hand side be replaced with any (finite) linear combination of monotone and separable terms with each term satisfying the hypothesis of the previous sections:

$$\sum_i S_i(\omega)\langle A_i^T(H(\omega)), F - H(\omega)\rangle \geq \sum_j R_j(\omega)\langle b_j, F - H(\omega), \ \forall F \in K(\omega) \tag{4.9}$$

Therefore, in (4.8) $R(\omega), S(\omega)$ can be replaced by a random vector and a random matrix, respectively. As a consequence, in the traffic network, we could consider the case where the random perturbation has a different weight for each path.

4.1.2 Numerical Results

In this test problem, we consider the so-called Dafermos' network consisting of one $O - D$ pair and 5 links (see Figure 4.1).

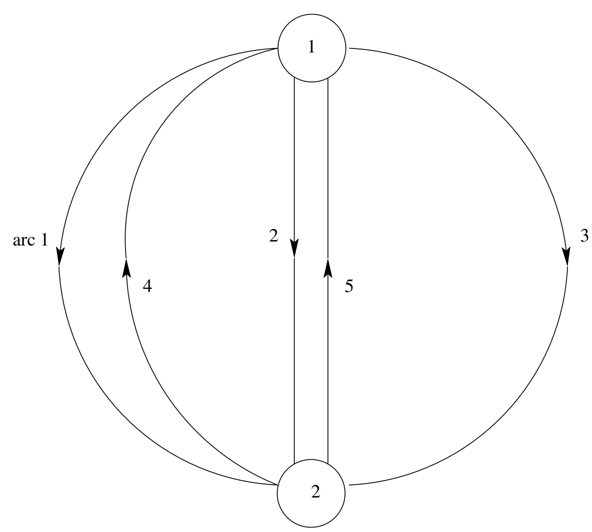

Figure 4.1: Dafermos' network

The travel cost function in this case is given by $t = Af + b$ where

$$A = \begin{pmatrix} 10 & 0 & 0 & 5 & 0 \\ 0 & 15 & 0 & 0 & 5 \\ 0 & 0 & 20 & 0 & 0 \\ 2 & 0 & 0 & 20 & 0 \\ 0 & 1 & 0 & 0 & 25 \end{pmatrix}, \quad b = \begin{pmatrix} 1000 \\ 950 \\ 3000 \\ 1000 \\ 1300 \end{pmatrix}.$$

Here the traffic demand vector is $(210, 120)$ which means

$$\begin{aligned} f_1 + f_2 + f_3 &= 210 \\ f_4 + f_5 &= 120. \end{aligned}$$

The above deterministic problem has a unique solution $f = (120, 90, 0, 70, 50)^T$. We consider the stochastic version of this problem considered in [6]. For this, we introduce two random variables ω_1 and ω_2 given by

$$\begin{aligned} \omega_1 &\sim 80 \leq N(210, 1200) \leq 340 \\ \omega_2 &\sim U(60, 180). \end{aligned}$$

This is, ω_1 and ω_2 follow truncated normal and uniform distributions, respectively. Travel demands are given by

$$\begin{aligned} f_1 + f_2 + f3 &= \omega_1 \\ f_4 + f_5 &= \omega_2. \end{aligned}$$

We define two cost coefficients (related to the fluctuation of travel demand) by

$$\begin{aligned} c_1(\omega) &= \frac{\omega_1 + \omega_2}{330} - 1 \\ c_2(\omega) &= \frac{\omega_1}{210} - 1. \end{aligned}$$

Define a new cost function $A(\omega)f + b$ with $A(\omega) = A + \tilde{A}(\omega)$ where

$$\tilde{A}(\omega) = \begin{pmatrix} 3c_1(\omega) & 0 & 0 & 0.5c_1(\omega) & \\ 0 & 4c_1(\omega) & 0 & 0 & c_1(\omega) \\ 0 & 0 & 0.5c_2(\omega) & 0 & 0 \\ 0.2c_1(\omega) & 0 & 0 & c_1(\omega) & 0 \\ 0 & 0.1c_1(\omega) & 0 & 0 & c_1(\omega) \end{pmatrix}.$$

We discretize the domain $[80, 340] \times [60, 180]$ using N_1 subintervals for ω_1, N_2 for ω_2. For each pair $(\omega_{1,i}, \omega_{2,j})$ a deterministic variational inequality is solved by using extragradient methods we consider. We evaluate the mean value of route flow using probability distribution functions of random variables ω_1 and ω_2. Table 4.3 shows the mean values of the route flows for four of the methods. As we see that all methods result very close results, and with

Methods	h_1	h_2	h_3	h_4	h_5
Marcotte	109.5881	82.38575	0.2099183	64.41517	45.78488
Marcotte1	109.5881	82.38577	0.20992	64.41524	45.78481
Marcotte2	109.5881	82.38575	0.2099193	64.41528	45.78477
Solodov-Tseng	109.5926	82.38117	0.210024	64.41455	45.78551

Table 4.1: Route flow solutions for $N_1 = 30, N_2 = 30$

Methods	$Var(h_1)$	$Var(h_2)$	$Var(h_3)$	$Var(h_4)$	$Var(h_5)$
Marcotte	1250.983	642.5301	1.7209138	636.23205	369.58061
Marcotte1	1250.9848	642.52891	1.7209352	636.22985	369.58247
Marcotte2	1250.9852	642.52868	1.7209265	636.22847	369.58361
Solodov-Tseng	1250.994	642.5123	1.7218559	636.22748	369.58339

Table 4.2: Variances for route flow for $N_1 = 30$, $N_2 = 30$

more discretization points (larger N_1 and N_2) the solutions get closer to the exact solution of the problem $(119.835, 89.869, 0.212, 69.992.50.008)$ given by

Methods	h_1	h_2	h_3	h_4	h_5
Marcotte	113.6137	85.32718	0.2459893	66.62013	47.45188
Marcotte 1	113.6137	85.3272	0.2459913	66.62019	47.45182
Marcotte 2	113.6137	85.32718	0.2459903	66.62024	47.45177
Solodov-Tseng	113.6181	85.3226	0.2461074	66.6195	47.45253

Table 4.3: Route flow solutions for $N_1 = 50, N_2 = 50$

Methods	$Var(h_1)$	$Var(h_2)$	$Var(h_3)$	$Var(h_4)$	$Var(h_5)$
Marcotte	949.2067	470.1265	2.041574	537.9744	321.7377
Marcotte1	949.2089	470.1251	2.041597	537.9718	321.7398
Marcotte2	949.2093	470.1249	2.041588	537.9702	321.7411
Solodov-Tseng	949.1832	470.1341	2.042672	537.9719	321.7389

Table 4.4: Variances for route flow for $N_1 = 50$, $N_2 = 50$

Methods	h_1	h_2	h_3	h_4	h_5
Marcotte	116.694	87.57599	0.2768794	68.29677	48.72123
Marcotte 1	116.694	87.576	0.2768815	68.29684	48.72116
Marcotte 2	116.694	87.57599	0.2768803	68.29688	48.72112
Solodov-Tseng	116.6984	87.57137	0.2770134	68.29614	48.72188

Table 4.5: Route flow solutions for $N_1 = 100, N_2 = 100$

Jadamba et al. in [23] (see Table 4.5). Table 4.6 shows variances for route flows. Comparing the CPU times, we find that for this particular example Solodov-Tseng method performs fastest given the same stopping criteria.

