Recursive Model Selection for GNSS-Combined Precise Point Positioning Algorithms

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by

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Abstract

The accuracy of Global Positioning algorithms can be improved by incorporating observations from the satellites of multiple Global Navigation Satellite Systems (GNSS). To best utilize these observations, inter-system biases must be modeled. A unified observational model is proposed which accounts for these factors for an arbitrary number of GNSS. The Bayesian Information Criterion (BIC) may be imposed upon the unified model to balance data-fitting degree with model complexity among candidate models for a given satellite configuration scenario. A simple formulation is derived for the change to the Weighted Sum Squared Residuals (WSSR) outcome caused by modifying the least-squares design matrix to accomodate additional ISB parameters. The process of updating WSSR is shown to be $O(n^2)$, allowing a low-cost determination of the information entropy between any two candidate models. With this computationally cheap parameter selection process and a set of GNSS-heterogeneous observations, the form of the unified model with the highest expected accuracy may be efficiently selected, at a stage before matrix inversion is performed.

1 Introduction

Three independent GNSS are in deployment today— the GPS System based in the United States, Russian-based GLONASS, and European-based Galileo. A fourth GNSS, COMPASS, is planned to expand out of the existing regional Chinese BeiDou navigation system. Several further GNSS and regional systems are in development. For the private user, each of these enterprises contributes to improving the accuracy of positioning algorithms— greater numbers of satellite observations become available, frequently providing more optimal combinations of observations for positioning algorithms[1].

Under some conditions, single-GNSS coverage does not provide sufficient accuracy or convergence speed for positioning purposes, particularly within natural ravines and in urban areas. The development of accurate multi-GNSS algorithms is expected to contribute to positioning accuracy for these situations. This improvement, however, requires a model to compensate for the inter-system biases (ISB) that exist due to differences among GNSS designs[2]. These biases produce data correlation and thus cannot be absorbed into another factor of the stochastic model[3].

For instance, consider the processing delay of a signal in the receiver's circuitry– the time elapsed from the start of an electromagnetic signal's absorption by the antenna to the time-stamping of the data. Signals of the same frequency coming from satellites of the same GNSS will experience this delay each time with the same outcome distribution. If this effect were fully compensated for by the receiver clock error parameter during least squares reduction, this delay would be effectively absorbed into the parameter. However, signals from different GNSS, which are not standardized, will undergo different mechanisms in the receiver circuitry before the data is time-stamped. This is just one factor of ISB. If the impact of all systematic influences on the final measurement is unaccounted for in the model, the overall differences in measurement outcomes will bias the time parameter, negatively affecting its accuracy. In stochastic terms, this means off-diagonal terms are occurring in the covariance matrix of the model. Such an unaccounted correlation between parameters will affect the convergence rate and accuracy of positioning algorithms[4].

The data encoded within each signal is the ephemeris— containing the precise position and velocity of the transmitting satellite. These orbits are known with very high accuracy, though the standard also varies for different GNSS. Positioning algorithms rely on measuring the exact difference between the timestamp in the ephemeris and the time it is received by the user. Using the best estimate for the time-of-flight from satellite to receiver, the light-time equation will determine the distance from receiver to satellite, as the signals travel at the speed of light. With a set of known satellite positions and estimated distances from these positions, it is possible to determine the receiver's position. Precise point positioning (PPP) algorithms utilize a set of measurements and, starting with a loose estimate of the receiver position, compute a least-squares regression

to produce an accurate solution vector. The application to PPP will be the primary concern here, although double-differencing (DD) positioning techniques, which measure baselines from other ground stations, may also benefit from the same principles.

In the process of measuring the time-of-flight, there are an array of physical effects which influence the delay. As the raw measurement data is a composite of physical effects, the measurements are referred to as pseudo-observables— pseudo-ranges in time units. The pseudo-range is defined as the difference of the timestamp of transmission (in satellite timescale) and the timestamp of reception (in receiver timescale). From there, corrections from the model will be applied to determine the actual time-of-flight. The observation equation for a pseudo-observable, relating all the delays involved, defines a functional model. The right-hand side of the observation equation is the sum of all effects, including the actual topocentric range. This model is defined in the next section. A set of observation equations from different frequencies and satellites forms one matrix equation. This system of equations leads to a design matrix familiar to the least-squares regression process.

The variance matrix of the individual pseudo-observables defines the corresponding stochastic model, which is designed to give greater weight to pseudo-observables of lower variance; this is defined in section 3. Off-diagonal terms in the variance matrix represent covariance between a given pair of pseudo-observables. These terms are increasingly prevalent for DD techniques such as real-time kinematic (RTK) positioning, which use data from additional ground stations in order to cancel atmospheric effects held in common. PPP techniques implicitly avoid this source of correlation, but ISB is another source of covariance among the pseudo-observables that will affect either method if unmodelled.

Methods for modeling other atmospheric effects will follow in the model definition of section 2, which are usually handled independently through external models. One delay effect that cannot be modeled this way is the clock error, which is the offset of the receiver's clock from the GNSS timescale on which the satellite clocks operate. The ground clock will not be synchronized with the GNSS clocks in any way and the offset must be parameterized in the design matrix as the fourth coordinate of the solution vector.