Methods	$Var(h_1)$	$Var(h_2)$	$Var(h_3)$	$Var(h_4)$	$Var(h_5)$
Marcotte	700.225	328.4678	2.320375	457.0569	281.8727
Marcotte 1	700.2273	328.4663	2.320401	457.0541	281.875
Marcotte 2	700.2276	328.4661	2.32039	457.0522	281.8764
Solodov-Tseng	700.1743	328.4948	2.321614	457.0557	281.8729

Table 4.6: Variances for route flow for $N_1 = 100$, $N_2 = 100$

4.2 Oligopolistic Market Equilibrium

We consider here the model in which m players are the producers of the same commodity. The quantity produced by firm i is denoted by q_i so that $q \in \mathbb{R}^m$ denotes the global production vector. Let (Ω, P) be a probability space and for every $i \in \{1, \ldots m,\}$ consider functions $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ and $p : \Omega \times \mathbb{R}^m \to \mathbb{R}$.
More precisely, for almost every $\omega \in \Omega$, (i.e. P-almost surely), $f_i(\omega, q_i)$ represents the cost of producing the commodity by firm i, and is assumed to be nonnegative, increasing, concave and C^1, while $p(\omega, q_1 + \ldots + q_m)$ represents the demand price associated with the commodity. For almost every $\omega \in \Omega$, p is assumed nonnegative, increasing, convex w.r.t. q_i and C^1. We also assume that all these functions are random variables w.r.t. ω, i.e. they are measurable with respect to the probability measure P on Ω. In this way, we have introduced the possibility that both the production cost and the demand price are affected by a certain degree of uncertainty, or randomness. Thus, the welfare (or utility) function of player i is given by:

$$w_i(\omega, q_1, \ldots, q_m) = p(\omega, q_1 + \ldots + q_n)q_i - f_i(\omega, q_i). \tag{4.10}$$

Although many authors assume no bounds on the production, in a more realistic model the production capability is bounded from above and we allow also for the upper bound being a random variable: $0 \leq q_i \leq \overline{q}_i(\omega)$,
Thus, the specific Nash equilibrium problem associated with this model takes

the following form. For a.e. $\omega \in \Omega$, find $q^*(\omega) = (q_1^*(\omega), \dots, q_m^*(\omega))$:

$$w_i(\omega, q^*) = \max_{0 \le q_i \le \overline{q}_i(\omega)} -f_i(\omega, q_i) + p(\omega, q_i + \sum_{j \neq i} q_j^*) q_i, \ \forall i \in \{1, \dots, m\}. \quad (4.11)$$

In order to write the equivalent variational inequality, consider the closed and convex subset of $\mathbb{R}^m$:

$$K(\omega) = \{(q_1, \dots, q_m) : 0 \le q_i \le \overline{q}_i(\omega), \forall i\}$$

for each ω and define the functions

$$F_i(\omega, q) := \frac{\partial f_i(\omega, q_i)}{\partial q_i} - \frac{\partial p(\omega, \sum_{j=1}^m q_j)}{\partial q_i} q_i - p(\omega, \sum_{j=1}^m q_j). \quad (4.12)$$

The Nash problem is then equivalent to the following variational inequality: for a.e. $\omega \in \Omega$, find $q^*(\omega) \in K(\omega)$ such that

$$\sum_{j=1}^m F_j[\omega, q^*(\omega)](q_j - q_j^*(\omega)) \ge 0, \ \forall q \in K(\omega). \quad (4.13)$$

Since $F(\omega, \cdot)$ is continuous, and $K(\omega)$ is convex and compact, problem (4.13) is solvable for almost every $\omega \in \Omega$, due to the Stampacchia's theorem. Moreover, we assume that $F(\omega, \cdot)$ is monotone, i.e.:

$$\sum_{i=1}^m (F_i(\omega, q) - F_i(\omega, q'))(q_i - q_i') \ge 0 \ \ \forall \omega \in \Omega, \forall q, q' \in \mathbb{R}^m.$$

F is said to be strictly monotone if the equality holds only for $q = q'$ and in this case (4.13) has a unique solution. In the sequel the following uniform strong monotonicity property will be useful:

$$\exists \alpha > 0 : \sum_{i=1}^m (F_i(\omega, q) - F_i(\omega, q'))(q_i - q_i') \ge \alpha \|q - q'\|^2 \ \ \forall \omega \in \Omega, \forall q, q' \in \mathbb{R}^m. \quad (4.14)$$

Although the uniform strong monotonicity property is quite demanding, nonetheless it is verified by some classes of utility functions frequently used in the literature (see e.g. sect. 4.2.2).

4.2.1 The Lebesgue space formulation

Since we are interested in computing statistical quantities associated with the solution $q^*(\omega)$, in particular its mean value, we introduce a Lebesgue space formulation of problems (4.11) and (4.13). In view of the numerical approximation of the solution, we also assume that the random and the deterministic part of the operator are separated. Thus, let:

$$w_i(\omega, q) = p(\sum_{j=1}^{m} q_j) + \beta(\omega) - \alpha(\omega) f_i(q_i) - g_i(q_i)$$

where α, β are real random variables, with $0 < \underline{\alpha} \leq \alpha(\omega) \leq \overline{\alpha}$, and the part of the cost which is affected by uncertainty is denoted now by f_i. As a consequence, the operator F takes the form:

$$F_i(\omega, q) = \alpha(\omega)\frac{\partial f_i(q_i)}{\partial q_i} + \frac{\partial g_i(q_i)}{\partial q_i} - p(\sum_{j=1}^{m} q_j) - \beta(\omega) - \frac{\partial p(\sum_{j=1}^{m} q_j)}{\partial q_i} q_i.$$

The separation of variables allows us to use the approximation procedure developed in [17]. Furthermore, we assume that F is uniformly strongly monotone according to (4.14) and satisfies the following growth condition:

$$|F_i(\omega, q)| \leq c(1 + |q|), \forall q \in \mathbb{R}^m, \forall \omega \in \Omega, \forall i \tag{4.15}$$

and $w_i(\omega, 0) \in L^1(\Omega)$. Moreover, we shall assume that $\alpha \in L^\infty(\Omega)$, while $\beta, \overline{q}_i \in L^2(\Omega)$. Under these assumptions the following Nash equilibrium problem can be derived (see [22] or [13] for a similar derivation which can be easily extended to our functional setting):

Find $u^* \in L^2(\Omega, P, \mathbb{R}^m)$ such that, $\forall i$

$$\int_\Omega w_i(\omega, u^*(\omega)) dP_\omega = \max_{0 \leq u_i \leq \overline{q}_i} \int_\Omega w_i(\omega, (u_i(\omega), u^*_{-i}(\omega)) dP_\omega, \tag{4.16}$$

where we used the notation: $(u_i, u^*_{-i}) := (u^*_1, \ldots, u^*_{i-1}, u_i, u^*_{i+1}, \ldots, u^*_m)$. Then, we define a closed and convex set K_P by

$$K_P = \{u \in L^2(\Omega, P, \mathbb{R}^m) : 0 \leq u_i(\omega) \leq \overline{q}_i(\omega), P - a.s., \forall i\}$$

and consider the variational inequality formulation of (4.16): Find $u^* \in K_P$ such that

$$\int_\Omega \sum_{j=1}^{m} F_j(\omega, u^*(\omega))(u_j(\omega) - u^*(\omega)) \geq 0, \forall u \in K_P. \tag{4.17}$$

4.2.2 A class of utility functions

In this subsection, we consider a random version of a class of utility functions widely used in the literature (see e.g. [34], chap. 6) and show that these functions satisfy the theoretical requirements stated previously.
Thus, let

$$f_i(\omega, q_i) = a(\omega)\, a_i q_i^2 + b_i q_i + c_i$$

$$p(\omega, \sum_{i=1}^{m} q_i) = -d \sum_{i=1}^{m} q_i + e(\omega)$$

where $0 < \underline{a} \leq a(\omega) \leq \overline{a}$, $a \in L^\infty(\Omega)$, $e \in L^2(\Omega)$, and a_i, b_i, d, c_i are positive real numbers. Thus, $w_i(\omega, q) = -[a(\omega)\, a_i q_i^2 + b_i q_i + c_i] - d \sum_{i=1}^{m} q_i + e(\omega)\,,$ and

$$F_i(\omega, q) = 2a(\omega)\, a_i q_i + b_i + d \sum_{i=1}^{m} q_i - e(\omega) = [2a(\omega)\, a_i + 2d] q_i + d \sum_{j \neq i} q_j + b_i - e(\omega) \tag{4.18}$$

For each ω the operator F consists of a linear part and a constant vector.