In the same manner, further parameters beyond this may be used to estimate inter-system biases, which are multi-faceted and more difficult to model[2]. In practice, it may be the case that the use of daily constants for ISB, determined empirically from an observatory, is sufficient[5] [6]. Such a substitution would allow the ISB parameter to be omitted from the constraint equation. The selection of the most appropriate model in this regard is the subject of section 4.

With regards to the ISB, different GNSS operate under different timescales and GNSS transmissions are not all operating within the same frequencies. The latter introduces a delay term to the ISB which is both satellite-dependent and receiver-dependent; as mentioned before, different frequencies have non-identical

processing times within the circuitry of a receiver, while various receivers (antenna type, firmware version) will introduce further frequency-dependent variability in processing times[6]. The effects of varying signal frequencies are particularly pronounced in the GLONASS system, as in that system there exist frequency differences between each satellite. This is because GLONASS uses Frequency Division Multiple Access (FDMA), requiring better-equipped hardware to receive all the frequencies used. This latter component of ISB is designated as Inter-Frequency Bias (IFB). The net ISB for a given GNSS and receiver hardware pair is typically defined to stem from these three phenomena, as reflected in (3) of section 2. While the receiver-dependence status of some terms is not a concern for PPP, which uses only one receiver at a time, DD techniques will experience further data correlation from these non-uniformities unless specifically modeled.

Central to the problem of positioning are the two measurements known as code observable and phase observable. The code observable provides a reliably accurate but noisy pseudorange. The phase observable gives a very precise track record of the change in distance since the first signal received, but with lower accuracy— the functional model for the phase observable contains an offset of an unknown integer number of wavelengths. Once an algorithm determines the correct ambiguity value, through a filtering process involving the code observable[7], the phase observable will be able to track the satellite distance very accurately in units of radians.

2 Observation Model

The unified observation model is composed of observation equations:

$$P_{i,sys}^{rs}(t) = \rho^{rs}(t) + c \cdot dt^r(t) + \Psi^s(ZTD^r, \theta^{rs}) + \mu_i I^s(t) + ISB_{i,sys}^{rs}(t) + e_i^{rs}(t) \tag{1}$$

$$L_{i,sys}^{rs}(t) = \rho^{rs}(t) + c \cdot dt^r(t) + \Psi^s(ZTD^r, \theta^{rs}) - \mu_i I^s(t) + \lambda_i N_i^s + ISB_{i,sys}^{rs}(t) + \epsilon_i^{rs}(t) \tag{2}$$

Where
i represents a given frequency, with $i = 1, ..., f$
r represents a given receiver.
s represents a given satellite.
sys represents a given GNSS.
$P_{i,sys}^{rs}(t)$ is the code observation.
$L_{i,sys}^{rs}(t)$ is the phase observation.
$\rho^{rs}(t)$ is the topocentric range from satellite s to receiver r.
c is the speed of light.
$dt^r(t)$ is the clock error of receiver r.
ZTD^r is the zenith tropospheric delay.
θ^{rs} is the elevation angle of satellite s.
$\Psi^s(\tau, \theta)$ is the mapping function.
λ_i is the carrier wavelength at frequency i.
$\mu_i = \lambda_i / \lambda_1$.
$I^s(t)$ is the ionospheric delay.
$N_i^s \in \mathbb{Z}$ is the carrier phase ambiguity.
$ISB_{i,sys}^{rs}(t)$ is the intersystem bias.

$e_i^{rs}(t), \epsilon_i^{rs}(t)$ are the unmodeled error terms.

The ISB term is further decomposed as follows:

$$ISB_{i,sys}^{rs} = TO_{sys} + DCB_{sys}^r + IFB_i^r \tag{3}$$

Where
TO_{sys} is the system time difference.
DCB_{sys}^r is the bias due to systematic processing differences.
IFB_i^r is the inter-frequency bias.

As mentioned previously, the observation equation gives a functional model for understanding the measurements. The right-hand side of each equation represents the model of contributions to the measurement. The most important distinction between the two is the ambiguity value on the phase observable.

The sign difference for $I^s(t)$ in (1) and (2) is due to the ionosphere being a dispersive medium. The phase velocity is increased above c in the medium. As the phase observable is measured in cycles, this pushing forward of phase results in a decrease to the measured delay. Within the GNSS frequency range, the ionospheric effect on group velocity is almost exactly opposite, down to 99.9 percent[8], for group velocity, so the code observable delay increases by the same value.

The ionosphere-free combination, a linear combination of pseudo-observables on separate frequencies, may be used to eliminate the I^s term[9]. This is only possible with multi-frequency receivers. The tropospheric delay may be modeled externally with the Hopfield tropospheric correction model, using the Vienna mapping function Ψ^s[10][11]. These two external models account for the simplification:

$$\tilde{P}_i^{rs}(t) \equiv P_i^{rs}(t) - \Psi^s(ZTD^r, \theta^{rs}) - \mu_i I^s(t) \tag{4}$$

$$\tilde{L}_i^{rs}(t) \equiv L_i^{rs}(t) - \Psi^s(ZTD^r, \theta^{rs}) + \mu_i I^s(t) \tag{5}$$

The two constraint equations will then take the form:

$$\tilde{P}_i^{rs}(t) = \rho^{rs}(t) + c \cdot dt^r(t) + ISB_{i,sys}^{rs}(t) + e_i^{rs}(t) \tag{6}$$

$$\tilde{L}_i^{rs}(t) = \rho^{rs}(t) + c \cdot dt^r(t) + \lambda_i N_i^s + ISB_{i,sys}^{rs}(t) + \epsilon_i^{rs}(t) \tag{7}$$

With GPS as the reference constellation, ISB_{GPS} will be chosen to be zero. Thus all other ISB system time offsets TO_{sys} will be pegged to the GPS system time, and DCB_{sys} likewise relative to the circuit delay DCB_{GPS}, the lattermost of which is absorbed by the clock error[12]. This approach is referred to as tight combination[13].