4.2.3 Numerical Results

As an examplem we take the random version of a classical oligopoly problem presented in [34] where 3 producers are involved in the production of a homogeneous commodity. In the nonrandom version of the problem, the cost f_i of producing the commodity by firm i, and the demand function p are given by

$$\begin{aligned} f_1(q_1) &= q_1^2 + q_1 + 1 \\ f_2(q_2) &= 0.5q_2^2 + 4q_2 + 2 \\ f_3(q_3) &= q_3^2 + 0.5q_3 + 5 \\ p\left(\sum_{i=1}^{3}\right) &= -\sum_{i=1}^{3} q_i + 5. \end{aligned}$$

Solution of the above problem $(q_1, q_2, q_3) = (23/30, 0, 14/15)$ is given in [34]. We consider a random version of the above problem where the cost f_i and demand p are given by

$$\begin{aligned} f_1(\omega, q_1) &= a(\omega)q_1^2 + q_1 + 1 \\ f_2(\omega, q_2) &= 0.5a(\omega)q_2^2 + 4q_2 + 2 \\ f_3(\omega, q_3) &= a(\omega)q_3^2 + 0.5q_3 + 5 \\ p\left(\omega, \sum_{i=1}^{3}\right) &= -\sum_{i=1}^{3} q_i + e(\omega) \end{aligned}$$

where $a(\omega)$ and $e(\omega)$ are random parameters that follow truncated normal distributions:

$$\begin{aligned} a &\sim 0.5 \leq N(1, 0.25) \leq 1.5 \\ e &\sim 4.5 \leq N(5, 0.25) \leq 5.5. \end{aligned}$$

We use the approximation procedure described in Chapter 2 to evaluate mean value of q. First, we choose a discretization of the parameter domain $[0.5, 1.5] \times [4.5, 5.5]$ using $N_1 \times N_2$ points and solve the problem for each pair $(a(i), e(j))$ using the extragradient methods described in Section 3. Then, we evaluate the mean value of q by using appropriate probability distribution functions. Approximate mean values of q_1, q_2 and q_3 are shown in the Tables 4.9 and 4.11.

Variances are summarized in Table 4.12, and a comparison of the CPU times is presented in Figure 4.2.

Methods	q_1	q_2	q_3
Marcotte	0.75843	$5.8345e-08$	0.92822
Marcotte1	0.75843	$2.8107e-08$	0.92822
Marcotte2	0.75837	$3.0205e-08$	0.92816
Solodov-Svaiter	0.75731	$3.2252e-08$	0.92711
Solodov-Tseng	0.75808	$-3.6382e-05$	0.92787

Table 4.7: Mean values of the production vector for $N_1 = 30, N_2 = 30$

Methods	$Var(q_1)$	$Var(q_2)$	$Var(q_3)$
Marcotte	0.021754	$3.3229e-14$	0.03428
Marcotte1	0.021754	$7.9103e-15$	0.03428
Marcotte2	0.021751	$9.5282e-15$	0.034276
Solodov-Svaiter	0.021764	$1.9773e-15$	0.034297
Solodov-Tseng	0.021843	$6.2711e-10$	0.034405

Table 4.8: Variances of the production vector for $N_1 = 30, N_2 = 30$

Methods	q_1	q_2	q_3
Marcotte	0.7648	$5.5928e{-}08$	0.93569
Marcotte 1	0.76479	$2.8404e{-}08$	0.93569
Marcotte 2	0.76473	$2.8992e{-}08$	0.93563
Solodov-Svaiter	0.76273	$3.6563e{-}08$	0.93363
Solodov-Tseng	0.76441	$3.131e{-}05$	0.93531

Table 4.9: Mean values of the production vector for $N_1 = 50, N_2 = 50$

Methods	$Var(q_1)$	$Var(q_2)$	$Var(q_3)$
Marcotte	0.01656	$3.1049e-14$	0.026447
Marcotte1	0.01656	$7.0627e-15$	0.026447
Marcotte2	0.016558	$7.241e-15$	0.026445
Solodov-Svaiter	0.016546	$1.2571e-14$	0.026435
Solodov-Tseng	0.016642	$3.4075e-10$	0.026561

Table 4.10: Variances of the production vector for $N_1 = 50, N_2 = 50$

Methods	q_1	q_2	q_3
Marcotte	0.76935	$5.8068e{-}08$	0.94103
Marcotte 1	0.76935	$2.7505e{-}08$	0.94103
Marcotte 2	0.76928	$2.9036e{-}08$	0.94096
Solodov-Svaiter	0.76791	$4.5022e{-}08$	0.93959
Solodov-Tseng	0.76902	$2.8959e{-}05$	0.9407

Table 4.11: Mean values of the production vector for $N_1 = 100, N_2 = 100$

Methods	$Var(q_1)$	$Var(q_2)$	$Var(q_3)$
Marcotte	0.012786	$3.4821e{-}14$	0.020762
Marcotte 1	0.012787	$6.0461e{-}15$	0.020762
Marcotte 2	0.012787	$7.1256e{-}15$	0.020763
Solodov-Svaiter	0.012769	$4.0207e{-}15$	0.020738
Solodov-Tseng	0.012869	$3.7517e{-}09$	0.020876

Table 4.12: Variances of the production vector for $N_1 = 100, N_2 = 100$

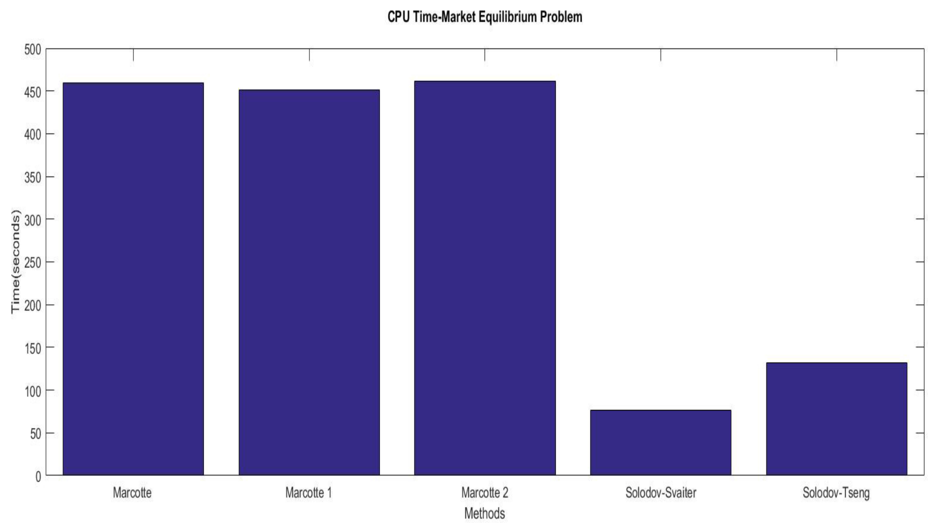

Figure 4.2: Comparison of CPU times for the market equilibrium problem

4.3 Environmental Games

In this section, we will introduce an application known as environmental games. In the fundamental paper [3], Breton et al. formulated a two players game to describe the joint implementation mechanism of the Kyoto Protocol, while in [46], Tidbal and Zaccour compared various models of an environmental problem using a large class of revenue and damage cost functions. The models proposed in [46] have been reformulated and extended in the paper [22] by using the variational inequalities theory. We will discuss stochastic noncooperative scenario where each player optimizes his/her welfare (defined as the difference between the revenue resulting form the production and the damage cost due to the corresponding pollution) under their individual environmental constraints. In this case, the players interact only through the damage cost which is a function of the total polluting emission and this scenario leads to a (stochastic) Nash equilibrium problem which, in turn, is formulated as a stochastic variational inequality in Lebesgue space. We will also describe the stochastic variational inequality that describes the cooperative scenario where the players agree to optimize the sum of their individual welfares under a joint environmental constraint. Next, we discuss the so-called *umbrella scenario* where the players act in a selfish manner but under a common environmental constraint; this scenario leads to a (stochastic) generalized Nash equilibrium problem (GNEP).

4.3.1 The stochastic noncooperative scenario

In this scenario, each player is a subject who produces, pollutes, and aims to maximize his/her welfare function under some environmental constraints. The welfare function is defined as the difference between the revenue resulting from the production and the damage cost due to pollution. In the model we consider the welfare function is not deterministic but can be affected by some random variables. We assume that pollution is proportional to the industrial output so that the revenue of player i, $i \in \{1, \dots, n\}$, can be expressed as a function of its polluting emission e_i. Let (Ω, P) be a probability space and for every $i \in \{1 \dots n\}$ consider functions $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ and $d_i : \Omega \times \mathbb{R}^n \to \mathbb{R}$. More precisely, for almost every $\omega \in \Omega$, (i.e. P-almost surely) $f_i(\omega, e_i)$ represents the revenue function of player i, which is assumed to be nonnegative, increasing, concave and C^1 while the cost of the environmental damage depends on the total emission and is denoted by $d_i(\omega, e_1 + \ldots + e_n)$. For almost every $\omega \in \Omega$, d_i are assumed to be nonnegative, increasing, convex w.r.t. e_i and C^1. We assume also that all these functions are random variables w.r.t. ω, i.e. they are measurable with respect to the probability measure P on Ω. Thus, the welfare function of player i is given by:

$$w_i(\omega, e_1, \ldots, e_n) = f_i(\omega, e_i) - d_i(\omega, e_1 + \ldots + e_n). \tag{4.19}$$

In the noncooperative scenario, each player has to satisfy the random environmental constraint: $0 \leq e_i \leq \overline{e}_i(\omega)$, while maximizing his/her welfare for every action of the other players. This situation naturally leads to the following Nash equilibrium problem:
For a.e. $\omega \in \Omega$, find $e^N(\omega) = \big(e_1^N(\omega), \ldots, e_n^N(\omega)\big)$:

$$w_i(\omega, e^N) = \max_{0 \leq e_i \leq \overline{e}_i(\omega)} f_i(\omega, e_i) - d_i(\omega, e_i + \sum_{j \neq i} e_j^N), \ \forall i \in \{1, \ldots, n\}. \tag{4.20}$$

This problem is equivalent to a variational inequality. Thus, for each ω consider the closed and convex subset of $\mathbb{R}^n$: $K^N(\omega) = \{(e_1, \ldots, e_n) : 0 \leq e_i \leq \overline{e}_i(\omega), \forall i\}$ and define the functions

$$F_i(\omega, e) := -\frac{\partial f_i(\omega, e_i)}{\partial e_i} + \frac{\partial d_i(\omega, \sum_{j=1}^n e_j)}{\partial e_i} \equiv -f'(\omega, e_i) + d_i'(\omega, \sum_{j=1}^n e_j). \tag{4.21}$$

The Nash problem is then equivalent to the following variational inequality: for a.e. $\omega \in \Omega$, find $e^N(\omega) \in K^N(\omega)$ such that

$$\sum_{j=1}^n F_j[\omega, e^N(\omega)](e_j - e_j^N(\omega)) \geq 0, \ \forall e \in K^N(\omega). \tag{4.22}$$

Since $F(\omega, \cdot)$ is continuous, and $K^N(\omega)$ is convex and compact, problem (4.22) is solvable, for almost every $\omega \in \Omega$. Moreover we assume that $F(\omega, \cdot)$ is monotone, i.e.: $\sum_{i=1}^n (F_i(\omega, e) - F_i(\omega, e'))(e_i - e'_i) \geq 0 \ \ \forall \omega \in \Omega, \forall e, e' \in \mathbb{R}^n$. F is said to be strictly monotone if the equality holds only for $e = e'$ and in this case (4.22) has a unique solution. We mention here following useful uniform strong monotonicity property:

$$\exists \alpha > 0 : \sum_{i=1}^n (F_i(\omega, e) - F_i(\omega, e'))(e_i - e'_i) \geq \alpha \|e - e'\|^2 \ \ \forall \omega \in \Omega, \forall e, e' \in \mathbb{R}^n. \tag{4.23}$$

As it was the case for earlier problems, we assume that the random and the deterministic part of the operator can be separated: $w_i(\omega, e) = a_i(\omega) f(e_i) - b_i(\omega) d_i(e_1 + \ldots e_n)$, where a_i, b_i are real valued random variables such that $0 < \underline{A}_i \leq a_i(\omega) \leq \overline{A}_i,\ 0 < \underline{B}_i \leq b_i(\omega) \leq \overline{B}_i$. The separation of variables now allows us to use the approximation procedure developed in [17]. We are interested in computing statistical quantities associated with the solution $e^N(\omega)$, in particular its mean value. For this purpose, we introduce a Lebesgue space formulation of problems (4.20) and (4.22). We assume that F is uniformly strongly monotone according to (4.23) and satisfies the following condition:

$$|F_i(\omega, e)| \leq c(1 + |e|), \forall e \in \mathbb{R}^n, \forall \omega \in \Omega, \ \forall i \tag{4.24}$$

and that $w_i(\omega, 0) \in L^1(\Omega)$. Moreover, we shall assume that $a_i, b_i \in L^\infty(\Omega)$, while $\overline{e}_i \in L^2(\Omega)$. Under these assumptions the following Nash equilibrium problem can be derived (see [22] for a similar derivation which can be easily extended to this functional setting):
Find $u^N \in L^2(\Omega, P, \mathbb{R}^n)$ such that, $\forall i$

$$\int_\Omega w_i(\omega, u^N(\omega)) dP_\omega = \max_{0 \leq u_i \leq \overline{e}_i} \int_\Omega w_i(\omega, (u_i(\omega), u^N_{-i}(\omega))) dP_\omega, \tag{4.25}$$

where we used the notation: $(u_i, u^N_{-i}) := (u^N_1, \ldots, u^N_{i-1}, u_i, u^N_{i+1}, \ldots, u^N_n)$. Then, we define the closed and convex set: $K^N_P = \{u \in L^2(\Omega, P, \mathbb{R}^n) : 0 \leq u_i(\omega) \leq \overline{e}_i(\omega), P - a.s., \forall i\}$ and consider the variational inequality formulation of (4.25): Find $u^N \in K_P^N$ such that:

$$\int_\Omega \sum_{j=1}^{n} F_j(\omega, u^N(\omega))(u_j(\omega) - u^N(\omega)) \geq 0, \forall u \in K_P^N. \qquad (4.26)$$