IFB_i^r accounts for biases arising from non-uniform frequency among signals within a FDMA GNSS;

DCB_{sys}^r accounts for the delay biases strictly on the inter-GNSS level. For CDMA-based GNSS, which keeps each transmission type to a fixed frequency, IFB_i^r is always zero. For each GNSS using FDMA, a value IFB_i^r is assigned for each occurring frequency in the GNSS; the zero-sum condition is imposed on each set of IFB_i^r.

It is important to note that for DCB_{sys}^r and IFB_i^r, modeling must be done separately for code and phase observations. In particular, phase IFB terms are known to vary linearly with frequency[14], but the code IFB terms generally do not, although this is beyond the scope here. The treatment of ISB given in section 4 will not break down the components individually, but will treat each full $ISB_{i,sys}^{rs}$ as a parameter in its own right for the minimization process.

To fit the pseudo-observations to a least-squares criteron, it is necessary to include an extra residual term in the functional model. The typical least-squares process is used to minimize the set of residuals; most generally:

$$Z_j = f_j(u_1, u_2, ..., u_k) + r_j \tag{8}$$

Where
Z_j is pseudo-observation/data point j.
f_j is the functional model [(6) or (7)] for observation j.
u_i is parameter i.
r_j is residual j.

Further define $\hat{u}$ as the vector of parameters such that $J = \sum\limits_{j=1}^{k} (r_j)^2$ is minimized. Then let $\hat{u} \equiv u_0 + \Delta u$.

With approximate values u_0 for all parameters, it is possible to linearize the observation equations. To reduce clutter, the r superscript is omitted.

$$\Delta b_i^s = -(\vec{v})^T \Delta w + (\vec{v})^T \Delta W + c\Delta dt + \Delta ISB_{i,sys}^s \tag{9}$$
$$\Delta \phi_i^s = -(\vec{v})^T \Delta w + (\vec{v})^T \Delta W + c\Delta dt + \lambda_i N_i^s + \Delta ISB_{i,sys}^s \tag{10}$$

Where
$\vec{v}$ is the line-of-sight (LOS) vector from receiver to satellite.
$\Delta w = (\Delta x, \Delta y, \Delta z)$ is the incremental receiver position.
$\Delta W = (\Delta X, \Delta Y, \Delta Z)$ is the incremental satellite position.

These equations are used to construct the matrix of partials, A. The breakdown of Δx gives the partial terms of the first three columns.

Let there be m_p pseudo-observables from GNSS p, $p = 1, ..., \kappa$. Let K be the total number of pseudo-observables: $K = \sum\limits_{p=1}^{\kappa} m_p$. Let the reference GNSS be $p = 1$. With (9), the design matrix for code observations takes the following form:

$$
A_c = \begin{bmatrix}
\dfrac{x_0 - X^1}{\rho_0^1} & \dfrac{y_0 - Y^1}{\rho_0^1} & \dfrac{z_0 - Z^1}{\rho_0^1} & 1 & 0 & \dots & 0 \\[2.5ex]
\dfrac{x_0 - X^2}{\rho_0^2} & \dfrac{y_0 - Y^2}{\rho_0^2} & \dfrac{z_0 - Z^2}{\rho_0^2} & 1 & 0 & \dots & 0 \\[2.5ex]
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\[2ex]
\dfrac{x_0 - X^{m_1}}{\rho_0^{m_1}} & \dfrac{y_0 - Y^{m_1}}{\rho_0^{m_1}} & \dfrac{z_0 - Z^{m_1}}{\rho_0^{m_1}} & 1 & 0 & \dots & 0 \\[2.5ex]
\dfrac{x_0 - X^{m_1+1}}{\rho_0^{m_1+1}} & \dfrac{y_0 - Y^{m_1+1}}{\rho_0^{m_1+1}} & \dfrac{z_0 - Z^{m_1+1}}{\rho_0^{m_1+1}} & 1 & 1 & \dots & 0 \\[2.5ex]
\dfrac{x_0 - X^{m_1+2}}{\rho_0^{m_1+2}} & \dfrac{y_0 - Y^{m_1+2}}{\rho_0^{m_1+2}} & \dfrac{z_0 - Z^{m_1+2}}{\rho_0^{m_1+2}} & 1 & 1 & \dots & 0 \\[2.5ex]
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\[2ex]
\dfrac{x_0 - X^K}{\rho_0^K} & \dfrac{y_0 - Y^K}{\rho_0^K} & \dfrac{z_0 - Z^K}{\rho_0^K} & 1 & 0 & \dots & 1
\end{bmatrix}
\qquad
\Delta u = \begin{bmatrix}
\Delta x \\[1ex]
\Delta y \\[1ex]
\Delta z \\[1ex]
\Delta dt \\[1ex]
\Delta ISB_2 \\[1ex]
\vdots \\[1ex]
\Delta ISB_K
\end{bmatrix}
$$

Matrix A_c is $K \times (\kappa + 3)$; K code observations by 4 coordinate parameters plus $\kappa - 1$ ISB parameters.

ρ_0^n is the approximate (pre-fit) topocentric range from satellite n to receiver.

(x_0, y_0, z_0) is the pre-fit position of the receiver.

(X^n, Y^n, Z^n) is the pre-fit position of satellite n.

Δu is the vector of corrections to the pre-fit receiver coordinates.

Δdt is the correction to the receiver clock error.

ΔISB_n is Eq. (3) for observable n.

Including the phase observations will expand A into the following block matrix, to include K additional

observations (rows) and K additional parameters— the ambiguities N_i^s.

$$A = \begin{bmatrix} A_c & 0 \\ A_c & I_K \end{bmatrix} \qquad \Delta u = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \\ \Delta dt \\ \Delta ISB_2 \\ \vdots \\ \Delta ISB_K \\ \Delta N_1 \\ \vdots \\ \Delta N_K \end{bmatrix}$$

The normal equations become

$$A^T \hat{r} = 0 \tag{11}$$

Where $\hat{r}$ is the minimized residual vector. For a solution to exist, $K \geq \kappa + 3$. Otherwise, the matrix is rank-deficient. The fitted receiver position $\hat{u} = u_0 + \Delta u$. Of course, $\hat{u}$ contains the solution for the receiver coordinates in the first three parameters.

$$\Delta u = (A^T A)^{-1} A^T r_0 \tag{12}$$

$$\hat{u} = u_0 + \Delta u \tag{13}$$

This is the most extensive form of the observation model to be considered here. See Appendix **A** for a simplified example of the fitting process. The general functional model can be made to exclude any ISB term from this process. This would mean the deletion of the corresponding column from A_c and removal of that parameter from the vector Δu. Defining which ISB terms should be included is the subject of Section 4. The alternative to the fitting of an ISB parameter would be the use of an external model, such as a daily constant.

In general purpose PPP, linear regression is run iteratively with successive signals, using the previous step's solution vectors as pre-fit vectors. With a powerful filtering process, this allows an algorithm to converge to superior results over time[7]. For purposes laid out in section 4, only the framework of the first iteration will be used here.

3   Stochastic Model

The stochastic model accounts for noise in the parameters, which are qualitatively distinct. In order to give more weight to data which is expected to contain less noise, the stochastic model provides the weight matrix W to the regression process.

The elevation-dependent exponential stochastic model[15] is employed to construct the covariance matrix.

$$\sigma_{c_i} = \sigma_{c_i}^z (1 + a_{obs} exp(-b_{obs}\theta_s)) \tag{14}$$

Where

σ_{c_i} is the standard deviation of the code observable on frequency i.

$\sigma_{c_i}^z$ is the zenith standard deviation of the code observable on frequency i.

a_{obs}, b_{obs} are observable-specific constants.

θ_s is the elevation angle.

This form contains empirical amplification constants in addition to the zenith variance for each type of observation and frequency. At zenith, the signals pass through the least amount of atmosphere. The elevation-dependent term adjusts this to account for the increased amount of atmosphere traversed when the line-of-sight makes an oblique angle to the Earth's surface. The stochastic model for phase observables is identical in form with the above, with symbol σ_{p_i}. From these definitions follows the covariance matrix:

$$\mathbf{W} = \begin{bmatrix} diag^{-1}(\sigma_c{}^2) & 0 \\ 0 & diag^{-1}(\sigma_p{}^2) \end{bmatrix}$$

Where $diag^{-1}(\sigma_c{}^2)$ is the inverse of the diagonal matrix, in this case for variances $\sigma_c{}^2$. The LSS can be adapted to this weighting scheme:

$$\hat{u} = (A^T W A)^{-1} A^T W r_0 \tag{15}$$

4   Model Selection

Some positioning models for multi-GNSS data have calibrated components of the ISB, externalizing the $TO[5]$, $IFB[6]$ or $DCB[16]$, respectively, as a daily constant. In these experiments, the accuracy of simu-

lations running each model was compared with known positions to gain insight into the behavior of these terms by controlling for antenna hardware, firmware and GNSS combination.

While code ISBs may reach into tens of meters and phase ISBs can approach a half cycle, the values remained relatively constant over time, drifting just a few nanoseconds per day[5][13]. Another study, however, which constructed a model to single out the TO component for four different GNSS, uncovered potential improvements in accuracy when closely monitoring these parameters in time[17]. One trial ran a model for single-frequency PPP using GPS and Galileo with fully parameterized ISB components[12], in a similar vein to the fully parameterized form of the model defined here. When enough pseudo-observations from a GNSS are available, it may become advantageous to model the ISB_{sys} for that GNSS as a parameter as opposed to using an external model to offset the bias indirectly.

The set of candidate models for a positioning scenario spans from the four-observable model— a model which fits just three receiver coordinates and the clock delay (k=4) with external models for ISB— to the modeling of some or all ISB errors as parameters. A candidate model will therefore be defined by a set of parameters, B, beyond the usual four, to be modeled as additional columns. In this case, B is a subset of all potential ISB_{sys} parameters (except for the reference GNSS ISB, which is zero). Each candidate model takes a unique form of the unified model as defined by the terms of A. We also define the set of observations S— As noted in section 5, it is also possible to perform a model selection on the observation set.