Random noncooperative scenario will be studied through (4.26)

4.3.2 The stochastic cooperative scenario

In the cooperative scenario, all players agree to optimize the sum of their individual welfares under a joint environmental constraint. Thus, for each $\omega \in \Omega$, we need to solve the optimization problem

$$\max_{e \in K^C(\omega)} \sum_{i=1}^{n} a_i(\omega) f(e_i) - b_i(\omega) d_i(e_1 + \ldots + e_n) \qquad (4.27)$$

where $K^C(\omega) = \{e \in \mathbb{R}^n : e_i \geq 0, \sum_{i=1}^n e_i \leq \sum_{i=1}^n \overline{e}_i(\omega) = \overline{e}(\omega)\}$.
Under the strict convexity hypothesis, the problem has a unique solution $e^C(\omega)$ for each $\omega \in \Omega$. Moreover, (4.27) is equivalent to the variational inequality associated to the convex set K^C and to the operator $F^C : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F_i^C(\omega, e) = -a_i(\omega)\frac{\partial f_i}{\partial e_i} + \frac{\partial \sum_i b_i(\omega) d_i(e_1 + \ldots + e_n)}{\partial e_i}.$$

Find $e^C(\omega) \in K^C(\omega)$: $\sum_{j=1}^n F_j^C[\omega, e^C(\omega)](e_j - e_j^C(\omega)) \geq 0$ for all $e \in K^C(\omega)$.
Analogously to the previous case, we can define the closed and convex set K_P^C by

$$K_P^C = \{u \in L^2(\Omega, P, \mathbb{R}^n) : 0 \leq u_i(\omega), \sum_{i=1}^n u_i(\omega) \leq \sum_{i=1}^n \overline{e}_i(\omega) = \overline{e}(\omega),\ P\text{−}a.s.\}$$

and derive the Lebesgue space optimization formulation for the cooperative scenario:

$$\int_\Omega \sum_{i=1}^{n} w_i(\omega, (u^C(\omega)))dP_\omega = \max_{u \in K_P^C} \int_\Omega \sum_{i=1}^{n} w_i(\omega, (u(\omega)))dP_\omega. \qquad (4.28)$$

The equivalent variational inequality in this case is to find $u^N \in K_P^C$ such that

$$\int_\Omega \sum_{j=1}^{n} F_j^C(\omega, u^C(\omega))(v_j(\omega) - u_j^C(\omega)) \geq 0, \forall v \in K_P^C. \tag{4.29}$$

4.3.3 The stochastic umbrella scenario

In the *umbrella* scenario, each player acts in a selfish manner and aims to optimize his/her individual welfare, for every choice of the rival's strategies. However, all the players agree to satisfy a common environmental constraint. Hence, in this model we are looking for a generalized Nash equilibrium, i.e. for a vector $e^R(\omega) = \big(e_1^R(\omega), \ldots, e_n^R(\omega)\big)$:

$$w_i(\omega, e^R) = \max_{e_i} a_i(\omega) f_i(e_i) - b_i(\omega) d_i(e_i + \sum_{j \neq i} e_j^R), \ \forall \omega \in \Omega \tag{4.30}$$

subject to $e_i + \sum_{j \neq i} e_j^R \leq \overline{e}(\omega)$, $e_i(\omega) \geq 0$, for all i, where we are using the superscript R to recall that equilibria of this kind were introduced for the first time by Rosen [39]. Let us note that a generalized Nash equilibrium problem (GNEP) has in general infinite solutions and it is equivalent to a quasivariational inequality as pointed out by Harker in a finite-dimensional setting [21]. However, as already noted by Rosen, among all possible solutions of a GNEP it is possible to select some equilibria with interesting properties. Quite recently, Facchinei et al. [12] reformulated the result of Rosen in the framework of variational inequalities (see also [33]). More precisely, they proved that one can associate a GNEP with common constraints to a variational inequality whose solutions also solve the original GNEP. Moreover, the solutions found in this way have the special property that all the players share a common vector of Lagrange multipliers. These solutions are considered to be "socially stable" due to the economic meaning usually given to Lagrange multipliers. We note that the result in [12] has been extended very recently to infinite dimensional spaces (see [13]).

Thus, we associate the following variational inequality to problem (4.30): For a.e. $\omega \in \Omega$, find $e^R(\omega) \in K^R(\omega)$ with

$$\sum_{i=1}^{n} F_i[\omega, e^R(\omega)](e_i - e_i^R(\omega)) \geq 0, \quad \forall e \in K^R(\omega), \tag{4.31}$$

where $K^R(\omega) = K^C(\omega)$. We can derive the Lebesgue formulation of the *umbrella* scenario analogously to the previous cases: Find $u^R \in L^2(\Omega, P, \mathbb{R}^n)$ such that

$$\int_\Omega w_i[\omega, u^R(\omega)]dP_\omega = \max_{u_i} \int_\Omega w_i[\omega, (u_i, u_{-i}^N)]dP_\omega, \tag{4.32}$$

where $u_i \geq 0, \forall i$, and $u_i(\omega) + \sum_{j \neq i} u_j^R(\omega) \leq \overline{e}(\omega)$. The Lebesgue formulation of (4.31) is:
Find $u^R \in K_P^R$ such that

$$\int_\Omega \sum_{j=1}^{n} F_j[\omega, u^R(\omega)](u_j(\omega) - u_j^R(\omega))dP_\omega \geq 0, \ \forall u \in K_P^R, \tag{4.33}$$

where $K_P^R = K_P^C$.

4.3.4 Numerical Results

In this section, we illustrate the three stochastic models through a two-player example. We consider a simple case of a two-player game where welfare functions are given by

$$w_i = A_i \log(1 + e_i) - B_i \frac{(e_1 + e_2)2}{2}.$$

Consider two random parameters A_1 and A_2 which follow uniform and truncated normal distributions, respectively,

$$\begin{aligned} A_1 &\sim U(0.6, 2) \\ A_2 &\sim 2 \leq N(3, 0.25) \leq 4 \end{aligned}$$

in all three scenarios. The following parameters are held fixed: $E_1 = 5.5$, $E_2 = 3$, $B1 = 1/30$, $B_2 = 1/30$. We choose a discretization of the interval $[0.6, 2]$ with N_1 points and a discretization of the interval $[2, 4]$ with N_2 points and solve the problem for each pair $(A_1(i), A_2(j))$ using extragradient methods introduced in Chapter 3. Comparisons of approximate mean values of the variables u_1 and u_2 are shown in the tables below. Table 4.17 shows the results for the noncooperative scenario, and tables 4.23 and 4.29 show the results for cooperative and umbrella scenarios respectively. A comparison of the CPU times is shown in Figure 4.3 (similar behavior is observed during comparisons for the noncooperative and cooperative scenarios).