The weighted sum of squared residuals (WSSR) for each candidate model is minimized by the least squares criterion (solution to the normal equations), giving solution vector $\hat{u}$. Let $\rho_{(S,B)}$ be the WSSR obtained from the least squares solution (LSS) of the normal equations, for model (S, B). In the context of model selection, "WSSR" will refer to $\rho_{(S,B)}$, the minimized LSS term for that model.

As the degrees of freedom of the model falls to zero, WSSR eventually tends to zero. Candidate models vary along these lines, so model selection simply on a basis of optimization of $\rho_{(S,B)}$ can fail. The idea for this model selection is a balance between weighing the statistical benefit of each particular additional parameter avoiding overfitting. One strong measure for determining the most parsimonious model is the Bayesian Information Criterion (BIC)[18].

$$BIC = n \cdot log(\rho_{(S,B)}/n) + k \cdot log(n) \tag{16}$$

Where $k = |B| + 4$ is the number of parameters and $n = |S|$ is the number of observations. The BIC balances the complexity of a model with its goodness-of-fit. For a given scenario, the model with the lowest BIC value is favorable. The ΔBIC for an alternative candidate model thus provides a measure of utility for a proposed alteration to design matrix A. In the context of (16), the model selection process will hinge upon

how to obtain $\rho_{(S,B)}$ for each candidate model. Once a model is selected as optimal, the full solution vector under that model may be computed. This process avoids the costly computation of an entire solution vector for each candidate model.

The WSSR may be computed partway through the process of solving the normal equations for LSS.

$$\rho_{(S,B)} = \min_{u}[(r_0 - Au)^T W(r_0 - Au)] \tag{17}$$

Where
$\rho_{(S,B)}$ is the WSSR.
u is a vector of parameters.
r_0 is the vector of pre-fit residuals.
A is the design matrix.
W is the weight matrix.

In contrast, computing the full solution vector requires the following:

$$\hat{u} = u_0 + (A^T W A)^{-1} A^T W r_0 \tag{18}$$

It is desirable to avoid computing a large matrix inversion for every candidate model being proposed. The four-observable model may be chosen to start the model selection process and a recursive method subsequently used to obtain ΔBIC for adjacent candidate models. Any modifications to A which produce a negative ΔBIC should be incorporated. The forthcoming method involves decomposing the system of equations to an upper triangular matrix with history information, allowing further changes to A and WSSR values to be examined thenceforth from each updated A. The process for deriving $\hat{u}$ once A is chosen will be explained in context, once the recursive method for selecting A has been elaborated.

4.1   Parameter Update

The following procedure allows the computation of ΔBIC for a candidate model consisting of one additional ISB parameter in the design matrix— updating from k to $k+1$ total parameters, including the necessary extra column in A_{k+1} defined in section 2 and the coefficients for the extra σ^{ISB} in W. Let the subscript k refer to terms for the initial model and subscript $k+1$ refer to terms for the updated model in the context of this section. Let W, u_0 and r_0 correspond to A in a given equation.

11

First, a few terms for the initial model will be defined. The least squares solution (LSS) for the candidate model with design matrix A_k ($n \times k$) is given by:

$$\hat{u}_k = u_0 + (N_k)^{-1} q_k \tag{19}$$

$$N_k \equiv A_k^T W A_k \tag{20}$$

$$q_k \equiv A_k^T W r_0 \tag{21}$$

By constructing an augmented matrix to perform Gaussian Elimination, the WSSR may be singled out in the diagonal. The bottom row represents the sum of residuals, which may be minimized by likewise applying the elimination operations[19]. Consider the $(k+1) \times (k+1)$ augmented cross-product matrix:

$$B_k^0 = \begin{bmatrix} N_k & q_k \\ (q_k)^T & (r_0)^T W r_0 \end{bmatrix} \tag{22}$$

The LDU decomposition of the normal equations may be expressed as[19]:

$$B_k^0 \equiv \begin{bmatrix} (R_k)^T & 0 \\ (s_k)^T & \rho_k \end{bmatrix} \begin{bmatrix} diag^{-1}(R_k) & 0 \\ 0 & \rho_k^{-1} \end{bmatrix} \begin{bmatrix} R_k & s_k \\ 0 & \rho_k \end{bmatrix} \tag{23}$$

The first two terms in (23) combine to form L in the LU decomposition. Further define:

$$B_k^k \equiv \begin{bmatrix} R_k & s_k \\ 0 & \rho_k \end{bmatrix} = L_k \dots L_2 L_1 B_k^0 \tag{24}$$

The decomposition components may be expressed $B_k^0 = LU = (L_k \dots L_2 L_1)^{-1} B_k^k$. The WSSR ρ_k is minimized and thus equal to $\rho_{(S,B)}$ when the rest of the bottom-row is zero[19], as in B_k^k.

L_i is defined as the transformation matrix which, for $j = i+1, i+2, \dots k$, replace row r_j with $r_j - r_i(g_{j,i}^{i-1}/g_{i,i}^{i-1})$, where $g_{m,n}^p$ is entry (m,n) of B_k^p. For example, $L_1 B_k^0 = B_k^1$ will appear in matrix form:

$$L_1 \cdot B_k^0 = L_1 \cdot \begin{bmatrix} g_{1,1}^0 & g_{1,2}^0 & \cdots & g_{1,k}^0 & g_{1,k+1}^0 \\ g_{2,1}^0 & g_{2,2}^0 & \cdots & g_{2,k}^0 & g_{2,k+1}^0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{k,1}^0 & g_{k,2}^0 & \cdots & g_{k,k}^0 & g_{k,k+1}^0 \\ g_{k+1,1}^0 & g_{k+1,2}^0 & \cdots & g_{k+1,k}^0 & \rho_k^0 \end{bmatrix} = \begin{bmatrix} g_{1,1}^0 & g_{1,2}^0 & \cdots & g_{1,k}^0 & g_{1,k+1}^0 \\ 0 & g_{2,2}^1 & \cdots & g_{2,k}^1 & g_{2,k+1}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & g_{k,2}^1 & \cdots & g_{k,k}^1 & g_{k,k+1}^1 \\ 0 & g_{k+1,2}^1 & \cdots & g_{k+1,k}^1 & \rho_k^1 \end{bmatrix} = B_k^1 \tag{25}$$

It follows that B_k^k is upper triangular and $\rho_k^k = \rho_k$. Since submatrices N_k and R_k occupy the same positions in B_k^0 and B_k^k respectively, and row operations of the form L_i have no effect on determinant; and

since B_k^k is square, placing ρ_k in an even parity position inside B_k^k, note the following relations:

$$det(N_k) = det(R_k) \tag{26}$$

$$det(B_k^0) = det(B_k^k) = \rho_k det(R_k) \tag{27}$$

Now consider the updated candidate model whose design matrix A_{k+1} has $k+1$ columns— those of A_k and one extra column, a_{k+1}, for the fitting of an additional ISB term. The new information appears in column $k+1$ and row $k+1$ of the updated LSS cross-product matrix:

$$B_{k+1}^0 = \left[\begin{array}{cc|c} N_k & A_k^T W a_{k+1} & q_k \\ a_{k+1}^T W A_k & a_{k+1}^T W a_{k+1} & a_{k+1}^T W r_0 \\ \hline q_k^T & r_0^T W a_{k+1} & r_0^T W r_0 \end{array} \right] = \left[\begin{array}{cc} N_{k+1} & q_{k+1} \\ q_{k+1}^T & r_0^T W r_0 \end{array} \right] \tag{28}$$

Define the vector $h_{k+1}^0 \equiv [A_k^T W a_{k+1}, r_0^T W a_{k+1}]^T \equiv [h_1^0, h_2^0, h_3^0, \ldots, h_k^0, h_{k+2}^0]^T$. This is column $k+1$ and the transpose of row $k+1$ of B_{k+1}^0, sans entry number $k+1$. It was shown that $B_k^k = L_k \ldots L_2 L_1 B_k^0 \equiv M B_k^0$. Applying the same transform matrix, M, with dimension $(k+1) \times (k+1)$ from the initial model, define the following:

$$M(h_{k+1}^0) \equiv [h_1^0, h_2^1, h_3^2, \ldots, h_k^{k-1}, h_{k+2}^M]^T \tag{29}$$

Let entries inherited from B_k^k still appear in the form $g_{i,j}^p$. Finally, let $\tilde{M}$ be defined to be equivalent to M except that whenever index j would be $k+1$, let it be $k+2$ instead. This way, instead of changing row $k+1$, $\tilde{M}$ will reduce row $k+2$ without altering the former. This allows the presence of $(h_{k+1}^0)^T$ in row $k+1$ without changing any of the $g_{i,j}^p$ values or ρ_k.

$$\tilde{M} B_{k+1}^0 = \left[\begin{array}{ccccc|c} g_{1,1}^0 & g_{1,2}^0 & \cdots & g_{1,k}^0 & h_1^0 & g_{1,k+1}^0 \\ 0 & g_{2,2}^1 & \cdots & g_{2,k}^1 & h_2^1 & g_{2,k+1}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & g_{k,k}^{k-1} & h_k^{k-1} & g_{k,k+1}^{k-1} \\ h_1^0 & h_2^0 & \cdots & h_k^0 & h_{k+1}^0 & h_{k+2}^0 \\ \hline 0 & 0 & \cdots & 0 & h_{k+2}^M & \rho_k \end{array} \right] \tag{30}$$

This matrix was obtained through determinant-conserving transformations of B_{k+1}^0 while preserving terms $g_{i,j}^{i-1}$ from B_k^k through application of a transform identical to M (for g entries) previously used thereto.

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For rows $i = 1, ..., k$, note that $L_k \ldots L_2 L_1$ also yields h_i^{i-1}. However, $\tilde{M}$ is not identical to $L_k \ldots L_2 L_1$ for row $k+2$, hence the term $h_{k+2}^M \neq 0$. This matrix may further be row-reduced by the transformation $L_k \ldots L_2 L_1$, of dimensions $(k+2) \times (k+2)$ for the updated model (index j goes to $k+1$ instead of k):

$$L_k \ldots L_2 L_1 \tilde{M} B_{k+1}^0 = \left[\begin{array}{ccccc|c} g_{1,1}^0 & g_{1,2}^0 & \cdots & g_{1,k}^0 & h_1^0 & g_{1,k+1}^0 \\ 0 & g_{2,2}^1 & \cdots & g_{2,k}^1 & h_2^1 & g_{2,k+1}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & g_{k,k}^{k-1} & h_k^{k-1} & g_{k,k+1}^{k-1} \\ 0 & 0 & \cdots & 0 & h_{k+1}^k & h_{k+2}^k \\ \hline 0 & 0 & \cdots & 0 & h_{k+2}^M & \rho_k \end{array} \right] = \tilde{B}_{k+1}^k \qquad (31)$$