Methods	u_1^N	u_2^N
Marcotte	5.8663	4.1712
Marcotte1	5.8667	4.1712
Marcotte2	5.8663	4.1712
Solodov-Tseng	5.8585	4.1712

Table 4.13: Noncooperative scenario: Mean values of u_1^N and u_2^N for $N_1 = 30, N_2 = 30$

Methods	$Var(u_1^N)$	$Var(u_2^N)$
Marcotte	5.746	2.0013
Marcotte1	5.7447	2.0013
Marcotte2	5.7459	2.0013
Solodov-Tseng	5.7731	2.0014

Table 4.14: Noncooperative scenario: Variances of u_1^N and u_2^N for $N_1 = 30, N_2 = 30$

Methods	u_1^N	u_2^N
Marcotte	5.9383	4.2179
Marcotte1	5.9385	4.2179
Marcotte2	5.9381	4.2179
Solodov-Tseng	5.9339	4.2179

Table 4.15: Noncooperative scenario: Mean values of u_1^N and u_2^N for $N_1 = 50, N_2 = 50$

Methods	$Var(u_1^N)$	$Var(u_2^N)$
Marcotte	5.77	2.0243
Marcotte1	5.7692	2.0243
Marcotte2	5.7707	2.0243
Solodov-Tseng	5.7855	2.0243

Table 4.16: Noncooperative scenario: Variances of u_1^N and u_2^N for $N_1 = 50, N_2 = 50$

Methods	u_1^N	u_2^N
Marcotte	5.9912	4.2522
Marcotte 1	5.9913	4.2522
Marcotte 2	5.9908	4.2522
Solodov-Tseng	5.9897	4.2522

Table 4.17: Noncooperative scenario: Mean values of u_1^N and u_2^N for $N_1 = 100, N_2 = 100$

Methods	$Var(u_1^N)$	$Var(u_2^N)$
Marcotte	5.7802	2.0383
Marcotte1	5.78	2.0383
Marcotte2	5.7817	2.0383
Solodov-Tsengt	5.7859	2.0383

Table 4.18: Noncooperative scenario: Variances of u_1^N and u_2^N for $N_1 = 100, N_2 = 100$

Methods	u_1^C	u_2^C
Marcotte	2.4128	7.3865
Marcotte1	2.4122	7.3871
Marcotte2	2.4123	7.3869
Solodov-Svaiter	2.4589	7.2804
Solodov-Tseng	2.4193	7.3731

Table 4.19: Cooperative scenario: Mean values of u_1^C and u_2^C for $N_1 = 30, N_2 = 30$

Methods	$Var(u_1^C)$	$Var(u_2^C)$
Marcotte	1.6081	7.2765
Marcotte1	1.6087	7.2741
Marcotte2	1.6086	7.2745
Solodov-Svaiter	1.5588	7.7037
Solodov-Tseng	1.6016	7.3315

Table 4.20: Cooperative scenario: Variances of u_1^C and u_2^C for $N_1 = 30, N_2 = 30$

Methods	u_1^C	u_2^C
Marcotte	2.4355	7.4882
Marcotte1	2.4349	7.4888
Marcotte2	2.435	7.4887
Solodov-Svaiter	2.4355	7.4883
Solodov-Tseng	2.4419	7.4749

Table 4.21: Cooperative scenario: Mean values of u_1^C and u_2^C for $N_1 = 50, N_2 = 50$

Methods	$Var(u_1^C)$	$Var(u_2^C)$
Marcotte	1.6134	7.2759
Marcotte1	1.6141	7.2733
Marcotte2	1.6139	7.2738
Solodov-Svaiter	1.5616	7.7337
Solodov-Tseng	1.6066	7.3333

Table 4.22: Cooperative scenario: Variances of u_1^C and u_2^C for $N_1 = 50, N_2 = 50$

Methods	u_1^C	u_2^C
Marcotte	2.4522	7.5631
Marcotte 1	2.4516	7.5638
Marcotte 2	2.4517	7.5636
Solodov-Svaiter	2.5037	7.419
Solodov-Tseng	2.4419	7.4749

Table 4.23: Cooperative scenario: Mean values of u_1^C and u_2^C for $N_1 = 100, N_2 = 100$

Methods	$Var(u_1^C)$	$Var(u_2^C)$
Marcotte	1.6165	7.2612
Marcotte1	1.6172	7.2585
Marcotte2	1.617	7.259
Solodov-Svaiter	1.5636	7.7384
Solodov-Tseng	1.6098	7.3181

Table 4.24: Cooperative scenario: Variances of u_1^C and u_2^C for $N_1 = 100, N_2 = 100$

Methods	u_1^U	u_2^U
Marcotte	2.9997	8.815
Marcotte1	2.999	8.8157
Marcotte2	2.9991	8.8156
Solodov-Svaiter	3.0947	8.7185
Solodov-Tseng	3.85009	10.4462

Table 4.25: Umbrella scenario: Mean values of u_1^U and u_2^U for $N_1 = 30, N_2 = 30$

Methods	$Var(u_1^U)$	$Var(u_2^U)$
Marcotte	1.88806	10.7016
Marcotte1	1.88914	10.6979
Marcotte2	1.88898	10.6984
Solodov-Svaiter	1.73548	11.1843
Solodov-Tseng	0.86584	2.1317

Table 4.26: Umbrella scenario: Variances of u_1^U and u_2^U for $N_1 = 30, N_2 = 30$

Methods	u_1^U	u_2^U
Marcotte	3.0244	8.9233
Marcotte1	3.0236	8.924
Marcotte2	3.0237	8.9239
Solodov-Svaiter	3.1209	8.8254
Solodov-Tseng	3.8805	10.6036

Table 4.27: Umbrella scenario: Mean values of u_1^U and u_2^U for $N_1 = 50, N_2 = 50$

Methods	$Var(u_1^U)$	$Var(u_2^U)$
Marcotte	1.89772	10.7739
Marcotte1	1.89885	10.77
Marcotte2	1.89868	10.7705
Solodov-Svaiter	1.73793	11.2852
Solodov-Tseng	0.7924	1.5658

Table 4.28: Umbrella scenario: Variances of u_1^U and u_2^U for $N_1 = 50, N_2 = 50$

Methods	u_1^U	u_2^U
Marcotte	3.0423	9.0028
Marcotte 1	3.0415	9.0036
Marcotte 2	3.0416	9.0035
Solodov-Tseng	3.0420	9.0037

Table 4.29: Umbrella scenario: Mean values of u_1^U and u_2^U for $N_1 = 100, N_2 = 100$

Methods	$Var(u_1^U)$	$Var(u_2^U)$
Marcotte	1.90412	10.8113
Marcotte1	1.90529	10.8072
Marcotte2	1.90512	10.8077
Solodov-Tseng	0.74137	1.0924

Table 4.30: Umbrella scenario: Variances of u_1^U and u_2^U for $N_1 = 100, N_2 = 100$

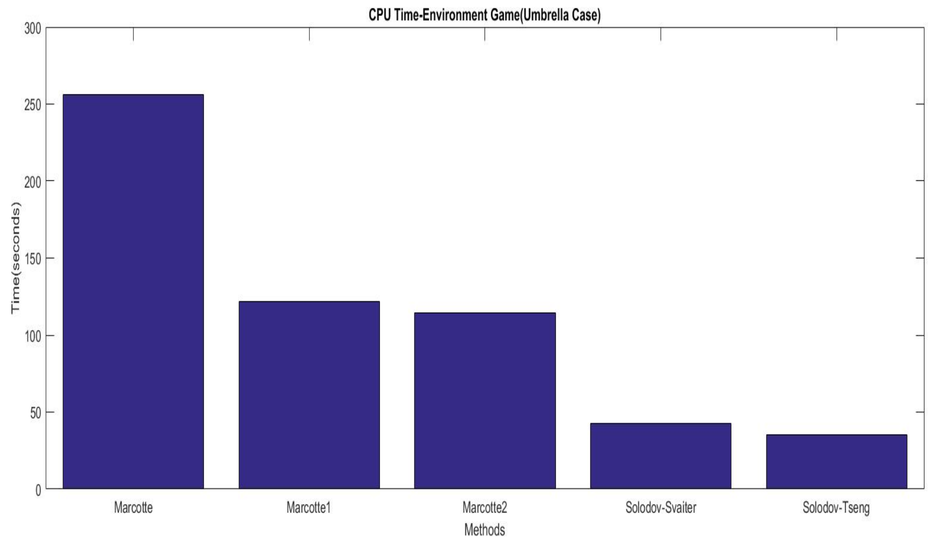

Figure 4.3: Comparison of CPU times for the environmental game (umbrella scenario)