From here the solution vector may be obtained by back-substitution, starting with value $\hat{u}_{k+1} = h_{k+2}^k/h_{k+1}^k$ in the second-bottom row. At this point, however, ρ_{k+1} may be derived without the need for this back-substitution. Recall (26), (27):

$$\rho_k = det(B_k^0)/det(N_k) \qquad (32)$$
$$det(B_{k+1}^0) = det(\tilde{B}_{k+1}^k) \qquad (33)$$
$$det(\tilde{B}_{k+1}^k) = \rho_k C - h_{k+2}^M D \qquad (34)$$

$$\text{where } C \equiv det \begin{bmatrix} g_{1,1}^0 & g_{1,2}^0 & \cdots & g_{1,k}^0 & h_1^0 \\ 0 & g_{2,2}^1 & \cdots & g_{2,k}^1 & h_2^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & g_{k,k}^{k-1} & h_k^{k-1} \\ 0 & 0 & \cdots & 0 & h_{k+1}^k \end{bmatrix}, D \equiv det \begin{bmatrix} g_{1,1}^0 & g_{1,2}^0 & \cdots & g_{1,k}^0 & g_{1,k+1}^0 \\ 0 & g_{2,2}^1 & \cdots & g_{2,k}^1 & g_{2,k+1}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & g_{k,k}^{k-1} & g_{k,k+1}^{k-1} \\ 0 & 0 & \cdots & 0 & h_{k+2}^k \end{bmatrix} \qquad (35)$$

From Cramer's rule,

$$\hat{u}_{k+1} = det(D)/det(C) \qquad (36)$$

And from the general case of (32),

$$\rho_{k+1} = det(B_{k+1}^0)/det(N_{k+1}) \qquad (37)$$
$$\rho_{k+1} = [\rho_k C - h_{k+2}^M D]/C \qquad (38)$$
$$\Rightarrow \rho_{k+1} = \rho_k - h_{k+2}^M \hat{u}_{k+1} \qquad (39)$$

Substituting for WSSR, and mindful that $\Delta k = 1$ and $\Delta n = 0$,

$$\Delta BIC = n \cdot log(-h_{k+2}^M \hat{u}_{k+1}/n) + log(n) \qquad (40)$$

When solving all the normal equations by finding $(A^T W A)^{-1}$, the computational cost is on the order of $O(k^3)$ multiplications plus divisions. This process, however, has a complexity of $3k(k+1)/2$: (30), (31), and

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the back-substitutions each take $k(k+1)/2$ operations.

When full back-substitutions are performed only for the optimal candidate model, the order of a recursive algorithm to select a candidate model within a given S observation set is $\sum_{k=4}^{\kappa}(k^2+k)$ operations.

The recursive method will compute the WSSR for the four-observable model, then update the set B to include any ISB parameters which reduce the BIC, and when this selection process for B is complete, obtain the solution vector by carrying out the back-substitutions from $\tilde{B}_{|B|+4}^{|B|+3}$ to actually solve for $\hat{u}_{|B|+4}$. See Appendix **B** for a simplified example detailing the model selection process.

5 Conclusion

For a modern PPP algorithm, the added benefit of increased model adaptability without increasing the computational order means that such a powerful improvement is available with very little concomitant downside. The simplicity through which the terms h_{k+1}^k, h_{k+2}^k and h_{k+2}^M, $\Delta\rho$ and Δ BIC are obtained allows an algorithm to converge on the estimated best model as a matter of vector operations. How this would be implemented together with the Kalman filtering mechanism is another matter of substance for general modeling purposes. When the full unified model would best be employed and how variable the outcome of this model selection is remains open to investigation through performance analyses. Regardless of the result or of specifics such as the form of information criterion used to perform the balancing, the principles used here fundamentally show that this functionality can come at a low cost.

An overlying possibility beyond the parameter analysis here lies in the observation update process. The motivation for such a process would include replacing the elevation mask with a more adaptable model selection dynamic by allowing added observations to be filtered through the BIC— Signals traveling through the atmosphere at lower elevations are subject to much higher variance. To address this, positioning algorithms can set a threshold (elevation mask) below which available satellite data will not be included. While this cuts out noise, it has the downside of weakening the geometry of the observations to some extent. A noisy but particularly well-positioned observation for the current overall geometry of observations may be blocked. This leaves some flexibility to be desired; the alternative would be a model selection process to determine which pseudo-observations to include on the basis of how much their inclusion affects the expected accuracy.

This type of model selection would require a recursive algorithm for updates to the set of observations $|S|$. The BIC would then be balancing the information entropy to weigh the benefits of improved linear independence of measurements while avoiding excess noise. Combined with the parameter update, an even greater amount of adaptibility becomes possible. Other parameters such as Ionospheric delay should also be possible to model similarly under the parameter update regime when under the restrictions of a single-

frequency receiver. Whether this would be necessitated and external models eschewed is one avenue of investigation, as unlike ISB, the Ionospheric delay is much easier to model directly (such as in[12]) without resorting to an empirical approach. This would only require another term in the linearized equation be determined for the model definition; otherwise the parameter update method is the same.