4.4 Stochastic Nonlinear Oligopoly Model

In this section, we introduce a model of oligopolistic market with uncertain data and show that the theoretical and numerical tools can be successfully applied to the model. The classical oligopolistic market equilibrium problem is a Nash game with a special structure and it was first introduced by A. Cournot a long time ago. Recent years have witnessed a renewed interest in oligopoly theory, and many specific cases of oligopolistic markets have been studied in detail, for instance the electricity market (see, e.g., [7, 8]).
We consider here the case in which m players are the producers of the same commodity. The quantity produced by firm i is denoted by q_i so that $q \in \mathbb{R}^m$ denotes the global production vector. Let $(\Omega, \mathcal{A}, P)$ be a probability space and for every $i \in \{1, \ldots m\}$ consider functions $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ and $p : \Omega \times \mathbb{R}^m \to \mathbb{R}$. More precisely, $f_i(\omega, q_i)$ represents the cost of producing the commodity for firm i, and is assumed to be, $P - a.s.$, nonnegative,

increasing and C^1, while $p(\omega, q_1 + \ldots + q_m)$ represents the demand price associated with the commodity. For $P-$almost every $\omega \in \Omega$, p is assumed nonnegative, increasing and C^1. The resulting welfare function w_i is assumed to be concave with respect to q_i. We also assume that all these functions are random variables w.r.t. ω, i.e. they are measurable with respect to the probability measure P on Ω. In this way, we cover the possibility that both the production cost and the demand price are affected by a certain degree of uncertainty, or randomness. Thus, the welfare (or utility) function of player i, representing the net revenue, is given by:

$$w_i(\omega, q_1, \ldots, q_m) = p(\omega, q_1 + \ldots + q_m)q_i - f_i(\omega, q_i). \tag{4.34}$$

Although many models assume no bounds on the production, in a more realistic model the production capability is bounded from above and we also allow these upper bounds to be random variables: $0 \le q_i \le \overline{q}_i(\omega)$. Thus, the specific Nash equilibrium problem associated with this model takes the following form:
For $P - a.e.\ \omega \in \Omega$, find $q^*(\omega) = (q_1^*(\omega), \ldots, q_m^*(\omega))$:

$$w_i(\omega, q^*(\omega)) = \max_{0 \le q_i \le \overline{q}_i(\omega)} \{p(\omega, q_i + \sum_{j \neq i} q_j^*(\omega)\,)q_i,\ -f_i(\omega, q_i)\},\ \forall i \in \{1, \ldots, m\}. \tag{4.35}$$

In order to write the equivalent variational inequality, consider,$\forall \omega$, a closed and convex subset of $\mathbb{R}^m$:

$$K(\omega) = \{(q_1, \ldots, q_m) : 0 \le q_i \le \overline{q}_i(\omega),\ \forall i\}$$

and define the functions

$$\begin{aligned} F_i(\omega, q) \ &:= \ \frac{\partial f_i(\omega, q_i)}{\partial q_i} - \frac{\partial p(\omega, \sum_{j=1}^m q_j)}{\partial q_i} q_i - p(\omega, \sum_{j=1}^m q_j) \qquad (4.36) \\ &= \ f_i'(\omega, q_i) - p'(\omega, Q)q_i - p(\omega, Q), \quad (Q = \sum_{j=1}^m q_j). \end{aligned}$$

The Nash problem is then equivalent to the following variational inequality: for $P-$a.e. $\omega \in \Omega$, find $q^*(\omega) \in K(\omega)$ such that

$$\sum_{i=1}^{m} \left[\frac{\partial f_i(\omega, q_i^*(\omega))}{\partial q_i} - \frac{\partial p(\omega, \sum_{j=1}^{m} q_j^*(\omega))}{\partial q_i} q_i - p(\omega, \sum_{j=1}^{m} q_j^*(\omega)) \right] (q_i - q_i^*(\omega)) \geq 0 \tag{4.37}$$

$\forall q \in K(\omega)$.

Now we are interested in computing statistical quantities associated with the solution $q^*(\omega)$, in particular its mean value. For this purpose, in accordance with the general scheme, we consider a Lebesgue space formulation of problems (4.37): Find $u^* \in K$ such that

$$\int_\Omega \sum_{i=1}^{m} \left[\frac{\partial f_i(\omega, u_i^*(\omega))}{\partial q_i} - \frac{\partial p(\omega, \sum_{j=1}^{m} u_j^*(\omega))}{\partial q_i} u_i - p(\omega, \sum_{j=1}^{m} u_j^*(\omega)) \right] \times$$
$$(u_i(\omega) - u_i^*(\omega)\,) dP_\omega \geq 0 \quad , (4.38)$$

where

$$K = \{u \in L^p(\Omega, P, \mathbb{R}^m) : 0 \leq u_i(\omega) \leq \overline{q}_i(\omega)\}, \ \overline{q}_i \in L^p(\omega, P).$$

Since the stochastic oligopolistic market problem will be studied through (4.38).

4.4.1 Numerical Results

In this subsection, we consider a modified and random version of a class of utility functions introduced by Murphy, Sheraly and Soyster in [32] and successively used by other scholars. These functions generate a nonlinear monotone variational inequality on a certain L^p space, where p is determined by the power law of the cost functions. The cost and demand price functions for the five-firm case in [32] are given by:

$$\begin{aligned} f_i(q_i) &= c_i q_i + \frac{b_i}{b_i+1} k_i^{-1/b_i} q_i^{\frac{b_i+1}{b_i}}, \ i = 1, \dots, 5 \\ p(Q) &= 5000^{1/1.1} Q^{-1/1.1}, \ \ Q = \sum_{i=1}^{5} q_i. \end{aligned}$$

The values of the parameters c_i, k_i, b_i in [32] alongwith our upper bounds for the q_i are given Table 4.31.
An approximate solution of the problem obtained by a projection method is given in [34] as $(q_1, q_2, q_3, q_4, q_5) = (36.937, 41.817, 43.706, 42.659, 39.179)$.
Before introducing random parameters in the above functions, we note that

Table 4.31: Parameter values for the nonlinear problem

i	1	2	3	4	5
c_i	10	8	6	4	2
k_i	5	5	5	5	5
b_i	1.2	1.1	1.0	0.9	0.8
$\overline{q}_i$	100	100	100	100	100

the demand price becomes unbounded when the total quantity Q approaches 0 (commodity is scarce). Although the solution $Q^* = 0$ is never met in most examples, in order to deal with a well behaved function we consider the functional form:

$$p(Q) = 5000^{1/1.1}(Q + e)^{-1/1.1},$$

where e is a small positive parameter which determines the maximum price the consumer can pay when the commodity is very scarce. We add a random perturbation $r(\omega)$ to c_i in the model, and also modulate the price function by a random function $S(\omega)$.
Thus, for the general case of m firms, we introduce cost functions given by:

$$f_i(\omega, q_i) = [c_i + r(\omega)]q_i + \frac{b_i}{b_i + 1} k_i^{-1/b_i} q_i^{\frac{b_i+1}{b_i}}, \tag{4.39}$$

where b_i, c_i, k_i are positive parameters, and demand price functions:

$$p(\omega, Q) = [S(\omega)]^a \; \frac{1}{(Q + e)^a}, \tag{4.40}$$

where $0 < \underline{s} < S(\omega) < \overline{s}$, and a is a parameter such that $0 < a < 1$ ($a = 1/1.1$ in [32]).