A   Appendix A

The following is an example PPP determination using eight code observations and no phase observations.

The initial approximation of the parameters:

$$
u_0 = \begin{bmatrix} -2296700.000 \\ -4472150.000 \\ 3915300.000 \\ 0 \end{bmatrix}
$$

The satellite positions (ECEF) at time of signal reception:

$$
\begin{bmatrix}
19262441.610 & -15434806.301 & 9733716.410 \\
-19490694.737 & 7415047.465 & 16248527.367 \\
-21033023.123 & -7470505.178 & 14254393.975 \\
7330621.079 & -23150466.670 & 10802063.143 \\
-16802097.343 & -2521269.811 & 20401432.453 \\
-17025373.271 & -2502338.714 & 20341182.610 \\
-25104512.100 & -8135496.048 & -4487195.511 \\
1240026.190 & -14854274.683 & 22164534.763
\end{bmatrix}
$$

The satellite velocities (ECEF) at time of signal reception:

$$
\begin{bmatrix}
-525.646 & 1147.819 & 2857.291 \\
959.956 & -1890.220 & 2001.685 \\
-915.741 & -1684.591 & -2144.544 \\
1084.705 & -881.603 & -2679.434 \\
-975.461 & -2405.9228 & -1098.457 \\
1580.145 & -2001.787 & 1134.642 \\
596.609 & -134.198 & -3082.001 \\
2726.169 & 462.537 & 145.213
\end{bmatrix}
$$

The pseudorange data:

$$
\begin{bmatrix}
24867111.479 \\
24252348.847 \\
21675579.500 \\
22070388.190 \\
21991390.108 \\
22267845.875 \\
24648421.852 \\
21234700.060
\end{bmatrix}
$$

Using the approximate time of flight given by the pseudorange, and assuming constant velocity of the satellite over that time, it is possible to determine the position of the satellite at time of transmission. This must also be translated into a non-inertial frame of reference to obtain the line-of-sight vector.

Range vectors; satellite positions (ECI) at time of transmission, and their magnitudes:

$$
\begin{bmatrix}
19262391.833 & -15435018.100 & 9733479.317 \\
-19490728.677 & 7415315.555 & 16248365.317 \\
-21032996.382 & -7470273.204 & 14254548.551 \\
7330416.548 & -23150441.071 & 10802260.782 \\
-16802039.130 & -2521002.791 & 200401513.229 \\
-17025503.500 & -2502099.084 & 20341098.778 \\
-25104609.660 & -8135334.944 & -4486942.809 \\
1239755.645 & -14854313.954 & 22164524.450
\end{bmatrix}
$$

Range vector magnitudes:

$$
\begin{bmatrix}
24866980.420 \\
24252361.051 \\
21675567.837 \\
22070272.419 \\
21991360.775 \\
22267811.087 \\
24642478.479 \\
21234574.647
\end{bmatrix}
$$

Residuals for initial approximation:

$$r_0 = \begin{bmatrix} 131.059 \\ -12.204 \\ 11.663 \\ 115.771 \\ 29.333 \\ 34.788 \\ -65.624 \\ 125.413 \end{bmatrix}$$

Matrix of partials:

$$A = \begin{bmatrix} -0.867 & 0.441 & -0.234 & 1 \\ 0.708 & -0.490 & -0.508 & 1 \\ 0.867 & 0.139 & -0.478 & 1 \\ -0.435 & 0.845 & -0.311 & 1 \\ 0.658 & -0.089 & -0.748 & 1 \\ 0.665 & -0.089 & -0.742 & 1 \\ 0.928 & 0.149 & 0.342 & 1 \\ -0.166 & 0.488 & -0.857 & 1 \end{bmatrix}$$

$$\hat{r} = r_0 - A(\hat{u} - u_0)$$

$$\hat{u} = u_0 + (A^T A)^{-1} A^T r_0 = \begin{bmatrix} -2296771.950 \\ -4472101.663 \\ 3915219.941 \\ 23.978 \end{bmatrix}$$

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B Appendix B

The following is a simplified example PPP determination using model selection on the basis of ISB paramaterization.

Take the above data and suppose the observations 5-7 originate from GNSS 2 and observation 8 originates from GNSS 3.

For the four-observable model, $\hat{\rho}$ is obtained by finding the upper trianglar matrix B_4^4 as defined in Section (**4.1**).

$$\hat{\rho} = 115.24838$$

$$BIC = 8log(\hat{\rho}/8) + 4log(8) = 29.65895$$

Now derive the WSSR for a five-observable model which includes a column in A for parameter ISB_2. Going through the process described in Section (**4.1**):

$$\Delta\hat{\rho} = -7.51058$$

$$\hat{\rho} = 107.7378$$

$$BIC = 31.19928$$

The BIC value increased, therefore the parameter ISB_2 should be modelled externally. Now repeat the process for ISB_3:

$$\Delta\hat{\rho} = -14.29296$$

$$\hat{\rho} = 100.9554$$

$$BIC = 30.67911$$

The BIC value increased with this candidate model as well. The algorithm concludes by performing the back-substitution for the 4-observable model, giving the same solution as in Appendix **A** (in this instance). Note that the external modelling assumption that ISB_2 and ISB_3 are zero is implicit to this simplified example. Using constant values closer to the ISB as external factors, as the suggested external models do, will also affect r_0 (and thus the BIC values) because r_0 depends on the range vectors.

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