With these functions we can build the Carathéodory function F which defines the variational inequality through:

$$F_i(\omega, q) = c_i + r(\omega) + k_i^{-1/b_i} q_i^{1/b_i} + a[S(\omega)]^a \frac{q_i}{(Q+e)^{a+1}} - \frac{[S(\omega)]^a}{(Q+e)^a}, \quad (4.41)$$

$i = 1 \dots m$.
We also use the notation $F_i(\omega, q) = G_i(\omega, q) + H_i(\omega, q)$, where G_i represents the sum of the first three terms in (4.41), while H_i is the rest of the sum, which contains the price function.
Now, let us consider the case $m = 5$ with the data as in Table 4.31. The function F, defines a Nemitsky operator between Lebesgue spaces, as explained in the previous sections. To be precise, since the exponents b_i in the cost functions vary from 0.8 to 1.2, we select $p = 1 + 1/0.8$ so that the Nemitsky operator associated to F maps functions $u \in L^{9/4}$ into $u \in L^{9/5}$. Moreover, we let random parameters $r(\omega)$ and $S(\omega)$ to have truncated normal distributions as follows:

$$\begin{aligned} r &\sim -0.5 \leq N(0, 0.25) \leq 0.5 \\ s &\sim 4950 \leq N(5000, 10) \leq 5050 \end{aligned}$$

while fixing parameter e at 0.0001. Mean values $E(u)$ of $u(r, s) = (u_1, u_2, u_3, u_4, u_5)$ obtained by numerical approximations are presented in Table **??** where n_r and n_s stand for number of discretization points for intervals $[-0.5, 0.5]$ and $[4950, 5050]$ respectively. Comparisons of variances are shown in Tables below and CPU times are shown in Figure 4.4.

Methods	u_1	u_2	u_3	u_4	u_5
Marcotte	35.3355	40.0073	41.8117	40.8079	37.4774
Marcotte1	35.3356	40.0073	41.8116	40.8079	37.4774
Marcotte2	35.3355	40.0073	41.8116	40.8078	37.4773
Solodov	35.3356	40.0074	41.8117	40.8079	37.4775
Solodov-Tseng	35.3284	40.0028	41.81	40.8083	37.4788

Table 4.32: Mean values of u_i, $i = 1, \dots, 5$ for $n_r = 30, n_s = 30$

Methods	$Var(u_1)$	$Var(u_2)$	$Var(u_3)$	$Var(u_4)$	$Var(u_5)$
Marcotte	57.2018	73.1324	79.7252	75.8389	63.9054
Marcotte1	57.2019	73.1324	79.7251	75.8388	63.9052
Marcotte2	57.2018	73.1323	79.725	75.8386	63.9048
Solodov-Svaiter	57.2021	73.1327	79.7254	75.839	63.9055
Solodov-Tseng	57.1786	73.1159	79.7188	75.8404	63.9101

Table 4.33: Variances of u_i, $i = 1, \dots, 5$ for $n_r = 30, n_s = 30$

Methods	u_1	u_2	u_3	u_4	u_5
Marcotte	35.9807	40.7377	$42,5752$	41.5533	38.1622
Marcotte1	35.9807	40.7377	$42,5751$	41.5533	38.1621
Marcotte2	35.9807	40.7377	$42,5751$	41.5532	38.162
Solodov-Svaiter	35.9808	40.7378	$42,5752$	41.5533	38.1622
Solodov-Tseng	35.9749	40.7379	$42,5736$	41.5534	38.1632

Table 4.34: Mean values of u_i, $i = 1, \dots, 5$ for $n_r = 50, n_s = 50$

Methods	$Var(u_1)$	$Var(u_2)$	$Var(u_3)$	$Var(u_4)$	$Var(u_5)$
Marcotte	34.9946	44.6658	48.6341	46.2236	38.9276
Marcotte1	34.9947	44.6659	48.6341	46.2236	38.9275
Marcotte2	34.9947	44.6658	48.634	46.2234	38.9273
Solodov-Svaiter	34.9948	44.666	48.6342	46.2237	38.9277
Solodov-Tseng	34.9831	44.6574	48.6304	46.2239	38.9296

Table 4.35: Variances of u_i, $i = 1, \ldots, 5$ for $n_r = 50, n_s = 50$

Methods	u_1	u_2	u_3	u_4	u_5
Marcotte	36.4616	41.2822	43.1442	42.1088	38.6725
Marcotte 1	36.4617	41.2822	43.1442	42.1088	38.6725
Marcotte 2	36.4616	41.2821	43.1442	42.1088	38.6724
Solodov-Svaiter	36.4529	41.2755	43.1398	42.1059	38.6697
Solodov-Tseng	36.4566	41.2788	43.1426	42.1086	38.6731

Table 4.36: Mean values of u_i, $i = 1, \ldots, 5$ for $n_r = 100, n_s = 100$

Methods	$Var(u_1)$	$Var(u_2)$	$Var(u_3)$	$Var(u_4)$	$Var(u_5)$
Marcotte	17.9041	22.7581	24.7062	23.4311	19.7038
Marcotte 1	17.9041	22.7581	24.7061	23.4311	19.7037
Marcotte 2	17.904	22.758	24.7009	23.4278	19.7009
Solodov-Tseng	17.8989	22.7542	24.7042	23.4308	19.7044

Table 4.37: Variances of u_i, $i = 1, \ldots, 5$ for $n_r = 100, n_s = 100$

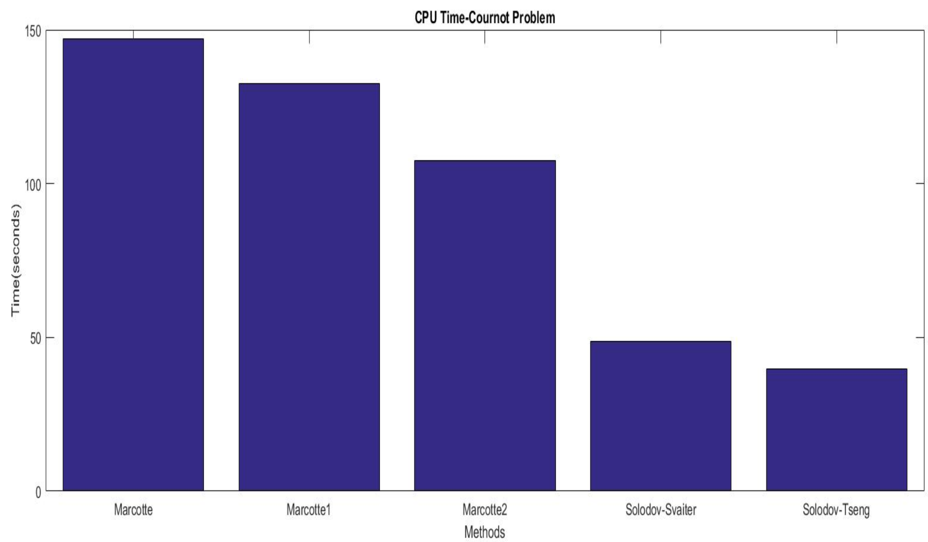

Figure 4.4: Comparison of CPU times for the nonlinear problem

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